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SOME GENERAL FRACTIONAL-ORDER KINETIC EQUATIONS ASSOCIATED WITH THE RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE

Abstract: In the current literature, various operators of fractional calculus (that is, fractional-order integrals and fractional-order derivatives) have been and continue to be successfully applied in the modeling and analysis of a remarkably large spectrum of applied scientific and real-world problems in the mathematical, physical, biological, engineering and statistical sciences, and indeed also in other scientific disciplines. In this article, we investigate a general family of fractional-order kinetic equations involving the Riemann-Liouville fractional derivative, which also includes a remarkably general class of functions as a part of the non-homogeneous term. The main results, which we have derived in this article, are capable of yielding solutions of a significantly large number of simpler fractional-order kinetic equations.

Keywords and Phrases: Riemann-Liouville and Related Fractional Derivative Operators, Riemann-Liouville Fractional Derivative Operator, Hypergeometric Functions, Special (or Higher Transcendental) Functions, Fox-Wright Hypergeometric Function, Mittag-Leffler Type Functions, General Fox-Wright Function, Zeta and Related Functions, Lerch Transcendent (or Hurwitz-Lerch Zeta function).

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1. Introduction and Motivation

In recent years, various operators of fractional calculus (that is, operators of integrals and derivatives of any real or complex order) has received considerable attention because mainly of their demonstrated applications in the modeling and analysis of applied problems and real-world situations occurring in numerous seemingly diverse and widespread fields of science and engineering. These operators do indeed provide several potentially useful tools and techniques for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables (see, for details, [8], [9], [11] and [12]; see also [4], [6] and [26]).

Traditionally (and by far the most commonly used), the operators of fractional-order integration and fractional-order differentiation are defined by means of the right-sided Riemann-Liouville fractional integral operator ${}^{\mathrm{RL}}I_{a+}^{\mu}$ and the left-sided Riemann-Liouville fractional integral operator ${}^{\mathrm{RL}}I_{a-}^{\mu}$, and the corresponding Riemann-Liouville fractional derivative operators ${}^{\mathrm{RL}}D_{a+}^{\mu}$ and ${}^{\mathrm{RL}}D_{a-}^{\mu}$, as follows (see, for example, [3, Chapter 13], [8, pp. 69-70] and [13]):

$$\left({}^{\mathrm{RL}}I_{a+}^{\mu}f\right)(x) = \frac{1}{\Gamma(\mu)}\int_{a}^{x}(x-t)^{\mu-1}f(t)dt \qquad (x > a; \Re(\mu) > 0), \tag{1}$$

$$\left({}^{\mathrm{RL}}I_{a-}^{\mu}f\right)(x) = \frac{1}{\Gamma(\mu)}\int_{x}^{a}(t-x)^{\mu-1}f(t)dt \qquad (x < a; \Re(\mu) > 0) \tag{2}$$

and

$$\left({}^{RL}D_{a\pm}^{\mu}f\right)(x) = \left(\pm\frac{d}{dx}\right)^{n}\left(I_{a\pm}^{n-\mu}f\right)(x) \qquad \left(\Re(\mu) \geqq 0; n = \left[\Re(\mu)\right] + 1\right). \tag{3}$$

Here, and in what follows, the function f is locally integrable, $\Re(\mu)$ denotes the real part of the complex number $\mu \in \mathbb{C}$ and $\left[\Re(\mu)\right]$ means the greatest integer in $\Re(\mu)$, and $\Gamma(z)$ denotes the classical (Euler's) Gamma function defined by

$$\Gamma(z) := \begin{cases} \int_0^\infty e^{-t} t^{z-1} dt & (\Re(z) > 0) \\ \dfrac{\Gamma(z+n)}{\prod_{j=0}^{n-1} (z+j)} & (z \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ n \in \mathbb{N}), \end{cases} \tag{4}$$

which happens to be one of the most fundamental and the most useful special functions of mathematical analysis, $\mathbb{N}$ and $\mathbb{Z}_0^-$ being the sets of *positive* and *non-positive* integers, respectively.

Our main object in this article is investigate some general families of fractional-order kinetic equations involving the Riemann-Liouville right-sided fractional derivative operator ${}^{\mathrm{RL}}(D_{0+}^{\mu} f)(x)$, which is given (for convenience) by (3) for $a = 0$, as well as including a remarkably general class of functions as a part of the non-homogeneous term. Our main results (Theorem 1, Theorem 2 and Theorem 3 in this article) are capable of yielding solutions of a significantly large number of simpler fractional-order kinetic equations.

