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SOME COMMON FIXED POINT THEOREMS IN COMPLEX VALUED METRIC SPACES FOR THREE SELF-MAPPINGS UNDER RATIONAL TYPE CONTRACTION

Abstract: In this paper we prove the common fixed point theorems in complex valued metric space for three self mappings. Banach's fixed point theorem plays a major role in fixed point theory. Because of its usefulness we have used Banach's contraction principle for the improvement and generalization of our result. Our result generalizes some recent results in the literature due to Azam *et al*. (2011) and Sintunavarat and Kumam (2012) by using the idea of two weakly compatible mappings. Azam *et al*. made a generalization by introducing a complex valued metric space using some contractive type conditions whereas Sintunavarat and Kumam generalized their result by replacing the constants of contraction by some control functions. Some concepts have been taken from the results obtained by Choi *et al*. (2017) and Jebril *et al*. (2019) to improve our results. The results of Choi *et al*. and Jebril *et al*. in bicoplex valued metric spaces are very effective tools to improve our result. Also an example is given to illustrate our obtained results.

Keywords and phrases: Complex Valued Metric Space, Common Fixed Point, Point of Coincidence, Weakly Compatible Mapping.

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1. Introduction, Definitions and Notations

The concept of fixed point theorem was first introduced by Poincare and Miranda [15] in 1883. After that Brouwer [4] published his famous fixed point

theorem in 1912. The theorem states that **"**If B is a closed unit ball in R^n and if $T : B \to B$ is continuous then T has a fixed point in B". In 1992 Banach [5] proved his famous fixed point theorem in which contraction principle is the main tools. Banach's fixed point theorem plays a major role in fixed point theory. It has applications in many branches of mathematics. Because of its usefulness, a lot of articles have been dedicated to the improvement and generalization of that result. Most of these generalizations have been made by considering different contractive type conditions in different spaces $\{cf.$[6] [19]$\}$. In 2011, Azam *et al*. [3] made a generalization by introducing a complex valued metric space using some contractive type conditions. Very recently, Sintunavarat *et al*. [20] generalized this result by replacing the constants of contraction by some control functions. The purpose of this work is to obtain a common fixed point result for three self mappings in complex valued metric spaces which generalizes the results of [2] and [20].

We write regular complex number as $z = x + iy$ where x and y are real numbers and $i^2 = -1$. Let $\mathbb{C}_1$ be the set of complex numbers and z_1 and $z_2 \in \mathbb{C}_1$. Define a partial order relation $\precsim$ on $\mathbb{C}_1$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$.

Thus, $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(iii) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,

(iv) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e. one of (ii), (iii) and (iv) is satisfied and $(z_1) \prec (z_2)$ if only (iv) is satisfied.

Taking this into account some fundamental properties of the partial order $\precsim$ on $\mathbb{C}_1$ as follows:

(1) If $0 \precsim z_1 \precsim z_2$ then $|z_1| < |z_2|$;

(2) If $z_1 \precsim z_2$, $z_2 \precsim z_3$ then $z_1 \precsim z_3$ and

(3) If $z_1 \precsim z_2$ and $0 < \lambda < 1$ is a real number then $\lambda z_1 \precsim z_2$.

Azam *et al*. defined the complex valued metric space in the following way:

Definition 1.1 [3]: *Let X be a nonempty set where as $\mathbb{C}_1$ be the set of complex numbers. Suppose that the mapping $d : X \times X \to \mathbb{C}_1$ satisfies the following conditions*:

$(d_1) : 0 \precsim d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

$(d_2) : d(x, y) = d(y, x)$ *for all* $x, y \in X$;

$(d_3) : d(x, y) \precsim d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Definition 1.2 [3]: *Let (X, d) be a complex valued metric space, let $\{x_n\}$ be a sequence in X and $x \in X$.*

(i) *If for every $c \in \mathbb{C}_1$ with $0 \prec c$, there exists an $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$ then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by* $\lim_{n\to\infty} x_n = x$ *as* $n \to \infty$.

