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TOPOLOGICAL SIMPLE ROUGH GROUPS

Abstract: In this paper, we introduce the concept of topological simple rough group using simple rough group. Also, we explore the filter of identity neighborhood in topological simple rough groups and we discuss some results related to these concepts.

Keywords: Topological Rough Group, Topological Rough Group Homomorphism, Topological Rough Normal Subgroup, Simple Rough Group, Topological Simple Rough Group, Filter of Identity Neighbourhood.

Mathematics subject Classification (2020) No.: 22A05, 54A05, 54D70, 20D05.

1. Introduction

The rough set theory, initially proposed by Pawlak (1982) [19], has been utilized as an effective mathematical tool for modeling and processing incomplete information. In recent years, rough sets have been integrated with various mathematical theories such as algebra and topology.

Algebraic structures of rough sets have been studied by several authors, including Bonikowaski. Z, Kuroki. N, Wang. PP and Li. Z *et al*. [5, 14, 15, 16]. In 1994, Biswas and Nanda [4] introduced the concept of rough group and rough subgroups, which are based on upper approximation and are independent of lower approximation. Miao *et al*. [7] have enhanced the definitions of rough group and rough subgroup, and have demonstrated their new properties. Conversely, Kuroki and Wang [15] outlined certain properties of lower and upper approximations in relation to the normal subgroups in 1996. Bagirmaz *et al*. [17] proposed the concept of topological rough groups, expanding the idea of a topological group to encompass the algebraic structures of rough groups.

In group theory, A group is simple if its only normal subgroups are the identity subgroup and the group itself [12]. The notion of a simple group was introduced by Galois about 180 years ago. Simple groups are the building blocks of all groups. In the concept of topological group, filters provide a powerful tool for understanding the topological properties like convergence, continuity, compactness, etc..

In this paper, we investigate the key principles of topological simple rough groups, which merge the structures of simple group and topological rough group. We give some examples to illustrate this concept and discuss the basis of topological simple rough group, which forms the foundation for studying their local properties, Also we explore the filter of identity neighborhoods, underscoring their role in analyzing the structure of topological simple rough groups.

2. Preliminaries

Definition 2.1 ([7]): Let U be a universe, C be a family of subsets of U, $C = \{X_1, X_2, \ldots, X_n\}$. C is called a classification of U if the following properties are satisfied:

1. $X_1 \cup X_2 \cup \ldots \cup X_n = U$;

2. $X_i \cap X_j = \phi, (i \neq j)$.

Definition 2.2 ([7]): Let $K = (U, R)$ be an approximation space and X be a subset of U. The sets

1. $\overline{X} = \{x \mid [x]_R \cap X \neq \phi\}$;

2. $\underline{X} = \{x \mid [x]_R \subseteq X\}$;

3. $BN(X) = \overline{X} - \underline{X}$

are called upper approximation, lower approximation and boundary region of X in K, respectively.

Definition 2.3 ([7]): Let $K = (U, R)$ be an approximation space and be a binary operation defined on U. A subset G of universe U is called a rough group if the following properties are satisfied:

1. $\forall x,\ y \in G, x * y \in \overline{G}$;

2. Association property holds in $\overline{G}$;

3. $\exists e \in \overline{G}$ such that $\forall x \in G, x * e = e * x = x$; e is called the rough identity element of rough group G;

4. $\forall x \in G, \exists y \in G$ such that $x * y = y * x = e$; y is called the rough inverse element of x in G;

Definition 2.4 ([7]): A non-empty subset H of rough group G is called its rough subgroup, if it is a rough group itself with respect to operation .

There is only one guaranteed trivial rough subgroup of rough group G, i.e., G itself. A necessary and sufficient condition for $\{e\}$ to be a trivial rough subgroup of rough group G is $e \in G$.

Definition 2.5 ([7]): A rough group is called a commutative rough group if for every $x, y \in G$, we have $x * y = y * x$.

Definition 2.6 ([7]): A rough subgroup N of rough group G is called a rough invariant subgroup, if $\forall a \in G, a * N = N * a$.

Definition 2.7 ([9]): Let G be a rough group and $A \subseteq G$. We say that A is symmetric if $A = A^{-1}$.

Definition 2.8 ([9]): Let $G_1 \subset U_1$ and $G_2 \subset U_2$ be rough groups. We say that G_1 and G_2 be rough homomorphism if there exists a surjection mapping $\varphi : \overline{G}_1 \to \overline{G}_2$ such that the following conditions (1)-(3) hold:

1. $\varphi \mid_{G_1}$ is a surjection mapping from G_1 to G_2;

2. For any $x, y \in G_1 \cup \{e\}$, we have $\varphi(x *_1 y) = \varphi(x) *_2 \varphi(y)$;

3. For any subset H of $G_1, \overline{H} = \varphi^{-1}\overline{(\varphi(H))}$.

If a rough homomorphism is a bijection, then we say that G_1 and G_2 are rough isomorphism.

Definition 2.9 ([3]): Let G be a topological group. A filter on G is a family η of non-empty subsets of G satisfying the two conditions:

1. If U and V are in η then $U \cap V$ is also in η;

2. If $U \in \eta$ and $U \subset W \subset G$, then $W \in \eta$.

Definition 2.10 ([3]): Let G be a topological group. A family ξ is called an open filter on G if there exists a filter η in G such that ξ is the intersection of η with the family of all open subsets of G.

Of course, this definition is equivalent to the following one: ξ is an open filter on G if ξ is a family of non-empty open subsets of G such that the intersection of any finite number of elements of ξ is also in ξ, and for each $U \in \xi$ and for every open subset W of G such that $U \subset W$, W also belongs to ξ.

Definition 2.11 ([17]): A topological rough group is a rough group $(G, *)$ together with a topology T on $\overline{G}$ satisfying the following two properties:

1. the mapping $f : G \times G \to \overline{G}$ defined by $f(x,y) = xy$ is continuous with respect to product topology on $G \times G$ and the topology T_G on $\overline{G}$ induced by T,

2. the inverse mapping $g : G \to G$ defined by $g(x) = x^{-1}$ is continuous with respect to the topology T_G on G induced by T.

Definition 2.12 ([17]): let G be a topological rough group and let H be a subgroup of G. Then, H is called a topological rough subgroup of G if

1. the mapping $f_H : H \times H \to \overline{H}$ defined by $f_H(x,y) = xy$ is continuous where $\overline{H}$ carries the topology induced by $\overline{G}$,

2. the inverse mapping $g_H : H \to H$ defined by $g_H(x) = x^{-1}$ is continuous.

Definition 2.13 ([1]): A mapping $\varphi : \overline{G_1} \to \overline{G_2}$ is called a topological rough group homomorphism, if φ is a rough homomorphism and continuous with respect to the topology τ_2 on $\overline{G_2}$ inducing τ_{G_2} on G_2 and the topology τ_1 on $\overline{G_1}$ inducing τ_{G_1} on G_1.

Definition 2.14 ([17]): Let G be a topological rough group and let N be a normal subgroup of G. Then, N is called a topological rough normal subgroup of G if $\forall a \in G, aN = Na$.

Throughout this paper, we consider X be the universal set, G_{R} be a rough group with identity e and $\overline{G}$ be the upper rough approximation of G .

3. Topological Simple Rough Group

Definition 3.1: A rough group G is called a *simple rough group* if it contains no proper non-trivial rough normal subgroups. That is, G has only the rough normal subgroups are $\{e\}$ and G .

Example 3.2: Let $X = \{1,2,3,4,5,6\}$ be the set all integers with respect to the multiplication modulo 7. A classification of X is $X/R = \{\{1,6\},\{2,3\},\{4,5\}\}$. Let $G = \{1,2,4\}$. Then $\overline{G} = X$ and $G = \phi$. Clearly, G is a rough group and it has no proper rough normal subgroups, hence G is a simple rough group.

Example 3.3: Let $X = S_4$ be the set of all permutations of {1, 2, 3, 4} with the multiplication operation of permutations. Consider a classification of X is $X/R = \{C1, C2, C3, C4\}$, where

C_1 = {(1), (12), (13), (14), (23), (24), (34)}

C_2 = {(123), (132), (124), (142), (134), (143), (234), (243)}

C_3 = {(1234), (1243), (1324), (1342), (1423), (1432)}

C_4 = {(12)(34), (13)(24), (14)(23)}

Let A_4 be the set all even permutations of S_4 that is, $A_4 = \{(1), C_2, C_4\}$. Then upper approximation of A_4, $\overline{A_4} = C_1 \cup C_2 \cup C_4$ and lower approximation of A_4, $\underline{A_4} = C_2 \cup C_4$. Hence, A_4 is a rough group.

Also we get some proper rough normal subgroups of A_4, like $\{(1)\}, \{(1), C_2\}$ and $\{(1), C_4\}$. Therefore, A_4 is not a simple rough group.

Definition 3.4: A *topological simple rough group* is a simple rough group $(G, *)$ together with a topology $\overline{\tau}$ on $\overline{G}$ satisfying the following two properties:

(i) The mapping $f : G \times G \to \overline{G}$ defined by $f(x, y) = xy,\ x, y \in G$ is continuous with respect to the product topology on $G \times G$ and the topology τ on G induced by $\overline{\tau}$

(ii) The inverse mapping $g : G \to G$ defined by $g(x) = x^{-1}$, $x \in G$ is continuous with respect to the topology τ on G induced by $\overline{\tau}$.

Example 3.5: Let $X = \{[0],[1],[2],[3],[4]\}$ be the set of residue classes of modulo 5 and $*$ be the binary operation of residue addition. A classification of X is $X/R = \{\{[0],[2]\},\{[3],[4]\}\}$. Let $G = \{[0],[1],[4]\}$, then $\overline{G} = X$ and $\underline{G} = \phi$. Obviously, G is a rough group and also G has no proper rough normal subgroups. Therefore, G is a simple rough group.

Let $\overline{\tau} = \{\phi, \overline{G}, \{[0]\}, \{[1],[2],[4]\}, \{[0],[1],[2],[4]\}$. Then we get the induced topology τ on G is $\{\phi, G, \{[0]\}, \{[1],[4]\}\}$. Hence, G is a topological simple rough group.

Proposition 3.6: Let G be a topological simple rough group and fix $x \in G$.

Then

(i) The map $L_x : G \to \overline{G}$ defined by $L_x(y) = xy$ is one-to-one and continuous, for all $y \in G$;

(ii) The map $R_x : G \to \overline{G}$ defined by $R_x(y) = yx$ is one-to-one and continuous, for all $y \in G$;

(iii) The map $f : G \to G$ defined by $f(a) = a^{-1}$ is homeomorphism, for all $a \in G$.

Proof: (i) Let $y_1, y_2 \in G$. Then $L_x(y_1) = L_x(y_2)$ implies $xy_1 = xy_2$. Since, G is a topological simple rough group and $x \in G$, $x^{-1} \in G \subseteq \overline{G}$. Thus, $x^{-1}(xy_1) = x^{-1}(xy_2)$ which implies $y_1 = y_2$. Hence L_x is one-to-one. Now let us prove L_x is continuous. Let U be an open set of xy in $\overline{G}$. Then there exists open sets V_1, V_2 of x, y in G such that $V_1V_2 \subseteq U$. Since, $xV_2 \subseteq V_1V_2 \subseteq U$, $L_x(V_2) = xV_2 \subseteq U$. Hence, L_x is continuous on G.

(ii) The proof of R_x is similar to L_x.

(iii) Since G is a topological simple rough group, the inverse mapping $f : G \to G$ is continuous. Therefore, f^{-1} is also continuous. Hence, the map f is homeomorphism of G into G. □

Proposition 3.7: Let G be a topological simple rough group. If $U \subseteq \overline{G}$ is an open set with $e \in U$, then there exists a symmetric open set V of e in G such that $VV \subseteq U$.

Proof: Since G is a topological simple rough group, the mapping $f : G \times G \to \overline{G}$ is continuous. Then $f^{-1}(U)$ is open in $G \times G$ and $(e, e) \in f^{-1}(U)$. Therefore, there exists open sets V_1, V_2 in G with $e \in V_1, e \in V_2$ such that $V_1V_2 \subseteq U$. Also the inverse mapping $g : G \to G$ is continuous, so V_1^{-1} and V_2^{-1} are open. Let $V_3 = V_1 \cap V_2$. Then V_3 is open in G and also $V_3V_3 \subseteq U$. Now we consider $V = V_3 \cap V_3^{-1}$ be an open set in G and $e \in V$. Hence, $V = V^{-1}$ and $VV \subseteq V_3V_3 \subseteq U$. □

Proposition 3.8: Let G be a topological simple rough group. Then for every open set W of e in G, there exists a symmetric open set V of e in G such that $VV \cap G \subseteq W$.

Proof: Let W be an open set of e in G. Then there exists an open set U of e in G such that $W = U \cap G$. Since the mapping $f : G \times G \to \overline{G}$ is continuous and the inverse mapping $g : G \to G$ is homeomorphism, there exists an open set V of e in G and $V = V^{-1}$ such that $VV \subseteq U$. Hence, $VV \cap G \subseteq W$. □

Proposition 3.9: Let G be a topological simple rough group. If G, $\{e\}$ are open sets of $\overline{G}$, then $\{e\}$ is open in G and G is a discrete space.

Proof: Since G is a topological simple rough group, the mapping

$f : G \times G \to \overline{G}$ is continuous and $e \in G$. Also, $\{e\}$ is open in $\overline{G}$. Thus, we get $f^{-1}(\{e\})$ is open in $G \times G$ and $ee = e \in \{e\} \in \overline{\tau}$ which implies $UV \subseteq \{e\}$, where U, V are open sets in G and $e \in U, e \in V$. Suppose $U \neq \{e\}$, we get $UU \not\subseteq \{e\}$. Hence, $U = V = \{e\}$. Let $x \in G$. Since the mapping $f : G \times G \to \overline{G}$ is continuous at (x, x^{-1}), there exists a neighborhood U of $x \in G$ such that $UU^{-1} \subseteq \{e\}$. So $UU^{-1} = \{e\}$. Hence, $U = \{x\}$. Hence, G is discrete. □

Proposition 3.10: Let G be a topological simple rough group. If G is open in $\overline{G}$, then $H = \cap \{U : U \in \tau(e)\}$ is a topological group.

Proof: Let $x, y \in H$. Then $x, y \in U$ and given $U \in \tau(e)$. Since G is a topological simple rough group and G is open in $\overline{G}$, there exists an open set $V \in \tau(e)$ such that $VV \subseteq U$. Thus, $xy \in VV \subseteq U$. Therefore, $xy \in H$. Since the inverse mapping $f : G \to G$ is homeomorphism, there exists an open set V of e in G such that $V = V^{-1}$. □

Theorem 3.11: Let X be a topological group and G be a topological simple rough group. If H is a topological rough subgroup of G, then the topological closure of H, $cl(H)$, in G is a topological rough group of G.

Proof: Let $x, y \in cl(H)$ and U be an open set of xy. Then there exists open sets V_1 and V_2 of x and y such that $V_1V_2 \subseteq U$. Since H is a topological rough subgroup of G, there exists an elements $a, b \in H$ such that $a \in V_1 \cap H$ and $b \in V_2 \cap H$. Thus, we get $ab \in V_1V_2$ and $ab \in \overline{H}$, that is $ab \in V_1V_2 \cap \overline{H}$. So, $V_1V_2 \cap \overline{H} \neq \phi$ and also $U \cap \overline{cl(H)} \neq \phi$. Hence, $xy \in \overline{cl(H)}$. Let W be an open set of x^{-1} in $cl(H)$. Then there exists an open set V of x such that $V^{-1} \subseteq W$. Since, $x \in cl(H)$, there exists an element a of H such that $a \in V$.

Then $a \in V \cap H$ which implies $a^{-1} \in V^{-1} \cap H$. So, $W \cap H \neq \phi$ and hence, $x^{-1} \in cl(H)$. □

Remark 3.12: The topological closure of H, $cl(H)$, in $\overline{G}$ is also a topological rough subgroup in $\overline{G}$.

Theorem 3.13: Let G be a topological simple rough group and G be an open set in $\overline{G}$. If S is a subset of G and U is an open set in G, then the sets $SU \cap G$ and $US \cap G$ are the open sets of S in G.

Proof: Let $x \in S$. Then there exists an open set $V \subseteq U$ of x in G such that $xV \subseteq xU \cap G$. Therefore, $\bigcup_{x \in S} xV \subseteq SU \cap G$. Hence, we get $SU \cap G$ is an open set of S in G. Likewise, $SU \cap G$ is an open set of S in G. □

Definition 3.14: Let G be a topological simple rough group and $\overline{\mathcal{B}} \subseteq \overline{\tau}$ be a base for $\overline{\tau}$. For $x \in G$, the family $\mathcal{B}x = \{U \cap G : U \in \overline{\mathcal{B}},\ x \in U\} \subseteq \overline{\mathcal{B}}$ is called a base at x in τ.

Theorem 3.15: Let G be a topological simple rough group and G be an open set in $\overline{G}$. Let ße be the family of base at e in G. Then, for every $x \in G$,

$$\mathcal{L}_x = \{(xU) \cap G : U \in \mathcal{B}_e\}, \qquad \mathcal{R}_x = \{(Ux) \cap G : U \in \mathcal{B}_e\}$$

are two families of bases at x in G.

Proof: Let $U \in \mathcal{B}_e$. Since G is a topological simple rough group and G is an open set in $\overline{G}$, $f : G \times G \to \overline{G}$ is continuous at (x, e). Then there exists an open set $V \in \mathcal{B}_e$ such that $V \subseteq U$ and $xV \subseteq G$. It is enough to prove that $xU \cap G$ and $Ux \cap G$ are open sets in G. Since the map $L_{x-1} : G \to G$ is one-to-one and continuous, $L_{x^{-1}}^{-1}(V)$ is open in G and $xV \subseteq G$. Then we get $L_{x^{-1}}^{-1}(V) = xV$ is open in G. Hence, $xV \subseteq xU$ and $xV \subseteq G$ which implies $xU \cap G$ is an open set in G. Similarly, $Ux \cap G$ is an open set in G. □

4. Filter of Identity Neighborhoods

In this section, the filter $\mathcal{F}_e$ be the set of all identity neighborhoods of G.

Proposition 4.1: Let G and H be topological simple rough groups. Then a rough homomorphism $f : \overline{G} \to \overline{H}$ is a topological rough group homomorphism if and only if it is continuous at the identity element.

Proof: Let e, e' be the identity elements in G and H respectively. Suppose f is a topological rough group homomorphism. That is, f is rough homomorphism and continuous. Since f is continuous, it is continuous at e in G. Conversely, suppose f is continuous at e. Let $a \in G$ and V be a neighborhood of $f(a)$ in H. Let us prove for any neighborhood U of a in G, $f(U) \subseteq V$. Since f is a rough homomorphism, $f(ax) = f(a).f(x)$, for all $x \in G$. Since f is continuous at e, there exists a neighborhood W of e such that $f(W) \subseteq V$. Then $U = aW$ is an open neighborhood of a and $f(U) \subseteq V$. □

Proposition 4.2: Let G and H be topological simple rough groups and $f : \overline{G} \to \overline{H}$ be a topological rough group homomorphism. Then the following conditions are equivalent:

(i) f is open

(ii) For each $W \in \mathcal{F}_e(G)$, the image $f(W)$ has a nonempty interior

(iii) There is a basis $\mathcal{B}_e$ of neighborhood W such that $f(W)$ has a nonempty interior

(iv) There is a basis $\mathcal{B}_e$ of neighborhood W in G such that $f(W)$ is an identity neighborhood in H

(v) For all $W \in \mathcal{F}_e(G)$, we have $f(W) \in \mathcal{F}_e(H)$

Proof: (i) $\Rightarrow$ (ii): Let W be an identity neighborhood in G. Since f is open, $f(W)$ is open. Also, $W \in \mathcal{F}_e(G)$. Hence, $\text{int}(f(W) \neq \phi$.

(ii) $\Rightarrow$ (iii): Suppose $f(W)$ has a nonempty interior, for each neighborhood W of e in G. Let $\mathcal{B}_e$ be a basis in G. Let $W \in \mathcal{B}_e$ and $W \in \mathcal{F}_e(G)$. Then $\text{int}(f(W)) \neq \phi$. Hence, the image $f(W)$ has a nonempty interior.

(iii) $\Rightarrow$ (iv): Let U, V be two identity neighborhoods in G such that $V \subseteq \text{int}(U)$. Then $\text{int}(f(V) \neq \phi$.

Consider $x \in V$ and $f(x) \in \text{int}(f(V)) \subseteq \text{int}\, f(\text{int}(U))$.

Let $W = \text{int}(U)x^{-1}$ and e, e' be identity elements in G, H.

Then $e = xx^{-1} \in \text{int}(U)x^{-1}$. So, W is an open neighborhood of identity element in G and

$$e' = f(x)f(x)^{-1} \in \text{int}(f(V))f(x)^{-1} \in \text{int}(f(\text{int}(U)))f(x)^{-1}$$

$$= \text{int}(f(\text{int}(U))f(x)^{-1})$$

$$= \text{int}(f(\text{int}(U)x^{-1})) .$$

$$= \text{int}(f(W)).$$

Hence, $f(W)$ is an identity neighbourhood of H.

(iv) $\Rightarrow$ (v): From the above proof, $f(W) \in \mathcal{F}_{e'}(H)$.

(v) $\Rightarrow$ (i): Obviously, this result follows from (v). □

Proposition 4.3: Let G and H be topological simple rough groups. Then the topological rough group homomorphism $f : \overline{G} \rightarrow \overline{H}$ is both continuous and open if and only if $f(\mathcal{F}_e) = \mathcal{F}_{e'}$, where $\mathcal{F}_e$ and $\mathcal{F}_{e'}$ are the filter of identity neighborhoods in G and H respectively.

