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SUMS INVOLVING EXTENDED GIBONACCI POLYNOMIALS

Abstract: We explore four sums involving gibonacci polynomial squares and their Pell versions.

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1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 4].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively [4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, and $b_n = p_n$ or q_n, $c_n = J_n$ or j_n, $\Delta = \sqrt{x^2+4}$, $2\alpha = x + \Delta$, $E = \sqrt{x^2+1}$, $\gamma = x + E$, and $D = \sqrt{4x+1}$ where $c_n = c_n(x)$.

It follows by the Binet-like formulas that $\lim_{m\to\infty} \frac{f_{m+1}}{l_m} = \frac{\alpha}{\Delta}$ and $\lim_{m\to\infty} \frac{l_{m+1}}{f_m} = \Delta\alpha$.

1.1 Fundamental Gibonacci Identities: Gibonacci polynomials satisfy the following properties:

$$\Delta^2 f_{n+1}f_n = l_{2n+1} - (-1)^n x; \tag{1}$$

$$l_{n+1}l_n = l_{2n+1} + (-1)^n x; \tag{2}$$

$$l_n f_{n+2} - l_{n+1}f_{n+1} = (-1)^n x; \tag{3}$$

$$l_n f_{n+2} + l_{n+1}f_{n+1} = 2f_{2n+2} + (-1)^n x; \tag{4}$$

$$f_n l_{n+2} - f_{n+1}l_{n+1} = (-1)^{n+1} x; \tag{5}$$

$$f_n l_{n+2} + f_{n+1}l_{n+1} = 2f_{2n+2} - (-1)^n x. \tag{6}$$

These properties can be confirmed using the Binet-like formulas.

It follows by identities (3) – (6) that

$$l_n^2 f_{n+2}^2 - l_{n+1}^2 f_{n+1}^2 = 2(-1)^n x f_{2n+2} + x^2; \tag{7}$$

$$f_n^2 l_{n+2}^2 - f_{n+1}^2 l_{n+1}^2 = 2(-1)^{n+1} x f_{2n+2} + x^2. \tag{8}$$

2. Telescoping Gibonacci Sums

We now establish two telescoping gibonacci sums, where $k \geq 0$ and $\lambda \geq 1$ are integers.

Lemma 1:

$$\sum_{n=1}^{\infty}\left(\frac{f_{n+k+2}^{\lambda}}{l_{n+k+1}^{\lambda}}-\frac{f_{n+k+1}^{\lambda}}{l_{n+k}^{\lambda}}\right)=\frac{\alpha^{\lambda}}{\Delta^{\lambda}}-\frac{f_{k+2}^{\lambda}}{l_{k+1}^{\lambda}}. \tag{9}$$

Proof: Since $\sum_{n=1}^{m}\left(\frac{f_{n+k+2}^{\lambda}}{l_{n+k+1}^{\lambda}}-\frac{f_{n+k+1}^{\lambda}}{l_{n+k}^{\lambda}}\right)$ is a telescoping sum, we have

$$\sum_{n=1}^{m}\left(\frac{f_{n+k+2}^{\lambda}}{l_{n+k+1}^{\lambda}}-\frac{f_{n+k+1}^{\lambda}}{l_{n+k}^{\lambda}}\right)=\frac{f_{m+k+2}^{\lambda}}{l_{m+k+1}^{\lambda}}-\frac{f_{k+2}^{\lambda}}{l_{k+1}^{\lambda}}.$$

This yields the desired result. □

Lemma 2:

$$\sum_{n=1}^{\infty}\left(\frac{l_{n+k+2}^{\lambda}}{f_{n+k+1}^{\lambda}}-\frac{l_{n+k+1}^{\lambda}}{f_{n+k}^{\lambda}}\right)=\Delta^{\lambda}\alpha^{\lambda}-\frac{l_{k+2}^{\lambda}}{f_{k+1}^{\lambda}}. \tag{10}$$

Proof: Using the fact that $\lim_{m\to\infty}\frac{l_{m+1}}{f_m}=\Delta\alpha$, the proof follows as above. So, in the interest of brevity, we omit the details. □

These two lemmas play a pivotal role in our discourse.