(ii) *If for every $c \in \mathbb{C}_1$ with $0 \prec c$, there exists an $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_m) \prec c$, where $m \in \mathbb{N}$ then $\{x_n\}$ is said to be Cauchy sequence.*

(iii) *If every Cauchy sequence in X is convergent then (X, d) is said to be a complete complex valued metric space.*

Definition 1.3 [1]: *Let T and S be self mappings of a set X. If $w = Tx = Sx$ for some x in X then x is called a coincidence point of T and S and w is called a point of coincidence of T and S.*

Definition 1.4 [11]: *Let S and T be self mappings of a nonempty set X. The mappitngs S and T are weakly compatible if $STx = TSx$ whenever $Sx = Tx$.*

The following theorem is established by Sintunavarat and Kumam in 2012.

Theorem 1.1 [20]: *Let (X, d) be a complete complex valued metric space and $S, T : X \to X$.*

If there exist mappings $\Lambda, \Xi : X \to \left[0, 1\right)$ such that for all $x,\ y \in X$:

(i) $\Lambda(Sx) \leq \Lambda(x)$ *and* $\Xi(Sx) \leq \Xi(x)$;

(ii) $\Lambda(Tx) \leq \Lambda(x)$ *and* $\Xi(Tx) \leq \Xi(x)$;

(iii) $(\Lambda + \Xi)(x) < 1$;

(iv) $$d(Sx, Ty) \precsim \Lambda(x)d(x, y) + \frac{\Xi(x)d(x, Sx)d(y, Ty)}{1 + d(x, y)}.$$

Then S and T have a unique common fixed point.

2. Lemma

In this section we introduce some lemmas which are the main tools for our results.

Lemma 2.1 [3]: *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.2 [3]: *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_m)| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

Lemma 2.3 [2]: *Let X be a nonempty set and the mappings $S, T, f : X \to X$ have a unique point of coincidence v in X. If (S, f) and (T, f) are weakly compatible then S, T and f have a unique common fixed point.*

3. Main Results

In this section we prove a common fixed point theorem and give an example to justify our obtained results.

Theorem 3.1 *Let (X, d) be a complex valued metric space and $f, S, T : X \to X$. Suppose there exist mappings $\alpha_1, \alpha_2, \alpha_3 : X \to [0, 1)$ such that for all $x, y \in X$:*

(i) $\alpha_i(Sx) \leq \alpha_i(fx)$, $\alpha_i(Tx) \leq \alpha_i(fx)$, $i = 1, 2, 3$;

(ii) $\alpha_1(fx) + \alpha_2(fx) + 2\alpha_3(fx) < 1$ *and*

(iii) $$d(Sx, Ty) \precsim \alpha_1(fx)d(fx, fy) + \alpha_2(fx)\frac{d(fx, Sx)d(fx, Ty)}{1 + d(fx, fy)} + \alpha 3(fx)\frac{d(fy, Sx) + d(fx, Ty)}{1 + d(fx, fy)}.$$

If $S(X) \cup T(X) \subseteq f(X)$ and $f(X)$ is complete then f, S and T have a unique point of coincidence. Moreover, if (S, f) and (T, f) are weakly compatible then f, S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be arbitrary. Choose a point $x_1 \in X$ such that $fx_1 = Sx_0$ which is possible as $S(X) \subseteq f(X)$. Also we may choose a point $x_2 \in X$ such that $fx_2 = Tx_1$ as $T(X) \subseteq f(X)$. Continuing this process we get

$$fx_n = \begin{cases} Sx_{n-1} & \text{if } n \text{ is odd} \\ Tx_{n-1} & \text{if } n \text{ is even.} \end{cases}$$