Proof: Suppose the homomorphism f is continuous and open. Let prove $f(\mathcal{F}_e) = \mathcal{F}_{e'}$. Since f is continuous, $f(\mathcal{F}_e) \subseteq \mathcal{F}_{e'}$. Since f is open, $\mathcal{F}_{e'} \subseteq f(\mathcal{F}e)$. Therefore $f(\mathcal{F}_e) = \mathcal{F}_{e'}$. Conversely, if $f(\mathcal{F}_e) = \mathcal{F}_{e'}$ then f is both open and continuous. Let $U \in \mathcal{F}_e$ and $V \in \mathcal{F}_{e'}$. Then $f(U) = V$ which implies $U \subseteq f(V)$. Since U is an open neighborhood of e in G, $f^{-1}(V)$ is an open set in G. Also $f(U) \in \mathcal{F}_{e'}$ which implies $f(U)$ is an open set containing the identity element e' in H. Hence f is both open and continuous. □

Lemma 4.4: If U is an open neighborhood of the identity element in topological simple rough group G, then $U \subseteq cl(U) \subseteq UU, cl(U)$ means closure of U.

Proof: We know that $U \subseteq cl(U)$. It is enough to prove that $cl(U) \subseteq UU$. Let $a \in cl(U)$. Then there exists a symmetric neighborhood W of e in G such that $WW \subseteq U$ that is, $W \subseteq U$. Also $a \in W$ and aW is an open neighborhood of a. So $aW \cap U \neq \phi$ and $a \in aW \cap cl(U) \neq \phi$. Let $b \in aW \cap U$. Then $b = aw$, for some $w \in W$ which implies $a = bw - 1$. Since W is symmetric and $b \in U$, $w^{-1} \in W^{-1} = W \subseteq U$ and $a \in UU$. □

Theorem 4.5: (*First closure lemma*) Let G be a topological simple rough group such that G is open in $\overline{G}$. If S is a subset of G, then

$$cl(S) = \bigcap_{U \in \mathcal{F}_e} SU = \bigcap_{U \in \mathcal{F}_e} cl(SU),$$

where $\mathcal{F}_e$ be the filter of identity neighborhood in G .

Proof: Let $a \in cl(S)$ and $U \in \mathcal{F}_e$. Then $aU \cap G$ is a neighborhood of a in G and $aU \cap S \neq \phi$. Since $U = U^{-1}$, $aU^{-1} \cap S$ is a neighborhood of a. So, let $b \in aU^{-1} \cap S$ that is $b \in aU^{-1}$ and $b \in S$. Then $b = au^{-1}$, for some $u \in U$. Therefore, $a = bu \in SU \subseteq cl(SU)$. Let $a \in \bigcap cl(SU)$. Let us prove $a \in \bigcap (SU)$. Then there exists an identity neighborhood $V \in \mathcal{F}_e$ such that $VV \subseteq U$. Therefore, $a \in cl(SV) \subseteq SVV \subseteq SU$. Proceeding this process we get, $\bigcap cl(SU) \subseteq \bigcap (SU)$. Suppose $a \in \bigcap (SU)$. Let W be an identity neighborhood of a in $\mathcal{F}_e$ and $W^{-1}a = U \in \mathcal{F}_e$. Since, $a \in SU$, $a = hu$, for some $h \in S, u \in U$. Thus, $h = au - 1 \in aU^{-1} = aa^{-1}W = W$ which implies $h \in S \cap W$. Hence, $a \in cl(S)$. □

Theorem 4.6: Let G be a finite topological rough group with identity e and let $\mathcal{F}_e$ be the filter of identity neighborhood in G . Then there exists a topological rough normal subgroup N in G such that $\mathcal{F}_e = \{U \subseteq G \ / N \subseteq U\}$ and the elements are symmetric.

Proof: Consider $N = \bigcap_{U \in \mathcal{F}e} U$. Since G is finite, N is non empty in $\mathcal{F}_e$ and $e \in N$. Let $a, b \in N$. Then there exists $W \in \mathcal{F}_e$ such that $WW^{-1} \subseteq N$. Since $N \subseteq W$, $a, b \in W$. Thus, $ab^{-1} \in UU^{-1} \subseteq N$. Hence, $ab^{-1} \in N$. Let $g \in G$. Since $N \in \mathcal{F}_e$, $gN g^{-1} \in \mathcal{F}_e$ and $g^{-1}N g \in \mathcal{F}_e$. That implies $N \subseteq gN g^{-1}$ and $N \subseteq gN g^{-1}$. Hence, N is a topological rough normal subgroup. □

5. Conclusion

In this paper, we studied topological simple rough group from the simple rough group structure and given some examples. Further we investigated the basis of topological simple rough groups and discussed the concept of filters in topological simple rough groups, which are essential for a deeper understanding of their topological properties.

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NEUTROSOPHIC SEMI δ-PRE IRRESOLUTE MAPPINGS

Abstract. This paper introduces a new class of mappings called neutrosophic semi *δ-pre irresolute* mappings in neutrosophic topological spaces and discusses some of its properties and characterizations.

Keywords: Neutrosophic Set, Neutrosophic Topology, Neutrosophic Semi δ-Pre Open Sets, Neutrosophic Semi δ-Pre Continuous and Neutrosophic Semi *δ-Pre Irresolute* mappings.

Mathematics Subject Classification No.: 54A.

1. Introduction

After the introduction of fuzzy sets [15] and intuitionistic fuzzy sets [4], Smarandache [10] created a neutrosophic set on a nonempty set by considering three components, namely membership, Indeterminacy,and non-membership whose sums lie between 0 and 3. In 2008, Lupiáñez [8] introduced the neutrosophic topology as an extension of intuitionistic fuzzy topology. Since, 2008 many authors such as Lupiáñez [8], Salama *et.al*. [10, 11], Acikgoz and his coworkers [1], Dhavaseelan *et.al*. [5], Al-Musaw [2], and others have contributed to neutrosophic topological spaces. Recently many weak and strong forms of neutrosophic open sets and neutrosophic continuity have been investigated by various authors [1, 2, 5, 6, 7, 12–14]. In this paper, we introduce a new class of mappings called neutrosophic fuzzy semi δ-pre irresolute mappings and obtain some of their characterizations and properties.

2. Preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequel.

Definition 2.1 [12]: A Neutrosophic set (**NS**) in X is a structure

$$A = \{< x, \mu_A(x), \varpi_A(x), \gamma_A(x) >: x \in X\}$$

where $\mu_A : X \to]^{-}0, 1^{+}[, \varpi A : X \to]^{-}0, 1^{+}[$, and $\gamma_A : X \to]^{-}0, 1^{+}[$ denote the membership, indeterminacy, and non-membership of A, satisfying the condition that $^{-}0 \leq \mu_A(x) + \varpi_A(x) + \gamma_A(x) \leq 3^{+}, \forall x \in X$.

In real-life applications in scientific and engineering problems, using a neutrosophic set with values from a real standard or a non-standard subset of $\left]-0,\ 1+\right[$ is difficult. Hence, we consider the neutrosophic set which takes the value from the closed interval [0,1] and the sum of membership, indeterminacy, and non-membership degrees of each element of the universe of discourse lies between 0 and 3.

Definition 2.2 [10]: Let X be a nonempty set and let the neutrosophic sets A and neutrosophic set B be in the form $A = \{< x, \mu_A(x), \varpi_A(x), \gamma_A(x) >: x \in X\}$, $B = \{< x, \mu_B(x), \varpi_B(x), \gamma_B(x) >: x \in X\}$ and let $\{A_i : i \in J\}$ be an arbitrary family of neutrosophic sets in X. Then:

(a) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x), \varpi_A(x) \leq \varpi_B(x)$, and $\gamma_A(x) \geq \gamma_B(x)$.

(b) $A = B$ if $A \subseteq B$ and $B \subseteq A$.

(c) $A^c = \{< x, \gamma_A(x), \varpi_A(x), \mu_A(x) >: x \in X\}$.

(d) $\cap A_i = \{< x, \wedge \mu_{Ai}(x), \wedge \varpi_{Ai}(x), \vee \gamma_{Ai}(x) >: x \in X\}$.

(e) $\cup A_i = \{< x, \vee \mu_{Ai}(x), \vee \varpi_{Ai}(x), \wedge \gamma_{Ai}(x) >: x \in X\}$.

(f) $\tilde{\mathbf{0}} = \{< x, 0, 0, 1 >: x \in X\}$ and $\tilde{\mathbf{1}} = \{< x, 1, 1, 0 >: x \in X\}$

Definition 2.3 [8, 9]: A neutrosophic topology on a nonempty set X is a family τ of neutrosophic sets in X that satisfies the following axioms.

(NT_1) $\tilde{\mathbf{0}}$ and $\tilde{\mathbf{1}} \in \tau$

(NT_2) Finite intersection of members of τ is a member of τ

(NT_3) Anyunion of members of τ is a member of τ

In this case, the pair (X, τ) is called a neutrosophic topological space and each neutrosophic set in τ is known as a neutrosophic open set in X. The complement A^c of a neutrosophic open set A is called a neutrosophic closed set in X.

Definition 2.4 [5]: Let $\alpha, \eta, \beta \in [0,1]$ and $0 \leq \alpha + \eta + \beta \leq 3$. A neutron-sophic point $x_{(\alpha,\eta,\beta)}$ of X is a neutrosophic set in X defined by

$$x_{(\alpha\eta,\beta)}(y) = \begin{cases} (\alpha, \eta, \beta) & if \ y = x \\ (0, 0, 1) & if \ y \neq x \end{cases}$$

Definition 2.5 [1]: Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point in X and $A = \{< x, \mu_A, \varpi_A, \gamma_A > : x \in X\}$ is a neutrosophic set in X. Then $x_{\ (\alpha,\eta,\beta)} \subseteq A$ if and only if $\alpha \subseteq \mu_A(x), \eta \subseteq \varpi_A$, and $\beta \supseteq \nu_A(x)$.

Definition 2.6 [1]: A neutrosophic point $x_{(\alpha,\eta,\beta)}$ is said to be quasi-coincident (q-coincident, for short) with A, denoted by $x_{(\alpha,\eta,\beta)}qA$ iff $x_{(\alpha,\eta,\beta)} \nsubseteq A^c$. If $x_{(\alpha,\eta,\beta)}$ is not quasi-coincident with A, we denote by $\urcorner(x(_{\alpha,\eta,\beta})qA)$.

Definition 2.7 [1]: Two neutrosophic set A and B of X are said to be q-coincident (denoted by $A_q B$) if $A \nsubseteq B^c$.

Lemma 2.8 [1]: *For any two neutrosophic sets* A *and* B *of* X, $\rceil(A_q B) \Leftrightarrow A \subset B^c$ *where* $\rceil(A_q B)$ *is not q-coincident with* B.

Definition 2.9 [9]: Let (X, τ) be a *NTS* and $F \in N(X)$. Then the neutrosophic interior and neutrosophic closure of A are defined by:

$$cl(F) = \cap \{H : H \in NC(X) \text{ and } F \subseteq H\}$$

$$\text{int}(F) = \cup \{K : K \in \tau \text{ and } K \subseteq F\}$$

Definition 2.10 [3]: A neutrosophic set A of a *NTS* (X, τ) is called neutrosophic regular open (resp.neutrosophic regular closed) if $A = \text{int}(cl(A))$ (resp. $A = cl(\text{int}(A))$).

Definition 2.11 [1]: The δ-interior (denoted by $\delta \text{ int}$) (resp.δ-closure (denoted by δcl)) of a neutrosophicset A of a $NTS(X, \tau)$ is the union of all neutrosophic regular open sets contained in (resp.intersection of all neutrosophic regular closed sets containing) A.

Definition 2.12 [3, 6, 13]: A neutrosophic set A of a *NTS* (X, τ) is called neutrosophic semi open (resp.neutrosophic pre open, neutrosophic α-open, neutrosophic semi preopen, neutrosophic δ-open, neutrosophic δ-preopen, neutrosophic δ-semi open, neutrosophic b-open) if

$$A \subseteq cl(\text{int}(A))(\text{resp.}A \subseteq \text{int}(cl(A)), A \subseteq \text{int}(cl(\text{int}(A))), A \subseteq cl(\text{int}(cl(A))),$$
$$A = \delta \text{ int}(A),\ A \subseteq \text{int}(\delta cl(A)), A \subseteq cl(\delta \text{ int}(A)), A \subseteq cl(\text{int}(A)) \cup \text{int}(cl(A)).$$

Definition 2.13 [11]: A neutrosophic set A of a neutrosophic topological space (X, τ) is called neutrosophic semi δ-preopen if there exists an eutrosophic δ-pre open set O in X such that $O \subseteq A \subseteq \delta cl(O)$.

The family of all neutrosophic semi δ-pre open set so fan neutrosophic topological space (X, τ) is denoted by $NS\delta PO(X)$.

Definition 2.14 [11]: A neutrosophic set A in a neutrosophic topological space (X, τ) is called neutrosophic semi δ-preclosed) if $A^c \in NS\delta PO(X)$. The family of all neutrosophic semi δ-preclosed) sets of an neutrosophictopological space (X, τ) is denoted by $NS\delta PC(X)$.

Remark 2.15 [11]: Every neutrosophic semi preopen (resp.neutrosophic δ-preopen) set is neutrosophic semi δ-preopen. But the separate converse may not be true.

Definition 2.16 [11]: Let (X, τ) be an neutrosophic topological space and A be an neutrosophic set of X. Then the neutrosophic semi δ-preinterior (denoted by $s\delta p\,\mathrm{int}$) and neutrosophic semi δ-preclosure (denoted by $s\delta pcl$) of A respectively defined as follows:

$$s\delta p\,\mathrm{int}(A) = \cup\,\{O : O \subseteq A; O \in NS\delta PO(X)\},$$

$$s\delta pcl(A) = \cap\,\{O : O \supseteq A; O \in NS\delta PC(X)\}.$$

Definition 2.17 [11]: Let A be an neutrosophic set A of an neutrosophic topological space (X, τ) and $x_{(\alpha,\eta,\beta)}$ be an neutrosophic point of X. A is called:

(a) Neutrosophic semi δ-pre neighborhood of $x_{(\alpha,\eta,\beta)}$ if there exists an neutrosophic set $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)} \in O \subseteq A$.

(b) Neutrosophic semi δ-pre Q-neighborhood of $x_{(\alpha,\eta,\beta)}$ if there exists an neutrosophic set $O \in NS\delta PO(X)$ such that $x(\alpha, \eta, \beta) \in O \subseteq A$.

Definition 2.19 [9,11]: A mapping $f : (X, \tau) \to (Y, \sigma)$ is called:

(a) Neutrosophic continuous if $f^{-1}(A)$ is a neutrosophic open set in X for each neutrosophic open set A of Y.

(b) Neutrosophic semi δ-pre continuous if $f^{-1}(A) \in NS\delta PO(X)$ for every neutrosophic open set A of Y.

3. Neutrosophic Semi δ-preir Resolute Mappings

In this section, we introduce the concept of neutrosophic semi δ-pre irresolute mappings and study some of their properties in neutrosophic topological spaces.

Definition 3.1: A mapping f from aneutrosophic topological space (X, τ) to another neutrosophic topological space (Y, σ) is said to be neutrosophic semi δ-pre irresolute if $f^{-1}(A) \in NS\delta PO(X)$ for every neutrosophic set $\delta \in NS\delta PO(Y)$.

Remark 3.2: Every neutrosophic semi δ-pre irresolute mapping is neutrosophic semi δ-pre continuous but the converse may not be true.

Example 3.3: Let $X = \{a, b\}, Y = \{p, q\}$, and neutrosophic sets U defined as follows:

$$U = \{< a, 0.5, 0.4, 0.5 > , < b, 0.4, 0.4, 0.6 >\}$$

let $\tau = \{\tilde{0}, U, \tilde{1}\}$ and $\sigma = \{\tilde{0}, \tilde{1}\}$ be neutrosophic topologies on X and Y respectively. Then the mapping $f : (X, \tau) \to (Y, \sigma)$ defined by $f(a) = p$ and $f(b) = q$ is neutrosophic semi δ-pre continuous and hence neutrosophic continuous but not neutrosophic semi δ-pre irresolute.

Consider the following example:

Example 3.4: Example 3.4. Let $X = \{a, b\}$, $Y = \{p, q\}$, and neutrosophic sets V defined as follows:

$$V = \{< a, 0.4, 0.3, 0.6 > , < b, 0.5, 0.3, 0.5 >\}$$

let $\tau = \{\tilde{0}, \tilde{1}\}$ and $\sigma = \{\tilde{0}, V, \tilde{1}\}$ be neutrosophic topologies on X and Y respectively. Then the mapping $g : (X, \tau) \to (Y, \sigma)$ defined by $g(a) = p$ and $g(b) = q$ is neutrosophic semi δ-*pre irresolute* but not neutrosophic continuous.

Remark 3.5: Example (3.3) and Example (3.4) show that the concepts of neutrosophic semi δ-*pre irresolute* and neutrosophic continuous mappings are independent.

Theorem 3.6: *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a mapping then the following statements are equivalent:*

(a) *F is neutrosophic semi δ-pre irresolute*

(b) *If $f^{-1}(A) \in NS\delta PC(X)$ for every neutrosophic set $A \in NS\delta PC(Y)$.*

(c) *For every neutrosophic point $x_{(\alpha,\eta,\beta)}$ in X and every neutrosophic set $A \in NS\delta PO(Y)$ such that $f(x_{(\alpha,\eta,\beta)}) \in A$ there is an neutrosophic set $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)} \in O$ and $f(O) \subseteq A$.*

(d) *For every neutrosophic point $x_{(\alpha,\eta,\beta)}$ of X and every neutrosophic semi δ-pre neighborhood A of $f(x_{(\alpha,\eta,\beta)})$, $f^{-1}(A)$ is an neutrosophic semi δ-pre neighborhood of $x_{(\alpha,\eta,\beta)}$.*

(e) *For every neutrosophic point $x_{(\alpha,\eta,\beta)}$ of X and every neutrosophic semi δ-pre neighbor hood A of $f(x_{(\alpha,\eta,\beta)})$, there is an neutrosophic semi δ-pre neighborhood U of $x_{(\alpha,\eta,\beta)}$ such that $f(U) \subseteq A$.*

(f) *For every neutrosophic point $x_{(\alpha,\eta,\beta)}$ of X and every neutrosophic set $A \in NS\delta PO(Y)$ such that $f(x_{(\alpha,\eta,\beta)})_q A$, there is an neutrosophic set $O \in NS\delta PO(X)$ such that $x(\alpha, \beta)qO$ and $f(O) \subseteq A$.*

(g) *for every neutrosophic point $x_{(\alpha,\eta,\beta)}$ of X and every neutrosophic semi δ-pre Q-neighborhood A of $f(x(\alpha,\eta,\beta))$, $f^{1}(A)$ is an neutrosophic semi δ-pre Q-neighborhood of $x_{(\alpha,\eta,\beta)}$.*

(h) *for every neutrosophic point $x_{(\alpha,\eta,\beta)}$ of X and every neutrosophic semi δ-pre Q-neighborhood A of $f(x_{(\alpha,\eta,\beta)})$, there is an neutrosophic semi pre Q-neighborhood U of $x_{(\alpha,\eta,\beta)}$ such that $f(U) \subseteq A$.*

(j) $f(s\delta pcl(A)) \subseteq s\delta pcl(f(A))$, *for every neutrosophicset A of X.*

(j) $s\delta pcl(f^{-1}(O)) \subseteq f^{-1}(s\delta pcl(O))$, *for every neutrosophic set O of Y.*

(k) $f^{-1}(s\delta p\operatorname{int}(O)) \subseteq s\delta p\operatorname{int}(f^{-1}(O))$, *for every neutrosophic set O of Y*.

Proof: (a) **(b)** Obvious.

(a) ⇒ (c) Let $x_{(\alpha,\eta,\beta)}$ be an eutrosophic point of X and $A \in NS\delta PO(Y)$ such that $f(x_{(\alpha,\eta,\beta)}) \in A$. Put $O = f^{-1}(A)$, then by $(a), O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)} \in O$ and $f(O) \subseteq A$.

(c) ⇒ (a) Let $A \in NS\delta PO(Y)$ and $x_{(\alpha,\eta,\beta)} \in f^{-1}(A)$. Then $f(x_{(\alpha,\eta,\beta)}) \in A$. Now by (c) there is an eutrosophic set $O \in NS\delta PO(X)$ such that $x(\alpha,\eta,\beta) \in O$ and $f(O) \subseteq A$. Then $x_{(\alpha,\eta,\beta)} \in O \subseteq f^{-1}(A)$. Hence, $f^{-1}(A) \in NS\delta PO(X)$.

(a) ⇒ (d) Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point of X, and let A be a semi δ-pre neighborhood of $f(x_{(\alpha,\eta,\beta)})$.Then there is an eutrosophic set $U \in NS\delta PO(X)$ such that $f(x_{(\alpha,\eta,\beta)}) \in U \subseteq A$. Now $f^{-1}(U) \in NS\delta PO(X)$ and $f^{-1}(U) \subseteq f^{-1}(A)$. Thus, $f^{-1}(A)$ is an eutrosophic semi δ-pre neighborhood of $x_{(\alpha,\eta,\beta)}$ in X.

(d) ⇒ (e) Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point of X, and let A be a semi δ-pre neighborhood of $f(x_{(\alpha,\eta,\beta)})$. Then $U = f^{-1}(A)$ is an eutrosophic semi δ-pre neighborhood of $x_{(\alpha,\eta,\beta)}$ and $f(U) = f(f^{-1}(A)) \subseteq A$.

(e) ⇒ (c) Let $x_{(\alpha,\eta,\beta)}$ be an neutrosophic point of X and $A \in NS\delta PO(Y)$ such that $f(x_{(\alpha,\eta,\beta)}) \in A$. So there is neutrosophic semi δ-pre neighborhood U of $x_{(\alpha,\eta,\beta)}$ in X such that $x_{(\alpha,\eta,\beta)} \in U$ and $f(U) \subseteq A$. Hence there is an eutrosophic set $O \in NS\delta PO(X)$ such that $x(\alpha,\eta,\beta) \in O \subseteq U$ and so $f(O) \subseteq f(U) \subseteq A$.

(a) ⇒ (f) Let $x_{(\alpha,\eta,\beta)}$ be an neutrosophic point of X and $A \in NS\delta PO(Y)$ such that $f(x_{(\alpha,\eta,\beta)})_q \in A$. Let $O = f^{-1}(A)$. Then $O \in NS\delta PO(X)$, $x_{(\alpha,\eta,\beta)q}O$ and $f(O) = f(f^{-1}(A)) \subseteq A$.

