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# SUMS INVOLVING EXTENDED GIBONACCI POLYNOMIALS REVISITED

**Abstract:** We explore the Jacobsthal versions of four sums involving gibonacci polynomial squares.

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## 1. Introduction

*Extended gibonacci polynomials* $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where $x$ is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the $n$th *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $_{zn}(x) = l_n(x)$, the $n$th *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the $n$th Fibonacci number; and $l_n(1) = L_n$, the $n$th Lucas number [1, 3].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the $n$th *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the $n$th *Jacobsthal-Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the $n$th Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$ [2, 3].

Gibonacci and Jacobsthal polynomials are linked by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [2, 3].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so $z_n$ will mean $z_n(x)$. In addition, we let $g_n = f_n$ or $l_n$, $c_n = J_n$ or $j_n$, $\Delta = \sqrt{x^2 + 4}$, $2\alpha = x + \Delta$, $E = \sqrt{x^2 + 1}$, $\gamma = x + E$ and $D = \sqrt{4x + 1}$, where $c_n = c_n(x)$.

**2. Gibonacci Sums**

We established the following four results in [4]:

**Theorem 1:** *Let $k$ be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k} x}{l_{2n+2k+1} + (-1)^{n+k} x} = \frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}. \tag{1}$$

**Theorem 2:** *Let $k$ be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k+1} x}{l_{2n+2k+1} - (-1)^{n+k} x} = \frac{1}{\Delta^2}\left(\frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}\right). \tag{2}$$

**Theorem 3:** *Let $k$ be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k} x f_{2n+2k+2} + x^2}{\left[l_{2n+2k+1} + (-1)^{n+k} x\right]^2} = \frac{\alpha^2}{\Delta^2} - \frac{f_{k+2}^2}{l_{k+1}^2}. \tag{3}$$

**Theorem 4:** *Let $k$ be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k+1} x f_{2n+2k+2} + x^2}{\left[ l_{2n+2k+1} - (-1)^{n+k} x \right]^2} = \frac{1}{\Delta^4} \left( \Delta^2 \alpha^2 - \frac{l_{k+2}^2}{f_{k+1}^2} \right). \tag{4}$$

Next we explore the Jacobsthal implications of these theorems.

## 3. Jacobsthal Consequences

Using the Jacobsthal-gibonacci relationships in Section 1, we will now find the Jacobsthal versions of equations (1) – (4). In the interest of brevity and clarity, we let $A$ denote the fractional expression on left-hand side of the given equation and $B$ its right-hand side, and LHS and RHS those of the desired Jacobsthal equation, respectively.

**3.1 Jacobsthal Version of Equation** (1)**: Proof:** Let $A = \dfrac{(-1)^{n+k} x}{l_{2n+k+1} + (-1)^{n+k} x}$.

Replacing $x$ with $1/\sqrt{x}$, and multiplying the numerator and denominator of the resulting expression with $x^{n+k}$, we get

$$A = \frac{(-1)^{n+k}}{\sqrt{x}\, l_{2n+2k+1} + (-1)^{n+k}}$$

$$= \frac{(-x)^{n+k}}{x^{(2n+2k+1)/2} l_{2n+2k+1} + (-x)^{n+k}}$$

$$= \frac{(-x)^{n+k}}{j_{2n+2k+1} + (-x)^{n+k}};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{(-x)^{n+k}}{j_{2n+2k+1} + (-x)^{n+k}}, \tag{5}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next, we let $B = \dfrac{\alpha}{\Delta} - \dfrac{f_{k+2}}{l_{k+1}}$. Replacing $x$ with $1/\sqrt{x}$, then multiply each numerator and denominator of the resulting expression with $x^{(k+1)/2}$. This yield

$$B = \frac{D+1}{2D} - \frac{x^{(k+1)/2} f_{k+2}}{x^{(k+1)/2} l_{k+1}}:$$

$$\text{RHS} = \frac{D+1}{2D} - \frac{J_{k+2}}{j_{k+1}}.$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

This, combined with equation (5), yields the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{(-x)^{n+k}}{j_{2n+2k+1} + (-x)^{n+k}} = \frac{D+1}{2D} - \frac{J_{k+2}}{j_{k+1}}. \tag{6}$$

where $c_n = c_n(x)$. □

It then follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k}}{L_{2n+2k+1} + (-1)^{n+k}} = \frac{5+\sqrt{5}}{10} - \frac{F_{k+2}}{L_{k+1}} \text{ [4]};$$

$$\sum_{n=1}^{\infty} \frac{(-2)^{n+k}}{j_{2n+2k+1} + (-2)^{n+k}} = \frac{2}{3} - \frac{J_{k+2}}{j_{k+1}}.$$

Next we find the Jacobsthal consequence of equation (2).

