Journal of Indian Acad. Math. ISSN: 0970-5120
Vol. 45, No. 1 (2023) pp. 1-10

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ADDITIONAL SUMS INVOLVING GIBONACCI POLYNOMIALS

Abstract: We explore three infinite sums involving a special class of gibonacci polynomials.

Keywords: Fibonacci Polynomials, Extended Gibonacci Polynomials, Lucas Pell Polynomials, Jacobsthal Polynomial, Jacobsthal Lucas Polynomials.

Mathematical Subject Classification (2020) No.: Primary 11B37, 11B39, 11C08.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $\Delta = \sqrt{x^2 + 4}$, $2\alpha = x + \Delta$, and $2\beta = x - \Delta$.

It follows by the Binet-like formulas that $\lim_{m\to\infty} \frac{1}{g_{m+r}} = 0$ and $\lim_{m\to\infty} \frac{g_{m+r}}{g_m} = \alpha^r$.

1.1 Fundamental Gibonacci Identities: Let *k* and *r* be positive integers, and

$$\mu^* = \begin{cases} \Delta^2, & if\ \ g_n = f_n \\ 1, & otherwise. \end{cases}$$

Gibonacci polynomials then satisfy the following properties:

$$\mu^* g_{(3n-2)k} g_{(3n+1)k} = \begin{cases} l_{(6n-1)k} - (-1)^{nk} l_{3k}, & \text{if } g_n = f_n \\ l_{(6n-1)k} + (-1)^{nk} l_{3k}, & \text{otherwise;} \end{cases} \tag{1}$$

$$\mu^* g_{(3n-1)k} g_{(3n+2)k} = \begin{cases} l_{(6n+1)k} - (-1)^{(n+1)k} l_{3k}, & \text{if } g_n = f_n \\ l_{(6n+1)k} + (-1)^{(n+1)k} l_{3k}, & \text{otherwise;} \end{cases} \tag{2}$$

$$\mu^* g_{3nk} g_{(3n+3)k} = \begin{cases} l_{(6n+3)k} - (-1)^{nk} l_{3k}, & \text{if } g_n = f_n \\ l_{(6n+3)k} + (-1)^{nk} l_{3k}, & \text{otherwise;} \end{cases} \tag{3}$$

$$g_{(3n-2)k+r} g_{(3n+1)k} - g_{(3n-2)k} g_{(3n+1)k+r} = \begin{cases} (-1)^{nk} f_r f_{3k}, & \text{if } g_n = f_n \\ (-1)^{nk+1} \Delta^2 f_r f_{3k}, & \text{otherwise;} \end{cases} \tag{4}$$

$$g_{(3n-1)k+r} g_{(3n+2)k} - g_{(3n-1)k} g_{(3n+2)k+r} = \begin{cases} (-1)^{(n+1)k} f_r f_{3k}, & \text{if } g_n = f_n \\ (-1)^{(n+1)k+1} \Delta^2 f_r f_{3k}, & \text{otherwise;} \end{cases} \tag{5}$$

$$g_{3nk+r}g_{(3n+3)k} - g_{3nk}g_{(3n+3)k+r} = \begin{cases} (-1)^{nk} f_r f_{3k}, & \text{if } g_n = f_n \\ (-1)^{nk+1}\Delta^2 f_r f_{3k}, & \text{otherwise.} \end{cases} \tag{6}$$

These properties can be confirmed using the Binet-like formulas. Identities (4)-(6) are gibonacci polynomial variations of *d'Ocagne identity* [2].

It follows by identities (1)-(3) that $l_{(6n-1)k} \equiv (-1)^{nk} l_{3k} (\text{mod } \Delta^2)$, $l_{(6n+1)k} \equiv (-1)^{(n+1)k} l_{3k} (\text{mod } \Delta^2)$, and $l_{(6n+3)k} \equiv (-1)^{nk} l_{3k} (\text{mod } \Delta^2)$ respectively. Consequently, $L_{(6n-1)k} \equiv (-1)^{nk} L_{3k} (\text{mod } 5)$, $L_{(6n+1)k} \equiv (-1)^{(n+1)k} L_{3k} (\text{mod } 5)$, and $L_{(6n+3)k} \equiv (-1)^{nk} L_{3k} (\text{mod } 5)$, again respectively.