(f) ⇒ (a) Let $A \in NS\delta PO(Y)$ and $x_{(\alpha,\eta,\beta)} \in f^{-1}(A)$ clearly $f(x_{(\alpha,\eta,\beta)}) \in A$ choose the Neutrosophic point x^c defined as

$$x^c{}_{(\alpha,\eta,\beta)}(z) = \begin{cases} (\beta, \eta, \alpha) & \text{if } z = x \\ (1, 1, 0) & \text{if } z \neq x \end{cases}$$

Then $f(x^c{}_{(\alpha,\eta,\beta)})_q \quad A$ and so by (f) there exists an neutrosophic set $O \in NS\delta PO(X)$ such that $x^c{}_{(\alpha,\eta,\beta)q} \quad O$ and $f(O) \subseteq A$. Now $x^c{}_{(\alpha,\eta,\beta)q}\, O$ implies $x_{(\alpha,\eta,\beta)} \in O$. Thus, $x_{(\alpha,\eta,\beta)} \subseteq f^{-1}A$. Hence, $f^{-1}(A) \in NS\delta PO(X)$.

(f) ⇒ (g) Let $x_{(\alpha,\eta,\beta)}$ be an neutrosophic point of X and A be semi δ-Q-neighborhood of $f(x(\alpha, \eta, \beta))$. Then there is a neutrosophic open set $A_1 \in NS\delta PO(Y)$ such that $f(x(\alpha, \eta, \beta))_q \subseteq A_1 \subseteq A$. By hypothesis, there is a neutrosophic set $O \in NS\delta PO(X)$ such that $x(\alpha, \eta, \beta)qO$ and $f(O) \subseteq A_1$. Thus, $x_{(\alpha,\eta,\beta)q}O \subseteq f^{-1}(A_1) \subseteq f^{-1}(A)$. Hence, $f^{-1}(A)$ is an neutrosophic semi δ-pre Q-neighborhood of $x_{(\alpha,\eta,\beta)}$.

(f) ⇒ (h) Let $x_{(\alpha,\eta,\beta)}$ be an eutrosophic point of X and A be a semi δ-pre-Q-neighborhood of $f(x(\alpha, \eta, \beta))$. Then $U = f^{-1}(A)$ is aneutrosophic semi δ-pre-Q-neighborhood of $x_{(\alpha,\eta,\beta)}$ and $f(U) = f(f^{-1}(A)) \subseteq A$.

(h) ⇒ (f) Let $x_{(\alpha,\eta,\beta)}$ be an eutrosophic point of X and $A \in NS\delta PO(Y)$ such that $f(x_{(\alpha,\eta,\beta)})_q A$. Then A is neutrosophic semi δ-pre-Q-neighborhood of $f(x(\alpha, \eta, \beta))$. So there is an eutrosophic semi δ-pre Q-neighborhood U of $x_{(\alpha,\eta,\beta)}$ such that $f(U) \subseteq A$. Now U being an eutrosophic semi δ-pre Q-neighborhood of $x_{(\alpha,\eta,\beta)}$. Then there exists an eutrosophicset $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)q}O \subseteq U$. Hence, $x_{(\alpha,\beta)q}O$ and $f(O) \subseteq f(U) \subseteq A$.

(b) ⇒ (i) Let A be an eutrosophic set of X . Since, $A = f^{-1}(f(A))$, we have

$A \subseteq f^{-1}(s\delta pcl(f^{-1}(A)))$. Now $s\delta pcl(f(A)) \in NS\delta PC(Y)$ and hence, $f^{-1}(s\delta pcl(f(A))) \in NS\delta PC(X)$. There for $es\delta pcl(A) \subseteq f^{-1}(s\delta pcl(f(A)))$ and $f(s\delta pcl(A)) \subseteq f(f^{-1}(s\delta pcl(A))) \subseteq s\delta pcl(f(A))$.

(i) ⇒ (b) Let $A \in NS\delta PC(Y)$ then $f(s\delta pcl(f^{-1}(A))) \subseteq s\delta pcl(f(f^{-1}(A))) \subseteq s\delta pcl(A) = A$. Hence, $s\delta pcl(f^{-1}(A)) \subseteq f^{-1}(A)$ and $so f^{-1}(A) \in NS\delta PC(X)$.

(i) ⇒ (j) Let O be any neutrosophic set of Y, then $f^{-1}(O)$ is an neutrosophic set of X. Therefore by hypothesis (i), $f(s\delta pcl(f^{-1}(O))) \subseteq s\delta pcl(f(f^{-1}(O))) \subseteq s\delta pcl(O)$. Hence, $s\delta pcl(f^{-1}(O)) \subseteq f^{-1}(s\delta pcl(O))$.

(j) ⇒ (i) Let A be any neutrosophic set of X, then $f^{-1}(A)$ is an neutrosophic set of Y ,and by (j), $s\delta pcl(f^{-1}(f(A))) \subseteq f^{-1}(s\delta pcl(f(A)))$. Hence, $f(s\delta pcl(A)) \subseteq s\delta pcl(f(A))$.

(a) ⇒ (k) Let O be any neutrosophic set of Y , then $s\delta p\,\mathrm{int}(O) \in NS\delta PO(Y)$ and $f^{-1}(s\delta p\,\mathrm{int}(O)) \in NS\delta PO(X)$. Since, $f^{-1}(s\delta p\,\mathrm{int}(O)) \subseteq f^{-1}(O)$, then $f^{-1}(s\delta p\,\mathrm{int}(O)) \subseteq s\delta p\,\mathrm{int}(f^{-1}(O))$.

(i) ⇒ (a) Let $O \in NS\delta PO(Y)$, then $s\delta p\,\mathrm{int}(O) = O$ and $f^{-1}(O) \subseteq s\delta p\,\mathrm{int}(f^{-1}(O))$. Thus, $f^{-1}(O) = s\delta p\,\mathrm{int}(f^{-1}(O))$ and $f^{-1}(O) \in NS\delta PO(X)$. Hence, f is neutrosophic semi δ-*pre irresolute*

Definition 3.7: A mapping $f : (X, \tau) \to (Y, \sigma)$ is called neutrosophic *R*-open if the image of every neutrosophic open set of X is neutrosophic δ-open in Y .

Theorem 3.8: *If* $f : (X, \tau) \to (Y, \sigma)$ *is neutrosophic* δ-*almost continuous and neutrosophic R-open mapping, then* f *is neutrosophic semi* δ-*pre irresolute.*

Proof: Let $A \in NS\delta PO(Y)$. Then there exist aneutrosophic set $O \in IF\delta PO(X)$ such that $O \subseteq A \subseteq \delta cl(O)$, therefore $f^{-1}(O) \subseteq f^{-1}(A) \subseteq f^{-1}(\delta cl(O)) \subseteq cl(f^{-1}(O))$ because f is neutrosophic R-open. Since f is neutrosophic δ-almost continuous and neutrosophic R-open, $f^{-1}(O) \in IF\delta PO(X)$. Hence, $f^{-1}(A) \in NS\delta PO(X)$.

Theorem 3.9: *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be neutrosophic semi δ-pre irresolute mappings then gof is neutrosophic semi δ-pre irresolute.*

Proof: Let $A \in NS\delta PO(Z)$. Since g is neutrosophic semi δ-preirresolute, $g^{-1}(A) \in NS\delta PO(Y)$. Therefore, $(gof)^{-1}(A) = f^{-1}(g^{-1}(A)) \in NS\delta PO(X)$, because f is neutrosophic semi δ-*pre irresolute*. Hence, gof is neutrosophic semi δ-*pre irresolute*.

Theorem 3.10: *Let $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic semi δ-pre irresolute and $g : (Y, \sigma) \to (Z, \eta)$ is neutrosophic semi δ-pre continuous mapping, then gof is neutrosophic semi δ-pre continuous.*

Proof: Let O be any neutrosophic open set of Z. Since g is neutrosophic semi δ-precontinuous $g^{-1}(O) \in NS\delta PO(Y)$. Therefore, $(gof)^{-1}(O) = f^{-1}(g^{-1}(O)) \in NS\delta PO(X)$ because f is neutrosophic semi δ-*pre irresolute*. Hence, gof is neutrosophic semi δ-precontinuous.

4. Conclusion

In this paper, a new class of mappings called neutrosophic fuzzy semi δ-pre irresolute mappings have been introduced, it is shown by examples that the concepts of neutrosophic fuzzy semi δ-pre irresolute mappings are stronger than the neutrosophic fuzzy semi δ-pre continuous mappings and independent of the neutrosophic fuzzy continuous mappings. Several characterizations and properties of this class of neutrosophic fuzzy mappings have been studied. In the future, we study the images and inverse images of neutrosophic compact, and neutrosophic connected spaces under these classes of mappings.

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A NOTE ON NANO FUZZY CLOSURE AND BICLOSURE SPACES

Abstract: The aim of this paper is to present, clarify and analyze Nano fuzzy closure spaces and Nano fuzzy bi-closure spaces in relation to Nano fuzzy topological spaces. We have tried to analyze the basic properties of these new types of spaces.

Keywords: Nano Fuzzy Topological Spaces, Nano Fuzzy Closure Space, Nano Fuzzy Biclosure Space.

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1. Introduction

Thivagar L. *at al*. [9, 6] introduced the concept of Nano topological spaces which were defined in terms of lower approximation, upper approximation and boundary region of a subset of a universe U using an equivalence relation on it and also defined Nano closed sets, Nano interior and Nano closure. Further, Bhuvneshwari K. *et al*. [2] introduced Nano generalized closed set in Nano topological spaces in 2014. B. A. Deole [5] has introduced Nano closure and Nano biclosure spaces in Nano topological spaces.

After the theory of fuzzy sets, given by L. Zadeh [11], fuzzfication of topological spaces was done. This work is done by C. L. Chang [4] and defined fuzzy topological spaces.

R. Navalakhe *et al*. [7] defined Nano fuzzy topological spaces with respect to a fuzzy subset λ of an universe which is defined in terms of lower and upper approximations of λ and studied Nano fuzzy closure and Nano fuzzy interior of a fuzzy subset. In this article we have presented the idea of Nano fuzzy closure spaces and Nano fuzzy bi-closure spaces and examined their characteristics.

2. Preliminaries

In this section we have narrated some of the important definition and results which are helpful in defining Nano fuzzy closure and Nano fuzzy bi-closure spaces in Nano fuzzy topological spaces.

Definition 2.1 [5]: Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U and $X \subseteq U$. Then Nano closure operator is a function: $Ncl_R: P(X) \rightarrow P(X)$ such that for all $A \subseteq X$

$$Ncl_R = \begin{cases} L_{iR}(X)\ if\ A \subseteq \quad L_{iR}(X) \\ B_{iR}(X)\ if\ A \subseteq \quad B_{iR}(X) \\ X; ot\ \ erwise\ and\ \phi\ if\ A = \phi \end{cases}$$

where L_i's are elements of $L_R(X)$ and B_i's are elements of $B_R(X)$. Which satisfies three conditions:

1. $Ncl_R(\phi)= \phi$

2. $A \subseteq Ncl_R(A)$

3. $Ncl_R(A \quad B) = Ncl_R(A) \quad Ncl_R(B)$

Hence, (X, Ncl_R) is called Nano closure space.

Definition 2.2 [7, 8]: Let U be the universe, R_1 and R_2 be equivalence relations on U. P_1 and P_2 are subsets of U. Then $\tau_{R1}(P_1)$ and $\tau_{R2}(P_2)$ satisfies the following axioms:

1. U and $\quad \in \tau_{R1}(P_1)$ and $\tau_{R2}(P_2)$.

2. The union of the elements of any sub collection of $\tau_{R1}(P_1)$ is in $\tau_{R1}(P_1)$ and $\tau_{R2}(P_2)$ is in $\tau_{R2}(P_2)$.

3. The intersection of the elements of any finite sub collection of $\tau_{R1}(P_1)$ is in $\tau_{R1}(P_1)$ and $\tau_{R2}(P_2)$ is in $\tau_{R2}(P_2)$.

Hence, $\tau_{R1}(P_1)$ and $\tau_{R2}(P_2)$ are called the Nano (τ_1, τ_2) bitopology on U with respect to $P_1$and P_2, $(U, \tau_{R_{1,2}}(X))$ is called Nano (τ_1, τ_2) bitopological space. Elements of the $\tau_{R_{1,2}}(X)$ are known as Nano $(1,2)$ open sets in U and elements of $[\tau_{R_{1,2}}(X)]^C$ are called Nano $(1,2)$ closed sets.

Definition 2.3 [1]: If $(U, \tau_{R_{1,2}}(X))$ is a Nano bitopological space with respect to X where $X \subseteq U$ and if $A \subseteq U$, then

1. The Nano $(1,2)$ closure of A is defined as the intersection of all Nano $(1,2)$ closed sets containing A and it is denoted by $N\tau_{R_{1,2}}cl(A)$. It is the smallest Nano $(1,2)$ closed set containing A.

2. The Nano $(1,2)$ interior of A is defined as the union of all Nano $(1,2)$ open subsets of A contained in A and it is denoted by $N\tau_{R_{1,2}}Int(A)$. It is the largest Nano $(1,2)$ open subset of A.

Definition 2.4 [5]: Let U be a non-empty finite set of objects called the universe and R_1 and R_2 be two equivalence relations on U and $X \subseteq U$. Then Nano closure operator is a function: $Ncl_{R_i}: P(X) \to P(X)$ where $i = \{1,2\}$ such that for all $A \subseteq X$

$$Ncl_{R_i} = \begin{cases} L_{iR}(X)\ if\ A \subseteq \quad L_{iR}(X) \\ B_{iR}(X)\ if\ A \subseteq \quad B_{iR}(X) \\ X; ot\ \ erwise\ and\ \phi\ if\ A = \phi \end{cases}$$

where L_i's are elements of $L_R(X)$ and B_i's are elements of $B_R(X)$. Which satisfies three conditions:

1. $Ncl_{R_1}(\phi) = \phi$ and $Ncl_{R_2}(\phi) = \phi$

2. $A \subseteq Ncl_{R_1}(A)$ and $A \subseteq Ncl_{R_2}(A)$

3. $Ncl_{R_1}(A \quad B) = Ncl_{R_1}(A) \quad Ncl_{R_1}(B)$ and

$Ncl_{R_2}(A \quad B) = Ncl_{R_2}(A) \quad Ncl_{R_2}(B)$.

That is there are two closure spaces (X, Ncl_{R_1}) and (X, Ncl_{R_2}). Hence, $(X, Ncl_{R_1}, Ncl_{R_2})$ is called Nano biclosure space.

Definition 2.5 [5]: Let $(X, Ncl_{R_1}, Ncl_{R_2})$ be a Nano biclosure space. A Nano biclosure space $(Y, Ncl_{R_3}, Ncl_{R_4})$ is called a Nano biclosure subspace of $(X, Ncl_{R_1}, Ncl_{R_2})$ if $Y \subseteq X$ and $Ncl_{R_j} = Ncl_{R_i} \quad Y$ for each $i = \{1,2\}$, $j = \{3,4\}$ and each subset $A \subseteq Y$.

Definition 2.6 Properties of Fuzzy Approximation Space [3, 10]: Let R be an arbitrary relation from X to Y. The lower and upper approximation operators of a fuzzy set $\underline{\mathrm{R}}$ and $\overline{\mathrm{R}}$ satisfies the following properties: for all $\alpha, \beta \in F(X)$,

(FL1) $\underline{R}(\alpha) = (\overline{R}(\alpha^C))^C$

(FU1) $\overline{R}(\alpha) = (\underline{R}(\alpha^C\ ))^C$

(FL2) $\underline{R}(\alpha \quad \beta) = \underline{R}(\alpha) \quad \underline{R}(\beta)$

(FU2) $\overline{R}(\alpha \quad \beta) = \overline{R}(\alpha) \quad \overline{R}(\beta)$

(FL3) $\alpha \leq \beta \Rightarrow \underline{R}(\alpha) \leq \underline{R}(\beta)$

(FU3) $\alpha \leq \beta \Rightarrow \overline{R}(\alpha) \leq \overline{R}(\beta)$

(FL4) $\underline{R}(\alpha \quad \beta) = \underline{R}(\alpha) \quad \underline{R}(\beta)$

(FU4) $\overline{R}(\alpha \quad \beta) = \overline{R}(\alpha) \quad \overline{R}(\beta)$

Definition 2.7 [7]: Let X be a non-empty finite set, R be an equivalence relation on *X*, $\lambda \leq X$ be a fuzzy subset and $\tau_R(\lambda) = \{1_\lambda, 0_\lambda, \underline{R}(\lambda), \overline{R}(\lambda), Bd(\lambda)\}$. Then by property (2.6), $\tau_R(\lambda)$ atisfies the following axioms

i. $0_\lambda, 1_\lambda \in \tau_{(R)}(\lambda)$ where $0_\lambda: \lambda \to I$ denotes the null fuzzy sets and $1_\lambda: \lambda \to I$ denotes the whole fuzzy set.

ii. Arbitrary union of members of $\tau_{(R)}(\lambda)$ is a member of $\tau_{(R)}(\lambda)$.

iii. Finite intersection of members of $\tau_{(R)}(\lambda)$ is a member of $\tau_{(R)}(\lambda)$.

That is, $\tau_{(R)}(\lambda)$ is a topology on X called the Nano fuzzy topology on X with respect to λ.

We call $(X, \tau_{(R)}(\lambda))$ as the Nano fuzzy topological space (NFTS). The elements of the Nano fuzzy topological space that is $\tau_{(R)}(\lambda)$, are called Nano fuzzy open sets and elements of $[\tau_{(R)}(\lambda)]^C$ are called Nano fuzzy closed sets.

Definition 2.8 [7]: Let $(X, \tau_{(R)}(\lambda))$ be a Nano fuzzy topological space with respect to λ where $\lambda \leq X$ and if $\mu \leq X$ then the Nano fuzzy interior of μ is defined as union of all Nano fuzzy open subsets of μ and it is denoted by $NfInt(\mu)$. That is, it is the largest Nano fuzzy open subset contained in μ.

Similarly, the Nano fuzzy closure of μ is defined as the intersection of all Nano fuzzy closed sets containing μ. It is denoted by $NfCl(\mu)$ and it is the smallest Nano fuzzy closed set containing μ.

Definition 2.9 [1]: Let X be a non-empty finite set, R_1 and R_2 be equivalence relations on X, $\lambda_1, \lambda_2 \leq X$ be fuzzy subsets and $\tau_{1,2_R}(\lambda) = \{\tau_{1_R}(\lambda_1), \tau_{2_R}(\lambda_2)\}$. Then $\tau_{1,2_R}(\lambda)$satisfies the following axioms:

1. $0_{\lambda_1}, 1_{\lambda_1} \in \tau_{1_R}(\lambda_1)$where $0_{\lambda_1}: \lambda_1 \to I$ denotes the null fuzzy sets and $1_{\lambda_1}: \lambda_1 \to I$ denotes the whole fuzzy set and $0_{\lambda_2}, 1_{\lambda_2} \in \tau_{2_R}(\lambda_2)$ where $0_{\lambda_2}: \lambda_2 \to I$ denotes the null fuzzy sets and $1_{\lambda_2}: \lambda_2 \to I$ denotes the whole fuzzy set.

2. Arbitrary union of members of $\tau_{1R}(\lambda_1)$ and $\tau_{2R}(\lambda_2)$ are in $\tau_{1R}(\lambda_1)$ and $\tau_{2R}(\lambda_2)$ respectively.

3. Finite intersection of members of $\tau_{1R}(\lambda_1)$ and $\tau_{2R}(\lambda_2)$ are in $\tau_{1R}(\lambda_1)$ and $\tau_{2R}(\lambda_2)$ respectively.

That is, $\tau_{1R}(\lambda_1)$ and $\tau_{2R}(\lambda_2)$ are called the Nano fuzzy bitopology $\tau_{1,2_R}(\lambda)$ on X with respect to λ_1 and λ_2. We call $\left(X, \tau_{1,2_R}(\lambda)\right)$ as the Nano fuzzy bitopological space (NFBTS). The elements of the Nano fuzzy bitopological space are called Nano fuzzy (1,2) open sets and elements of $[\tau_{1,2_R}(\lambda)]^C$ are called Nano fuzzy (1,2) closed sets.

3. Nano fuzzy Closure Spaces

Definition 3.1: Let X be a non-empty finite set of objects which called the universe and R be an equivalence relation defined on X and λ be an fuzzy subset of X. Then Nano fuzzy closure operator is a function $NfCl_R: F(X) \to F(X)$ where $F(X)$ is the set of all fuzzy subsets of X, such that for all $\mu \leq \lambda$

$$NfCl_R = \begin{cases} \overline{R}_i(\lambda)\ if\ \mu \leq \ \ \overline{R}_i(\lambda) \\ Bd_i(\lambda)\ if\ \mu \leq \ \ Bd_i(\lambda) \\ \lambda; ot\ \ erwise\ and\ 0_\lambda\ if\ \mu = 0_\lambda \end{cases}$$

Where R_i's are elements of $\overline{R}(\lambda)$ and Bd_i's are elements of $Bd(\lambda)$. Which satisfies three conditions:

1. $NfCl_R(0_\lambda) = 0_\lambda$

2. $\mu \leq NfCl_R(\mu)$

3. $NfCl_R(\mu \quad \beta) = NfCl_R(\mu) \quad NfCl_R(\beta)$

Hence, $(X, NfCl_R)$ is called Nano fuzzy closure space.

Definition 3.2: The elements of Nano fuzzy closure space are called Nano fuzzy open sets in Nano fuzzy closure spaces. The complement of Nano fuzzy open sets is called Nano fuzzy closed sets with respect to the Nano fuzzy closure space.

Definition 3.3: A fuzzy subset α of a Nano fuzzy closure space $(X, NfCl_R)$ is called Nano fuzzy closed if $\left(NfCl_R(\alpha)\right) = \alpha$.

The complement of Nano fuzzy closed set is called Nano fuzzy open.