If $n \in \mathbb{N}$ is odd then by using the hypothesis we obtain that

$$d(fx_n, fx_{n+1}) = d(Sx_{n-1}, Tx_n)$$

$$\precsim \alpha_1(fx_{n-1})d(fx_{n-1}, fx_n) + \alpha_2(fx_{n-1}) \frac{d(fx_{n-1}, Sx_{n-1})d(fx_n, Tx_n)}{1 + d(fx_{n-1}, fx_n)}$$

$$+ \alpha_3(fx_{n-1}) \frac{d(fx_n, Sx_{n-1}) + d(fx_{n-1}, Tx_n)}{1 + d(fx_{n-1}, fx_n)}$$

$$= \alpha_1(fx_{n-1})d(fx_{n-1}, fx_n) + \alpha_2(fx_{n-1}) \frac{d(fx_{n-1}, fx_n)d(fx_n, fx_{n+1})}{1 + d(fx_{n-1}, fx_n)}$$

$$+ \alpha_3(fx_{n-1}) \frac{d(fx_n, fx_n) + d(fx_{n-1}, fx_{n+1})}{1 + d(fx_{n-1}, fx_n)}$$

Therefore,

$$| d(fx_n, fx_{n+1}) | \leq \alpha_1(fx_{n-1}) | d(fx_{n-1}, fx_n) |$$

$$+ \alpha_2(fx_{n-1}) \left| \frac{d(fx_{n-1}, fx_n)d(fx_n, fx_{n+1})}{1 + d(fx_{n-1}, fx_n)} \right|$$

$$+ \alpha_3(fx_{n-1}) \left| \frac{d(fx_n, fx_n) + d(fx_{n-1}, fx_{n+1})}{1 + d(fx_{n-1}, fx_n)} \right|$$

$$\leq \alpha_1(Tx_{n-2}) \mid d(fx_{n-1}, fx_n) \mid$$

$$+ \alpha_2(Tx_{n-1}) \left| \frac{d(fx_{n-1}, fx_n)}{1 + d(fx_{n-1}, fx_n)} \right| \mid d(fx_n, fx_{n+1}) \mid$$

$$+ \alpha_3(Tx_{n-2}) \mid d(fx_{n-1}, fx_{n+1}) \mid$$

$$\leq \alpha_1(fx_{n-2}) \mid d(fx_{n-1}, fxn) \mid + \alpha_2(fx_{n-2}) \mid d(fx_n, fx_{n+1}) \mid$$

$$+ \alpha_3(fx_{n-2}) \mid d(fx_{n-1}, fx_n) \mid + \alpha_3(fx_{n-2}) \mid d(fx_n, fx_{n+1}) \mid$$

$$= \alpha_1(Sx_{n-3}) \mid d(fx_{n-1}, fx_n) \mid + \alpha_2(Sx_{n-3}) \mid d(fx_n, fx_{n+1}) \mid$$

$$+ \alpha_3(Sx_{n-3}) \mid d(fx_{n-1}, fx_n) \mid + \alpha_3(Sx_{n-3}) \mid d(fx_n, fx_{n+1}) \mid$$

$$\leq \alpha_1(fx_{n-3}) \mid d(fx_{n-1}, fx_n) \mid + \alpha_2(fx_{n-3}) \mid d(fx_n, fx_{n+1}) \mid$$

$$+ \alpha_3(fx_{n-3}) \mid d(fx_{n-1}, fx_n) \mid + \alpha_3(fx_{n-3}) \mid d(fx_n, fx_{n+1}) \mid$$

$$\vdots$$

$$\leq \alpha_1(fx_0) \mid d(fx_{n-1}, fx_n) \mid + \alpha_2(fx_0) \mid d(fx_n, fx_{n+1}) \mid$$

$$+ \alpha_3(fx_0) \mid d(fx_{n-1}, fx_n) \mid + \alpha_3(fx_0) \mid d(fx_n, fx_{n+1}) \mid,$$

which implies that

$$\mid d(fx_n, fx_{n+1}) \mid \leq \frac{(\alpha_2(fx_0) + \alpha_3(fx_0))}{(1 - \alpha_1(fx_0) - \alpha_3(fx_0))} \mid d(fx_{n-1}, fx_n) \mid \qquad (1)$$