2. Telescoping Gibonacci Sums

The following lemmas play a major role in our exploration of gibonacci polynomial sums.

Lemma 1: *Let k, r, and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g^{\lambda}_{(3n-2)k+r}}{g^{\lambda}_{(3n-2)k}} - \frac{g^{\lambda}_{(3n+1)k+r}}{g^{\lambda}_{(3n+1)k}} \right] = \frac{g^{\lambda}_{k+r}}{g^{\lambda}_{k}} - \alpha^{\lambda r} \ . \tag{7}$$

Proof: With recursion [2], we will first establish that

$$\sum_{n=1}^{m} \left[\frac{g^{\lambda}_{(3n-2)k+r}}{g^{\lambda}_{(3n-2)k}} - \frac{g^{\lambda}_{(3n+1)k+r}}{g^{\lambda}_{(3n+1)k}} \right] = \frac{g^{\lambda}_{k+r}}{g^{\lambda}_{k}} - \frac{g^{\lambda}_{(3m+1)k+r}}{g^{\lambda}_{(3m+1)k}} . \tag{8}$$

To this end, we let A_m denote the left-hand side of this equation and B_m its right-hand side.

Then

$$B_m - B_{m-1} = A_m - A_{m-1} .$$

Using recursion, this implies

$$A_m - B_m = A_{m-1} - B_{m-1} = \cdots = A_1 - B_1$$

$$= 0 ,$$

confirming the validity of equation (8).

Since, $\lim_{m\to\infty} \frac{g_{m+r}}{g_m} = \alpha^r$, equation (8) yields the desired result. □

The next two lemmas can be established similarly. So, in the interest of brevity, we omit their proofs.

Lemma 2: *Let k, r, and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g^{\lambda}_{(3n-1)k+r}}{g^{\lambda}_{(3n-1)k}} - \frac{g^{\lambda}_{(3n+2)k}}{g^{\lambda}_{(3n+2)k+r}} \right] = \frac{g^{\lambda}_{2k+r}}{g^{\lambda}_{k}} - \alpha^{\lambda r} . \tag{9}$$

Lemma 3: *Let k, r, and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g^{\lambda}_{3nk+r}}{g^{\lambda}_{3nk}} - \frac{g^{\lambda}_{(3n+3)k+r}}{g^{\lambda}_{(3n+3)k}} \right] = \frac{g^{\lambda}_{3k+r}}{g^{\lambda}_{3k}} - \alpha^{\lambda r} . \tag{10}$$

3. Gibonacci Sums

Our discourse hinges on identities (1)-(6) and the above lemmas with $\lambda = 1$. To this end, in the interest of conciseness, we let

$$\mu = \begin{cases} 1, & if\ g_n = f_n \\ \Delta^2, & otherwise; \end{cases} \qquad \upsilon = \begin{cases} -1, & if\ g_n = f_n \\ 1, & otherwise; \end{cases}$$

and $\upsilon^* = -\upsilon$.

We begin our explorations with the following result, an application of Lemma 1.

Theorem 1: *Let k and r be positive integers. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{nk} \upsilon^* \Delta^2 f_r f_{3k}}{l_{(6n-1)k} + (-1)^{nk} \upsilon l_{3k}} = \frac{g_{k+r}}{g_k} - \alpha^r . \tag{11}$$

Proof: Let $g_n = f_n$. With identities (1) and (4), Lemma 1 then yields

$$\frac{(-1)^{nk} \Delta^2 f_r f_{3k}}{l_{(6n-1)k} - (-1)^{nk} l_{3k}} = \frac{f_{(3n-2)k+r} f_{(3n+1)k} - f_{(3n-2)k} f_{(3n+1)k+r}}{f_{(3n+1)k} f_{(3n-2)k}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{nk} \Delta^2 f_r f_{3k}}{l_{(6n-1)k} - (-1)^{nk} l_{3k}} = \sum_{n=1}^{\infty} \left[\frac{f_{(3n-2)k+r}}{f_{(3n-2)k}} - \frac{f_{(3n+1)k+r}}{f_{(3n+1)k}} \right]$$