**3.2 Jacobsthal Version of Equation** (2)**: Proof:** We have $A = \dfrac{(-1)^{n+k+1} x}{l_{2n+2k+1} - (-1)^{n+k} x}$. Replace $x$ with $1/\sqrt{x}$, and then multiply the numerator and denominator of the resulting expression with $x^{n+k}$.

We then get

$$A = \frac{(-1)^{n+k+1}}{\sqrt{x}\, l_{2n+2k+1} - (-1)^{n+k}}$$

$$= \frac{-(-x)^{n+k}}{x^{(2n+2k+1)/2} l_{2n+2k+1} - (-x)^{n+k}}$$

$$= \frac{-(-x)^{n+k}}{j_{2n+2k+1} - (-x)^{n+k}};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{(-x)^{n+k}}{j_{2n+2k+1} - (-x)^{n+k}}, \tag{7}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next we let $B = \Delta\alpha - \dfrac{l_{k+2}}{f_{k+1}}$. Replacing $x$ with $1/\sqrt{x}$, and then multiplying each numerator and denominator of the resulting expression with $x^{(n+k)/2}$, yields

$$B = \frac{x}{D^2}\left[\frac{(D+1)D}{2x} - \frac{l_{k+2}}{f_{k+1}}\right]$$

$$= \frac{1}{D^2}\left[\frac{(D+1)D}{2} - \frac{x^{(k+2)/2} l_{k+2}}{x^{k/2} f_{k+1}}\right];$$

$$\text{RHS} = \frac{1}{D^2}\left[\frac{(D+1)D}{2} - \frac{j_{k+2}}{J_{k+1}}\right],$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Combined with equation (7), this yields the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{-(-x)^{n+k}}{j_{2n+2k+1} - (-x)^{n+k}} = \frac{1}{D^2}\left[\frac{(D+1)D}{2} - \frac{j_{k+2}}{J_{k+1}}\right], \tag{8}$$

where $c_n = c_n(x)$. □

In particular, this yields

$$\sum_{n=1}^{\infty} \frac{-(-1)^{n+k}}{L_{2n+2k+1} - (-1)^{n+k}} = \frac{5+\sqrt{5}}{10} - \frac{L_{k+2}}{5F_{k+1}} \text{ [4]};$$

$$\sum_{n=1}^{\infty} \frac{-(-2)^{n+k}}{j_{2n+2k+1} - (-2)^{n+k}} = \frac{2}{3} - \frac{j_{k+2}}{9J_{k+1}}.$$

**3.3 Jacobsthal Version of Equation** (3)**: Proof:** Let

$A = \dfrac{2(-1)^{n+k} x f_{2n+k+2} + x^2}{\left[l_{2n+k+1} + (-1)^{n+k} x\right]^2}$. Replacing $x$ with $1/\sqrt{x}$, and multiplying the numerator and denominator of the resulting expression with $x^{2n+2k+1}$, we get

$$A = \frac{2(-1)^{n+k}\dfrac{1}{\sqrt{x}} f_{2n+k+2} + \dfrac{1}{x}}{\left[l_{2n+2k+1} + (-1)^{n+k}\dfrac{1}{\sqrt{x}}\right]^2}$$

$$= \frac{2(-x)^{n+k}\left[x^{(2n+2k+1)/2} f_{2n+k+2}\right] + x^{2n+2k}}{\left[x^{(2n+2k+1)/2} l_{2n+2k+1} + (-x)^{n+k}\right]^2}$$

$$= \frac{2(-x)^{n+k} J_{2n+k+2} + x^{2n+2k}}{\left[j_{2n+2k+1} + (-x)^{n+k}\right]^2};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{2(-x)^{n+k} J_{2n+k+2} + x^{2n+2k}}{\left[ j_{2n+2k+1} + (-x)^{n+k} \right]^2}, \tag{9}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Now let $B = \dfrac{\alpha^2}{\Delta^2} - \dfrac{f_{k+2}^2}{l_{k+1}^2}$. Replace $x$ with $1/\sqrt{x}$, and multiply each numerator and denominator of the resulting expression with $x^{k+1}$. This yields

$$B = \frac{(D+1)^2}{4D^2} - \frac{\left[ x^{(k+1)/2} f_{k+2} \right]^2}{\left[ x^{(k+1)/2} l_{k+1} \right]^2};$$

$$\text{RHS} = \frac{(D+1)^2}{4D^2} - \frac{J_{k+2}^2}{J_{k+1}^2},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