Again if n is even then we have

$$\begin{aligned} d(fx_n, fx_{n+1}) &= d(Tx_{n-1}, Sx_n) = d(Sx_n, Tx_{n-1}) \\ &\precsim \alpha_1(fx_n) d(fx_n, fx_{n-1}) \\ &\quad + \alpha_2(fx_n) \frac{d(fx_n, Sx_n) d(fx_{n-1}, Tx_{n-1})}{1 + d(fx_n, fx_{n-1})} \\ &\quad + \alpha_3(fx_n) \frac{d(fx_{n-1}, Sx_n) + d(fx_n, Tx_{n-1})}{1 + d(fx_n, fx_{n-1})} \\ &= \alpha_1(fx_n) d(fx_n, fx_{n-1}) \\ &\quad + \alpha_2(fx_n) \frac{d(fx_n, fx_{n+1}) d(fx_{n-1}, fx_n)}{1 + d(fx_n, fx_{n-1})} \\ &\quad + \alpha_3(fx_n) \frac{d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n)}{1 + d(fx_n, fx_{n-1})}. \end{aligned}$$

Therefore,

$$\begin{aligned} |\, d(fx_n, fx_{n+1}) \,| &\leq \alpha_1(fx_n) \,|\, d(fx_n, fx_{n-1}) \,| \\ &\quad + \alpha_2(fx_n) \left| \frac{d(fx_n, fx_{n+1}) d(fx_{n-1}, fx_n)}{1 + d(fx_n, fx_{n-1})} \right| \\ &\quad + \alpha_3(fx_n) \frac{d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n)}{1 + d(fx_n, fx_{n-1})} \\ &\leq \alpha_1(fx_n) \,|\, d(fx_n, fx_{n-1}) \,| \\ &\quad + \alpha_2(fx_n) \left| \frac{d(fx_{n-1}, fx_n)}{1 + d(fx_n, fx_{n-1})} \right| |\, d(fx_n, fx_{n+1}) \,| \end{aligned}$$

$$+ \alpha_3(fx_n) \mid d(fx_{n-1}, fx_{n+1}) \mid$$

$$\leq \alpha_1(fx_n) \mid d(fx_n, fx_{n-1}) \mid + \alpha_2(fx_n) \mid d(fx_n, fx_{n+1}) \mid$$

$$+ \alpha_3(fx_n) \mid d(fx_{n-1}, fx_n) \mid + \alpha_3(fx_n) \mid d(fx_n, fx_{n+1}) \mid$$

$$= \alpha_1(Tx_{n-1}) \mid d(fx_{n-1,}\ fx_n) \mid + \alpha_2(Tx_{n-1}) \mid d(fx_n, fx_{n+1}) \mid$$

$$+ \alpha_3(Tx_{n-1}) \mid d(fx_{n-1,}\ fx_n) \mid + \alpha_3(Tx_{n-1}) \mid d(fx_n, fx_{n+1}) \mid$$

$$\leq \alpha_1(fx_{n-1}) \mid d(fx_{n-1}, fx_n) \mid + \alpha_2(fx_{n-1}) \mid d(fx_n, fx_{n+1}) \mid$$

$$+ \alpha_3(fx_{n-1}) \mid d(fx_{n-1}, fx_n) \mid + \alpha_3(fx_{n-1}) \mid d(fx_n, fx_{n+1}) \mid$$

$$= \alpha_1(Sx_{n-2}) \mid d(fx_{n-1,}\ fx_n) \mid + \alpha_2(Sx_{n-2}) \mid d(fx_n, fx_{n+1}) \mid$$

$$+ \alpha_3(Sx_{n-2}) \mid d(fx_{n-1}, fx_n) \mid + \alpha_3(Sx_{n-2}) \mid d(fx_n, fx_{n+1}) \mid$$

$$\leq \alpha_1(fx_{n-2}) \mid d(fx_{n-1}, fx_n) \mid + \alpha_2(fx_{n-2}) \mid d(fx_n, fx_{n+1}) \mid$$

$$+ \alpha_3(fx_{n-2}) \mid d(fx_{n-1}, fx_n) \mid + \alpha_3(fx_{n-2}) \mid d(fx_n, fx_{n+1}) \mid$$

$$\vdots$$

$$\leq \alpha_1(fx_0) \mid d(fx_{n-1}, fx_n) \mid + \alpha_2(fx_0) \mid d(fx_n, fx_{n+1}) \mid$$

$$+ \alpha_3(fx_0) \mid d(fx_{n-1}, fx_n) \mid + \alpha_3(fx_0) \mid d(fx_n, fx_{n+1}) \mid ,$$

which implies that

$$\mid d(fx_n, fx_{n+1}) \mid \leq \frac{(\alpha_2(fx_0) + \alpha_3(fx_0))}{(1 - \alpha_1(fx_0) - \alpha_3(fx_0))} \mid d(fx_{n-1}, fx_n) \mid . \tag{2}$$