4. Nano fuzzy Bi-closure Spaces

Definition 4.1: Let X be a non-empty finite set of objects which is called the universe and R_1 and R_2 be two equivalence relations on X and λ be any fuzzy subset of X. Then Nano fuzzy closure operator is a function: $NfCl_{R_i}: F(X) \to F(X)$, where $i = \{1,2\}$, and $F(X)$ is the set of all fuzzy subsets of X, such that for all $\mu \leq \lambda$

$$NfCl_R = \begin{cases} \overline{R}_i(\lambda)\ if\ \mu \leq \ \ \overline{R}_i(\lambda) \\ Bd_i(\lambda)\ if\ \mu \leq \ \ Bd_i(\lambda) \\ \lambda; ot\ \ erwise\ and\ 0_\lambda\ if\ \mu = 0_\lambda \end{cases}$$

Where R_i's are elements of $\overline{R}(\lambda)$ and Bd_i's are elements of $Bd(\lambda)$. Which satisfies three conditions:

1. $NfCl_{R_1}(0_\lambda) = 0_\lambda$ and $NfCl_{R_2}(0_\lambda) = 0_\lambda$

2. $\mu \leq NfCl_{R_1}(\mu)$ and $\mu \leq NfCl_{R_2}(\mu)$

3. $NfCl_{R_1}(\mu \quad \beta) = NfCl_{R_1}(\mu) \quad NfCl_{R_1}(\beta)$

and

$NfCl_{R_2}(\mu \quad \beta) = NfCl_{R_2}(\mu) \quad NfCl_{R_2}(\beta)$

That is there are two fuzzy closure spaces $(X, NfCl_{R_1})$ and $(X, NfCl_{R_2})$.

Hence, $(X, NfCl_{R_1}, NfCl_{R_2})$ is called Nano fuzzy biclosure space.

Definition 4.2: The elements of Nano fuzzy biclosure space are called Nano fuzzy open sets in Nano fuzzy bi-closure spaces. The complement of Nano fuzzy open sets is called Nano fuzzy closed sets with respect to the Nano fuzzy biclosure space.

Definition 4.3: Let $(X, NfCl_{R_1}, NfCl_{R_2})$ be a Nano fuzzy biclosure space. A Nano fuzzy biclosure space $(Y, NfCl_{R_3}, NfCl_{R_4})$ is called a Nano fuzzy biclosure subspace of $(X, NfCl_{R_1}, NfCl_{R_2})$ if $Y \subseteq X$ and $NfCl_{R_j} = NfCl_{R_i} \quad Y$ for each $i = \{1,2\}$, $j = \{3,4\}$ and each fuzzy subset $\lambda \leq Y$.

Remark 4.4: 1. Nano fuzzy open sets of Nano fuzzy bi-closure space are open in both Nano fuzzy closure spaces.

2. A fuzzy subset α of a Nano fuzzy bi-closure space $(X, NfCl_{R_1}, NfCl_{R_2})$ is called Nano fuzzy closed if $NfCl_{R_1}\left(NfCl_{R_2}(\alpha)\right) = \alpha$.

The complement of Nano fuzzy closed set is called Nano fuzzy open.

3. α is a Nano fuzzy closed subset of Nano fuzzy biclosure space $(X, NfCl_{R_1}, NfCl_{R_2})$ if and only if α is Nano fuzzy closed subset of both $(X, NfCl_{R_1})$ and $(X, NfCl_{R_2})$.

4. Let α be a Nano fuzzy closed subset of a Nano fuzzy biclosure space $(X, NfCl_{R_1}, NfCl_{R_2})$

The following conditions are equivalent.

1. $NfCl_{R_2}\left(NfCl_{R_1}(\alpha)\right) = \alpha$

2. $NfCl_{R_1}(\alpha) = \alpha, NfCl_{R_2}(\alpha) = \alpha$

Remark 4.5: Let α be a fuzzy subset of a Nano fuzzy biclosure space $(X, NfCl_{R_1}, NfCl_{R_2})$. If α is a Nano fuzzy open set in $(X, NfCl_{R_1}, NfCl_{R_2})$, then

$$NfCl_{R_1}\left(NfCl_{R_2}(1_\lambda \quad \alpha)\right) = NfCl_{R_2}\left(NfCl_{R_1}(1_\lambda \quad \alpha)\right)$$

Proposition 4.6: Let $(X, NfCl_{R_1}, NfCl_{R_2})$ be a Nano fuzzy biclosure space and let $\alpha \leq X$. Then

1. α is Nano fuzzy open if and only if $\alpha = 1_\lambda \quad \left(NfCl_{R_1}\left(NfCl_{R_2}(1_\lambda \quad \alpha)\right)\right)$

2. If α is Nano fuzzy open and $\alpha \leq \mu$, then $\alpha \leq 1_\lambda \quad \left(NfCl_{R_1}\left(NfCl_{R_2}(1_\lambda \quad \alpha)\right)\right)$.

Proof: 1. Let $(X, NfCl_{R_1}, NfCl_{R_2})$ be a Nano fuzzy biclosure space and let $\mu \leq X$ and μ is Nano fuzzy open then $1_\lambda \quad \alpha$ is Nano fuzzy closed in Nano fuzzy biclosure space. So, by definition, $NfCl_{R_1}\left(NfCl_{R_2}(1_\lambda \quad \alpha)\right) = 1_\lambda \quad \alpha$. This implies that $\alpha = 1_\lambda \quad \left(NfCl_{R_1}\left(NfCl_{R_2}(1_\lambda \quad \alpha)\right)\right)$.

2. By part (1) obvious.

Proposition 4.7: Let $(X, NfCl_{R_1}, NfCl_{R_2})$ is a Nano fuzzy biclosure space. If α and β are two Nano fuzzy closed subsets of $(X, NfCl_{R_1}, NfCl_{R_2})$. Then $\alpha \quad \beta$ is also Nano fuzzy closed in $(X, NfCl_{R_1}, NfCl_{R_2})$.

Proposition 4.8: Let $(X, NfCl_{R_1}, NfCl_{R_2})$ is a Nano fuzzy biclosure space. If α and β are two Nano fuzzy closed subsets of $(X, NfCl_{R_1}, NfCl_{R_2})$ then $\alpha \quad \beta$ is Nano fuzzy closed if $NfCl_{R_1}$ and $NfCl_{R_2}$ are disjoint.

Proof: Let α and β are two Nano fuzzy closed subsets of $(X, NfCl_{R_1}, NfCl_{R_2})$.

Then $NfCl_{R_1}\left(NfCl_{R_2}(\alpha)\right) = \alpha$ and $NfCl_{R_1}\left(NfCl_{R_2}(\beta)\right) = \beta$
Now,

$$NfCl_{R_1}\left(NfCl_{R_2}(\alpha \quad \beta)\right) = NfCl_{R_1}\left(NfCl_{R_2}(\alpha) \quad NfCl_{R_2}(\beta)\right)$$
$$= NfCl_{R_1}\left(NfCl_{R_2}(\alpha)\right) \quad NfCl_{R_1}\left(NfCl_{R_2}(\beta)\right) = \alpha \quad \beta$$

Therefore $\alpha \quad \beta$ is Nano closed if $NfCl_{R_1}$ and $NfCl_{R_2}$ are disjoint.

Proposition 4.9: If $(Y, NfCl_{R_3}, NfCl_{R_4})$ is a Nano fuzzy biclosure subspace of $(X, NfCl_{R_1}, NfCl_{R_2})$, then for every Nano fuzzy open subset ϑ of $(X, NfCl_{R_1}, NfCl_{R_2})$, $\vartheta \quad Y$ is an Nano fuzzy open set in $(Y, NfCl_{R_3}, NfCl_{R_4})$.

Proof: Let ϑ be a Nano fuzzy open set in $(X, NfCl_{R_1}, NfCl_{R_2})$, then by property we can say that ϑ is Nano fuzzy open in both $NfCl_{R_1}$ and $NfCl_{R_2}$.

Thus,

$NfCl_{R_j}(Y \quad \vartheta \quad Y) = NfCl_{R_i}(Y \quad \vartheta \quad Y) \quad Y \leq NfCl_{R_i}(X \quad \vartheta) \quad Y = (X \quad \vartheta) \quad Y = Y \quad (\vartheta \quad Y)$ for each $i = \{1,2\}, j = \{3,4\}$. Consequently, $\vartheta \quad Y$ is Nano fuzzy open in both $(Y, NfCl_{R_3})$ and $(Y, NfCl_{R_4})$. Therefore, $\vartheta \quad Y$ is Nano fuzzy open in $(Y, NfCl_{R_3}, NfCl_{R_4})$.

Proposition 4.10: Let $(X, NfCl_{R_1}, NfCl_{R_2})$ be a Nano fuzzy biclosure space and let $(Y, NfCl_{R_3}, NfCl_{R_4})$ be a Nano fuzzy biclosure subspace of $(X, NfCl_{R_1}, NfCl_{R_2})$. If α is a Nano fuzzy closed subset of $(Y, NfCl_{R_3}, NfCl_{R_4})$, then α is also a Nano fuzzy closed subset of $(X, NfCl_{R_1}, NfCl_{R_2})$.

Proof: Let α be a Nano fuzzy closed subset of $(Y, NfCl_{R_3}, NfCl_{R_4})$. Then $NfCl_{R_3}(\alpha) = \alpha$ and $NfCl_{R_4}(\alpha) = \alpha$. Since α is Nano fuzzy closed subset of both $(X, NfCl_{R_1})$ and $(X, NfCl_{R_2})$.

Consequently, α is a Nano fuzzy closed subset of both $(X, NfCl_{R_1})$ and $(X, NfCl_{R_2})$. Therefore, α is a Nano fuzzy closed subset of $(X, NfCl_{R_1}, NfCl_{R_2})$.

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A COMPARATIVE ANALYSIS OF SELJE TOPOLOGICAL SPACE WITH OTHER TOPOLOGICAL SPACES

Abstract: In recent years, numerous topologies have emerged, including the newly discovered Selje topology, which builds on micro and nano topologies. This paper offers a comparative analysis of Selje topology, emphasizing its real-world applications, particularly in analyzing dynamic systems such as climate change. The fundamental principles that link Selje, Micro and Nano topologies are discussed. The analysis demonstrates that Selje topology provides a more refined and flexible framework, allowing for greater precision in understanding complex, multifactorial systems. Key findings highlight Selje's ability to handle intricate interdependencies and scalability challenges more effectively than nano and micro topologies, making it especially valuable for studying large datasets and highly interconnected systems.

Keywords: Selje Topological Space, Micro Topology, Nano Topology, Scalability, Precision, Inclusion.

Mathematics Subject Classification: 54A05, 54B05.

1. Introduction

Topology, a branch of mathematics focused on studying properties of space that remain invariant under continuous deformations, has evolved significantly with the development of specialized topological structures. These structures have become essential in analyzing complex, multifactorial systems across diverse fields, such as engineering, medical sciences and, more recently, climate change analysis. Among the notable advancements in topological spaces are nano topology, micro topology and the newly introduced Selje topology. Each of these topologies offers unique frameworks for examining spatial relationships, continuity and the interaction of critical variables within complex systems.

Nano topology [12] introduced by Lellis Thivagar in 2013, relies on lower and upper approximations, providing a binary classification system that identifies whether elements belong to a critical or non-critical set. This straightforward structure excels in isolating key spatial elements in relatively simple systems. However, nano topology struggles with more complex and interdependent systems, as it cannot fully capture the wide range of possible relationships between elements. This limitation becomes especially pronounced in systems where variables interact dynamically and change over time.

To overcome these limitations, micro topology [10] was developed by Sakkraiveeranan in 2019. Micro topology builds on the framework of nano topology by incorporating Levine's generalized closed sets, which allow for more flexible and detailed approximations. This extension provides a deeper exploration of open and closed sets, making micro topology better suited for dynamic systems with greater complexity. While this approach offers a more refined understanding of spatial relationships, it still encounters difficulties when handling highly multifactorial systems with overlapping interdependent variables.

In 2023, Selje topology [5] introduced by Jeyanthi and Selva Nandhini, emerged as a further refinement of nano and micro topologies. It was developed to address the challenges posed by complex systems where multiple variables interact in intricate ways. Selje topology builds on the strengths of its predecessors, incorporating Selje-open and Selje-closed sets that provide even finer approximations of spatial elements. This enhanced framework allows for better handling of set intersections and scalability, making it particularly effective in studying systems that involve intricate dependencies and relationships among multiple variables.

While climate change analysis represents a key application of Selje topology, its usefulness extends beyond this field. The topology's ability to manage multifactorial systems makes it suitable for other domains as well, including biological systems where gene interactions and cellular processes are interdependent. Similarly, in network analysis, Selje topology can provide insights into the intricate relationships within social or communication networks, where multiple layers of connection and influence must be considered. By offering a more refined and adaptable approach to spatial relationships, Selje topology demonstrates significant potential for analyzing dynamic, interconnected systems across various disciplines.

This paper presents a comparative analysis of nano, micro and Selje topologies, focusing on their respective strengths and limitations in addressing the complexities of climate change. By examining the foundational theorems of each topology and applying them to climate change impact analysis, this study aims to show how Selje topology offers a deeper, more flexible understanding of multifactorial processes. The analysis highlights the critical role of topological methods in detecting and analyzing the intricate patterns and relationships that define dynamic systems, emphasizing the practical utility of these frameworks in contemporary scientific research.

Preliminaries

Definition 2.1: *Let $\mathfrak{V}$ denote a non-empty finite set of objects referred to as the universe and let $\mathcal{R}$ represent an equivalence relation on $\mathfrak{V}$ known as the indiscernibility relation. Elements within the same equivalence class are considered indiscernible from each other. This pair, denoted as $(\mathfrak{V}, \mathcal{R})$, constitutes the approximation space.*

Let $\mathfrak{E}$ be a subset of $\mathfrak{V}$.

1. *The lower approximation of $\mathfrak{E}$ with respect to $\mathcal{R}$, denoted as $\mathcal{L}_{\mathcal{R}}(\mathfrak{E})$, consists of all objects that can definitively be classified as belonging to $\mathfrak{E}$ under the influence of $\mathcal{R}$. In mathematical terms, $\mathcal{L}_{\mathcal{R}}(\mathfrak{E}) = \cap\{\mathcal{R}(\mathcal{X}) : \mathcal{R}(\mathcal{X}) \subseteq \mathfrak{E}\}$ where $\mathcal{R}$ signifies the equivalence class determined by $\mathfrak{E}$.*

2. *The upper approximation of $\mathfrak{E}$ with respect to $\mathcal{R}$, denoted as $\mathcal{U}_{\mathcal{R}}(\mathfrak{E})$, comprises all objects that could potentially be classified as $\mathfrak{E}$ under the influence of $\mathcal{R}$. Mathematically, $\mathcal{U}_{\mathcal{R}}(\mathfrak{E}) = \cap\{\mathcal{R}(\mathcal{X}) : \mathcal{R}(\mathcal{X}) \cap \mathfrak{E} \neq \Phi\}$*

3. *The boundary region of $\mathfrak{E}$ with respect to $\mathcal{R}$, denoted as $\mathcal{B}_{\mathcal{R}}(\mathfrak{E})$, includes all objects that cannot be definitively classified as either belonging to $\mathfrak{E}$ or not belonging to $\mathfrak{E}$ under the influence of $\mathcal{R}$. In mathematical terms, $\mathcal{B}_{\mathcal{R}}(\mathfrak{E}) = \mathcal{U}_{\mathcal{R}}(\mathfrak{E}) - \mathcal{L}_{\mathcal{R}}(\mathfrak{E})$*

Definition 2.2: *Let* $\mathfrak{V}$ *represent the universe, an equivalence relation on* $\mathfrak{V}$ *denote* $\mathcal{R}$ *and* $T_{\mathcal{R}}(\mathfrak{E}) = \{\mathfrak{E}, \Phi, \mathcal{L}_{\mathcal{R}}(\mathfrak{E}), \mathcal{U}_{\mathcal{R}}(\mathfrak{E}), \mathcal{B}_{\mathcal{R}}(\mathfrak{E})\}$, *where* $\mathfrak{E} \subseteq \mathfrak{V}$. *Under these conditions,* $\mathcal{R}(\mathcal{X})$ *Proceeding with the given postulates*:

1. Φ *and* $\mathfrak{V}$ *belong to* $\mathcal{T}_{\mathcal{R}}(\mathfrak{E})$.

2. *Any subset of the union of elements* $\mathcal{T}_{\mathcal{R}}(\mathfrak{E})$ *remains within* $\mathcal{T}_{\mathcal{R}}(\mathfrak{E})$.

3. *Any finite subset of the intersection of elements* $\mathcal{T}_{\mathcal{R}}(\mathfrak{E})$ *is contained in* $\mathcal{T}_{\mathcal{R}}(\mathfrak{E})$. *In other words,* $\mathcal{T}_{\mathcal{R}}(\mathfrak{E})$ *forms a topology on* $\mathfrak{V}$ *known as the nano topology on* $\mathfrak{V}$ *concerning* $\mathfrak{E}$. $(\mathfrak{V}, \mathcal{T}_{\mathcal{R}}(\mathfrak{E}))$ *constitutes the nano topological space. The sets within* $\mathcal{T}_{\mathcal{R}}(\mathfrak{E})$ *are denoted as nano open sets and the dual nano topology of* $[\mathcal{T}_{\mathcal{R}}(\mathfrak{E})]$ *is represented by* $[\mathcal{T}_{\mathcal{R}}(\mathfrak{E})]^c$.

In this context, $\mathfrak{T}_{\mathcal{R}}(\mathfrak{E})$ *is termed the Nano Topology* [5] *of the universal set* $\mathfrak{V}$ *with respect to the subset* $\mathfrak{E}$. *The pair* $(\mathfrak{V}, \mathfrak{T}_{\mathcal{R}}(\mathfrak{E}))$ *constitutes a nano topological space and its constituent elements are referred to as nano-open sets.*

Definition 2.3: $(\mathfrak{V}, \mathfrak{T}_{\mathcal{R}}(\mathfrak{E}))$ *creates a nanotopological space. In this case, the set* $\mathfrak{Y}\mathcal{R}(\mathfrak{E})$ *consists of two groups, namely* $\{\mathfrak{N} \cup (\mathfrak{N}' \cap \mathfrak{Y}) : \mathfrak{N}, \mathfrak{N}' \in \mathfrak{T}(\mathfrak{E})\}$. *The combination* $\mathfrak{T}_{\mathcal{R}}(\mathfrak{E})$ *is expressed as the microtopology* $\mathfrak{Y}$; *where* $\mathfrak{Y}$ *is not nanotopology elements of* $\mathfrak{T}_{\mathcal{R}}(\mathfrak{E})$.

Definition 2.4: *Micro Topology* $\mathfrak{Y}\mathcal{R}(\mathfrak{E})$ *adheres to the following postulates*:

1. *Both the universal set* $(\mathfrak{E})$ *and the empty set* Φ *are elements of* $\mu_{\mathcal{R}}(\mathfrak{E})$.
2. *Any subset of the union of elements of* $\mu_{\mathcal{R}}(\mathfrak{E})$ *remains within* $\mu_{\mathcal{R}}(\mathfrak{E})$.

3. *Any finite subset of the intersection of elements of* $\mu_{\mathcal{R}}(\mathfrak{E})$ *is contained within* $\mu_{\mathcal{R}}(\mathfrak{E})$. *Thus, the Micro topology* $\mu_{\mathcal{R}}(\mathfrak{E})$ *is defined as* $\mu_{\mathcal{R}}(\mathfrak{E}) = \{\mathfrak{N} \cup (\mathfrak{N}' \cap \mu)\}$ *for* $\mathfrak{N}$ *and* $\mathfrak{N}' \in \mu_{\mathcal{R}}(\mathfrak{E})$, *where* $\mu \notin \mathfrak{T}_{\mathcal{R}}(\mathfrak{E})$. *This constitutes the Micro topology on the set* $\mathfrak{V}$ *concerning* $\mathfrak{E}$.

The trio $(\mathfrak{V}, \mathfrak{T}_{\mathcal{R}}(\mathfrak{E}),\ \mu_{\mathcal{R}}(\mathfrak{E}))$ *is denoted as the Micro topological space and the elements of* $\mu_{\mathcal{R}}(\mathfrak{E})$ *are known as Mic-open sets. Moreover, the complement of a Mic-open set is defined as a Mic-closed set.*

Next, $\mathfrak{Y}_{\mathcal{R}}(\mathfrak{E})$ *is called the microtopology of* $\mathfrak{E}$ *and* $\mathfrak{V}$. *Triple* $(\mathfrak{E}, \mathfrak{T}_{\mathcal{R}}(\mathfrak{E}), \mathfrak{Y}_{\mathcal{R}}(\mathfrak{E}))$ *called micro-topological space. Elements in* $\mathfrak{Y}_{\mathcal{R}}(\mathfrak{E})$ *are slightly open and their complements are slightly off.*

Definition 2.5: *Consider the microtopological space* $(\mathfrak{V}, \mathfrak{Y}_{\mathcal{R}}(\mathfrak{E}))$ *and Selje topology be defined as* $SJ_{\mathcal{R}}(\mathfrak{E}) = \{(S - \boldsymbol{J}) \cup (S - J') : S \in \mathfrak{Y}_{\mathcal{R}}(\mathfrak{E})$ *and for fixed* $\boldsymbol{J}, J' \notin \mathfrak{Y}_{\mathcal{R}}(\mathfrak{E}), \boldsymbol{J} \cup J' = \mathfrak{V}\}$

Definition 2.6: *The Selje topology* $SJ_{\mathcal{R}}(\mathfrak{E})$ *satisfies the following axioms*

1. *Both the universal set* $\mathfrak{V}$ *and the empty set* Φ *are elements of* $\mathfrak{T}_{\mathcal{R}}(\mathfrak{E})$.

2. *Any subset of the union of elements from* $SJ_{\mathcal{R}}(\mathfrak{E})$ *remains within* $SJ_{\mathcal{R}}(\mathfrak{E})$.

3. *Any finite subset of the intersection of elements within* $SJ_{\mathcal{R}}(\mathfrak{E})$ *is contained within* $SJ_{\mathcal{R}}(\mathfrak{E})$.