2. Definitions and Preliminaries

First of all, it is easily observed that most (if not all) of the various claimed one-variable and multi-parameter (or multi-index) "generalizations" of the familiar Mittag-Leffler function $E_\alpha(z)$ and its two-parameter extension $E_{\alpha,\beta}(z)$, which are defined as follows:

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \qquad \text{and} \qquad E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{5}$$

are no more than fairly obvious special or limit cases of the substantially much more general Fox-Wright function ${}_p\Psi_q\ (p, q \in \mathbb{N}_0)$ or ${}_p\Psi_q^*\ (p, q \in \mathbb{N}_0)$, which happens to be the Fox-Wright generalization of the relatively more familiar hypergeometric function ${}_pF_q\ (p, q \in N_0)$, with p numerator parameters $a_1, \cdots, a_p$ and q denominator parameters $b_1, \cdots, b_q$ such that

$$a_j \in \mathbb{C}\ (j = 1, \cdots,\ p) \qquad \text{and} \qquad b_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad (j = 1, \cdots, q)\ .$$

These general Fox-Wright functions ${}_p\Psi_q$ $(p, q \in \mathbb{N}_0)$ and ${}_p\Psi_q^*$ $(p, q \in \mathbb{N}_0)$ are indeed defined by (see, for details, [2, p. 183] and [25, p. 21]; see also [7, p. 65], [8, p. 56] and [14])

$$
{}_p\Psi_q^* \left[\begin{matrix} (a_1, A_1), \cdots, (a_p, A_p); \\ \\ (b_1, B_1), \cdots, (b_q, B_q); \end{matrix} z \right]
$$

$$
:= \sum_{n=0}^{\infty} \frac{(a_1)_{A_1 n} \cdots (a_p)_{A_p n}}{(b_1)_{B_1 n} \cdots (b_p)_{B_q n}} \frac{z^n}{n!}
$$

$$
= \frac{\Gamma(b_1) \cdots \Gamma(b_q)}{\Gamma(a_1) \cdots \Gamma(a_p)} \, {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \cdots, (a_p, A_p); \\ \\ (b_1, B_1), \cdots, (b_q, B_q); \end{matrix} z \right] \tag{6}
$$

$$
\left(\Re(A_j) > 0 \ (j = 1, \cdots, p); \Re(B_j) > 0 \ (j = 1, \cdots, q); 1 + \Re \left(\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \right) \geqq 0 \right),
$$

where, and in what follows, $(\lambda)_\nu$ denotes the general Pochhammer symbol or the *shifted* factorial, since

$$
(1)_n = n! \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \ \mathbb{N} := \{1, 2, 3, \cdots\}),
$$

which is defined (for $\lambda, \nu \in \mathbb{C}$ and in terms of the above-defined familiar Gamma function in the equation (4)) by

$$
(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases} \tag{7}
$$

it being assumed *conventionally* that $(0)_0 := 1$ and understood *tacitly* that the Γ-quotient exists. Here we suppose, in general, that

$$a_j, A_j \in \mathbb{C}\ (j = 1, \cdots, p) \quad \text{and} \quad b_j, B_j \in \mathbb{C}\ (j = 1, \cdots, q)$$

and that the equality in the convergence condition in the definition (6) holds true only for suitably bounded values of $|z|$ given by

$$|z| < \nabla := \left(\prod_{j=1}^{p} A_j^{-A_j} \right) \cdot \left(\prod_{j=1}^{q} B_j^{B_j} \right).$$

We remark in passing that the above-mentioned generalized hypergeoemtric function ${}_pF_q\,(p, q \in \mathbb{N}_0)$, with p numerator parameters $a_1, \cdots, a_p$ and q denominator parameters $b_1, \cdots, b_q$, is a widely- and extensively-investigated and potentially useful special case of the general Fox-Wright function ${}_p\Psi_q\,(p, q \in \mathbb{N}_0)$ when

$$A_j = 1\ (j = 1, \cdots, p) \quad \text{and} \quad B_j = 1\ (j = 1, \cdots, q).$$

We now turn to a series of monumental works (see, for example, [28], [29] and [30]) by Sir Edward Maitland Wright (1906-2005), with whom I had the privilege to meet and discuss researches emerging from his publications on hypergeometric and related functions during my visit to the University of Aberdeen in the year 1976, introduced and systematically studied the asymptotic expansion of the following Taylor-Maclaurin series (see [28, p. 424]):

$$\mathfrak{E}_{\alpha,\beta}(\phi; z) := \sum_{n=0}^{\infty} \frac{\phi(n)}{\Gamma(\alpha n + \beta)}\, z^n \qquad (\alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0), \tag{8}$$

where $\phi(t)$ is a function satisfying suitable conditions. In fact, it was my proud privilege to have also met many times and discussed mathematical researches, especially on various families of higher transcendental functions and related topics, with my Canadian colleague, Charles Fox (1897-1977) of birth and education in England, both at McGill University and Sir George Williams University (*now* Concordia University) in Montréal, mainly during the 1970s (see, for details, [14]).

The above-cited contributions by Wright were motivated essentially by the earlier developments reported for simpler cases by Magnus Gustaf (Gösta) Mittag-Leffler (1846-1927) in 1905, Anders Wiman (1865-1959) in 1905, Ernest William Barnes (1874-1953) in 1906, Godfrey Harold Hardy (1877-1947) in 1905,