From (1) and (2) we can conclude that for any integer

$$| d(fx_n, fx_{n+1}) | \leq \frac{(\alpha_2(fx_0)+ \alpha_3(fx_0))}{(1 - \alpha_1(fx_0) - \alpha_3(fx_0))} | d(fx_{n-1}, fx_n) |. \tag{3}$$

We set $\gamma = \frac{(\alpha_2(fx_0)+ \alpha_3(fx_0))}{(1 - \alpha_1(fx_0) - \alpha_3(fx_0))}$. Then by condition (ii), $\gamma < 1$. So by repeated application of (3) we obtain that

$$\begin{aligned} | d(fx_n, fx_{n+1}) | &\leq \gamma | d(fx_{n-1}, fx_n) | \\ &\leq \gamma^2 | d(fx_{n-2}, fx_{n-1}) | \\ &\vdots \\ &\leq \gamma^n | d(fx_0, fx_1) | . \end{aligned}$$

Now for all $m, n \in \mathbb{N}$ and $m > n$ we have

$$d(fx_n, fx_m) \precsim d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2}) + \ldots d(fx_{m-1}, fx_m).$$

Therefore,

$$\begin{aligned} | d(fx_n, fx_m) | &\leq | d(fx_n, fx_{n+1}) | + | d(fx_{n+1}, fx_{n+2}) | + \ldots + | d(fx_{m-1}, fx_m) | \\ &\leq (\gamma^n + \gamma^{n+1} + \ldots + \gamma^{m-1}) | d(fx_0, fx_1) | \\ &\leq \frac{\gamma^n}{1 - \gamma} | d(fx_0, fx_1) |. \end{aligned}$$

Since $\gamma < 1$, taking limit as $n, m \to \infty$ we have $| d(fx_n, fx_m) | \to 0$, which implies that $\{fx_n\}$ is a Cauchy sequence in $f(X)$. By completeness of $f(X)$, there exist $u, v \in X$ such that $fx_n \to v = fu$.

Now

$$d(fu, Tu) \precsim d(fu, f_{2n+1}) + d(fx_{2n+1}, Tu)$$

$$= d(fu, f_{2n+1}) + d(Sx_{2n}, Tu)$$

$$\precsim d(fu, f_{2n+1}) + \alpha_1(fx_{2n})d(fx_{2n}, fu)$$

$$+ \alpha_2(fx_{2n}) \frac{d(fx_{2n}, Sx_{2n})d(fu, Tu)}{1 + d(fx_{2n}, fu)}$$

$$+ \alpha_3(fx_{2n}) \frac{d(fu, Sx_{2n}) + d(fx_{2n}, Tu)}{1 + d(fx_{2n}, fu)}.$$

Which implies that

$$| d(fu, Tu) | \leq | d(fu, f_{2n+1}) | + \alpha_1(fx_{2n}) | d(fx_{2n}, fu) |$$

$$+ \alpha_2(fx_{2n}) | d(fx_{2n}, Sx_{2n}) | | d(fu, Tu) |$$

$$+ \alpha_3(fx_{2n})\{| d(fu, Sx_{2n}) | + | d(fx_{2n}Tu) |\}, \left[\text{as } \frac{1}{1 + d(fx_{2n}, fu)} < 1\right]$$

$$\leq | d(fu, \ f_{2n+1}) | + \alpha_1(fx_0) | d(fx_{2n}, fu) |$$

$$+ \alpha_2(fx_0) | d(fx_{2n}, Sx_{2n}) | | d(fu, Tu) |$$

$$+ \alpha_3(fx_0)\{| d(fu, Sx_{2n}) | + | d(fx_{2n}, Tu) |\}.$$

Taking $n \to \infty$, it follows that $| d(fu, Tu) | = 0$ and hence, $d(fu, Tu) = 0$.