$$= \frac{f_{k+r}}{f_k} - \alpha^r . \tag{12}$$

Now let $g_n = l_n$. Lemma 1, coupled with identities (1) and (4), then yields

$$\frac{(-1)^{nk+1} \Delta^2 f_r f_{3k}}{l_{(6n-1)k} + (-1)^{nk} l_{3k}} = \frac{l_{(3n-2)k+r} l_{(3n+1)k} - l_{(3n-2)k} l_{(3n+1)k+r}}{l_{(3n+1)k} l_{(3n-2)k}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{nk+1} \Delta^2 f_r f_{3k}}{l_{(6n-1)k} + (-1)^{nk} l_{3k}} = \sum_{n=1}^{\infty} \left[\frac{l_{(3n-2)k+r}}{l_{(3n-2)k}} - \frac{l_{(3n+1)k+r}}{l_{(3n+1)k}} \right]$$

$$= \frac{l_{k+r}}{l_k} - \alpha^r .$$

By combining the two cases, we get the desired result. □

It then follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{6n-1} - 4(-1)^n} = \frac{1}{20} - \frac{\sqrt{5}}{20}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{6n-1} + 4(-1)^n} = -\frac{1}{4} + \frac{\sqrt{5}}{20};$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2(6n-1)} - 18} = \frac{3}{80} - \frac{\sqrt{5}}{80}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2(6n-1)} + 18} = -\frac{1}{48} + \frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{3(6n-1)} - 76(-1)^n} = \frac{1}{170} - \frac{\sqrt{5}}{340}; \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{3(6n-1)} + 76(-1)^n} = -\frac{1}{136} + \frac{\sqrt{5}}{340}.$$

The next theorem invokes Lemma 2.

Theorem 2: *Let k and r be positive integers. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)k} \upsilon^* \Delta^2 f_r f_{3k}}{l_{(6n+1)k} + (-1)^{(n+1)k} \upsilon l_{3k}} = \frac{g_{2k+r}}{g_{2k}} - \alpha^r. \tag{13}$$

Proof: Let $g_n = f_n$. Using identities (2) and (5), and Lemma 2, we get

$$\frac{(-1)^{(n+1)k} \Delta^2 f_r f_{3k}}{l_{(6n+1)k} - (-1)^{(n+1)k} l_{3k}} = \frac{f_{(3n-1)k+r} f_{(3n+2)k} - f_{(3n-1)k} f_{(3n+2)k+r}}{f_{(3n-1)k} f_{(3n+2)k}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)k} \Delta^2 f_r f_{3k}}{l_{(6n+1)k} - (-1)^{(n+1)k} l_{3k}} = \sum_{n=1}^{\infty} \left[\frac{f_{(3n-1)k+r}}{f_{(3n-1)k}} - \frac{f_{(3n+2)k+r}}{f_{(3n+2)k}} \right]$$

$$= \frac{f_{2k+r}}{f_{2k}} - \alpha^r.$$

On the flip side, let $g_n = l_n$. With identities (2) and (5), Lemma 2 then yields

$$\frac{(-1)^{(n+1)k+1} \Delta^2 f_r f_{3k}}{l_{(6n+1)k} + (-1)^{(n+1)k} l_{3k}} = \frac{l_{(3n-1)k+r} l_{(3n+2)k} - l_{(3n-1)k} l_{(3n+2)k+r}}{l_{(3n-1)k} l_{(3n+2)k}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)k+1} \Delta^2 f_r f_{3k}}{l_{(6n+1)k} + (-1)^{(n+1)k} l_{3k}} = \sum_{n=1}^{\infty} \left[\frac{l_{(3n-1)k+r}}{l_{(3n-1)k}} - \frac{l_{(3n+2)k+r}}{l_{(3n+2)k}} \right]$$

$$= \frac{l_{2k+r}}{l_{2k}} - \alpha^r.$$

By merging the two cases, we then get the given result, as desired. □

In particular, this theorem yields

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{6n+1} + 4(-1)^n} = -\frac{3}{20} + \frac{\sqrt{5}}{20}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{6n+1} - 4(-1)^n} = \frac{1}{12} - \frac{\sqrt{5}}{20};$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2(6n+1)} - 18} = \frac{7}{240} - \frac{\sqrt{5}}{80}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2(6n+1)} + 18} = -\frac{3}{112} + \frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{3(6n+1)} + 76(-1)^n} = -\frac{9}{1,360} + \frac{\sqrt{5}}{340}; \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{3(6n+1)} + 76(-1)^n} = \frac{1}{153} - \frac{\sqrt{5}}{340}.$$

Finally, we showcase an application of Lemma 3.

