Journal of Indian Acad. Math. ISSN: 0970-5120
Vol. 45, No. 1 (2023) pp. 29-39

Y. Therese Sunitha Mary[1] *and R. Kala*[2]

SPLIT DOMATIC NUMBER OF A GRAPH

Abstract: A dominating set $D \subseteq V(G)$ is a split dominating set if the induced subgraph $<V - D>$ is disconnected. The minimum cardinality of a split dominating set is called the split domination number of G, denoted by $\gamma_s(G)$. The maximum order of a partition of $V(G)$ into split dominating sets of G is called the split domatic number of G and is denoted by $d_s(G)$. In this paper, we study several aspects of these two parameters and find certain classes of graphs that are domatically full.

Key words and phrases: Domination, Domination Number, Split Domination, Split Domination Number, Domatic Number, Split Domatic Number.

Mathematical Subject Classification No.: 05C69.

1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively.

For graph theoretical terms we refer to Harary [2] and for terms related to domination we refer Haynes *et al*. [3] and [4]. A subset D of V is said to be a

dominating set in G if every vertex in $V - D$ is adjacent to at least one vertex in D. The maximum order of a partition of V into dominating sets of G is called the *domatic number* of G and is denoted by $d(G)$. G is *domatically full* if $d(G) = \delta(G) + 1$. A dominating set D of a graph $G = (V, E)$ is a split dominating set if the induced subgraph $< V - D >$ is disconnected. The split domination number $\gamma_s(G)$ of G is the minimum cardinality of a split dominating set. The maximum order of a partition of $V(G)$ into split dominating sets of G is called the split domatic number of G and is denoted by $d_s(G)$. The *corona* of two graphs G_1 and G_2 is the graph $G = G_1 \ o \ G_2$ formed from one copy of G_1 and $| V(G_1) |$ copies of G_2 , where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2. In this paper, we study several aspects of these parameters and find certain classes of graphs that are domatically full.

Kulli and Janakiram introduced the concept of split domination in graphs [5]. The following results are very useful in the subsequent sections.

Theorem 1.1 [1]: (*If* $\delta(G) \geq 1$ *then* $\gamma(G) + d(G) \leq \left\lfloor \frac{p}{2} \right\rfloor + 2$ *and equality requires that* $\{\gamma(G), d(G)\} = \{\left\lfloor \frac{p}{2} \right\rfloor, 2\}$.

Theorem 1.2 [2]: *For any graph* G, $\chi(G) \leq 1 + \Delta(G)$.

Theorem 1.3 [6]: *Let* G *be any unicyclic graph with cycle* C_p. *Then* $\gamma(G) = \gamma_{ns}(G)$ *if and only if* $G \cong G_1, G_2, G_3$ *or* G_4 *where* $G_1 \cong H \circ K_1$ *where* H *is any unicyclic graph,* $G_2 \cong$ *any unicyclic graph with cycle* $C_p (p \geq 4)$ *in which every support is adjacent to exactly one pendent vertex, every vertex not on the cycle is a pendent vertex and exactly three consecutive vertices on the cycle have degree* 2 *and* G_3 , G_4 *are as given in* Figure 1.

Figure 1

Example 1.4: *For* $G \cong G_1$, *where* G_1 *is given in* Figure 2, $\gamma_s(G) = 3$.

Figure 2

It is interesting to observe that the property 'split' is one among very few properties which are neither hereditary nor super hereditary. Motivated by this feature, we study several aspects of this parameter.

2. Split Domination Number of a Graph

The following are immediate from the definition:

Proposition 2.1: (1) $\gamma_s(G)$ *does not exist if and only if* $G \cong K_p$.

(2) *If* G *is any graph with* $\Delta(G) = p - 1$ *and* $\delta(G) = 1$ *then* $\gamma_s(G) = 1$. *Converse is not true. If* G *is the graph given in* Figure 3, $\gamma_s(G) = 1$, $\Delta(G) = p - 1$ *but* $\delta(G) \neq 1$.

Figure 3

(3) *For any connected graph* G *which is not isomorphic to complete graph,* $\gamma_s(G) \leq p - 2$.

(4) *If* H *is a connected spanning subgraph of* G, *then* $\gamma_s(H) \leq \gamma_s(G)$.

(5) *If* $G \cong K_{m,n}$, *then* $\gamma(\overline{G}) = \gamma_s(\overline{G}) = 2$.

(6) *Let* G *be any connected graph* $\ncong P_3$ *such that either* $\Delta(G) = \delta(G) = p - 2$ *or* $\Delta(G) = p - 2 =$ *number of pendent vertices. Then* $\gamma_s(G) \leq 2$.

(7) *If* G *is any disconnected graph without isolated vertices, then* $\gamma_s(G) = p - 2$ *if and only if* $G \cong 2K_2$.