The triplet $(\mathfrak{E}, \mathfrak{Y}_{\mathcal{R}}(\mathfrak{E}), SJ_{\mathcal{R}}(\mathfrak{E}))$ *is labeled as Selje topological space. Then, the components of Selje topology are Selje-Open* ($S\boldsymbol{J}$ *-Open) sets and* *their complements are Selje-closed* (SJ *-closed*) *sets. The collection of Selje closed sets of Selje topology is denoted as* $SJCL(\mathfrak{C})$.

3. Theoretical Foundations and Comparative Analysis of Nano, Micro and Selje Topologies

The theorems compare nano, micro and Selje Topological Spaces, showing that Selje Topology offers finer approximations, better scalability for complex systems and generalizes the other two. They demonstrate why Selje Topological Space is superior for handling complex, multifactorial applications with improved precision and flexibility.

Theorem 3.1 establishes a hierarchical relationship between nano, micro and Selje Topological Spaces, showing that Selje topological space provides the most refined approximations, followed by micro and nano topologies. The inclusions between closures and interiors reflect the increasing precision of each space.

Theorem 3.1: Inclusion in Nano, Micro and Selje Topologies: *Let* $X \subseteq U$ *be a subset in the universe U. The relationships between the approximations in nano, micro and Selje topologies are given by*:

$$L_R(X) \subseteq Mic-cl(X) \subseteq SJ_R-cl(X)$$

and

$$SJ_R-\text{int}(X) \subseteq Mic-\text{int}(X) \subseteq U_R(X)$$

where $LR(X)$ *and* $UR(X)$ *are the lower and upper approximations in nano topology,* $Mic-cl(X)$ *and* $Mic-\text{int}(X)$ *are the micro closure and interior in micro topology and* $SJ_{\mathcal{R}}-cl(X)$ *and* $SJ_{\mathcal{R}}-\text{int}(X)$ *are the Selje closure and interior, respectively.*

Proof: In nano topology, $L_{\mathcal{R}}(X) \subseteq X \subseteq U_{\mathcal{R}}(X)$.

In micro topology, $L_{\mathcal{R}}(X) \subseteq Mic-cl(X)$ and $Mic-\text{int}(X) \subseteq UR(X)$.

In Selje topology, $Mic-cl(X) \subseteq SJ_{\mathcal{R}}-cl(X)$

and $SJ_{\mathcal{R}}-\mathrm{int}(X) \subseteq Mic-\mathrm{int}(X)$.

Thus, the theorem follows. □

Lemma 3.2 states that if a function is continuous in nano topology, it will also be continuous in both micro and Selje topologies. This is because micro and Selje topologies generalize the structures of nano topology, preserving the continuity of functions across these spaces.

Lemma 3.2 Preservation of Continuity in Micro and Selje Topologies: *If* $f : U \to V$ *is continuous in nano topology, then f is continuous in both micro and Selje topological spaces.*

Proof: In nano topology, $f^{-1}(V') \in \tau_R(U)$ for any nano-open set $V' \subseteq V$. In micro topology, since micro-open sets are unions or intersections of nano-open sets, $f^{-1}(W') \in \mu_R(U)$. Similarly, in Selje topology, $f^{-1}(S') \in SJ_R(U)$. Hence, f is continuous in both micro and Selje topologies. □

Theorem 3.3 demonstrates that Selje Topological Space scales better than nano and micro topologies. As system complexity increases, Selje retains higher precision in approximating sets, making it ideal for complex systems.

Scalability here refers to how well the different topologies handle an increase in system complexity. As the complexity of the dataset (e.g., the number of variables, the amount of data) increases, the precision of approximations made by each topology changes.

Theorem 3.3 Scalability of Approximations: *For any subset* $A \subseteq U$, *we have*:

$$\lim_{complexity(A)\to\infty} precision(SJ_R-cl(A)) > precision(Mic-cl(A)) > precision(L_R(A))$$

Proof: 1. In nano topology, precision$(L_R(A)) = \dfrac{|L_R(A)|}{|A|}$ tends to 0 as $|A| \to \infty$.

2. In micro topology, precision$(Mic-cl(A)) = \dfrac{|Mic-cl(A)|}{|A|}$, which is more precise than in nano topology.

3. In Selje topology, precision $(SJ-cl(A)) = \dfrac{|SJ_R-cl(A)|}{|A|}$, which remains precise even as complexity increases. □

Theorem 3.4 shows that the intersection of Selje-open sets provides a finer approximation than nano-open or micro-open sets. Selje Topology captures more intricate relationships, making it more powerful for handling complex data.

Theorem 3.4 Finer Set Operations in Selje Topology: *For any subsets* $A, B \subseteq U$ *, the intersection of Selje-open sets provides a finer approximation than the intersection of micro-open or nano-open sets*:

$$SJ_R - int(A \cap B) \subseteq Mic{-}int(A \cap B) \subseteq L_R(A \cap B)$$

Proof: 1. In nano topology, $L_R(A \cap B) = \{x \in U \mid x \in L_R(A) \cap L_R(B)\}$.
2. In micro topology, $Mic{-}int(A \cap B) = \{x \in U \mid x \in Mic{-}int(A) \cap Mic{-}int(B)\}$.
3. In Selje topology, $SJ_R{-}int(A \cap B) = \{x \in U \mid x \in SJ_R{-}int(A) \cap SJ_R{-}int(B)\}$, thus, providing a finer approximation. □

The below corollary states that Selje Topological Space generalizes both nano and micro topologies, but not all Selje-open sets are nano-open or micro-open, offering a broader and more flexible structure.

Corollary: (Generalization of Nano and Micro Topologies). *Selje Topological Space generalizes both nano and micro topologies. Every nano-open and micro-open set is a Selje-open set, but not every Selje-open set is nano-open or micro-open.*

Proof: By the definition of Selje Topology, $\tau_R(U) \subseteq \mu_R(U) \subseteq SJ_R(U)$, meaning all nano-open and micro-open sets belong to the Selje Topology. However, Selje-open sets can contain additional elements that nano and micro topologies cannot capture. □

4. Topological Analysis of Climate Change Impact: A Comparative Study Using Nano, Micro and Selje Topologies

This application focuses on differentiating three topological spaces-nano topology, micro topology and Selje Topology-through the lens of climate change impact analysis. Climate change, a multifactorial process, affects various sectors like agriculture, health and the economy, with factors such as temperature rise, rainfall patterns and sea level rise influencing different regions in diverse ways.

By modeling these factors within each topological space, we aim to identify which regions and sectors are most affected. The process involves analyzing key climate-related variables, applying each topological method to assess their significance and comparing the results to determine how each topology captures critical factors. The comparison highlights the strengths of each topology, with special focus on how Selje Topology refines the relationships between variables, offering a more detailed and precise analysis compared to nano and micro topologies. In the end, the betterment of each topological space is analyzed, showing how they differ in precision, scalability and flexibility in identifying the most impactful factors of climate change on different regions.

4.1 Methodology for Topological Analysis of Climate Change Impact: The following structured steps outline the methodology used for applying nano, micro and Selje topologies to analyze climate change impacts:

- ***Data Preparation*****:** Collected and standardized climate data, focusing on critical factors such as temperature rise, rainfall patterns, sea level rise, greenhouse gas emissions, deforestation and other socio-economic variables across various regions and sectors. This data was organized to ensure consistency and comparability across different regions.

- **Topological Space Application:** Applied nano, micro and Selje Topological Spaces to the climate data to assess the relationships between the key factors. The topologies were used to study how these factors interact and influence one another in various regions, allowing for the identification of underlying patterns in the data. Special attention was paid to how the different topological spaces handle these relationships, particularly their set approximations and scalability.

- ***Critical Factor Identification*:** Determined the most significant climate-related factors for each region by analyzing the topological spaces. The analysis focused on identifying which variables-such as temperature rise, rainfall variability, or deforestation-had the greatest impact on environmental, economic and health outcomes in specific regions.

- ***Visualization and Analysis*:** Generated diagrams, tables and comparative metrics to visualize the relationships between climate factors and the regions they affect. These visualizations highlight the differences in performance between nano, micro and Selje topologies. (Add visual aids such as graphs comparing factor influence across regions for each topology to show how Selje provides deeper insights.)

- ***Comparison of Topologies*:** Compared the efficiency and flexibility of nano, micro and Selje Topologies in analyzing the climate change impact. This comparison focused on determining which topology provided the most accurate and scalable analysis for multi- factorial climate systems. Results showed that while all three topologies identified key variables, Selje topology allowed for more detailed insights into variable interactions, offering superior scalability and precision in the analysis of complex datasets.

4.2 Topological Analysis of Climate Change Impact: The table below presents the data collected for climate change impact analysis. This data is then processed to compare the performance of nano, micro and Selje Topological Spaces.

Region	Temp-rature (Te)	Rainfall (Ra)	Sea Level (Sl)	GHG Emis-sions (GHG)	Defores-tation(D)	Agri. Prod. (Ap)	Health Impacts (Hi)	Eco-nomic Costs (Ec)	Impact Rate(Ir)
Coastal Regions (Cr)									Medium
Agri cultural Lands (Al)									Medium
Forested Areas (Fa)									High
Urban Areas (Ua)									High

Region	Temp-rature (Te)	Rainfall (Ra)	Sea Level (Sl)	GHG Emis-sions	Defores-tation(D)	Agri. Prod. (Ap)	Health Impacts (Hi)	Eco-nomic Costs	Impact Rate(Ir)
Island Nations (In)									Medium
River Basins (Rb)									High
Energy Sector (Es)									High
Fisheries (Fs)									High
Tourism Indus-try (Ti)									High
Health care Sys-tems (Hs)									Medium

Table 1: Impact of Climate Change on Various Regions and Sectors

Let the set of region be $\mathfrak{E} = \{\text{Cr,Al,F,Ua,In,Rb,Es,Fs,Ti,Hs}\}$

and

$$\mathfrak{G} = \{\text{Te,Ra,Sl,GHG,De,Ap,Hi,Ec,Ir}\}.$$

It splits into two cases where

$$\mathfrak{H} = \{\text{Te,Ra,Sl,GHG,De,Ap,Hi,Ec}\} \text{ and } \mathfrak{I} = \{\text{Ir}\}$$

The group of Equivalence types $\mathfrak{V}/\mathfrak{H}$ corresponding to $\mathfrak{H}$ is given by

$$\mathfrak{V}/\mathfrak{H} = \{\{\text{Cr}\},\{\text{Al,In}\},\{\text{Ua}\},\{\text{Fa,Rb,Es}\},\{\text{Ti}\},\{\text{Fs,Hs}\}\},$$

$$\mathfrak{E} = \{\text{Fa,Ua,Rb,Es,Fs,Ti}\}$$

Case 1: When Impact level is High

$\mathfrak{T}_{\mathcal{R}}(\mathfrak{E})$ = {Φ,$\mathfrak{V}$,{Fa,Ua,Rb,Es,Ti},{Fs,Hs},{Fa,Ua,Rb,Es,Fs,Ti,Hs}}

$\mu_{\mathcal{R}}(\mathfrak{E})$ = {Φ,$\mathfrak{V}$,{Al},{Fa,Ua,Rb,Es,Ti},{Al,Fa,Ua,Rb,Es,Ti},{Fs,Hs},{Al,Fs,Hs},
{Fa,Ua,Rb,Es,Fs,Ti,Hs},{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs}}

$SJ_{\mathfrak{H}}(\mathfrak{E})$ = {Φ,$\mathfrak{V}$,{Cr,Fa,In,Es,Ti},{Fa,Es,Ti},{Al,Ua,Rb,Fs,Hs},
{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs},{Al},{Al,Fa,Ti},
{Cr,Al,Fa,In,Es,Ti},{Ua,Rb},{Cr,Fa,Ua,In,Rb,Es,Ti},{Fa,Ua,Rb,Es,Ti},
{Al,Ua,In},{Cr,Al,Fa,Ua,In,Rb,Es,Ti},{Al,Fa,Ua,Rb,Es,Ti},{Fs,Hs},
{Cr,Fa,In,Es,Fs,Ti,Hs},{Fa,Es,Fs,Ti,Hs},
{Al,Fs,Hs},{Cr,Al,Fa,In,Es,Fs,Hs},{Al,Fa,Es,Fs,Ti,Hs},
{Ua,Rb,Fs,Hs},{Cr,Fa,Ua,In,Rb,Es,Ti,Hs},{Fa,Ua,Rb,Es,Fs,Ti,Hs}}

Phase I: Te is removed

$\mathfrak{T}_{\mathcal{R}}(\mathfrak{E})$ = {Φ,$\mathfrak{V}$,{Fa,Ua,Rb,Es},{Al,Fa,Ua,In,Rb,Es,Fs,Ti,Hs},{Al,In,Fs,Ti,Hs}}

$\mu_{\mathcal{R}}(\mathfrak{E})$ = {Φ,$\mathfrak{V}$,{Al},{Fa,Ua,Rb,Es},{Al,Fa,Ua,Rb,Es},
{Al,Fa,Ua,In,Rb,Es,Fs,Ti,Hs}}

$SJ_{\mathfrak{H}}(\mathfrak{E})$ = {Φ,$\mathfrak{V}$,{Al},{Al,Fa,Es},{Al,Fa,Es,Ti},{Cr,Al,Fa,In,Es,Ti},{Ua,Rb},
{Fa,Ua,Rb,Es},{Fa,Ua,Rb,Es,Ti},{Cr,Fa,Ua,In,Rb,Es,Ti},{Al,Ua,Rb},
{Al,Fa,Ua,Rb,Es},{Fa,Es},{Fa,Es,Ti},{Cr,Fa,In,Es,Ti},
{Al,Ua,Rb,Fs,Hs},{Al,Fa,Ua,Rb,Es,Fs,Hs},{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs},
{Fa,Es},{Fa,Es,Ti},{Al,Fa,Ua,Rb,Es,Ti},{Cr,Al,Fa,Ua,In,Rb,Es,Ti}}

Phase II: Ra is removed

$\mathfrak{T}_{\mathcal{R}}(\mathfrak{E})$ = {Φ,$\mathfrak{V}$, {Fa,Ua,Rb,Es,Ti},{Fa,Ua,Rb,Es,Fs,Ti,Hs},{Fs,Hs}}

$\mu_{\mathcal{R}}(\mathfrak{E})$ = {Φ,$\mathfrak{V}$,{Al},}Fa,Ua,Rb,Es,Ti},{Al,Fa,Ua,Rb,Es,Ti},
{Fa,Ua,Rb,Es,Fs,Ti,Hs},{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs},{Fs,Hs},{Al,Fs,Hs}}

$SJ_{\mathfrak{H}}(\mathfrak{E})$ = {Φ,$\mathfrak{V}$,{Cr,Fa,In,Es,Ti},{Fa,Es,Ti},{Al,Ua,Rb,Fs,Hs},
{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs},{Al},{Al,Fa,Es,Ti},{Cr,Al,Fa,In,Es,Ti},
{Ua,Rb},{Cr,Fa,Ua,In,Rb,Es,Ti},{Fa,Ua,Rb,Es,Ti},{Al,Ua,Rb},
{Cr,Al,Fa,Ua,In,Rb,Es,Ti},{Al,Fa,Ua,Rb,Es,Ti},{Fs,Hs},
{Cr,Fa,In,Es,Fs,Ti,Hs},{Fa,Es,Fs,Ti,Hs},{Al,Fs,Hs},
{Cr,Al,Fa,In,Es,Fs,Ti,Hs}

Phase III: Sl is removed

$\mathfrak{T}_{\mathcal{R}}(\mathfrak{E}) =$ {Φ,𝔙, {Fa,Rb,Es,Ti},{Cr,Fa,Ua,Rb,Es,Fs,Ti,Hs},{Cr,Ua,Fs,Hs}}

$\mu_{\mathcal{R}}(\mathfrak{E}) =$ {Φ,𝔙, {Al},{Fa,Rb,Es,Ti},{Al,Fa,Rb,Es,Ti},{Cr,Fa,Ua,Rb,Es,Fs,Ti,Hs},
{Cr,Al,Fa,Ua,Rb,Es,Fs,Ti,Hs},{Cr,Ua,Fs,Hs},{Cr,Al,Ua,Fs,Hs}}

$SJ_{\mathfrak{H}}(\mathfrak{E}) =$ {Φ,𝔙,{Al},{Al,Fa},{Al,Fa,Es,Ti},{Cr,Al,Fa,Es,Ti},{Cr,Al},
{Cr,Al,Fa,In,Es,Ti},{Rb},{Fa,Rb},{Fa,Rb,Es,Ti},{Cr,Fa,Rb,Es,Ti},
{Al,Rb},{Al,Fa,Rb},{Al,Fa,Rb,Es,Ti},{Cr,Al,Fa,Rb,Es,Ti},
{Cr,Al,Rb},{Cr,Al,Fa,In,Rb,Es,Ti},{Al,Ua,Rb,Fs,Hs},
{Al,Fa,Ua,Rb,Fs,Hs},{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs},
{Cr,Al,Fa,Ua,Rb,Es,Fs,Ti,Hs},{Ua,Fs,Hs},{Fa,Ua,Fs,Hs},
{Fa,Ua,Es,Fs,Hs},{Cr,Fa,Ua,Es,Fs,Ti,Hs},{Cr,Ua,Fs,Hs},
{Cr,Fa,Ua,In,Es,Fs,Ti,Hs},{Al,Ua,Fs,Hs} {Al,Fa,Ua,Fs,Hs}
{Al,Fa,Ua,Es,Fs,Ti,Hs},{Cr,Al,Fa,Ua,Es,Fs,Ti,Hs},{Cr,Al,Ua,Fs,Hs},
{Cr,Al,Fa,Ua,In,Es,Fs,Ti,Hs},{Fa},{Fa,Es,Ti},{Cr,Fa,Es,Ti},{Cr},
{Cr,Fa,In,Es,Ti},{Cr,Al,Ua,Rb,Fs,Hs},{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs}}

Phase IV: GHG is removed

$\mathfrak{T}_{\mathcal{R}}(\mathfrak{E}) =$ {Φ,𝔙,{Fa,Rb,Es,Ti},{Fa,Ua,In,Rb,Es,Fs,Ti,Hs},{Fa,Ua,Rb,Fs,Hs}}

$\mu_{\mathcal{R}}(\mathfrak{E}) =$ {Φ,𝔙,{Al},{Fa,Rb,Es,Ti},{Al,Fa,Rb,Es,Ti},{Fa,Ua,Rb,Es,Fs,Ti,Hs},
{Al,Fa,Rb,Es,Fs,Ti,Hs},{Fa,Ua,Rb,Es,Fs,Ti,Hs},{Al,Fa,Ua,Rb,Es,Fs,Hs}}

$SJ_{\mathfrak{H}}(\mathfrak{E}) =$ {Φ,𝔙, {Cr,Fa,In,Es,Ti},{Fa,Es,Ti},{Fa,Es},{Al,Ua,Rb,Fs,Hs},
{Al,Fa,Ua,Rb,Es,Fs,Hs},{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs},{Al},
{Cr,Al,Fa,In,Es,Ti},{Al,Fa,Es,Ti},{Al,Fa,Es},{Rb},{Cr,Fa,In,Rb,Es,Ti},
{Fa,Rb,Es,Ti},{Fa,Rb,Es},{Al,Rb},{Cr,Al,Fa,In,Rb,Es,Ti},
{Al,Fa,Rb,Es,Ti},{Al,Fa,Rb,Es},{Ua,Rb,Fs,Hs},
{Cr,Al,Fa,In,Rb,Es,Fs,Ti,Hs},{Al,Fa,Rb,Es,Fs,Ti,Hs},
{Al,Fa,Rb,Es,Fs,Hs}}

Phase V: De is removed

$\mathfrak{T}_{\mathcal{R}}(\mathfrak{E}) =$ {Φ,𝔙,{Fa,Ua,Rb,Es,Ti},{Fa,Ua,Rb,Es,Fs,Ti,Hs},{Fs,Hs}}

$\mu_{\mathcal{R}}(\mathfrak{E})$ = {Φ,𝔙,{Al},{Fa,Ua,Rb,Es,Ti},{Al,Fa,Ua,Rb,Es,Ti},
{Fa,Ua,Rb,Es,Fs,Ti,Hs},{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs},{Fs,Hs},{Al,Fs,Hs}}

$\mathcal{SJ}_{\mathfrak{H}}(\mathfrak{E})$ = {Φ,𝔙,{Cr,Fa,In,Es,Ti},{Fa,Es,Ti},{Al,Ua,Rb,Fs,Hs},
{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs},{Al},{Al,Fa,Es,Ti},{Cr,Al,Fa,In,Es,Ti},
{Ua,Rb},{Cr,Fa,Ua,In,Rb,Es,Ti},{Fa,Ua,Rb,Es,Ti},{Al,Ua,Rb},
{Cr,Al,Fa,Ua,In,Rb,Es,Ti},{Al,Fa,Ua,Rb,Es,Ti},{Fs,Hs},
{Cr,Fa,In,Es,Fs,Ti,Hs},{Fa,Es,Fs,Ti,Hs},{Al,Fs,Hs},
{Cr,Al,Fa,In,Es,Fs,Ti,Hs}}

Phase VI: Ap is removed

$\mathfrak{T}_{\mathcal{R}}(\mathfrak{E})$ = {Φ,𝔙,{Fa,Ua,Rb,Es,Ti},{Fa,Ua,Rb,Es,Fs,Ti,Hs},{Fs,Hs}}

$\mu_{\mathcal{R}}(\mathfrak{E})$ = {Φ,𝔙,{Al},{Fa,Ua,Rb,Es,Ti},{Al,Fa,Ua,Rb,Es,Ti},
{Fa,Ua,Rb,Es,Fs,Ti,Hs},{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs},{Fs,Hs},{Al,Fs,Hs}}