George Neville Watson (1886-1965) in 1913, Charles Fox (1897-1977) in 1928, and other authors. In particular, the aforementioned work [1] by *Bishop* Ernest William Barnes (1874-1953) of the Church of England in Birmingham considered the asymptotic expansions of functions in the class defined below:

$$E_{\alpha,\beta}^{(\kappa)}(s;\, z) := \sum_{n=0}^{\infty} \frac{z^n}{(n+\kappa)^s\ \Gamma(\alpha n+\beta)} \qquad (\alpha,\, \beta \in \mathbb{C};\, \Re(\alpha) > 0) \tag{9}$$

for suitably-restricted parameters κ and s. Clearly, we have the following relationship:

$$\lim_{\alpha\to\infty} \left\{ E_{\alpha,\beta}^{(\kappa)}(s;\, z) \right\} = \frac{1}{\Gamma(\beta)}\, \Phi(z,\, s,\, \kappa)$$

with the classical Lerch transcendent (or the Hurwitz-Lerch zeta function) $\Phi(z,\, s,\, \kappa)$ defined by (see, for example, [2, p. 27, Eq. 1.11 (1)]; see also [23] and [24])

$$\Phi(z,\, s,\, \kappa) := \sum_{n=0}^{\infty} \frac{z^n}{(n+\kappa)^s} \tag{10}$$

$$(k \in \mathbb{C} \setminus \mathbb{Z}_0^-;\, s \in \mathbb{C} \text{ when } |z| < 1;\, \Re(s) > 1 \text{ when } |z| = 1).$$

The Hurwitz-Lerch zeta function $\Phi(z,\, s,\, \kappa)$ defined by (10) contains, as its *special* cases, not only the Riemann zeta function $\zeta(s)$ and the Hurwitz (or generalized) zeta function $\zeta(s,\, \kappa)$:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \Phi(1,\, s,\, 1) \quad \text{and} \quad \zeta(s,\, \kappa) := \sum_{n=0}^{\infty} \frac{1}{(n+\kappa)^s} = \Phi(1,\, s,\, \kappa) \tag{11}$$

and the Lerch zeta function $\ell_s(\xi)$ defined by (see, for details, [2, Chapter I] and [23, Chapter 2])

$$\ell_s(\xi) := \sum_{n=1}^{\infty} \frac{e^{2n\pi \mathrm{i}\xi}}{n^s} = e^{2\pi \mathrm{i}\xi}\, \Phi(e^{2\pi \mathrm{i}\xi},\, s,\, 1) \tag{12}$$

$$(\mathrm{i} = \sqrt{-1};\, \xi \in \mathbb{R};\, \Re(s) > 1),$$

but also such other important functions of *Analytic Number Theory* as the Polylogarithmic function (or *de Jonquière's function*) $\mathrm{Li}_s(z)$:

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\,\Phi(z, s, 1) \tag{13}$$

$$(s \in \mathbb{C} \quad \text{when} \quad |z| < 1;\ \Re(s) > 1 \quad \text{when} \quad |z| = 1)$$

and the Lipschitz-Lerch zeta function (see [23, p. 122, Eq. 2.5 (11)]):

$$\phi(\xi, \kappa, s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi \mathrm{i}\xi}}{(n+\kappa)^s} = \Phi\left(e^{2\pi \mathrm{i}\xi}, s, \kappa\right) =: L(\xi, s, \kappa) \tag{14}$$

$$(\kappa \in \mathbb{C} \setminus \mathbb{Z}_0^-; \Re(s) > 0 \quad \text{when} \quad \xi \in \mathbb{R} \setminus \mathbb{Z}; \Re(s) > 1 \quad \text{when} \quad \xi \in \mathbb{Z}),$$

which was first studied by Rudolf Lipschitz (1832-1903) and Matyáš Lerch (1860-1922) in connection with Dirichlet's famous theorem on primes in arithmetic progressions (see, for details, [17] and [18]).

A natural unification and generalization of the Fox-Wright function ${}_p\Psi_q^*$ defined by (6) as well as the Hurwitz-Lerch zeta function $\Phi(z, s, \kappa)$ defined by (10) was indeed accomplished by introducing essentially arbitrary numbers of numerator and denominator parameters in the definition (10). For this purpose, in addition to the symbol ∇^* defined by

$$\nabla^* := \left(\prod_{j=1}^{p} \rho_j^{-\rho_j}\right) \cdot \left(\prod_{j=1}^{q} \sigma_j^{-\sigma_j}\right), \tag{15}$$

the following notations will be employed:

$$\Delta := \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j \quad \text{and} \quad \Xi := s + \sum_{j=1}^{q} \mu_j - \sum_{j=1}^{p} \lambda_j + \frac{p-q}{2}. \tag{16}$$

Then the extended Hurwitz-Lerch zeta function

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}(z,s,\kappa)$$

is defined by [27, p. 503, Equation (6.2)] (see also [15] and [24])

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}(z,s,\kappa) := \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{n!\cdot\prod_{j=1}^{q}(\mu_j)_{n\sigma_j}} \frac{z^n}{(n+\kappa)^s} \tag{17}$$

$$\left(p,q \in \mathbb{N}_0;\ \lambda_j \in \mathbb{C}\ (j=1,\cdots,p);\ \kappa,\mu_j \in \mathbb{C}\setminus\mathbb{Z}_0^-(j=1,\cdots,q);\right.$$

$$\rho_j,\sigma_k \in \mathbb{R}^+(j=1,\cdots,p;\ k=1,\cdots,q);\ \Delta > -1 \quad \text{when} \quad s,z \in \mathbb{C};$$

$$\Delta = -1 \quad \text{and} \quad s \in \mathbb{C} \quad \text{when} \quad |z| < \nabla^*;$$

$$\left.\Delta = -1 \ \text{and} \ \Re(\Xi) > \frac{1}{2} \ \text{when} \ |z| = \nabla^*\right).$$