3. Gibonacci Sums

With the above identities and lemmas at our disposal, we are now ready for further explorations.

The next two theorems invoke the lemmas with $\lambda = 1$.

Theorem 1: *Let k be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k}x}{l_{2n+2k+1} + (-1)^{n+k}x} = \frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}. \tag{11}$$

Proof: It follows by identities (2) and (3) that

$$l_{n+k+1}l_{n+k} = l_{2n+2k+1} + (-1)^{n+k}x;$$

$$l_{n+k}f_{n+k+2} - l_{n+k+1}f_{n+k+1} = (-1)^{n+k}x.$$

By Lemma 1, we then have

$$\frac{(-1)^{n+k}x}{l_{2n+2k+1} + (-1)^{n+k}x} = \frac{l_{n+k}f_{n+k+2} - l_{n+k+1}f_{n+k+1}}{l_{n+k+1}l_{n+k}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k}x}{l_{2n+2k+1} + (-1)^{n+k}x} = \sum_{n=1}^{\infty} \left(\frac{f_{n+k+2}}{l_{n+k+1}} - \frac{f_{n+k+1}}{l_{n+k}} \right)$$

$$= \frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}},$$

as desired. □

It then follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+1} + (-1)^n} = -\frac{1}{2} + \frac{\sqrt{5}}{10}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+3} - (-1)^n} = \frac{1}{6} - \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+5} + (-1)^n} = -\frac{1}{4} + \frac{\sqrt{5}}{10}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+7} - (-1)^n} = \frac{3}{14} - \frac{\sqrt{5}}{10}.$$

The next result invokes Lemma 2 with $\lambda = 1$.

Theorem 2: *Let k be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k+1}x}{l_{2n+2k+1} - (-1)^{n+k}x} = \frac{1}{\Delta^2}\left(\frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}\right). \tag{12}$$

Proof: Using identities (1) and (4), we get

$$\Delta^2 f_{n+k+1} f_{n+k} = l_{2n+2k+1} - (-1)^{n+k}x;$$

$$f_{n+k} l_{n+k+2} - f_{n+k+1} l_{n+k+1} = (-1)^{n+k+1}x.$$

By Lemma 2, we then have

$$\frac{(-1)^{n+k+1}x}{l_{2n+2k+1} - (-1)^{n+k}x} = \frac{f_{n+k} l_{n+k+2} - f_{n+k+1} l_{n+k+1}}{\Delta^2 f_{n+k+1} f_{n+k}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k+1}x}{l_{2n+2k+1} - (-1)^{n+k}x} = \frac{1}{\Delta^2} \sum_{n=1}^{\infty} \left(\frac{l_{n+k+2}}{f_{n+k+1}} - \frac{l_{n+k+1}}{f_{n+k}}\right)$$

$$= \frac{1}{\Delta^2} - \left(\Delta\alpha - \frac{l_{k+2}}{f_{k+1}}\right),$$

as desired. □

Consequently, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+1} - (-1)^n} = \frac{1}{10} - \frac{\sqrt{5}}{10}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+3} + (-1)^n} = -\frac{3}{10} + \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+5} - (-1)^n} = \frac{1}{5} - \frac{\sqrt{5}}{10}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+7} + (-1)^n} = -\frac{7}{30} + \frac{\sqrt{5}}{10}.$$

Gibonacci Delights: By combining these two theorems, we can extract interesting dividends.