$\mathcal{SJ}_{\mathfrak{H}}(\mathfrak{E})$ = {Φ,𝔙, {Cr,Fa,In,Es,Ti},{Fa,Es,Ti},{Al,Ua,Rb,Fs,Hs},
{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs},{Al},{Al,Fa,Es,Ti},{Cr,Al,Fa,In,Es,Ti},
{Ua,Rb},{Cr,Fa,Ua,In,Rb,Es,Ti},{Fa,Ua,Rb,Es,Ti},{Al,Ua,Rb},
{Cr,Al,Fa,Ua,In,Rb,Es,Ti},{Al,Fa,Ua,Rb,Es,Ti},{Fs,Hs},
{Cr,Fa,In,Es,Fs,Ti,Hs},{Fa,Es,Fs,Ti,Hs},{Al,Fs,Hs},
{Cr,Al,Fa,In,Es,Fs,Ti,Hs}}

Phase VII: Hi is removed

$\mathfrak{T}_{\mathcal{R}}(\mathfrak{E})$ = {Φ,𝔙,{Fa,Rb,Es,Ti},{Fa,Ua,Rb,Es,Fs,Ti,Hs},{Ua,Fs,Hs}}

$\mu_{\mathcal{R}}(\mathfrak{E})$ = {Φ,𝔙,{Al},{Fa,Rb,Es,Ti},{Al,Fa,Rb,Es,Ti},{Fa,Ua,Rb,Es,Fs,Ti,Hs},
{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs},{Ua,Fs,Hs},{Al,Ua,Fs,Hs}}

$\mathcal{SJ}_{\mathfrak{H}}(\mathfrak{E})$ = {Φ,𝔙,{Al},{Al,Fa,Es,Ti},{Cr,Al,Fa,In,Es,Ti},{Rb},{Fa,Rb,Es,Ti},
{Cr,Fa,In,Rb,Es,Ti},{Al,Rb},{Al,Fa,Rb,Es,Ti},Cr,Al,Fa,Ua,In,Rb,Es,Ti},
{Ua,Rb,Fs,Hs},{Fa,Ua,Rb,Es,Fs,Ti,Hs},{Cr,Fa,Ua,In,Rb,Es,Fs,Ti,Hs},
{Al,Ua,Rb,Fs,Hs},{Al,Fa,Ua,Es,Fs,Ti,Hs},{Cr,Al,Fa,Ua,In,Es,Fs,Ti,Hs},
{Fa,Es,Ti},{Cr,Fa,In,Es,Ti}}

Phase VIII: Ec is removed

$\mathfrak{T}_{\mathcal{R}}(\mathfrak{E}) = \{\Phi, \mathfrak{V}, \{Fa,Ua,Rb,Es,Ti\}, \{Fa,Ua,Rb,Es,Fs,Ti,Hs\}, \{Fs,Hs\}\}$

$\mu_{\mathcal{R}}(\mathfrak{E}) = \{\Phi, \mathfrak{V}, \{Al\}, \{Fa,Ua,Rb,Es,Ti\}, \{Al,Fa,Ua,Rb,Es,Ti\},$
$\{Fa,Ua,Rb,Es,Fs,Ti,Hs\}, \{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs\}, \{Fs,Hs\}, \{Al,Fs,Hs\}\}$

$SJ_{\mathfrak{H}}(\mathfrak{E}) = \{\Phi, \mathfrak{V}, \{Cr,Fa,In,Es,Ti\}, \{Fa,Es,Ti\}, \{Al,Ua,Rb,Fs,Hs\},$
$\{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs\}, \{Al\}, \{Al,Fa,Es,Ti\}, \{Cr,Al,Fa,In,Es,Ti\},$
$\{Ua,Rb\}, \{Cr,Fa,Ua,In,Rb,Es,Ti\}, \{Fa,Ua,Rb,Es,Ti\}, \{Al,Ua,Rb\},$
$\{Cr,Al,Fa,Ua,In,Rb,Es,Ti\}, \{Al,Fa,Ua,Rb,Es,Ti\}, \{Fs,Hs\},$
$\{Cr,Fa,In,Es,Fs,Ti,Hs\}, \{Fa,Es,Fs,Ti,Hs\}, \{Al,Fs,Hs\},$
$\{Cr,Al,Fa,In,Es,Fs,Ti,Hs\}\}$

Following the aforementioned analysis of CrRase Cr, it has been determined that the principal factors affecting climate change impact are Rainfall, Deforestation, Algricultural Productivity and Economic Crosts.

Case 2: When Impact level is Normal

$\mathfrak{T}_{\mathcal{R}}(\mathfrak{E}) = \{\Phi, \mathfrak{V}, \{Cr,Al,In\}, \{Cr,Al,In,Fs,Hs\}, \{Fs,Hs\}\}$

$\mu_{\mathcal{R}}(\mathfrak{E}) = \{\Phi, \mathfrak{V}, \{Al\}, \{Cr,Al,In\}, \{Cr,Al,In,Fs,Hs\}, \{Fs,Hs\}, \{Al,Fs,Hs\}\}$

$SJ_{\mathfrak{H}}(\mathfrak{E}) = \{\Phi, \mathfrak{V}, \{Al\}, \{Cr,Al\}, \{Cr,Al,In,Ti\}, \{Cr,Al,Fa,In,Es,Ti\}, \{Al,Fs,Hs\},$
$\{Cr,Al,Fs,Hs\}, \{Cr,Al,In,Fs,Ti,Hs\}, \{Cr,Al,Fa,In,Es,Fs,Ti,Hs\},$
$\{Fs,Hs\}, \{Cr,Fs,Hs\}, \{Cr,In,Fs,Ti,Hs\}, \{Cr,Fa,In,Es,Fs,Ti,Hs\},$
$\{Cr\}, \{Cr,In,Ti\}, \{Cr,Fa,In,Es,Ti\}, \{Al,Ua,Rb,Fs,Hs\},$
$\{Cr,Al,Ua,Rb,Fs,Hs\}, \{Cr,Al,Ua,In,Rb,Fs,Ti,Hs\}\}$

Phase I: Te is removed

$\mathfrak{T}_{\mathcal{R}}(\mathfrak{E}) = \{\Phi, \mathfrak{V}, \{Cr\}, \{Cr,Al,In,Fs,Ti,Hs\}, \{Al,In,Fs,Ti,Hs\}$

$\mu_{\mathcal{R}}(\mathfrak{E}) = \{\Phi, \mathfrak{V}, \{Al\}, \{Cr\}, \{Cr,Al\}, \{Cr,Al,In,Fs,Ti,Hs\}, \{Al,In,Fs,Ti,Hs\}$

$SJ_{\mathfrak{H}}(\mathfrak{E}) = \{\Phi, \mathfrak{V}, \{Cr,Fa,In,Es,Ti\}, \{Cr\}, \{Cr,In,Ti\}, \{In,Ti\}, \{Al,Ua,Rb,Fs,Hs\},$

{Cr,Al,Ua,Rb,Fs,Hs},{Cr,Al,Ua,In,Rb,Fs,Ti,Hs},
{Al,Ua,In,Rb,Fs,Ti,Hs},{Al},{Cr,Al,Fa,In,Es,Ti},{Cr,Al},{Cr,Al,In,Ti},
{Al,In,Ti},{Al,Fs,Hs} {Cr,Al,Fa,In,Es,Fs,Ti,Hs},{Cr,Al,In,Fs,Ti,Hs},
{Al,In,Fs,Ti,Hs},{Cr,Al,Fs,Hs}}

Phase II: Ra is removed

$\mathfrak{T}_{\mathcal{R}}(\mathfrak{E})$ = {Φ,$\mathfrak{V}$,{Fa,Ua,Rb,Es,Ti},{Fa,Ua,Rb,Es,Fs,Ti,Hs},{Fs,Hs}

$\mu_{\mathcal{R}}(\mathfrak{E})$ = {Φ,$\mathfrak{V}$,{Al},{Fa,Ua,Rb,Es,Ti},{Al,Fa,Ua,Rb,Es,Ti},
{Fa,Ua,Rb,Es,Fs,Ti,Hs},{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs},{Fs,Hs},{Al,Fs,Hs}}

$SJ_{\mathfrak{H}}(\mathfrak{E})$ = {Φ,$\mathfrak{V}$,{Cr,Fa,In,Es,Ti},{Fa,Es,Ti},{Al,Ua,Rb,Fs,Hs},
{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs},{Al},{Al,Fa,Es,Ti},{Cr,Al,Fa,In,Es,Ti},
{Ua,Rb},{Cr,Fa,Ua,In,Rb,Es,Ti},{Fa,Ua,Rb,Es,Ti},
{Al,Ua,Rb},{Cr,Al,Fa,Ua,In,Rb,Es,Ti},{Al,Fa,Ua,Rb,Es,Ti},
{Fs,Hs},{Cr,Fa,In,Es,Fs,Ti,Hs},{Fa,Es,Fs,Ti,Hs},{Al,Fs,Hs},
{Cr,Al,Fa,In,Es,Fs,Ti,Hs}}

Phase III: Sl is removed

$\mathfrak{T}_{\mathcal{R}}(\mathfrak{E})$ = {Φ,$\mathfrak{V}$,{Al,In}{Cr,Ua,Fs,Hs},{Cr,Al,Ua,In,Fs,Hs}}

$\mu_{\mathcal{R}}(\mathfrak{E})$ ={Φ,$\mathfrak{V}$,{Al},{Al,In},{Cr,UaFs,Hs},{Cr,Al,Ua,Fs,Hs},{Cr,Al,Ua,In,Fs,Hs}}

$SJ_{\mathfrak{H}}(\mathfrak{E})$ = {Φ,$\mathfrak{V}$,{Cr,Fa,In,Es,Ti},{Cr},{In},{Cr,In},{Al,Ua,Rb,Fs,Hs},
{Al,Ua,In,Rb,Fs,Hs},{Cr,Al,Ua,Rb,Fs,Hs},{Cr,Al,Ua,In,Rb,Fs,Hs},
{Al},{Cr,Al,Fa,In,Es,Ti},{Al,In},{Cr,Al},{Cr,Al,In},{Ua,FsHs},
{Cr,Fa,Ua,In,Es,Fs,Ti,Hs},{Ua,In,Fs,Hs},{Cr,Ua,Fs,Hs},
{Cr,Ua,In,Fs,Hs},{Al,Ua,Fs,Hs},{Cr,Al,Fa,Ua,In,Es,Fs,Ti,Hs},
{Al,Ua,In,Fs,Hs},{Cr,Al,Ua,Fs,Hs},{Cr,Al,Ua,In,Fs,Hs}}

Phase IV: GHG is removed

$\mathfrak{T}_{\mathcal{R}}(\mathfrak{E})$ = {Φ,$\mathfrak{V}$,{Cr,Al,In},{Fa,Rb,Es,Fs,Hs},{Cr,Al,Fa,In,Rb,Es,Fs,Ti,Hs}}

$\mu_{\mathcal{R}}(\mathfrak{E})$ = {Φ,$\mathfrak{V}$,{Al},{Cr,Al,In},{Fa,Rb,Es,Fs,Hs},{Al,Fa,Rb,Es,Fs,Hs},
{Cr,Al,Fa,In,Rb,Es,Fs,Hs}}

$\mathcal{SJ}_{\mathfrak{H}}(\mathfrak{E}) =$ {Φ,𝔙,{Al},{Cr,Al,In},{Al,Fa,Es},{Al,Fa,In,Es},{Cr,Al,Fa,In,Es,Ti},
{Rb,Fs,Hs},{Cr,In,Rb,Fs,Hs},{Fa,Rb,Es,Fs,Hs},{Fa,In,Rb,Fs,Hs},
{Cr,Fa,In,Rb,Es,Fs,Ti,Hs},{Al,Rb,Fs,Hs},{Cr,Al,In,Rb,Fs,Ti,Hs},
{Al,Fa,Rb,Es,Fs,Hs},{Al,Fa,In,Rb,Es,Fs,Hs},
{Cr,Al,Fa,In,Rb,Es,Fs,Ti,Hs},{Cr,In},{Fa,In},{Fa,In,Ti},
{Cr,Fa,In,Es,Ti},{Al,Ua,Rb,Fs,Hs},{Cr,Al,Ua,Fs,Rb,Fs,Hs},
{Al,Fa,Rb,Es,Fs,Hs},{Al,Fa,In,Rb,Es,Fs,Hs}}

Phase V: De is removed

$\mathfrak{T}_{\mathcal{R}}(\mathfrak{E}) =$ {Φ,𝔙,{Fa,Ua,Rb,Es,Ti},{Fa,Ua,Rb,Es,Fs,Ti,Hs},{Fs,Hs}

$\mu_{\mathcal{R}}(\mathfrak{E}) =$ {Φ,𝔙,{Al},{Fa,Ua,Rb,Es,Ti},{Al,Fa,Ua,Rb,Es,Ti},{Fa,Ua,Rb,Es,Fs,Ti,Hs},
{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs},{Fs,Hs},{Al,Fs,Hs}}

$\mathcal{SJ}_{\mathfrak{H}}(\mathfrak{E}) =$ {Φ,𝔙,{Cr,Fa,In,Es,Ti},{Fa,Es,Ti},{Al,Ua,Rb,Fs,Hs},
{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs},{Al},{Al,Fa,Es,Ti},{Cr,Al,Fa,In,Es,Ti},
{Ua,Rb},{Cr,Fa,Ua,In,Rb,Es,Ti},{Fa,Ua,Rb,Es,Ti},{Al,Ua,Rb},
{Cr,Al,Fa,Ua,In,Rb,Es,Ti},{Al,Fa,Ua,Rb,Es,Ti},{Fs,Hs},
{Cr,Fa,In,Es,Fs,Ti,Hs},{Fa,Es,Fs,Ti,Hs},{Al,Fs,Hs},
{Cr,Al,Fa,In,Es,Fs,Ti,Hs}}

Phase VI: Ap is removed

$\mathfrak{T}_{\mathcal{R}}(\mathfrak{E}) =$ {Φ,𝔙,{Fa,Ua,Rb,Es,Ti},{Fa,Ua,Rb,Es,Fs,Ti,Hs},{Fs,Hs}}

$\mu_{\mathcal{R}}(\mathfrak{E}) =$ {Φ,𝔙,{Al},{Fa,Ua,Rb,Es,Ti},{Al,Fa,Ua,Rb,Es,Ti},
{Fa,Ua,Rb,Es,Fs,Ti,Hs},{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs},{Fs,Hs},{Al,Fs,Hs}}

$\mathcal{SJ}_{\mathfrak{H}}(\mathfrak{E}) =$ {Φ,𝔙,{Cr,Fa,In,Es,Ti},{Fa,Es,Ti},{Al,Ua,Rb,Fs,Hs},
{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs},{Al},{Al,Fa,Es,Ti},{Cr,Al,Fa,In,Es,Ti},
{Ua,Rb},{Cr,Fa,Ua,In,Rb,Es,Ti},{Fa,Ua,Rb,Es,Ti},
{Al,Ua,Rb},{Cr,Al,Fa,Ua,In,Rb,Es,Ti},{Al,Fa,Ua,Rb,Es,Ti},{Fs,Hs},
{Cr,Fa,In,Es,Fs,Ti,Hs},{Fa,Es,Fs,Ti,Hs},{Al,Fs,Hs},
{Cr,Al,Fa,In,Es,Fs,Ti,Hs}}

Phase VII: Hi is removed

$\mathfrak{T}_{\mathcal{R}}(\mathfrak{E}) = \{\Phi, \mathfrak{V}, \{Cr,Al,In\}, \{Cr,Al,Ua,In,Fs,Hs\}, \{Ua,Fs,Hs\}\}$

$\mu_{\mathcal{R}}(\mathfrak{E}) = \{\Phi, \mathfrak{V}, \{Al\}, \{Cr,Al,In\}, \{Cr,Al,Ua,In,Fs,Hs\}, \{Ua,Fs,Hs\}, \{Al,Ua,Fs,Hs\}\}$

$$\begin{aligned} SJ_{\mathfrak{H}}(\mathfrak{E}) = \{&\Phi, \mathfrak{V}, \{Cr,Fa,In,Es,Ti\}, \{Cr\}, \{Cr,In\}, \{Al,Ua,Rb,Fs,Hs\}, \\ &\{Cr,Al,Ua,Rb,Fs,Hs\}, \{Cr,Al,Ua,In,Rb,Fs,Hs\}, \{Al\}, \{Cr,Al,Fa,In,Es,Ti\}, \\ &\{Cr,Al\}, \{Cr,Al,In\}, \{Al,Ua,Fs,Hs\}, \{Cr,Al,Fa,Ua,In,Es,Fs,Ti,Hs\}, \\ &\{Cr,Al,Ua,Fs,Hs\}, \{Cr,Al,Ua,In,Fs,Hs\}, \\ &\{Ua,Fs,Hs\}, \{Cr,Fa,Ua,In,Es,Fs,Ti,Hs\}, \{Cr,Ua,Fs,Hs\}, \\ &\{Cr,Ua,In,Fs,Hs\}\} \end{aligned}$$

Phase VIII: Ec is removed

$\mathfrak{T}_{\mathcal{R}}(\mathfrak{E}) = \{\Phi, \mathfrak{V}, \{Fa,Ua,Rb,Es,Ti\}, \{Fa,Ua,Rb,Es,Fs,Ti,Hs\}, \{Fs,Hs\}\}$

$$\begin{aligned} \mu_{\mathcal{R}}(\mathfrak{E}) = \{&\Phi, \mathfrak{V}, \{Al\}, \{Fa,Ua,Rb,Es,Ti\}, \{Al,Fa,Ua,Rb,Es,Ti\}, \\ &\{Fa,Ua,Rb,Es,Fs,Ti,Hs\}, \{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs\}, \{Fs,Hs\}, \{Al,Fs,Hs\}\} \end{aligned}$$

$$\begin{aligned} SJ_{\mathfrak{H}}(\mathfrak{E}) = \{&\Phi, \mathfrak{V}, \{Cr,Fa,In,Es,Ti\}, \{Fa,Es,Ti\}, \{Al,Ua,Rb,Fs,Hs\}, \\ &\{Al,Fa,Ua,Rb,Es,Fs,Ti,Hs\}, \{Al\}, \{Al,Fa,Es,Ti\}, \{Cr,Al,Fa,In,Es,Ti\}, \\ &\{Ua,Rb\}, \{Cr,Fa,Ua,In,Rb,Es,Ti\}, \{Fa,Ua,Rb,Es,Ti\}, \{Al,Ua,Rb\}, \\ &\{Cr,Al,Fa,Ua,In,Rb,Es,Ti\}, \{Al,Fa,Ua,Rb,Es,Ti\}, \{Fs,Hs\}, \\ &\{Cr,Fa,In,Es,Fs,Ti,Hs\}, \{Fa,Es,Fs,Ti,Hs\}, \{Al,Fs,Hs\}, \\ &\{Cr,Al,Fa,In,Es,Fs,Ti,Hs\}\} \end{aligned}$$

From both **Case 1** and **Case 2**, it is clear that Rainfall, Deforestation, Agricultural Productivity and Economic Costs play a crucial role in driving climate change outcomes.

Visualization and Analysis

To provide a clear comparison of the performance of nano, micro and Selje topologies in identifying critical climate factors, a heat map was generated (see Figure 1). This visual repre- sentation compares the ability of each topology to detect key factors, such as temperature rise, rainfall variability and deforestation, across various regions including Coastal, Agricultural, Urban, Forested and Island regions.

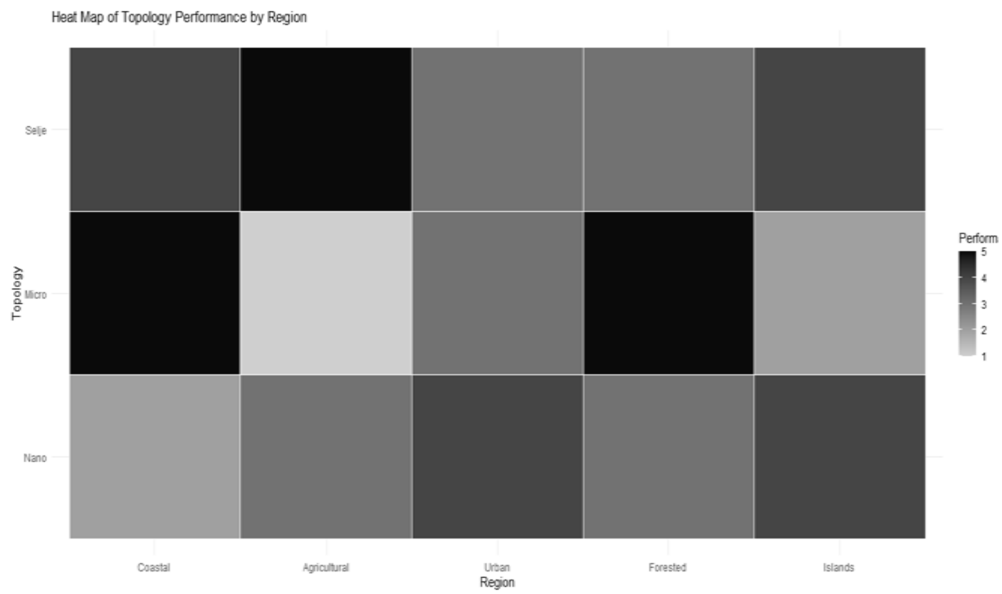

Figure 1: Heat Map of Topology Performance by Region

The heat map shows the performance score of each topology, with darker shades representing better performance in terms of accurately identifying impactful factors. As seen in the heat map, the Selje topology consistently demonstrates superior performance across all regions, particularly in complex environments like urban and forested areas, where multifactorial dependencies are prevalent.

5. Results and Discussion

Comparison of Nano, Micro and Selje Topological Spaces

In this analysis, nano topology, micro topology and Selje topology were applied to climate change impact factors to assess their efficiency in identifying critical variables. While all three topologies consistently identified Rainfall, Deforestation, Agricultural Productivity and Economic Costs as major factors, the depth of analysis, precision and flexibility differed significantly across the topologies.

Nano Topology

Strengths: Nano topology provides a simple binary classification of critical climate factors, making it effective for identifying whether a factor is part of a critical set.

Weaknesses: Its binary approach cannot capture the complexities of dynamic systems, leading to limitations in handling multifactorial relationships, scalability and interdependencies.

Micro Topology

Strengths: Micro topology refines nano topology by introducing micro-open and micro closed sets, allowing for more nuanced classifications and adaptable relationships between factors.