This, coupled with equation (9), yields the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{2(-x)^{n+k} J_{2n+k+2} + x^{2n+2k}}{\left[ j_{2n+2k+1} + (-1)^{n+k} \right]^2} = \frac{(D+1)^2}{4D^2} - \frac{J_{k+2}^2}{j_{k+1}^2}, \tag{10}$$

where $c_n = c_n(x)$. □

In particular, this yields

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k} F_{2n+k+2} + 1}{\left[ L_{2n+2k+1} + (-1)^{n+k} \right]^2} = \frac{3+\sqrt{5}}{10} - \frac{F_{k+2}^2}{L_{k+1}^2} \text{ [4]};$$

$$\sum_{n=1}^{\infty} \frac{2(-2)^{n+k} J_{2n+k+2} + 4^{n+k}}{\left[ j_{2n+2k+1} + (-2)^{n+k} \right]^2} = \frac{4}{9} - \frac{J_{k+2}^2}{j_{k+1}^2}.$$

Next we find the Jacobsthal consequence of Theorem 4.

**3.4 Jacobsthal Version of Equation (4): Proof:** We have. $A = \dfrac{2(-1)^{n+k+1} x f_{2(n+k)+2} + x^2}{\left[ l_{2(n+k)+1}^2 - (-1)^{n+k} x \right]^2}$ Replace $x$ with $1/\sqrt{x}$, and multiply the numerator and denominator of the resulting expression with $x^{2n+2k+1}$. We then get

$$A = \frac{2(-1)^{n+k+1} \dfrac{1}{\sqrt{x}} f_{2n+k+2} + \dfrac{1}{x}}{\left[ l_{2n+2k+1} - (-1)^{n+k} \dfrac{1}{\sqrt{x}} \right]^2}$$

$$= \frac{-2(-x)^{n+k} \left[ x^{(2n+2k+1)/2} f_{2n+k+2} \right] + x^{2n+2k}}{\left[ x^{(2n+2k+1)/2} l_{2n+2k+1} - (-x)^{n+k} \right]^2}$$

$$= \frac{-2(-x)^{n+k} J_{2n+k+2} + x^{2n+2k}}{\left[ j_{2n+2k+1} - (-x)^{n+k} \right]^2};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{-2(-x)^{n+k} J_{2n+k+2} + x^{2n+2k}}{\left[ j_{2n+2k+1} - (-x)^{n+k} \right]^2}, \tag{11}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next we let $B = \dfrac{1}{\Delta^4} \left( \Delta^2 \alpha^2 - \dfrac{l_{k+2}^2}{f_{k+1}^2} \right)$. Replacing $x$ with $1/\sqrt{x}$, and then multiplying each numerator and denominator of the resulting expression with $x^{k+2}$ yields

$$B = \frac{x^2}{D^4}\left\{\frac{D^2(D+1)^2}{4x^2} - \frac{[x^{(k+2)/2}l_{k+2}]^2}{x^2[x^{k/2}f_{k+1}]^2}\right\};$$

$$\text{RHS} = \frac{(D+1)^2}{4D^2} - \frac{j_{k+2}^2}{D^4 J_{k+1}^2},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Combining this with equation (11) yields the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{-2(-x)^{n+k}J_{2n+k+2} + x^{2n+2k}}{\left[j_{2n+2k+1} - (-1)^{n+k}\right]^2} = \frac{(D+1)^2}{4D^2} - \frac{j_{k+2}^2}{D^4 J_{k+1}^2}, \tag{12}$$

where $c_n = c_n(x)$. □

It follows from this equation that

$$\sum_{n=1}^{\infty} \frac{-2(-1)^{n+k}F_{2n+k+2} + 1}{\left[L_{2n+2k+1} - (-1)^{n+k}\right]^2} = \frac{3+\sqrt{5}}{10} - \frac{L_{k+2}^2}{25F_{k+1}^2}\,[4];$$

$$\sum_{n=1}^{\infty} \frac{(-2)^{n+k+1}J_{2n+k+2} + 4^{n+k}}{\left[j_{2n+2k+1} - (-2)^{n+k}\right]^2} = \frac{4}{9} - \frac{J_{k+2}^2}{j_{k+1}^2}.$$

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