Adding equations (11) and (12), we get

$$\sum_{n=1}^{\infty} \frac{2x^2}{l_{2n+2k+1}^2 - x^2} = -\frac{\Delta^2 + 1}{\Delta^2}\left(\frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}\right).$$

In particular, this yields

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n+1}^2 - 1} = \frac{3}{10} - \frac{3\sqrt{5}}{50}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n+3}^2 - 1} = \frac{1}{10} - \frac{3\sqrt{5}}{50};$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n+5}^2 - 1} = \frac{3}{20} - \frac{3\sqrt{5}}{50}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n+7}^2 - 1} = \frac{9}{70} - \frac{21\sqrt{5}}{350}.$$

Likewise, subtraction of the two equations yield

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k} x l_{2n+2k+1}}{l_{2n+2k+1}^2 - x^2} = \frac{\Delta^2 - 1}{\Delta^2}\left(\frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}\right).$$

This implies

$$\sum_{n=1}^{\infty} \frac{(-1)^n L_{2n+1}}{L_{2n+1}^2 - 1} = \frac{2\sqrt{5}}{25}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n L_{2n+3}}{L_{2n+3}^2 - 1} = -\frac{2}{15} + \frac{2\sqrt{5}}{15};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n L_{2n+5}}{L_{2n+5}^2 - 1} = \frac{1}{10} + \frac{2\sqrt{5}}{25}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n L_{2n+7}}{L_{2n+7}^2 - 1} = -\frac{4}{35} + \frac{2\sqrt{5}}{25}.$$

The next two theorems employ the lemmas with $\lambda = 2$.

Theorem 3: *Let k be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k} x f_{2n+2k+2} + x^2}{\left[l_{2n+2k+1} + (-1)^{n+k} x\right]^2} = \frac{\alpha^2}{\Delta^2} - \frac{f_{k+2}^2}{l_{k+1}^2}. \tag{13}$$

Proof: Lemma 1, coupled with identities (2) and (7), yields

$$\frac{2(-1)^{n+k} x f_{2n+2k+2} + x^2}{\left[l_{2n+2k+1} + (-1)^{n+k} x\right]^2} = \frac{l_{n+k}^2 f_{n+k+2}^2 - l_{n+k+1}^2 f_{n+k+1}^2}{l_{n+k+1}^2 l_{n+k}^2}$$

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k} f_{2n+2k+2} + x^2}{\left[l_{2n+2k+1} + (-1)^{n+k} x\right]^2} = \sum_{n=1}^{\infty} \left(\frac{f_{n+k+2}^2}{l_{n+k+1}^2} - \frac{f_{n+k+1}^2}{l_{n+k}^2} \right)$$

$$= \frac{\alpha^2}{\Delta^2} - \frac{f_{k+2}^2}{l_{k+1}^2}.$$

as desired. □

In particular, we then get

$$\sum_{n=1}^{\infty} \frac{2(-1)^n F_{2n+2} + 1}{\left[L_{2n+1} + (-1)^n\right]^2} = -\frac{7}{10} + \frac{\sqrt{5}}{10}; \qquad \sum_{n=1}^{\infty} \frac{2(-1)^{n+1} F_{2n+4} + 1}{\left[L_{2n+3} - (-1)^n\right]^2} = -\frac{23}{90} + \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{2(-1)^n F_{2n+6} + 1}{\left[L_{2n+5} + (-1)^n\right]^2} = -\frac{21}{80} + \frac{\sqrt{5}}{10}; \qquad \sum_{n=1}^{\infty} \frac{2(-1)^{n+1} F_{2n+8} + 1}{\left[L_{2n+7} - (-1)^n\right]^2} = -\frac{103}{490} + \frac{\sqrt{5}}{10}.$$

The next result invokes Lemma 2.

Theorem 4: *Let k be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k+1} x f_{2n+2k+2} + x^2}{\left[l_{2n+2k+1} - (-1)^{n+k} x\right]^2} = \frac{1}{\Delta^4} \left(\Delta^2 \alpha^2 - \frac{l_{k+2}^2}{f_{k+1}^2} \right). \tag{14}$$

Proof: With identities (1) and (8), Lemma 2 yields

$$\frac{2(-1)^{n+k+1} x f_{2n+2k+2} + x^2}{\left[l_{2n+2k+1} - (-1)^{n+k} x\right]^2} = \frac{f_{n+k}^2 l_{n+k+2}^2 - f_{n+k+1}^2 l_{n+k+1}^2}{\Delta^4 f_{n+k+1}^2 f_{n+k}^2}$$