For an interesting and potentially useful family of λ-generalized Hurwitz-Lerch zeta functions, which *further* extend the multi-parameter Hurwitz-Lerch zeta function

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}(z,s,\kappa)$$

defined by (17), was introduced and investigated systematically in a recent paper by Srivastava [16], who also discussed their potential application in Number Theory by appropriately constructing a presumably new continuous analogue of Lippert's Hurwitz measure and also considered some other statistical applications of these families of the λ-generalized Hurwitz-Lerch zeta functions in probability distribution theory (see also the references to several related earlier works cited by Srivastava [16]).

We now introduce some general families of the Riemann-Liouville type fractional integrals and fractional derivatives by making use of the following interesting unification of the definitions in (8) and (17) for a suitably-restricted function $\varphi(\tau)$ given by

$$\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa) := \sum_{n=0}^{\infty} \frac{\varphi(n)}{(n+\kappa)^s\ \Gamma(\alpha n + \beta)}\, z^n \qquad (\alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0), \tag{18}$$

where the parameters α, β, s and κ are appropriately constrained as above. The resulting general right-sided fractional integral operator $\mathcal{I}_{a+}^{\mu}(\varphi; z, s, \kappa, \nu)$ and the general left-sided fractional integral operator $\mathcal{I}_{a-}^{\mu}(\varphi; z, s, \kappa, \nu)$, and the corresponding fractional derivative operators $\mathcal{D}_{a+}^{\mu}(\varphi; z, s, \kappa, \nu)$ and $\mathcal{D}_{a-}^{\mu}(\varphi; z, s, \kappa, \nu)$, each of the Riemann-Liouville type, are defined by (see, for details, [20], [21] and [22])

$$\left(\mathcal{I}_{a+}^{\mu}(\varphi; z, s, \kappa, \nu) f\right)(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} \mathcal{E}_{\alpha,\beta}(\varphi; z(x-t)^{\nu}, s, \kappa)\, f(t)\, dt \tag{19}$$
$$(x > a; \Re(\mu) > 0),$$

$$\left(\mathcal{I}_{a-}^{\mu}(\varphi; z, s, \kappa, \nu) f\right)(x) = \frac{1}{\Gamma(\mu)} \int_x^a (t-x)^{\mu-1} \mathcal{E}_{\alpha,\beta}(\varphi; z(t-x)^{\nu}, s, \kappa)\, f(t)\, dt \tag{20}$$
$$(x < a; \Re(\mu) > 0)$$

and

$$\left(\mathcal{D}_{a\pm}^{\mu}(\varphi; z, s, \kappa, \nu) f\right)(x) = \left(\pm \frac{d}{dx}\right)^n \left(\mathcal{I}_{a\pm}^{n-\mu}(\varphi; z, s, \kappa, \nu) f\right)(x) \tag{21}$$
$$(\Re(\mu) \geqq 0;\ n = [\Re(\mu)] + 1),$$

where the function f is in the space $L(\mathfrak{a}, \mathfrak{b})$ of Lebesgue integrable functions on a finite closed interval $[\mathfrak{a}, \mathfrak{b}]$ $(\mathfrak{b} > \mathfrak{a})$ of the real line $\mathbb{R}$ given by

$$L(\mathfrak{a}, \mathfrak{b}) = \left\{ f : \|f\|_1 = \int_{\mathfrak{a}}^{\mathfrak{b}} |f(x)|\, dx < \infty \right\}, \tag{22}$$

it being *tacitly* assumed that, in situations such as those occurring in conjunction with the usages of the definitions in (19), (20) and (21), the point $\mathfrak{a}$ in all such function spaces as (for example) the function space $L(\mathfrak{a}, \mathfrak{b})$ coincides precisely with the *lower* terminal a in the integrals involved in the definitions (19), (20) and (21).

Next, in terms of the operator $\mathcal{L}$ of the Laplace transform given by

$$\mathcal{L}\{f(\tau) : \mathfrak{s}\} := \int_0^\infty e^{-s\tau} f(\tau)\, d\tau =: F(\mathfrak{s}) \qquad (\Re(\mathfrak{s}) > 0), \tag{23}$$

where the function $f(\tau)$ is so constrained that the integral exists, it is easily seen for the function $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$, defined above by (18), that

$$\mathcal{L}\{\tau^{\mu-1}\mathcal{E}_{\alpha,\beta}(\varphi; z\tau^{\nu}, s, \kappa) : \mathfrak{s}\} = \frac{1}{\mathfrak{s}^{\mu}} \sum_{k=0}^{\infty} \frac{\varphi(k)\Gamma(\nu k + \mu)}{(k+\kappa)^{s}\Gamma(\alpha k + \beta)} \left(\frac{z}{\mathfrak{s}^{\nu}}\right)^{k} \tag{24}$$

$$(\Re(\mathfrak{s}) > 0; \Re(\mu) > 0; \Re(\nu) > 0; \Re(\alpha) > 0),$$

provided that each member of (24) exists. Obviously, upon setting $\mu = \beta$ and $\nu = \alpha$, the Laplace transform formula (24) simplifies to the following form:

$$\mathcal{L}\{\tau^{\beta-1}\mathcal{E}_{\alpha,\beta}(\varphi; z\tau^{\alpha}, s, \kappa) : \mathfrak{s}\} = \frac{1}{\mathfrak{s}^{\mu}} \sum_{k=0}^{\infty} \frac{\varphi(k)}{(k+\kappa)^{s}} \left(\frac{z}{\mathfrak{s}^{\alpha}}\right)^{k} \tag{25}$$

$$(\Re(\mathfrak{s}) > 0; \Re(\alpha) > 0; \Re(\beta) > 0).$$

In case we apply the following limit formula:

$$\mathfrak{E}_{\alpha,\beta}(\phi; z) = \lim_{s\to 0}\{\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)\} \mid_{\varphi\equiv\phi} . \tag{26}$$

or, alternatively, if we make use of the definitions in (8) and (23), we find for Wright's function $\mathfrak{E}_{\alpha,\beta}(\Phi; z)$ that

$$\mathcal{L}\{\tau^{\mu-1}\mathfrak{E}_{\alpha,\beta}(\phi; z\tau^{\nu}) : \mathfrak{s}\} = \frac{1}{\mathfrak{s}^{\mu}} \sum_{k=0}^{\infty} \frac{\phi(k)\Gamma(\nu k + \mu)}{\Gamma(\alpha k + \beta)} \left(\frac{z}{\mathfrak{s}^{\nu}}\right)^{k} \tag{27}$$

$$(\Re(\mathfrak{s}) > 0; \Re(\mu) > 0; \Re(\nu) > 0; \Re(\alpha) > 0),$$

which, in the special case when $\nu = \alpha$ and $\mu = \beta$, yields

$$\mathcal{L}\{\tau^{\beta-1}\mathfrak{E}_{\alpha,\beta}(\phi; z\tau^{\alpha}) : \mathfrak{s}\} = \frac{1}{\mathfrak{s}^{\beta}} \sum_{k=0}^{\infty} \phi(k) \left(\frac{z}{\mathfrak{s}^{\alpha}}\right)^{k} \tag{28}$$

$$\left(\Re(\mathfrak{s}) > 0; \Re(\alpha) > 0; \Re(\beta) > 0\right).$$

Moreover, in the case when the sequence $\{\varphi(n)\}_{n=0}^{\infty}$ is given by

$$\varphi(n) = \frac{\Gamma(\alpha n + \beta) \prod_{j=1}^{p} (\lambda_j)_{np_j}}{n\,! \cdot \prod_{j=1}^{q} (\mu_j)_{n\sigma_j}} \quad (n \in \mathbb{N}_0), \tag{29}$$

then the Laplace transformation formula (25) would yield the following result:

$$\mathcal{L}\left\{\tau^{\mu-1}\, \Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q}(z\tau^{\nu}, s, k) : \mathfrak{s}\right\}$$

$$= \frac{\Gamma(\mu)}{\mathfrak{s}^{\mu}} \Phi_{\mu,\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\nu,\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)} \left(\frac{z}{\mathfrak{s}^{\nu}}, s, k\right) \tag{30}$$

$$\left(\Re(\mathfrak{s}) > 0; \Re(\mu) > 0; \Re(\nu) > 0; \Re(\alpha) > 0\right)$$

for the extended Hurwitz-Lerch zeta function

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}(z, s, k)$$

defined by (17).

Finally, for the right-sided Riemann-Liouville fractional derivative operator $\mathcal{D}_{0+}^{\mu}$ of order μ in the definition (3), it is easily observed that (see, for example, [12, p. 105, Eq. (2.248)])

$$\mathcal{L}\{\left({}^{\mathrm{RL}}\mathcal{D}_{0+}^{\mu} f\right)(t) : \mathfrak{s}\} = \mathfrak{s}^{\mu} F(\mathfrak{s}) - \sum_{k=0}^{n-1} \mathfrak{s}^{k} \left({}^{RL}\mathcal{D}_{0+}^{\mu-k-1} f\right)(0+) \tag{31}$$

$$(n-1 < \Re(\mu) < n; n \in \mathbb{N})$$

or, equivalently, that (see, for example, [8, p. 84, Eq. (2.2.37)])

$$\mathcal{L}\{\left({}^{\mathrm{RL}}\mathcal{D}_{0+}^{\mu}f\right)(t):\mathfrak{s}\} = \mathfrak{s}^{\mu}F(\mathfrak{s}) - \sum_{k=0}^{n-1}\mathfrak{s}^{n-k-1}\,\frac{d^k}{dt^k}\left\{\left({}^{\mathrm{RL}}I_{0+}^{n-\mu}f\right)(t)\right\}\Bigg|_{t=0}$$

$$= \mathfrak{s}^{\mu}F(\mathfrak{s}) - \sum_{k=0}^{n-1}\mathfrak{s}^{k}\,\frac{d^{n-k-1}}{dt^{n-k-1}}\left\{\left({}^{\mathrm{RL}}I_{0+}^{n-\mu}f\right)(t)\right\}\Bigg|_{t=0} \tag{32}$$