(8) *If* G *is a galaxy then* $\gamma(G) = \gamma_s(G) =$ *number of components of* G. *Converse is not true. If* G *is the graph given in* Figure 4, $\gamma(G) = \gamma_s(G) = 2 =$ *number of components of* G, *but* G *is not a galaxy.*

Figure 4

Theorem 2.2: *Let* T *be any tree.* $\gamma_s(T) = 2$ *if and only if* T *has exactly two supports and* $diam(T) \leq 5$.

Proof: Suppose $\gamma_s(T) = 2$. If T has exactly one support then $T \cong K_{1,p}$. But $\gamma_s(K_{1,p}) = 1$ and so T has at least two supports. If T has 3 or more supports then clearly $\gamma_s(T) \geq 3$ and so T has exactly two supports. If $diam(T) > 6$, then again $\gamma_s(T) > 2$ and so $diam(T) \leq 5$. Converse is obvious. □

The following is immediate.

Corollary 2.3: *Let T be any tree with exactly two supports. Then $\gamma_s(T) = 3$ if and only if $diam(T) = 6, 7$ or 8.*

Theorem 2.4: *Let T be a tree. Every minimum dominating set is a split dominating set if and only if $T \ncong P_p \circ K_1$.*

Proof: Assume that every minimum dominating set is a split dominating set. If $T \ncong P_p \circ K_1$, the set of all pendent vertices is a minimum dominating set which is not a split dominating set and so $T \ncong P_p \circ K_1$. If $T \ncong P_p \circ K_1$, every minimum dominating set contains a non-pendent vertex and so is a split dominating set. □

Theorem 2.5: *Let T be any tree such that $T \ncong K_{1,p}$. Then*

$$\gamma_s(\overline{T}) = \begin{cases} p - \Delta(T) & if \ \ diam(T) = 3 \\ p - \Delta(T) - 1 & if \ \ diam(T) \geq 4. \end{cases}$$

Proof: Let $v \in V(T)$ with $deg\ v = \Delta(T)$.

Case (i): $diam(T) = 3$.

Since, $diam(T) = 3$, there exists a pendent vertex u adjacent to v and a pendent vertex w non-adjacent to v. Let $S = N[v] - \{u\}$.

Claim: $V - S$ is a minimum split dominating set of $\overline{T}$.

$w \in V - S$ dominates v and every vertex of $N(v) - \{u\}$ is dominated by u. v is an isolated vertex of $< S >$ in $\overline{T}$ and so $V - S$ is a split dominating set of $\overline{T}$. The vertices of $V(T) - N[v]$ are all adjacent to a single vertex of $N(v)$ since otherwise T has a cycle. Hence, $u \in N[v]$ is essential to dominate that vertex in $\overline{T}$. If any other vertex of $V(T) - N[v]$ lies in S, then that vertex is adjacent to v in $\overline{T}$ so that S is not a split dominating set. Hence, $V - S$ is a minimum split dominating set of $\overline{T}$ so that $\gamma_s(\overline{T}) = p - \Delta(T)$.

Case (ii): $diam(T) \geq 4$.

Let $S' = N[v]$. As $diam(T) \geq 4$, either there exists $x \in V(T)$ with $d(v, x) \geq 3$ or there exists $x, y \in V(T)$ with $d(v, x) = 2$ and $d(v, y) = 2$. In the former case, x is adjacent to all vertices of S' in $\overline{T}$ and in the latter case $\{x, y\}$ dominates all vertices of S' in $\overline{T}$. Also v is an isolated vertex in $< S' >$ in $\overline{T}$. So $V - S'$ is a split dominating set of $\overline{T}$. As in case (i), it is minimum. Hence, $\gamma_s(\overline{T}) = p - (\Delta(T) + 1)$. □

Corollary 2.6: *For any tree* $T \ncong K_{1,p}$, $\gamma_s(T) + \gamma_s(\overline{T}) = p - \Delta(T) + 1$ *if and only if* T *is obtained from* P_5 *or* P_6 *by adding zero or more number of pendents to the supports.*

Proof: If $diam(T) = 3$ then $\gamma_s(\overline{T}) = p - \Delta(T)$ and so $\gamma_s(T) + \gamma_s(\overline{T}) = p - \Delta(T) + 1 \Rightarrow \gamma_s(T) = 1$ which is impossible as $T \ncong K_{1,p}$.

Suppose $diam(T) = 4$. Then by Theorem 2.5, $\gamma_s(\overline{T}) = p - \Delta(T) - 1$ and so $\gamma_s(T) = 2$. By Theorem 2.2, T has exactly two supports and $diam(T) \leq 5$. Thus, $diam(T) = 4$ or 5 and T is obtained from P_5 or P_6 by adding zero or more number of pendents to the supports.