Therefore, $fu = Tu = v$.

Similarly, we can show that $fu = Su = v$.

Thus, $fu = Su = Tu = v$ and so v becomes a common point of coincidence of *f, S* and *T*.

Uniqueness:

For uniqueness suppose there exists another point $w(f \neq v) \in X$ such that $fx = Sx = Tx = w$ for some $x \in X$.

Thus,

$$d(v, w) = d(Su, Tx)$$

$$\precsim \alpha_1(fu)d(fu, fx) + \alpha_2(fu)\frac{d(fu, Su)d(fx, Tx)}{1 + d(fu, fx)}$$

$$+ \alpha_3(fu)\frac{d(fx, Su) + d(fu, Tx)}{1 + d(fu, fx)}$$

$$= \alpha_1(v)d(v, w) + \alpha_2(v)\frac{d(v, v)d(w, w)}{1 + d(v, w)}$$

$$+ \alpha_3(v)\frac{d(w, v) + d(v, w)}{1 + d(v, w)}$$

$$= \alpha_1(v)d(v, w) + 2\alpha_3(v)d(v, w).$$

Which implies that

$$|\, d(v, w)\,| \leq \alpha_1(v)\,|\, d(v, w)\,| + 2\alpha_3(v)\,|\, d(v, w)\,|.$$

Since, $0 \leq \alpha_1(v) + 2\alpha_3(v) < 1$, it follows that $|\, d(v, w)\,| = 0$ and so $v = w$. If (S, f) and (T, f) are weakly compatible then by Lemma 2.3 *f, S* and T have a unique common fixed point in *X*. ■

Corollary 3.1: *Let* (X, d) *be a complex valued metric space and* $S, T : X \to X$. *Suppose there exist mappings* $\alpha_1, \alpha_2, \alpha_3 : X \to \left[0, 1\right)$ *such that for all* $x, y \in X$

(i) $\alpha_i(Sx) \leq \alpha_i(x)$, $\alpha_i(Tx) \leq \alpha_i(x)$ *for all* $i = 1, 2, 3$;

(ii) $\alpha_1(x) + \alpha_2(x) + 2\alpha_3(x) < 1$;

(iii) $$d(Sx, Ty) \precsim \alpha_1(x)d(x, y) + \alpha_2(x) \frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)}$$

$$+ \alpha_3(x) \frac{d(y, Sx) + d(x, Ty)}{1 + d(x, y)}.$$

Then S *and* T *have a unique common fixed point in* X.

Proof: The result follows from Theorem 3.1 by taking $f = I$, the identity mapping. ■

Corollary 3.2: *Let* (X,d) *be a complex valued metric space and* $S, T : X \to X$. *If* S *and* T *satisfy*

$$d(Sx, Ty) \precsim \lambda d(x, y) + \mu \frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + \eta \frac{d(y, Sx) + d(x, Ty)}{1 + d(x, y)}$$

for all $x, y \in X$, *where* λ, μ, η *are nonnegative reals with* $\lambda + \mu + 2\eta < 1$ *then* S *and* T *have a unique common fixed point.*

Proof: The desired result can be obtained from Theorem 3.1 by setting $\alpha_1(x) = \lambda$, $\alpha_2(x) = \mu$, $\alpha_3(x) = \eta$ and $f = I$. ■

Corollary 3.3: *Let* (X, d) *be a complex valued metric space and* $f, T : X \to X$ *be such that* $T(X) \subseteq f(X)$ *and* $f(X)$ *is complete. Suppose there exist mappings* $\alpha_1, \alpha_2, \alpha_3 : X \to \left[0, 1\right)$ *such that for all* $x, y \in X$:

(i) $\alpha_i(Tx) \leq \alpha_i(fx)$ *for all* $i = 1, 2, 3$;

(ii) $\alpha_1(fx) + \alpha_2(fx) + 2\alpha_3(fx) < 1$;

(iii) $$d(Tx, Ty) \precsim \alpha_1(fx)\, d(fx, fy) + \alpha_2(fx)\, \frac{d(fx, Tx)d(fy, Ty)}{1 + d(fx, fy)} + \alpha_3(fx)\, \frac{d(fy, Tx) + d(fx, Ty)}{1 + d(fx, fy)}.$$

Then f *and* T *have a unique point of coincidence. Moreover, if* f *and* T *are weakly compatible then* f *and* T *have a a unique common fixed point in* X.