$$(n-1 < \Re(\mu) < n;\, n \in \mathbb{N})$$

where, for convenience

$$\left({}^{\mathrm{RL}}\mathcal{D}_{0+}^{\mu-k-1}f\right)(0+) := \lim_{t\to 0+}\left\{\left({}^{\mathrm{RL}}\mathcal{D}_{0+}^{\mu-k-1}f\right)(t)\right\} =: \left({}^{\mathrm{RL}}\mathcal{D}_{0+}^{\mu-k-1}f\right)(t)\Big|_{t=0}$$

and

$$\frac{d^k}{dt^k}\left\{\left({}^{\mathrm{RL}}I_{0+}^{n-\mu}f\right)(t)\right\}\Bigg|_{t=0} := \lim_{t\to 0+}\frac{d^k}{dt^k}\left\{\left({}^{\mathrm{RL}}I_{0+}^{n-\mu}f\right)(t)\right\}$$

$$=: \frac{d^k}{dt^k}\left\{\left({}^{\mathrm{RL}}I_{0+}^{n-\mu}f\right)(0+)\right\} \qquad (k \in \{0,1,2,\cdots,n-1\}).$$

Indeed, for the *ordinary* derivative $f^{(n)}(t)$ of order $n \in \mathbb{N}_0$, it is known that

$$\mathcal{L}\{f^{(n)}(t):\mathfrak{s}\} = \mathfrak{s}^{n}F(\mathfrak{s}) - \sum_{k=0}^{n-1}\mathfrak{s}^{k}f^{(n-k-1)}(t)\Big|_{t=0} \qquad (n \in \mathbb{N}_0) \tag{33}$$

or, equivalently, that

$$\mathcal{L}\{f^{(n)}(t):\mathfrak{s}\} = \mathfrak{s}^{n}F(\mathfrak{s}) - \sum_{k=0}^{n-1}\mathfrak{s}^{n-k-1}\,f^{(k)}(0+) \quad (n \in \mathbb{N}_0), \tag{34}$$

where, as well as in all of such situations in this paper, an *empty* sum is to be interpreted as 0.

3. A General Family of Fractional-Order Kinetic Equations

For an arbitrary reaction, which is characterized by a time-dependent quantity $N = N(t)$, it is possible to calculate the rate of change $\frac{dN}{dt}$ to be a balance between the destruction rate $\mathfrak{d}$ and the production rate $\mathfrak{p}$ of N, that is,

$$\frac{dN}{dt} = -\mathfrak{d} + \mathfrak{p}.$$

By means of feedback or other interaction mechanism, the destruction and the production depend on the quantity N itself, that is,

$$\mathfrak{d} = \mathfrak{d}(N) \qquad \text{and} \qquad \mathfrak{p} = \mathfrak{p}(N).$$

Since the destruction or the production at a time t depends not only on $N(t)$, but also on the past history $N(\eta)$ $(\eta < t)$ of the variable N, such dependence is, in general, complicated. This may be formally represented by the following equation (see [5]):

$$\frac{dN}{dt} = -\mathfrak{d}\,(N_t) + \mathfrak{p}\,(N_t)\,, \tag{35}$$

where N_t denotes the function defined by

$$N_t(t^*) = N(t - t^*) \qquad (t^* > 0)\ .$$

Haubold and Mathai [5] studied a special case of the equation (35) in the following form:

$$\frac{dN_j}{dt} = -c_j N_j(t), \tag{36}$$

that is,

$$\frac{dN_j(t)}{N_j(t)} = -c_j\, dt, \tag{37}$$

with the initial condition that

$$N_j\,(t)\Big|_{t=0} = N_0,$$

is the number density of species j at time $t = 0$ and the constant $c_j > 0$. This is known as a standard kinetic equation. The solution of the equation (36) (*without* the subscript j) is readily seen to be given by

$$N_j(t) = N_0 e^{-c_j t}, \tag{38}$$

which, upon integration, yields the following alternative form of the solution of the equation (36) (*without* the subscript j):

$$N(t) - N_0 = c \cdot {}_0D_t^{-1}\{N(t)\}, \tag{39}$$

where ${}_0D_t^{-1}$ is the standard (ordinary) integral operator and c is a constant of integration.

The fractional-order generalization of the equation (39) is given as in the following form (see [5]):

$$N(t) - N_0 = c^{\nu}\left({}^{\mathrm{RL}}I_{0+}^{\nu}N\right)(t) \tag{40}$$

in terms of the familiar right-sided Riemann-Liouville fractional integral operator ${}^{\mathrm{RL}}I_{0+}^{\nu}$ of order ν defined, as in (1), by (see, for example, [8])

$$\left({}^{\mathrm{RL}}I_{0+}^{\nu}f\right)(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-u)^{\nu-1} f\,u\,(du) \qquad (t > 0; \Re(\nu) > 0). \tag{41}$$

For a considerably large number of extensions and further generalizations of the fractional-order kinetic equation (40), the interested reader should refer (for example) to [10], [19] and [20] as well as the other relevant references which are cited in each of these earlier publications. We propose here to investigate the solution of a general family of fractional-order kinetic equations which are associated with the function $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$ defined by (18), which we have introduced in this article, as well as the Riemann-Liouville fractional derivative operator ${}^{\mathrm{RL}}D_{0+}^{\sigma}$ defined by (3). The results presented here are sufficiently general in character and are indeed capable of being specialized appropriately to include solutions of the corresponding (known or new) fractional-order kinetic equations associated with simpler functions.