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k+1} x f_{2n+2k+2} + x^2}{\left[l_{2n+2k+1} - (-1)^{n+k} x\right]^2} = \frac{1}{\Delta^4} \sum_{n=1}^{\infty} \left(\frac{l_{n+k+2}^2}{f_{n+k+1}^2} - \frac{l_{n+k+1}^2}{f_{n+k}^2} \right)$$

$$= \frac{1}{\Delta^4} \left(\Delta^2 \alpha^2 \frac{l_{k+2}^2}{f_{k+1}^2} \right),$$

confirming the given result. □

It then follows that

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1} F_{2n+2} + 1}{\left[L_{2n+1} - (-1)^n\right]^2} = -\frac{3}{50} + \frac{\sqrt{5}}{10}; \quad \sum_{n=1}^{\infty} \frac{2(-1)^n F_{2n+4} + 1}{\left[L_{2n+3} + (-1)^n\right]^2} = -\frac{17}{50} + \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1} F_{2n+6} + 1}{\left[L_{2n+5} - (-1)^n\right]^2} = -\frac{19}{100} + \frac{\sqrt{5}}{10}; \quad \sum_{n=1}^{\infty} \frac{2(-1)^n F_{2n+8} + 1}{\left[L_{2n+7} + (-1)^n\right]^2} = -\frac{107}{450} + \frac{\sqrt{5}}{10}.$$

Finally, we explore the Pell versions of the theorems.

4. Pell Implications

Using the relationship $b_n(x) = g_n(2x)$, we can find the Pell versions of equations (11) – (14):

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k} x}{q_{2n+2k+1} + 2(-1)^{n+k} x} = \frac{\gamma}{4E} - \frac{p_{k+2}}{2q_{k+1}};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k} x}{q_{2n+2k+1} - 2(-1)^{n+k} x} = \frac{1}{8E^2} \left(\frac{\gamma}{2E} - \frac{p_{k+2}}{q_{k+1}} \right);$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k} x p_{2n+2k+2} + x^2}{\left[q_{2n+2k+1} + 2(-1)^{n+k} x\right]^2} = \frac{1}{4} \left(\frac{\gamma^2}{4E^2} - \frac{p_{k+2}^2}{q_{k+1}^2} \right);$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k+1} x p_{2n+2k+2} + x^2}{\left[q_{2n+2k+1} - 2(-1)^{n+k} x\right]^2} = \frac{1}{64E^4} \left(4E^2\gamma^2 - \frac{q_{k+2}^2}{p_{k+1}^2} \right).$$

They yield

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k}}{Q_{2n+2k+1} + (-1)^{n+k}} = \frac{2+\sqrt{2}}{4} - \frac{P_{k+2}}{Q_{k+1}};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k+1}}{Q_{2n+2k+1} - (-1)^{n+k}} = \frac{2+\sqrt{2}}{32} - \frac{P_{k+2}}{16Q_{k+1}};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k} P_{2n+2k+2} + 1}{\left[Q_{2n+2k+1} + (-1)^{n+k}\right]^2} = \frac{3+2\sqrt{2}}{8} - \frac{P_{k+2}^2}{4Q_{k+1}^2};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k+1} P_{2n+2k+2} + 1}{\left[Q_{2n+2k+1} - (-1)^{n+k}\right]^2} = \frac{3+2\sqrt{2}}{8} - \frac{Q_{k+2}^2}{16P_{k+1}^2}.$$

respectively.

5. Chebyshev and Vieta Consequences

Chebyshev polynomials T_n and U_n, Vieta polynomials V_n and v_n, and gibonacci polynomials are linked by the relationships $V_n(x) = i^{n-1} f_n(-ix)$, $v_n(x) = i^n l_n(-ix)$, $V_n(x) = U_{n-1}(x/2)$, and $v_n(x) = 2T_n(x/2)\left[2,\ 3,\ 4\right]$, where $i = \sqrt{-1}$; they can be employed to find the Chebyshev and Vieta versions of the theorems. In the interest of brevity, we omit them; but we encourage gibonacci enthusiasts to explore them.

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