Theorem 3: *Let k and r be positive integers. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{nk} \upsilon^* \Delta^2 f_r f_{3k}}{l_{(6n+3)k} + (-1)^{nk} \upsilon l_{3k}} = \frac{g_{3k+r}}{g_{3k}} - \alpha^r. \tag{14}$$

Proof: Suppose $g_n = f_n$. With identities (3) and (6), Lemma 3 then yields

$$\frac{(-1)^{nk} \Delta^2 f_r f_{3k}}{l_{(6n+3)k} - (-1)^{nk} l_{3k}} = \frac{f_{3nk+r} f_{(3n+3)k} - f_{3nk} f_{(3n+3)k+r}}{f_{3nk} f_{(3n+3)k}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{nk} \Delta^2 f_r f_{3k}}{l_{(6n+3)k} - (-1)^{nk} l_{3k}} = \sum_{n=1}^{\infty} \left[\frac{f_{3nk+r}}{f_{3nk}} - \frac{f_{(3n+3)k+r}}{f_{(3n+3)k}} \right]$$

$$= \frac{f_{3k+r}}{f_{3k}} - \alpha^r.$$

On the other hand, let $g_n = l_n$. Using identities (3) and (6), and Lemma 3, we then get

$$\frac{(-1)^{nk+1}\Delta^2 f_r f_{3k}}{l_{(6n+3)k} + (-1)^{nk} l_{3k}} = \frac{l_{3nk+r} l_{(3n+3)k} - l_{3nk} l_{(3n+3)k+r}}{l_{3nk} l_{(3n+3)k}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{nk+1}\Delta^2 f_r f_{3k}}{l_{(6n+2)k} + (-1)^{nk} l_{3k}} = \sum_{n=1}^{\infty} \left[\frac{l_{3nk+r}}{l_{3nk}} - \frac{l_{(3n+3)k+r}}{l_{(3n+3)k}} \right]$$

$$= \frac{l_{3k+r}}{l_{3k}} - \alpha^r .$$

Combining the two cases yields the desired result. □

This theorem yields

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{6n+3} - 4(-1)^n} = \frac{1}{10} - \frac{\sqrt{5}}{20}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{6n+3} + 4(-1)^n} = -\frac{1}{8} + \frac{\sqrt{5}}{20};$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2(6n+3)} - 18} = \frac{9}{320} - \frac{\sqrt{5}}{80}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2(6n+3)} + 18} = -\frac{1}{36} + \frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{3(6n+3)} - 76(-1)^n} = \frac{19}{2,890} - \frac{\sqrt{5}}{340}; \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{3(6n+3)} + 76(-1)^n} = -\frac{1}{152} + \frac{\sqrt{5}}{340}.$$

4. Gibonacci Consequences

Using the proof techniques in the lemmas, we can establish the following results:

$$\sum_{n=1}^{\infty} \left[\frac{g^{\lambda}_{(3n-2)k}}{g^{\lambda}_{(3n-2)k+r}} - \frac{g^{\lambda}_{(3n+1)k}}{g^{\lambda}_{(3n+1)k+r}} \right] = \frac{g^{\lambda}_k}{g^{\lambda}_{k+r}} - (-\beta)^{\lambda r}; \qquad (15)$$

$$\sum_{n=1}^{\infty}\left[\frac{g^{\lambda}_{(3n-1)k}}{g^{\lambda}_{(3n-1)k+r}}-\frac{g^{\lambda}_{(3n+2)k+r}}{g^{\lambda}_{(3n+2)k}}\right]=\frac{g^{\lambda}_{k}}{g^{\lambda}_{k+r}}-(-\beta)^{\lambda r};\qquad(16)$$

$$\sum_{n=1}^{\infty}\left[\frac{g^{\lambda}_{3nk}}{g^{\lambda}_{3nk+r}}-\frac{g^{\lambda}_{(3n+3)k}}{g^{\lambda}_{(3n+3)k+r}}\right]=\frac{g^{\lambda}_{3k}}{g^{\lambda}_{3k+r}}-(-\beta)^{\lambda r}\ ,\qquad(17)$$

where k, r, and λ are positive integers.