Weaknesses: While an improvement, micro topology still struggles with highly multifactorial systems, lacking the precision needed to fully address the complex, interconnected nature of climate factors.

Selje Topology

Strengths: Selje topology generalizes both nano and micro topologies, providing superior flexibility and precision. It uses Selje-open and Selje-closed sets to capture intricate relationships between climate factors, even in dynamic and multifactorial systems.

Theorem 1: Demonstrates finer approximations through better handling of closures and interiors.

Theorem 2: Highlights Selje's superior scalability, enabling it to handle complex systems more effectively.

Theorem 3: Proves Selje topology's ability to capture interdependencies through finer approximations of set intersections.

Better Performance: Selje topology offers deeper insights into the variability of climate impacts across regions. Unlike nano and micro, which treat factors as static, Selje allows for a dynamic understanding of how these factors fluctuate under different conditions and regions.

Weaknesses: The complexity of Selje topology may be unnecessary for simpler systems where its precision is not required.

Selje Topology's Superiority

While all three topologies identified the same major factors, Selje topology stands out due to its enhanced precision, scalability and ability to capture complex relationships.

Precision in Complex Systems: It handles intricate, multifactorial environments like climate change, providing a finer analysis of the interactions between key factors.

Scalability: As demonstrated in Theorem 3.3, Selje topology scales well with system complexity, retaining accuracy even as more variables are introduced.

Handling Nuanced Relationships: Theorem 3.4 shows that Selje topology excels in analyzing overlapping and interdependent factors, offering a more detailed understanding of cumulative impacts.

Flexibility: Unlike nano's rigid binary classification, Selje topology adapts to uncertainties and changing conditions, making it more versatile for dynamic systems.

6. Conclusion

While nano, micro and Selje topologies all identified the same key climate factors, Selje topology offers greater analytical power due to its flexibility, precision and scalability. These qualities make it the optimal choice for analyzing complex, multifactorial systems like climate change, where relationships between factors are dynamic and interdependent. Future research could explore Selje topology's application in other fields, such as optimizing smart grids or analyzing healthcare systems, where multifactorial interactions are critical. Its adaptability and precision make it well-suited for real-world applications in dynamic environments, providing deeper insights and better handling of complex systems.

Conflict of Interest

The authors affirm that there is no conflicts of interest pertaining to the research presented in this paper. No financial or personal relationships with any organizations or individuals that could potentially bias the findings or interpretations are reported.

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EXPLORING HEPTAPARTITIONED NEUTROSOPHIC PYTHAGOREAN TOPOLOGICAL SPACES

Abstract: The aim of this paper is to introduce a novel concept known as the Heptapartitioned Neutrosophic Pythagorean Topological spaces and discussed the fundamental aspects and key properties. This new concept integrates with existing mathematical structure and its significance in the broader field of Topology.

Keywords: Heptapartitioned Neutrosophic Set, Heptapartitioned Neutrosophic Topological Space, Heptapartitioned Neutrosophic Pythagorean Topological Space.

Mathematics Subject Classification: 54C50, 54G99.

1. Introduction

The fuzzy [15] set concept was introduced by Zadeh in 1965. Later, F. Smarandache introduced the neutrosophic set, which is a mathematical tool designed to address problems involving imprecise, indeterminate and inconsistent data. Smarandache's neutrosophic set allows the indeterminacy membership function to operate independently from the truth and falsity membership functions. This theory has been extensively explored by researchers and has been applied to various real-life situations that involve uncertainty. Rajesh Chatterjee pioneered the concept of quadripartitioned single-valued neutrosophic sets. Recently, Das [1] and his team introduced Quadripartitioned Neutrosophic Topological Spaces by applying topology to these quadripartitioned neutrosophic sets. Rama Malik [5] and Surapati Pramanik introduced the concept of the pentapartitioned neutrosophic set and its properties. In this set, indeterminacy is divided into three components contradiction, ignorance and unknown membership functions.

In 2021, R. Radha and A. Stanis Arul Mary [7,8] expanded on the concepts of pentapartitioned and quadripartitioned neutrosophic sets to develop the heptapartitioned neutrosophic set [6]. This advancement brought a new dimension to handle complex indeterminate data by introducing a seven-part partitioning system. Building on this foundation, V. Jeyanthi and T. Mythili [4] made further strides in 2023 by introducing heptapartitioned neutrosophic topological spaces. Their work applied topological principles to the heptapartitioned neutrosophic sets, enhancing their utility in various scientific and mathematical applications. These developments mark significant progress in the field, offering more sophisticated tools for dealing with uncertainty and indeterminacy. As a result, researchers now have better methods to address real-world problems involving complex data. In 1995, F. Smarandache [14] introduced Seven Symbol-Valued Neutrosophic Logic. When the elements T_A, T_R, F_A, F_R, U, C, and G are considered as subsets of [0, 1], this logic evolves into a numerical system with seven distinct values. This system provides the foundation for defining the Heptapartitioned Neutrosophic Set and examining its characteristics. Each of these symbols corresponds to a specific type of membership: absolute truth, relative truth, contradiction, unknown, ignorance, relative falsity, and absolute falsity, respectively.

Building on Heptapartitioned Neutrosophic Topological Spaces, the authors have extended their research to the Heptapartitioned Neutrosophic Pythagorean Set, incorporating it into the framework of topological spaces. This extension allows us to explore the properties and implications of this set within the broader context of topology. Our work now integrates these concepts, offering new insights into their interaction and application in topological settings.

2. Preliminaries

2.1 Basic Concepts

Definition 2.1.1: Let X be a universe. A Neutrosophic set A on X can be defined as follows:

$$A = \{\langle \zeta, T_A(\zeta), I_A(\zeta), F_A(\zeta) \rangle : \zeta \in X\}$$

Where $T_A, I_A, F_A : X \to [0,1]$ and $0 \le T_A(\zeta) + I_A(\zeta) + F_A(\zeta) \le 3$.

Here, $T_A(\zeta)$ is the degree of membership, $I_A(\zeta)$ is the degree of indeterminacy, and $F_A(\zeta)$ is the degree of nonmembership.

Moreover, $T_A(\zeta)$ and $F_A(\zeta)$ are dependent neutrosophic components, while $I_A(x)$ is an independent component.

Definition 2.1.2: Let X be a universe. A Quadripartitioned Neutrosophic Set A with independent neutrosophic components on X is defined as follows:

$$A = \{\langle \zeta, T_A(\zeta), C_A(\zeta), U_A(\zeta), F_A(x) \rangle : \zeta \in X\}$$

where $T_A, C_A, U_A, F_A : X \to [0,1]$ and $0 \le T_A(\zeta) + C_A(\zeta) + U_A(\zeta) + F_A(\zeta) \le 4$.

In this context, $T_A(\zeta)$ represents the degree of truth membership, $C_A(\zeta)$ represents the degree of contradiction membership, $U_A(\zeta)$ represents the degree of ignorance membership, and $F_A(\zeta)$ represents the degree of false membership.

Definition 2.1.3: Let X be a non-empty set. A PNS A over X characterizes each element ζ in X by a truth-membership function T_A, a contradiction membership function C_A, an ignorance membership function U_A, an unknown membership function K_A, and a falsity membership function F_A. These functions satisfy the condition:

$$0 \le T_A(\zeta) + C_A(\zeta) + K_A(\zeta) + U_A(\zeta) + F_A(\zeta) \le 5$$

for each $\zeta \in X$.

Definition 2.1.4: Consider R to be a universe. Then G, a HNS over R is defined as:

$$G = \{(\zeta, T_G(\zeta), M_G(\zeta), C_G(\zeta), U_G(\zeta), I_G(\zeta), K_G(\zeta), F_G(\zeta)) : \zeta \in R\},$$

where the values $T_G(\zeta), M_G(\zeta), C_G(\zeta), U_G(\zeta), I_G(\zeta), K_G(\zeta), F_G(\zeta)$ correspond to the absolute truth membership, relative truth membership, contradiction membership, unknown membership, ignorance membership, relative falsity membership, and absolute falsity membership of ζ, respectively. Here, ζ is an element of the set R and each membership value belongs to the interval [0, 1]. Thus,

$$0 \le T_G(\zeta) + M_G(\zeta) + C_G(\zeta) + U_G(\zeta) + I_G(\zeta) + K_G(\zeta) + F_G(\zeta) \le 7, \ \ \forall \zeta \in R.$$

Definition 2.1.5: Let X be a universe. A Heptapartitioned Neutrosophic Pythagorean Set G with T_G, M_G, C_G and U_G as dependent neutrosophic components and I_G, K_G, and F_G as independent components for G on X is an object of the form:

$$G = \{\langle \zeta, T_G(\zeta), M_G(\zeta), C_G(\zeta), U_G(\zeta), I_G(\zeta), K_G(\zeta), F_G(\zeta)\rangle : \zeta \in X\}$$

where $T_G(\zeta) + F_G(\zeta) \le 1, M_G(\zeta) + K_G(\zeta) \le 1$, and

$$(T_G(\zeta))^2 + (M_G(\zeta))^2 + (C_G(\zeta))^2 + (U_G(\zeta))^2 + (I_G(\zeta))^2 + (K_G(\zeta))^2 + (F_G(\zeta))^2 \le 3$$

Here, $TG(\zeta)$ represents the degree of absolute truth membership, $M_G(\zeta)$ represents the degree of relative truth membership, $C_G(\zeta)$ represents the degree of contradiction membership, $U_G(\zeta)$ represents the degree of unknown membership, $I_G(\zeta)$ represents the degree of ignorance membership, $K_G(\zeta)$ represents the degree of relative falsity membership, and $F_G(\zeta)$ represents the degree of absolute false membership.

Definition 2.1.6: A Heptapartitioned Neutrosophic Pythagorean Set (HNPS) A is contained in another Heptapartitioned Neutrosophic Pythagorean Set B (denoted as $A \subseteq B$) if and only if the following conditions hold for every element
$\zeta \in X : T_A(\zeta) \le T_B(\zeta)$, $M_A(\zeta) \le M_B(\zeta)$, $C_A(\zeta) \le C_B(\zeta)$, $U_A(\zeta) \ge U_B(\zeta)$, $I_A(\zeta) \ge I_B(\zeta)$, $K_A(\zeta) \le K_B(\zeta)$ and $F_A(\zeta) \le F_B(\zeta)$

Definition 2.1.7: The complement of a Heptapartitioned Neutrosophic Pythagorean Set (F, G) on X, denoted by $(F, G)^c$, is defined as:

$$(F, A)^c(\zeta) = \{\langle \zeta, F_G(\zeta), U_G(\zeta), 1 - I_G(\zeta), C_G(\zeta), T_G(\zeta), M_G(\zeta), K_G(\zeta)\rangle\} : \zeta \in X\}$$

Definition 2.1.8: Let X be a non-empty set, A and B are two Heptapartitioned Neutrosophic Pythagorean sets. Then

$$A \cup B = [\zeta(\max(T_A, T_B),\ \max(M_A, M_B),\ \max(C_A, C_B),\ \min(U_A, U_B),\ \min(I_A, I_B),\ \min(K_A, K_B),\ \min(F_A, F_B) : \zeta \in X]$$

$$A \cap B = [\zeta(\min(T_A, T_B),\ \min(M_A, M_B),\ \min(C_A, C_B),\ \max(U_A, U_B),\ \max(I_A, I_B),\ \max(K_A, K_B),\ \max(F_A, F_B) : \zeta \in X]$$

Definition 2.1.9: A Heptapartitioned neutrosophic set G is called an absolute Heptapartitioned neutrosophic set if and only if it's absolute truth-membership, relative truth-membership, contradiction-membership, ignorance-membership, unknown-membership, absolute falsity-membership, and relative falsity-membership are defined as follows:

$T_G(\zeta) = 1$, $M_G(\zeta) = 1$, $C_G(\zeta) = 1$, $U_G(\zeta) = 0$, $I_G(\zeta) = 0$, $K_G(\zeta) = 0$, and $F_G(\zeta) = 0$

Definition 2.1.10: A Heptapartitioned neutrosophic set G is called a relative Heptapartitioned neutrosophic set if and only if its absolute truth-membership, relative truth-membership, contradiction-membership, ignorance-membership, unknown-membership, absolute falsity-membership, and relative falsity-membership are defined as follows:

$T_G(\zeta) = 0$, $M_G(\zeta) = 0$, $C_G(\zeta) = 0$, $U_G(\zeta) = 1$, $I_G(\zeta) = 1$, $K_G(\zeta) = 1$, and $F_G(\zeta) = 1$

3. Heptapartitioned Neutrosophic Pythagorean Topological Spaces

Definition 3.0.1: A Heptapartitioned Neutrosophic Pythagorean topology on a non-empty set W is a τ of Heptapartitioned Neutrosophic Pythagorean sets satisfying the following axioms.

(i) $0_W, 1_W \in \tau$

(ii) The union of the elements of any sub collection of τ is in τ.

(iii) The intersection of the elements of any finite sub collection τ is in τ .

The pair (W,τ) is called an Heptapartitioned Neutrosophic Pythagorean Topological Space over W.

Note 3.1: 1. Every member of τ is called a HNP open set in W.

2. The set A_W is called a HNP closed set in W if $A_W \in \tau^c$, where $\tau^c = \{A_W{}^c : A_W \in \tau\}$

Example 3.1: *Let* $W = \{c_1, c_2, c_3\}$ *and Let* A_W, B_W, C_W *be Heptapartitioned Neutrosophic Pythagorean sets where*

$$A_W = \{\langle c_1, 0.4, 0.2, 0.5, 0.3, 0.1, 0.6, 0.2\rangle \langle c_2, 0.6, 0.4, 0.3, 0.2, 0.5, 0.7, 0.1\rangle \langle c_3, 0.5, 0.3, 0.4, 0.1, 0.2, 0.6, 0.3\rangle\}$$

$$B_W = \{\langle c_1, 0.3, 0.5, 0.2, 0.4, 0.6, 0.2, 0.7\rangle \langle c_2, 0.7, 0.3, 0.5, 0.1, 0.4, 0.6, 0.2\rangle \langle c_3, 0.6, 0.2, 0.3, 0.5, 0.1, 0.4, 0.3\rangle\}$$

$$C_W = \{\langle c_1, 0.5, 0.4, 0.6, 0.2, 0.3, 0.7, 0.1\rangle \langle c_2, 0.4, 0.6, 0.5, 0.3, 0.2, 0.1, 0.7\rangle \langle c_3, 0.7, 0.5, 0.3, 0.6, 0.4, 0.2, 0.1\rangle\}$$

In this example, $\tau = \{A_W, B_W, C_W, 0_W, 1_W\}$ *forms a Heptapartitioned Neutrosophic Pythagoreantopology on W.*

Proposition 3.2: *Let* (W,τ_1) *and* (W,τ_2) *be two Heptapartitioned Neutrosophic Pythagorean topological space on W, Then* $\tau_1 \cap \tau_2$ *is an Heptapartitioned Neutrosophic Pythagorean topology on W where* $\tau_1 \cap \tau_2 = \{A_W : A_W \in \tau_1 \text{ and } A_W \in \tau_2\}$

Obviously $0_W, 1_W \in \tau$.

Let $A_W, B_W \in \tau_1 \cap \tau_2$

Then $A_W, B_W \in \tau_1$ *and* $A_W, B_W \in \tau_2$

We know that τ_1 *and* τ_2 *are two Heptapartitioned Neutrosophic Pythagorean topological space W.*

Then $A_W \cap B_W \in \tau_1$ *and* $A_W \cap B_W \in \tau_2$

Hence, $A_W \cap B_W \in \tau_1 \cap \tau_2$

Let τ_1 *and* τ_1 *are two Heptapartitioned Neutrosophic Pythagorean topological spaces on W.*

Denote $\tau_1 \vee \tau_1 = \{A_W \cup B_W : A_W \in \tau_1 \text{ and } A_W \in \tau_2\}$ $\tau_1 \wedge \tau_1 = \{A_W \cap B_W : A_W \in \tau_1 \text{ and } A_W \in \tau_2\}$.

Example 3.3: *Let* A_W *and* B_W *be two Heptapartitioned Neutrosophic Pythagorean topological space on W.*

Define $\tau_1 = \{0_W, 1_W, A_W\}$

$\tau_2 = \{0_W, 1_W, B_W\}$

Then $\tau_1 \cap \tau_2 = \{0_W, 1_W\}$ *is a Heptapartitioned Neutrosophic Pythagorean topological space on W.*

But $\tau_1 \cup \tau_2 = \{0_W, A_W, B_W, 1_W\}$, $\tau_1 \wedge \tau_2 = \{0_W, A_W, B_W, 1_W, A_W \cup B_W\}$ *and* $\tau_1 \vee \tau_2 = \{0_W, A_W, B_W, 1_W, A_W \cap B_W\}$ *are not Heptapartitioned Neutrosophic Pythagorean topological space on W.*

4. Properties of Heptapartitioned Neutrosophic Pythagorean Topological Spaces

Definition 4.0.1: Let (W, τ) be a Heptapartitioned Neutrosophic Pythagorean topological space on W and let A_W belongs to Heptapartitioned Neutrosophic Pythagorean set on W. Then the interior of A_W is denoted as $\text{HNPInt}(A_W)$. It is defined by $\text{HNPInt}\ (A_W) = \cup \{B_W \in \tau : A_W \subseteq B_W\}$

Definition 4.0.2: Let (W, τ) be a Heptapartitioned Neutrosophic Pythagorean topological space on W and let A_W belongs to Heptapartitioned Neutrosophic Pythagorean set W. Then the clo sure of A_W is denoted as $\text{HNPC}(A_W)$. It is defined by $\text{HNPC}(A_W) = \cap \{B_W \in \tau^c : A_W \subseteq B_W\}$

Theorem 4.1: *Let (W, τ) be a Heptapartitioned Neutrosophic Pythagorean topological space over W. Then the following properties are hold.*

(i) 0_W *and* 1_W *are Heptapartitioned Neutrosophic Pythagorean closed sets over W.*

(ii) *The intersection of any number of Heptapartitioned Neutrosophic Pythagorean closed set is a Heptapartitioned Neutrosophic Pythagorean closed set over W.*

(iii) *The union of any two Heptapartitioned Neutrosophic Pythagorean closed set is an Heptapartitioned Neutrosophic Pythagorean closed set over W.*

Proof: It is obviously true. □

Theorem 4.2: *Let (W, τ) be a Heptapartitioned Neutrosophic Pythagorean topological space over W and Let $A_W \in$ Heptapartitioned Neutrosophic Pythagorean topological space. Then the following properties hold.*

(i) $HNP\mathrm{Int}(A_W) \subseteq A_W$

(ii) $A_W \subseteq B_W$ *implies* $HNP\mathrm{Int}(A_W) \subseteq HNP\mathrm{Int}(B_W)$.

(iii) $HNP\mathrm{Int}(A_W) \in \tau$.

(iv) A_W *is a HNP open set implies* $HNP\mathrm{Int}(A_W) = A_W$.

(v) $HNP\mathrm{Int}(HNP\mathrm{Int}(A\,W)) = HNP\mathrm{Int}(A_W)$

(vi) $HNP\mathrm{Int}(0_W) = 0_W$, $HNP\mathrm{Int}(1_W) = 1_W$.

Proof: (i) and (ii) are obviously true.

(iii) obviously $\cup\{B_W \in \tau : B_W \subseteq A_w\} \in \tau$

Note that $\cup\{B_W \in \tau : B_W \subseteq A_w\} = \mathrm{HNPInt}(A_W)$

Therefore, $\mathrm{HNPInt}(A_W) \in \tau$

(iv) Necessity: Let A_W be a HNP open set. ie., $A_W \in \tau$ By (i) and (ii) $\mathrm{HNPInt}(A_W) \subseteq A_w$.

Since $A_W \in \tau$ and $A_W \subseteq A_w$

Then $A_W \cup \{B_W \in \tau : B_W \subseteq A_w\} = \mathrm{HNPInt}(A_W)$

$A_W \subseteq \mathrm{HNPInt}(A_W)$

Thus, $\mathrm{HNPInt} = A_w$.

Sufficiency: Let $\mathrm{HNPInt}(A_w) = A_w$

By (iii) $\mathrm{HNPInt}(A_w) \in \tau$ ie., A_w is a HNP open set.

(v) To prove $\mathrm{HNPInt}(\mathrm{HNPInt}(A_w)) = \mathrm{HNPInt}(A_w)$

By (iii) $\mathrm{HNPInt}(A_w) \in \tau$.

By (iv) $\mathrm{HNPInt}(\mathrm{HNPInt}(A_w)) = \mathrm{HNPInt}(A_w)$.

We know that 0_W and 1_W are in τ

By (iv) $\mathrm{HNPInt}(0_W) = 0_W$, $\mathrm{HNPInt}(1_W) = 1_W$.

Hence, the result. □

Theorem 4.3: *Let (W, τ) be a Heptapartitioned Neutrosophic Pythagorean topological space over W and Let A_W is in the Heptapartitioned Neutrosophic Pythagorean topological space. Then the following properties hold.*

(i) $A_W \subseteq HNPCl(A_W)$

(ii) $A_W \subseteq B_W$ *implies* $HNPCl(A_W) \subseteq HNPCl(B_W)$.

(iii) $HNPCl(A_W)^C \in \tau$.

(iv) A_W *is a HNP closed set implies* $HNPCl(A_W) = A_W$.

(v) $HNPCl(HNPCl(A_W)) = HNPCl(A_W)$

(vi) $HNPCl(0_W) = 0_W$, $HNPCl(1_W) = 1_W$.

Proof: (i) and (ii) are obviously true.

(iii) By theorem, $\mathrm{HNPCl}(A_{W^c}) \in \tau$.

Therefore, $[HNPCl(A_W)]^c = (\cap\{B_W \in \tau^c : B_W \subseteq A_w\})^c$

$= \cup\{B_W \in \tau : B_W \subseteq A_w{}^c\} = \mathrm{HNPInt}(A_W{}^c)$.

Therefore, $[\mathrm{HNPCl}(A_W)]^c \in \tau$.