Proof: The proof of the corollary follows from Theorem 3.1 by considering $S = T$. ■

Corollary 3.4: *Let* (X, d) *be a complete complex valued metric space and* $T : X \to X$. *Suppose there exist mappings* $\alpha_1, \alpha_2, \alpha_3 : X \to \left[0, 1\right)$ *such that for all* $x, y \in X$:

(i) $\alpha_i(Tx) \leq \alpha_i(x)$ *for all* $i = 1, 2, 3$;

(ii) $\alpha_1(x) + \alpha_2(x) + 2\alpha_3(x) < 1$;

(iii) $$d(Tx, Ty) \precsim \alpha_1(x)\, d(x, y) + \alpha_2(x)\, \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} + \alpha_3(x)\, \frac{d(y, Tx) + d(x, Ty)}{1 + d(x, y)}.$$

Then T has a unique fixed point in X.

Proof: The proof of the corollary follows from Theorem 3.1 by considering $S = T$ and $f = I$. ■

Corollary 3.5: *Let (X, d) be a complex valued metric space and $T : X \to X$. Suppose T satisfies*

$$d(Tx, Ty) \precsim \lambda d(fx, fy) + \mu \frac{d(fx, Sx)d(fy, Ty)}{1 + d(fx, fy)}$$

$$+ \eta \frac{d(fy, Sx) + d(fx, Ty)}{1 + d(fx, fy)}.$$

for all $x, y \in X$, where λ, μ, η are nonnegative reals with $\lambda + \mu + 2\eta < 1$. If $T(X) \subseteq f(X)$ and $f(X)$ is complete then f and T have a unique point of coincidence. Moreover, if f and T are weakly compatible then f and T have a unique common fixed point in X.

Proof: Putting $S = T$, $\alpha_1(x) = \lambda$, $\alpha_2(x) = \mu$, $\alpha_3(x) = \eta$ in Theorem 3.1 we can prove this result. ■

Example 3.1: *Let $X = [1, \infty)$. Define $T, f : X \to X$ b y $Tx = \frac{3x-1}{2}$ and $fx = \frac{5x-2}{3}$ in X. If d_u is the usual metric on X then T and f are not the contraction mappings on X as for all $x, y \in X$, $d_u(Tx, Ty) = \frac{3}{2} | x - y |$ and $d_u(fx, fy) = \frac{5}{3} | x - y |$.*

So we can not apply Banach contraction theorem to find the unique fixed point of T and f.

Now we consider a complex valued metric $d : X \times X \to \mathbb{C}_1$ by

$$d(x, y) = | x - y | + i | x - y |$$

Then (X, d) *is a complete complex valued metric space.*

Now,

$$d(Tx, Ty) = \frac{3}{2}\left[\left| x - y \right| + i_1 \left| x - y \right|\right]$$

$$= \frac{9}{10}\, d(fx, fy) \leq hd(fx, fy), \quad where \;\; 0 < \;\; h = \frac{9}{10} < 1 \; .$$

Since, $T(X) = f(X) = X$ *, we have all the conditions of* Corollary 3.5 *with* $\lambda = h, \;\; \mu = 0 = \eta$ *.*

So applying Corollary 3.5 *we can obtain a unique fixed point* 1 *of* T *and* f *in* X*.*

4. Future Prospect

In the line of the works as carried out in the paper one may think of the deduction of fixed point theorems using fuzzy metric, quasi metric, partial metric and other different types of metrics under the flavour of bicomplex analysis. This may be an active area of research to the future workers in this branch.

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