Converse is obvious. □

Theorem 2.7: *Let* T *be any tree with* $diam(T) = 3$. *Then*

$$\gamma_s(T)\gamma_s(\overline{T}) \leq 2(p - \Delta(T)).$$

Proof: By Theorem 2.5, $\gamma_s(\overline{T}) = p - \Delta(T)$ if $diam(T) = 3$. If $diam(T) = 3$ then T has exactly 2 supports and so by Theorem 2.2, $\gamma_s(T) = 2$. Hence, $\gamma_s(T)\gamma_s(\overline{T}) \leq 2(p - \Delta(T))$. □

Remark 2.8: *Converse of Theorem* 2.7 *is not true. Consider* P_5. $\gamma_s(P_5)\gamma_s(\overline{P}_5) = 4 < 6$. *But* $diam(P_5) = 4$.

We now relate $\gamma_s(G)$ with other graph theoretic parameters.

Theorem 2.9: *For any connected graph* $G, \gamma_s(G) + \chi(G) \leq p + \Delta(G) - 2$. *Equality holds for* $G \cong C_4,\ P_4$.

Proof: By Theorem 1.2, and by Proposition **??**(3) we have $\gamma_s(G) + \chi(G) \leq p + \Delta(G) - 1$. By Theorem 1.2, if $\chi(G) = \Delta(G) + 1$ then G is either an odd cycle or a complete graph. For a complete graph, $\gamma_s(G)$ is not defined. For an odd cycle C_p, $\gamma_s(C_p) < p - 2$. Hence, the above bound can be improved as $\gamma_s(G) + \chi(G) \leq p + \Delta(G) - 2$. Equality holds for $G \cong C_4,\ P_4$. □

Theorem 2.10: *For any connected graph* $G, \gamma_s(G) + k(G) \leq p + \Delta(G) - 2$. *Equality holds if* $G \cong C_4$.

Proof: For any connected graph G, $\gamma_s(G) \leq p - 2$ and $k(G) \leq \Delta(G)$ so that $\gamma_s(G) + k(G) \leq p + \Delta(G) - 2$. Equality holds if $\gamma_s(G) = p - 2$ and $k(G) = \Delta(G)$. If $G \cong C4$, $\gamma_s(G) = 2$, $k(G) = 2$ and so the bound is sharp.

Theorem 2.11: *For any connected graph* $G, \gamma_s(G) + diam(G) \leq 2p - 3$. *Equality holds if* $G \cong P_4$.

Proof: For any connected graph G, $\gamma_s(G) \leq p - 2$ and $diam(G) \leq p - 1$ so that $\gamma_s(G) + diam(G) \leq 2p - 3$. Clearly equality holds if $G \cong P_4$. □

Theorem 2.12: *Let* G *be any unicyclic graph with cycle* C_p. *Then* $\gamma(G) = \gamma_{ns}(G) = \gamma_s(G)$ *if and only if* $G \cong G_1,\ G_2,\ G_3$ *or* G_4 *where* $G_1 \cong H \circ K_1$ *for any unicyclic graph* H, $G_2 \cong$ *any unicyclic graph with cycle* C_p $(p \geq 4)$ *in which every support is adjacent to exactly one pendent vertex,*

every vertex not on the cycle is a pendent vertex and exactly three consecutive vertices on the cycle have degree 2 *and* G_3, G_4 *are as given in* Figure 5.

Figure 5

Proof: Follows by Theorem 1.3 since for every such graph there exists a split dominating set with cardinality $\gamma(G)$. □

Theorem 2.13: *For any tree* T, $\gamma(T) = \gamma_s(T) = \gamma_{ns}(T) = \gamma_{sns}(T) = 2$ *if and only if* $T \cong P4$.

Proof: Since a tree cannot contain a cycle, $\gamma_{sns}(T) = 2$ which implies $p = 4$. So $T \cong K_{1,3}$ or P_4. If $T \cong K_{1,3}$ then $\gamma(T) = 1$ and so $T \cong P_4$.

Converse is obvious. □

Proposition 2.14: *Let* G *be any connected graph and* $G' = G \circ K_1$, *where* $G \circ K_1$ *is the corona of* G *and* K_1. *Then* $\gamma(G') = \gamma_s(G') = \gamma_{ns}(G') = p$, *where* $p = |V(G)|$.

Proof: Since G is connected, the set of all pendent vertices of G' forms a γ_{ns}-set of G and so $\gamma_{ns}(G') = p$. Similarly $V(G)$ forms a γ_s-set of G' which is also a γ-set of G'. Hence, $\gamma(G') = \gamma_s(G') = \gamma_{ns}(G') = p$. □

3. Split Domatic number of a Graph

Definition 3.1: Let $G = (V, E)$ be a graph. The maximum order of a partition of $V(G)$ into split dominating sets of G is called the split domatic number

of G and is denoted by $d_s(G)$.