With the Binet-like formulas, we can confirm that

$$\mu^{*}g_{(3n-2)k+r}g_{(3n+1)k+r}=\begin{cases}l_{(6n-1)k+2r}-(-1)^{nk+r}l_{3k}, & \text{if } g_n=f_n\\ l_{(6n-1)k+2r}+(-1)^{nk+r}l_{3k}, & \text{otherwise;}\end{cases}\qquad(18)$$

$$\mu^{*}g_{(3n-1)k+r}g_{(3n+2)k+r}=\begin{cases}l_{(6n+1)k+2r}-(-1)^{(n+1)k+r}l_{3k}, & \text{if } g_n=f_n\\ l_{(6n+1)k+2r}+(-1)^{(n+1)k+r}l_{3k}, & \text{otherwise;}\end{cases}\qquad(19)$$

$$\mu^{*}g_{3nk+r}g_{(3n+3)k+r}=\begin{cases}l_{(6n+3)k+2r}-(-1)^{nk+r}l_{3k}, & \text{if } g_n=f_n\\ l_{(6n+3)k+2r}+(-1)^{nk+r}l_{3k}, & \text{otherwise.}\end{cases}\qquad(20)$$

In the interest of concision, we omit the proofs of all; but we encourage the gibonacci enthusiasts to confirm them.

It follows from these identities that $l_{(6n-1)k+2r}\equiv(-1)^{nk+r}l_{3k}(\mathrm{mod}\ \Delta^2)$, $l_{(6n+1)k+2r}\equiv(-1)^{(n+1)k+r}l_{3k}(\mathrm{mod}\ \Delta^2)$, and $l_{(6n+3)k+2r}\equiv(-1)^{nk+r}l_{3k}(\mathrm{mod}\ \Delta^2)$ respectively. These congruences yield $L_{(6n-1)k+2r}\equiv(-1)^{nk+r}L_{3k}(\mathrm{mod}\ 5)$, $L_{(6n+1)k+2r}\equiv(-1)^{(n+1)k+r}L_{3k}(\mathrm{mod}\ 5)$, and $L_{(6n+3)k+2r}\equiv(-1)^{nk+r}L_{3k}(\mathrm{mod}\ 5)$, again respectively.

4.1 Additional Gibonacci Sums: Sums (15)-(17), in concert with the above identities, yield the following counterparts of the gibonacci formulas in the theorems:

$$\sum_{n=1}^{\infty} \frac{(-1)^{nk} \upsilon \Delta^2 f_r f_{3k}}{l_{(6n-1)k+2r} + (-1)^{nk+r} \upsilon l_{3k}} = \frac{g_k}{g_{k+r}} - (-\beta)^r \ ; \tag{21}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)k} \upsilon \Delta^2 f_r f_{3k}}{l_{(6n+1)k+2r} + (-1)^{(n+1)k+r} \upsilon l_{3k}} = \frac{g_{2k}}{g_{2k+r}} - (-\beta)^r ; \tag{22}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{nk} \upsilon \Delta^2 f_r f_{3k}}{l_{(6n+3)k+2r} + (-1)^{nk+r} \upsilon l_{3k}} = \frac{g_{3k}}{g_{3k+r}} - (-\beta)^r \ , \tag{23}$$

respectively.

In particular, they yield

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{6n+1} + 4(-1)^n} = -\frac{3}{20} + \frac{\sqrt{5}}{20} ; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{6n+1} - 4(-1)^n} = \frac{1}{12} - \frac{\sqrt{5}}{20} ;$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{6n+3} - 4(-1)^n} = \frac{1}{10} - \frac{\sqrt{5}}{20} ; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{6n+3} + 4(-1)^n} = -\frac{1}{8} + \frac{\sqrt{5}}{20} ;$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{6n+5} + 4(-1)^n} = -\frac{7}{60} + \frac{\sqrt{5}}{20} ; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{6n+5} - 4(-1)^n} = \frac{3}{28} - \frac{\sqrt{5}}{20} ,$$

again respectively.

REFERENCES

[1] M. Bicknell (1970): A Primer for the Fibonacci Numbers: Part VII, *The Fibonacci Quarterly*, Vol. 8(4), pp. 407-420.

[2] T. Koshy (2019): Fibonacci and Lucas Numbers with Applications, Volume II, Wiley, Hoboken, New Jersey.

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(*Received*, January 18, 2023)