Theorem 1: *Let* $c, \mu, \nu, \rho \in \mathbb{R}^+$ *and* $0 < \sigma < 1$. *Suppose also that the general function-order* $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$, *defined by* (18), *exists. If we set*

$$\chi_0(\sigma) := \left({}^{\mathrm{RL}} I_{0+}^{1-\sigma} f \right)(0+), \tag{42}$$

then the solution of the following generalized fractional-order kinetic equation:

$$N(t) - N_0\, t^{\mu-1}\, \mathcal{E}_{\alpha,\beta}(\varphi; zt^{\nu}, s, \kappa) = -c^{\rho} \left({}^{\mathrm{RL}} D_{0+}^{\sigma} N \right)(t) \tag{43}$$

is given by

$$N(t) = N_0\, t^{\mu-1} \sum_{r=0}^{\infty} (-1)^r \left(\frac{t^{\sigma}}{c^{\rho}} \right)^{r+1} \sum_{k=0}^{\infty} \frac{\varphi(k)\Gamma(\nu k + \mu)}{(k+\kappa)^s\, \Gamma(\alpha k + \beta)\Gamma(\nu k + (r+1)\sigma + \mu)} (zt^{\nu})^k$$

$$+ \chi_0(\sigma) \sum_{r=0}^{\infty} (-1)^r \frac{t^{\sigma(r+1)-1}}{c^{\rho r}\Gamma\big(\sigma(r+1)\big)} \quad (t > 0), \tag{44}$$

provided that the right-hand side of the solution asserted by (44) *exists.*

Proof: Since, by hypothesis, $0 < \sigma < 1$, we can make use of the Laplace transform formula (32) in the following form:

$$\mathcal{L}\{ \left({}^{\mathrm{RL}} D_{0+}^{\sigma} N \right)(t) : \mathfrak{s} \} = \mathfrak{s}^{\sigma} \mathcal{N}(\mathfrak{s}) - \chi_0(\sigma) \qquad (0 < \sigma < 1), \tag{45}$$

where

$$\mathcal{N}(\mathfrak{s}) := \mathcal{L}\{ N(t) : \mathfrak{s} \} = \int_0^{\infty} e^{-\mathfrak{s}t} N(t)\, dt \tag{46}$$

and $\chi_0(\sigma)$ is defined by (42).

Now, by applying the formulas (24) and (45), if we take the Laplace transforms of both sides of the fractional-order kinetic equation (43), we find that

$$\mathcal{N}(\mathfrak{s}) - \frac{N_0}{\mathfrak{s}^{\mu}} \sum_{k=0}^{\infty} \frac{\varphi(k)\Gamma(\nu k + \mu)}{(k+\kappa)^s \Gamma(\alpha k + \beta)} \left(\frac{z}{\mathfrak{s}^{\nu}}\right)^k$$

$$= -c^{\rho}[\mathfrak{s}^{\sigma}\mathcal{N}(\mathfrak{s}) - \left({}^{\mathrm{RL}}I_{0+}^{1-\sigma}N\right)(0+)]$$

$$= -c^{\rho}\mathfrak{s}^{\sigma}\mathcal{N}(\mathfrak{s}) + c^{\rho}\chi_0(\sigma), \tag{47}$$

which readily yields

$$\mathcal{N}(\mathfrak{s}) = \frac{N_0}{1 + c^{\rho}\mathfrak{s}^{\sigma}} \sum_{k=0}^{\infty} \frac{\varphi(k)\Gamma(\nu k + \mu)}{(k+\kappa)^s \Gamma(\alpha k + \beta)} \frac{z^k}{\mathfrak{s}^{\nu k + \mu}} + \frac{c^{\rho}\chi_0(\sigma)}{1 + c^{\rho}\mathfrak{s}^{\sigma}}. \tag{48}$$

In view of the following series expansion:

$$\frac{1}{1 + c^{\rho}\mathfrak{s}^{\sigma}} = \sum_{r=0}^{\infty} \frac{(-1)^r}{(c^{\rho}\mathfrak{s}^{\sigma})^{r+1}} \qquad (|c^{\rho}\mathfrak{s}^{\sigma}| > 1),$$

this last equation (48) can be rewritten as follows:

$$\mathcal{N}(\mathfrak{s}) = N_0 \sum_{r=0}^{\infty} \frac{(-1)^r}{c^{\rho(r+1)}} \sum_{k=0}^{\infty} \frac{\varphi(k)\Gamma(\nu k + \mu)}{(k+\kappa)^s \Gamma(\alpha k + \beta)} \frac{z^k}{\mathfrak{s}^{\nu k + \mu + \sigma(r+1)}}$$

$$+ \chi_0(\sigma) \sum_{r=0}^{\infty} \frac{(-1)^r}{c^{\rho r}\mathfrak{s}^{\sigma(r+1)}}. \tag{49}$$