Remark 3.2: $d_s(G)$ cannot be determined for any graph with $\Delta(G) = p - 1$.

In the following proposition we summarize a number of elementary results which determine $d_s(G)$ for special classes of graphs. The proofs of these results are simple and are omitted.

Proposition 3.3: (1) *If* $G \cong P_p (p \geq 4)$ *then* $d_s(P_p) = 2$.

(2) *If* $G \cong K_{m,n}$ *then* $d_s(K_{m,n}) = 2$.

(3) *If* $G \cong G \circ K_1$ *then* $d_s(G \circ K_1) = 2$.

(4) *If* $G \cong \overline{C}_p$ *where* $p = 4, 5$ *or* 6 *then* $d_s(\overline{C}_p) = 2$.

(5) *If* $G \cong \overline{P}_p$ *where* $p = 4, 5$ *or* 6 *then* $d_s(\overline{P}_p) = 2$.

(6) *For any graph* G, $d_s(G) \leq \delta(G) + 1$.

Definition 3.4: A graph G is split domatically full if $d_s(G) = \delta(G) + 1$.

Proposition 3.5: *If* $G \cong C_p (p = 3k, \ k > 1)$, *then* $C_p^{'}s$ *are domatically full.*

Proof: Let $V(C_p) = \{v_1, v_2, \ldots, v_{3k}\}$. The sets $\{v_1, v_4, v_7, \ldots, v_{3k-2}\}$, $\{v_2, v_5, \ldots, v_{3k-1}\}$ and $\{v_3, v_6, \ldots, v_{3k}\}$ form a partition of $V(G)$ into split dominating sets and so $d_s(C_p) = 3 = \delta(C_p) + 1$. □

Corollary 3.6: *Let* $G \cong C_p (p \geq 4)$. *Then* G *is split domatically full if and only if* $p \equiv 0 (\mathrm{mod} 3)$.

Proof: By Theorem 1.1, $d(C_p) = 2$ if $p \equiv 1, 2(\mathrm{mod}3)$ and hence, the result follows. □

Using cycles, we now construct a special class of split domatically full graphs.

Example 3.7: Let $I = \{C_p : C_p$ is a cycle on p vertices, $p = 3k, k > 1\}$. An operation $\circ$ is defined on I as follows:

If $C_i, C_j, C_k \in I$, then $C_i \circ C_j$ is obtained by joining any one vertex of C_i to any one vertex of C_j by an edge. $(C_i \circ C_j) \circ C_k$ is obtained by joining any one vertex of $C_i \circ C_j$ to any one vertex of C_k by an edge. The process is repeated finite number of times and we define G^* to be the collection of all such graphs. As each C_p in I is domatically full and union of split dominating sets of the individual cycles give split dominating sets of the newly constructed graphs, every element of G^* is split domatically full.

Proposition 3.8: *If T is any tree other than a star, then T is split domatically full.*

Proof: By Theorem 1.1, $d(T) = 2$ for any tree with at least two vertices. In any tree with at least two distinct supports, it is easy to observe that there exist two disjoint dominating sets which are also split dominating sets. If T is a star then $d_s(T) = 1$.

Hence, if $T \ncong K_{1,p-1}$ then $d_s(T) = 2 = \delta(G) + 1$. □

REFERENCES

[1] E. J. Cockayne and S. T. Hedetniemi (1977): Towards a theory of domination in graphs, *Networks*, Vol. 7, pp. 247-261.

[2] F. Harary (1969): Graph theory, Addison Wesley, Reading Mass.

[3] T. W. Haynes, S. T. Hedetniemi and P. J. Slater (1998): Fundamentals of Domination in graphs, Marcel Dekker, Inc..

[4] T. W. Haynes, S. T. Hedetniemi and P. J. Slater (1998): Domination in Graphs, Advanced Topics, Marcel Dekker.

[5] V. R. Kulli and B. Janakiram, (1997): The split domination number of a graph, *Graph Theory Notes of New York*, Vol. XXX11, pp. 16-19.

[6] Y. Therese Sunitha Mary and R. Kala (2007): The nonsplit domination number of a graph, Proceedings of the National Conference on Mathematical and Computational Models, December, PSG College of Technology, Coimbatore, INDIA, 218-224.

1. Department of Mathematics,
St. Xavier's College,
Palayamkottai - 627 002, Tamil Nadu, India
E-mail: sunithanepolian@gmail.com

2. Department of Mathematics,
Manonmaniam Sundaranar University,
Tirunelveli - 627 012, Tamil Nadu, India
E-mail: karthipyi91@yahoo.co.in

(*Received*, April 6, 2023)