Finally, we invert the Laplace transforms occurring in (49) by using the following well-known identity:

$$\mathcal{L}\{t^{\lambda} : \mathfrak{s}\} = \frac{\Gamma(\lambda+1)}{\mathfrak{s}^{\lambda+1}}$$

$$\mathcal{L}^{-1}\left(\frac{1}{\mathfrak{s}^{\lambda+1}}\right) = \frac{t^{\lambda}}{\Gamma(\lambda+1)} \qquad \left(\Re(\lambda) > -1; \Re(\mathfrak{s}) > 0\right). \tag{50}$$

We are thus led to the solution (44) asserted by Theorem 1. This evidently completes the proof of Theorem 1. □

The distinct advantage of using the general function $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$, defined by (18), in the non-homogeneous term of the fractional-order kinetic equation (43) lies in its generality so that solutions of other kinetic equations involving relatively simpler non-homogeneous terms can be derived by appropriately specializing the solution (44) asserted by Theorem 1. We find it to be worthwhile to record the following relatively simpler versions of Theorem 1.

Theorem 2: *Let* $c, \mu, \nu, \rho \in \mathbb{R}^+$ *and* $0 < \sigma < 1$. *Suppose also that the general function* $\mathfrak{E}_{\alpha,\beta}(\phi; z)$, *defined by* (8), *exists. If* $\chi_0(\sigma)$ *is given by* (42), *then the solution of the following generalized fractional kinetic equation*:

$$N(t) - N_0\, t^{\mu-1}\, \mathfrak{E}_{\alpha,\beta}(\phi; zt^{\nu}) = -c^{\rho}({}^{\mathrm{RL}}D_{0+}^{\sigma}N)(t) \tag{51}$$

is given by

$$N(t) = N_0\, t^{\mu-1}\sum_{r=0}^{\infty}(-1)^r\left(\frac{t^{\sigma}}{c^{\rho}}\right)^{r+1}$$

$$\cdot\sum_{k=0}^{\infty}\frac{\phi(k)\Gamma(\nu k+\mu)}{\Gamma(\alpha k+\beta)\Gamma(\nu k+(r+1)\sigma+\mu)}(zt^{\nu})^k$$

$$+\,\chi_0(\sigma)\sum_{r=0}^{\infty}(-1)^r\,\frac{t^{\sigma(r+1)-1}}{c^{\rho r}\Gamma(\sigma(r+1))}\,(t>0), \tag{52}$$

provided that the right-hand side of the solution asserted by (52) *exists*.

Proof: Our demonstration of Theorem 2 would run parallel to that of Theorem 1. Use is made; in this case, of the definition (8) and the Laplace transform formula (27). The details are being omitted here. □

Theorem 3: *For* $c, \mu, \nu, \rho \in \mathbb{R}^+$ *and* $0 < \sigma < 1$, *let the extended Hurwitz-Lerch zeta function*:

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}(z, s, \kappa),$$

defined by (17), *exist. If* $\chi_0(\sigma)$ *is given by* (42), *then the solution of the following generalized fractional kinetic equation*:

$$N(t) - N_0\, t^{\mu-1}\, \Phi^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(zt^{\nu}, s, \kappa) = -c^{\rho}\left({}^{\mathrm{RL}}D^{\sigma}_{0+}N\right)(t) \qquad (53)$$

is given by

$$N(t) = N_0\, t^{\mu-1}\sum_{r=0}^{\infty}(-1)^r\left(\frac{t^{\sigma}}{c^{\rho}}\right)^{r+1}\frac{\Gamma(\mu)}{\Gamma(\sigma(r+1)+\mu)}$$

$$\cdot\,\Phi^{(\nu,\rho_1,\cdots,\rho_p;\nu,\sigma_1,\cdots,\sigma_q)}_{\mu,\lambda_1,\cdots,\lambda_p;\sigma(r+1)+\mu,\mu_1,\cdots,\mu_q}(zt^{\nu}, s, \kappa)$$

$$+\,\chi_0(\sigma)\sum_{r=0}^{\infty}(-1)^r\,\frac{t^{\sigma(r+1)-1}}{c^{\rho r}\Gamma(\sigma(r+1))}\,(t>0),$$

provided that the right-hand side of the solution asserted by (54) *exists.*

Proof: Theorem 3 can be proven, along the lines analogous to those of our demonstrations of Theorem 1 and Theorem 3, by applying the definition (17) and the Laplace transform formula (30). We choose to skip the details involved. □

4. Concluding Remarks and Observations

In our present investigation, we have established the explicit solution of some significantly general families of fractional-order kinetic equations involving the Riemann-Liouville right-sided fractional derivative operator $\left({}^{\mathrm{RL}}D^{\mu}_{0+}f\right)(x)$, which is given (for convenience) by (3) for $a = 0$, as well as a remarkably general class of functions as a part of the non-homogeneous term. Our main results (Theorem 1, Theorem 2 and Theorem 3 in this article) include, as a part of the non-homogeneous term, such general functions as $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$, $\mathfrak{E}_{\alpha,\beta}(\phi; z)$ and

$$\Phi^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z, s, \kappa),$$

which are defined by (18), (8) and (17), respectively. Each of these main results is

indeed capable of yielding solutions of a significantly large number of (known or new) simpler fractional-order kinetic equations.

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