(iv) Necessity:

By theorem, $A_W \subseteq \mathrm{HNPCl}(A_W)$

Let A_W be a HNP closed set. ie., $A_W \in \tau^c$

Since, $A_W \in \tau$ and $A_W \subseteq A_w$

HNPCl $(A_W) = \cap \{B_W \in \tau^c : A_W \subseteq A_w\}$

HNPCl $(A_W) \subseteq A_w$

Thus, $A_w = \mathrm{HNPCl}(A_w)$

Sufficiency: This is obviously true by (iii)

(v) and (vi) can be proved by (iii) and (iv). □

Theorem 4.4: *Let* (W, τ) *be a Heptapartitioned Neutrosophic Pythagorean topological space over* W *and Let* A_W, B_W *are in Heptapartitioned Neutrosophic Pythagorean topological space W. Then the following properties hold.*

(i) $HNPInt(A_W) \cap HNPInt(B_W) = HNPInt(A_W \cap B_W)$

(ii) $HNPInt(A_W) \cup HNPInt(B_W) \subseteq HNPInt(A_W \cup B_W)$

(iii) $HNPCl(A_W) \cup HNPCl(B_W) \subseteq HNPCl(A_W \cup B_W)$

(iv) $HNPCl(A_W \cup B_W) \subseteq HNPCl(A_W) \cap HNPCl(B_W)$

(v) $(HNPInt(A_W))^c = HNPCl(A_W{}^c)$

(vi) $(HNPCl(A_W))^c = HNPInt(A_W{}^c)$

Proof: (i) Since $A_W \cap B_W \subseteq A_w$ for any w in W

By theorem, $\mathrm{HNPInt}(A_W \cap B_W) \subseteq \mathrm{HNPInt}(A_W)$

Similarly, $\mathrm{HNPInt}(A_W \cap B_W) \subseteq \mathrm{HNPInt}(B_W)$

$\mathrm{HNPInt}(A_W \cap B_W) \subseteq \mathrm{HNPInt}(A_W) \cap \mathrm{HNPInt}(B_W)$

By theorem, $\mathrm{HNPInt}(A_W) \subseteq A\,W$ and $\mathrm{HNPInt}(B_W) \subseteq B_W$

Thus, $\mathrm{HNPInt}(A_W \cap B_W) \subseteq A_W \cap B_W$

Therefore, $\mathrm{HNPInt}(A_W) \cap \mathrm{HNPInt}(B_W) = \mathrm{HNPInt}(A_W \cap B_W)$

Similarly we can prove (ii),(iii) and (iv).

(v) $(\mathrm{HNPInt}(A_W))^c = (\cap\{B_W \in \tau : B_W \subseteq A_w\})A_w$

$= \cap\{B_W \in \tau^c : A_W{}^c \subseteq B_w\}$

$= \mathrm{HNPCl}(A_w{}^c)$

Similarly we can prove (vi). □

Example 4.5: *Let* $W = \{c_1, c_2\}$ *and Let* A_W, B_W, C_W *be Heptapartitioned Neutrosophic Pythagorean sets where*

$A_W = \{\langle c_1, 0.3, 0.2, 0.1, 0.4, 0.3, 0.2, 0.1\rangle \langle c_2, 0.4, 0.3, 0.2, 0.3, 0.2, 0.1, 0.2\rangle\}$

$B_W = \{\langle c_1, 0.2, 0.3, 0.2, 0.3, 0.2, 0.1, 0.2\rangle \langle c_2, 0.3, 0.4, 0.1, 0.4, 0.3, 0.2, 0.1\rangle\}$

$C_W = \{\langle c_1, 0.4, 0.3, 0.2, 0.3, 0.2, 0.1, 0.3\rangle \langle c_2, 0.3, 0.4, 0.1, 0.3, 0.2, 0.1, 0.2\rangle\}$

$\tau = \{A_W, B_W, C_W, 0_W, 1_W\}$ *is an Heptapartitioned Neutrosophic Pythagorean topology on W.*

(i) $HNP\text{Int}(A_W) = 0_W = HNP\text{Int}(A_W)$

Then $A_W \cup B_W = C_W$

$HNP\text{Int}(A_W) \cup HNP\text{Int}(B_W) = 0_W \cup 0_W = 0_W$

And $HNP\text{Int}(A_W \cup B_W) = HNP\text{Int}(C_W) = C_W$

$HNP\text{Int}(A_W) \cup HNP\text{Int}(B_W) \neq HNP\text{Int}(A_W \cup B_W)$

(ii) $HNPCl(B_W)^c = (HNPCl(B_W))^c = 0_W{}^c = 1_W$

$HNP\text{Int}(A_W)^c \cap HNP\text{Int}(B_W{}^c) = 1_W \cap 1_W = 1_W$

Similarly, $HNPCl(A_W{}^c \cap B_W{}^c) = HNPCl(A_W{}^c \cap B_W{}^c)$

$= HNPCl(A_W{}^c \cup B_W{}^c)$

$$= {C_W}^c$$

$$HNPCl({A_W}^c \cap {B_W}^c) \neq HNP\text{Int}(A_W)^c \cap HNP\text{Int}(BW)^c$$

5. Conclusion

Here, the authors explore the properties of Heptapartitioned Neutrosophic Pythagorean Topological Spaces. They delve into the theoretical aspects of these spaces, examining their unique characteristics and behavior. Here also applied in real life problems, demonstrating its practical utility. By integrating these topological spaces into various real world scenarios, they showcase the versatility and effectiveness of Heptapartitioned Neutrosophic Topological Spaces in solving complex issues. The research highlights the potential of this novel approach in both theoretical and applied contexts.

Conflict of Interest

The authors of this paper declare that they have no conflicts of interest.

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AXION FIXED POINT THEOREM: A NEW FRAMEWORK BRIDGING HILBERT MANIFOLDS AND HILBERT SPACES

Abstract: This paper presents a novel fixed-point framework on Hilbert manifolds, called Axion. The local and global structure of manifolds can be better understood by using contraction mappings to define axion points. By using an Axion structure (a, Ψ, Γ), where Ψ is a diffeomorphism and Γ is its inverse meeting a contraction condition, the Axion Fixed Point Theorem extends conventional fixed-point findings to infinite-dimensional spaces. By establishing the existence and uniqueness of axion points, this approach advances our knowledge of fixed points in functional spaces.

Key Words and Phrases: Axion Set, Embedding, Hilbert Manifold, Hilbert Space.

Mathematics Subject Classification (2020) No.: Primary: 46T10, 57R40; Secondary: 46C05.

1. Introduction and Preliminaries

A Hilbert space is an infinite-dimensional generalization of Euclidean space, equipped with an inner product that induces a norm and a complete metric topology. Fixed point theorems are essential in analysis, topology, and geometry, providing fundamental results in nonlinear functional analysis, differential equations, and dynamical systems. The Banach Fixed Point Theorem, one of the most well-known results, guarantees the existence and uniqueness of fixed points under contraction mappings in complete metric spaces. In 1956, Nash established the fundamental theory for embedding abstract Riemannian manifolds into Euclidean spaces, which remains a cornerstone in differential geometry [10]. A few years later, Hamilton (1982) contributed critical insights into curvature evolution, significantly influencing modern perspectives in differential geometry and general relativity [6]. In 2006, Chavel provided an extensive treatment of modern Riemannian geometry, focusing on embedding theorems and geometric flows [5]. Lee (2013) presented a contemporary perspective on smooth manifolds and Lie groups, which has been instrumental in advancing research in differential structures [8]. Between 2020 and 2024, significant progress was made in isometric embeddings and Hilbert manifold structures. Chattopadhyay *et al*. (2020) investigated the isometric embeddability of S_q^m into S_p^n, contributing to a deeper understanding of embeddings between finite-dimensional spaces [4]. In 2024, Capdeville examined the isometric embeddings of n-point spaces for $n \leq 4$, laying the groundwork for further studies in discrete metric spaces [3]. Looking ahead to 2025, Madhan Velayuthan and Jeyanthi Venkatapathy have extended embedding theories by addressing diffeomorphic embeddings of higher-dimensional Hilbert manifolds into Hilbert spaces. Their work introduces innovative techniques for handling infinite-dimensional structures and preserving geometric and topological properties [9].

Definition 1.1 ([8]): *A topological space* $\mathcal{M}$ *is called an* n***-dimensional manifold*** *if*:

1. ***Local Euclidean Property***: $\forall p \in \mathcal{M}, \exists$ *a neighborhood* $U \subset \mathcal{M}$ *and a homeomorphism* $\sigma : U \to V \subset \mathbb{R}^n$, *such that* σ *and* $\sigma - 1$ *are continuous*.

2. ***Hausdorff Property***: $\mathcal{M}$ *is Hausdorff, i.e.,* $\forall p,\ q \in \mathcal{M},\ p \neq q, \exists$ *disjoint open sets* $U_p,\ U_q$ *such that* $p \in U_p$ *and* $q \in U_q$.

3. ***Second-Countability***: *The topology of* $\mathcal{M}$ *has a countable basis.*

If, in addition, $\mathcal{M}$ *is equipped with an atlas* $\{(U_j, \sigma_j)\}_{j \in \mathcal{J}}$ *such that for any two overlapping charts* (U_j, σ_j) *and* (U_k, σ_k), *the transition maps*

$\sigma_k \circ \sigma_j^{-1} : \sigma_j(U_j \cap U_k) \to \sigma_k(U_j \cap U_k)$ *are infinitely differentiable* (C^∞), *then* $\mathcal{M}$ *is called a **smooth manifold**.*

Definition 1.2 ([5]): *A **Hilbert manifold*** $\mathbb{H}_\mathcal{M}$ *is a smooth manifold modeled on an Hilbert space* $\mathcal{H}$. *Specifically,* $\mathbb{H}_\mathcal{M}$ *satisfies*:

1. $\exists$ *an atlas* $\{(U_\alpha, \sigma_\alpha)\}_{\alpha \in \mathcal{A}}$ *such that each chart* $\sigma_\alpha : U_\alpha \to \sigma_\alpha(U_\alpha) \subset \mathcal{H}$ *is a bijective homeomorphism mapping onto an open subset of* $\mathcal{H}$.

2. *Transition maps between overlapping charts,* $\sigma_\beta \circ \sigma_\alpha^{-1} : \sigma_\alpha(U_\alpha \cup U_\beta) \to \sigma_\beta(U_\alpha \cap U_\beta)$, *are infinitely differentiable* (C^∞).

3. *The topology of* $\mathbb{H}_\mathcal{M}$ *is induced by* $\mathcal{H}$, *i.e.,* $A \subset \mathbb{H}_\mathcal{M}$ *is open if and only if* $\sigma_\alpha(A)$ *is open in* $\mathcal{H}$ *for each chart* σ_α.

Definition 1.3 [9]: *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be Hilbert manifolds. A mapping* $\Psi : \mathcal{M} \to \mathcal{N}$ *is called a **diffeomorphism** if it satisfies the following conditions*:

1. ***Bijectivity***: *The map* Ψ *is a bijection, meaning it is both injective and surjective.*

2. ***Smoothness***: *The map* Ψ *is infinitely differentiable, i.e.,* $\Psi \in C^\infty(\mathcal{M}, \mathcal{N})$.

3. ***Smooth Inverse***: *The inverse mapping* $\Psi^{-1} : \mathcal{N} \to \mathcal{M}$ *exists and is also smooth, ensuring that* Ψ *establishes a smooth one-to-one correspondence between* $\mathcal{M}$ *and* $\mathcal{N}$.

If such a map exists, we say that $\mathcal{M}$ *and* $\mathcal{N}$ *are* ***diffeomorphic****, denoted as* $\mathcal{M} \cong \mathcal{N}$.

Definition 1.4 [8]: *Given a local chart* $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathcal{H}$, *the induced metric on* U_α *is defined by* $d_{\mathcal{M}}(x, y) = \| \varphi_\alpha(x) - \varphi_\alpha(y) \|_{\mathcal{H}}$, *where* $\|\cdot\|_{\mathcal{H}}$ *denotes the norm in the Hilbert space* $\mathcal{H}$.

Theorem 1.5 ([8]): *Let* $\mathbb{S}_d$ *be a complete metric space and let* $C : \mathbb{S}_d \to \mathbb{S}_d$ *be a contraction mapping, i.e.,* $\exists c \in [0,1]$ *such that* $d(\mathcal{C}(h), \mathcal{C}(t)) \leq c \cdot d(h,t), \forall h, \ t \in \mathbb{S}_d$. *Then,* $\mathcal{C}$ *has a unique fixed point* $h^* \in \mathbb{S}_d$.

2. Axion Fixed Point Theorem

This section presents new definition Axion and Axion fixed point theorem.

Definition 2.1: *Let* $\mathcal{M}$ *be a Hilbert manifold modeled on a Hilbert space* $\mathcal{H}$, *with an atlas* $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$. *An* ***Axion*** *is an ordered triplet* $(a, \ \Psi, \Gamma)$ *satisfying*:

1. ***Accumulation*****:** $a \in \mathcal{M}$ *is an accumulation point of* $S \subset M$, *i.e.,* $\forall U' \subset \mathcal{M}, a \in U' \Rightarrow U' \cap S \ = \varnothing$.

2. ***Smooth Chart*****:** $\exists$ *a chart* $(U_\alpha, \varphi_\alpha)$ *with* $a \in U_\alpha$ *and a smooth diffeomorphism* $\Psi : U_\alpha \to \Psi(U_\alpha) \subset \mathcal{H}$.

3. ***Inverse Mapping*****:** $\Gamma = \Psi^{-1} : \Psi(U_\alpha) \to U_\alpha$ *is smooth.*

The set of all Axions $\mathcal{X} = \{(a, \Psi, \Gamma) \mid \Psi : U\alpha \to \Psi(U_\alpha)$ *is a diffeomorphism,* $\Gamma = \Psi^{-1}\}$.

Theorem 2.2 (Axion Fixed Point Theorem): *Let* (a, Ψ, Γ) *be an Axion in a Hilbert manifold* $\mathcal{M}$ *with respect to a chart* $(U_\alpha, \varphi_\alpha)$, *where*:

1. $\Psi : U_\alpha \to \Psi(U_\alpha) \subset \mathcal{H}$ *is a smooth diffeomorphism.*

2. $\Gamma : \Psi(U_\alpha) \to U_\alpha$ *is the inverse of* Ψ, *i.e.,* $\Gamma = \Psi^{-1}$.

3. Γ *satisfies the contraction condition*: $\exists$ *a constant* $c \in [0,1)$ *such that* $d_{\mathcal{M}}(\Gamma(x), \Gamma(y)) \le c \cdot d_{\mathcal{M}}(x, y), \ \forall x, \ y \in U_\alpha$.

Then, $\exists$ *a unique fixed point* $a^* \in U_\alpha$ *such that* $\Gamma(a^*) = a^*$.

Proof: The local chart $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathcal{H}$ induces a metric on U_α defined by: $d_{\mathcal{M}}(x, y) = \| \varphi_\alpha(x) - \varphi_\alpha(y) \|_{\mathcal{H}}$, where $\|\cdot\|_{\mathcal{H}}$ is the norm in $\mathcal{H}$. This metric provides a distance measure for elements of U_α. To apply the Banach Fixed Point Theorem, we must show that $(U_\alpha, d_{\mathcal{M}})$ is a complete metric space. Let (x_n) be a Cauchy sequence in U_α with respect to $d_{\mathcal{M}}$. By definition, $d_{\mathcal{M}}(x_n, x_m) = \| \varphi_\alpha(x_n) - \varphi_\alpha(x_m) \|_{\mathcal{H}}$.

Since, (x_n) is Cauchy in U_α, the sequence $\varphi_\alpha(x_n)$ is Cauchy in $\mathcal{H}$. Since, $\mathcal{H}$ is a Hilbert space, it is complete, and thus, $\exists$ a limit point $y \in \mathcal{H}$ such that: $\varphi_\alpha(x_n) \to y$ as $n \to \infty$. Since, $\Psi = \varphi_\alpha^{-1}$ is a diffeomorphism. By the continuity of Ψ, $x_n = \Psi(\varphi_\alpha(x_n)) \to \Psi(y)$ as $n \to \infty$. Since $y \in \Psi(U_\alpha)$, we have $\Psi(y) \in U_\alpha$, proving that U_α is complete. By assumption, $\exists\, c \in [0,1)$ such that: $d_{\mathcal{M}}(\Gamma(x), \Gamma(y)) \le c \cdot d_{\mathcal{M}}(x, y), \ \forall x, \ y \in U_\alpha$. This confirms that Γ is a strict contraction mapping.

Since $(U_\alpha, d_{\mathcal{M}})$ is a complete metric space and Γ is a contraction, the Banach Contraction Theorem guarantees the existence of a unique fixed point

$a^* \in U_\alpha$ such that: $\Gamma(a^*) = a^*$. Suppose there exist two fixed points $a_1,\ a_2 \in U_\alpha$ such that $\Gamma(a_1) = a_1$ and $\Gamma(a_2) = a_2$.

Then, $d_{\mathcal{M}}(a_1, a_2) = d_{\mathcal{M}}(\Gamma(a_1), \Gamma(a_2)) \leq c \cdot d_{\mathcal{M}}(a_1, a_2)$.

Since, $c \in [0,1)$, it follows that $d_{\mathcal{M}}(a_1, a_2) = 0$, implying $a_1 = a_2$.

Thus, the fixed point is unique. □

3. Conclusion

The Axion triplet (a, Ψ, Γ) is introduced in the Axion fixed point Theorem, which extends the standard Banach fixed point theorem to Hilbert manifolds. This framework preserves the underlying geometric structure of the manifold while enabling the analysis of contraction mappings inside local charts. In order to provide stability under smooth transformations, the theorem ensures that fixed points for such mappings exist and are unique. The Banach contraction theorem is used in the proof to demonstrate convergence, taking use of the contraction quality of Γ and the completeness of the induced metric space.

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SOLITONS ON VARIOUS GEOMETRIC STRUCTURES OF PARA-SASAKIAN MANIFOLDS ADMITTING SCHOUTEN-VAN KEMPEN CONNECTION

Abstract. In this paper we investigate properties of para-Sasakian manifold by the help of Schouten-van Kampen connection. We also study para-Sasakian manifolds of various equivalent structures admitting conformal Ricci soliton and conformal η-Ricci soliton with respect to Schouten-van Kampen connection.

Key words and phrases: Para-Sasakian Manifold, Schouten-van Kampen Connection, Conformal Ricci Soliton, Conformal η-Ricci Soliton.

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1. Introduction

In 1979, the notion of para-Sasakian (briefly, P-Sasakian) and special para-Sasakian (briefly, SP-Sasakian) manifolds were introduced by Sato and Matsumoto [28]. Later, Adati and Matsumoto investigate some interesting results on P-Sasakian manifolds and SP-Sasakian manifolds in [1]. The properties of para-Sasakian manifold have been studied by many authors. For instance, we see [2, 16, 17, 19, 27, 30] and their references.

The notion of Schouten-van Kampen connection (shortly, SVK-connection) was introduced in the third decade of last century for a study of non-holomorphic manifolds [29, 37]. In 2006, Bejancu [3] studied Schouten-van Kampen connection on Foliated manifolds. Recently, Biswas and Baisya [4, 5] investigated some properties of pesudo symmetric Sasakian manifolds with respect to SVK-connectiopn. Most recently, this connection has been introduced on para-Sasakian manifold by Sundriyal and Upreti [31]. They studied projective curvature tensor, concircular curvature tensor and Nijenhuis tensor for the para-sasakian manifold with respect to this connection. SVK-connection $(\overline{\nabla})$ for an n-dimensional almost contact metric manifold M equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a (1, 1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, is defined by

$$\overline{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi\,, \tag{1.1}$$

for all $X, Y \in \chi(M)$, where $\chi(M)$ is the set of all vector fields on M and ∇ being the Levi-Civita connection on M.

The concept of Ricci flow was first introduced by R. S. Hamilton in the early 1980s. Hamilton [13] observed that the Ricci flow is an excellent tool for simplifying the structure of a manifold. It is the process which deforms the metric of a Riemannian manifold by smoothing out the irregularities. The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S\,, \tag{1.2}$$

where g is a Riemannian metric, S is Ricci tensor and t is time. The solitons for the Ricci flow is the solutions of the above equation, where the metrices at different times differ by a diffeomorphism of the manifold. A Ricci soliton is represented by a triple (g, V, λ), where V is a vector field and λ is a scalar, which satisfies the equation

$$L_V g + 2S + 2\lambda g = 0\,, \tag{1.3}$$

where S is Ricci curvature tensor and $L_V g$ denotes the Lie derivative of g along the vector field V. A Ricci soliton is said to be shrinking, steady, expanding according as $\lambda < 0$, $\lambda = 0$, $\lambda > 0$, respectively. The vector field V is called potential vector field and if it is a gradient of a smooth function, then the Ricci soliton (g, V, λ) is called a gradient Ricci soliton and the associated function is called the potential function. Ricci soliton was further studied by many researchers. For instance, we see [18, 25, 35, 36] and their references.

In 2005, Fischer [12] introduced conformal Ricci flow which is a generalisation of the Ricci flow equation that modifies the unit volume constraint to a scalar curvature constraint. The conformal Ricci flow equation is given by

$$\frac{\partial g}{\partial t} + 2\left(S + \frac{g}{n} \right) = -pg \tag{1.4}$$

$$r(g) = -1 \tag{1.5}$$

where $r(g)$ is the scalar curvature of the manifold, p is a non-dynamical scalar field and n is the dimension of the manifold. In 2015, corresponding to the conformal Ricci flow equation, Basu and Bhattacharyya [7] introduced the notion of conformal Ricci soliton as a generalisation of Ricci soliton and it is given by

$$L_V g + 2S + \left[2\lambda - \left(p + \frac{2}{n} \right) \right] g = 0, \tag{1.6}$$

where λ is a constant.

As a generalization of Ricci soliton, the η-Ricci soliton was introduced by Cho and Kimura [9]. This notion has also been studied by Cälin and Crasmarearu [10]. Later, remarkable studies on η-Ricci soliton have been made by Blaga [6] and Prakasha [24]. Let M be a Riemannian manifold with structure (ϕ, ξ, η, g). Consider the equation

$$L_V g + 2S + 2\lambda g + 2\mu \eta \otimes \eta = 0 , \tag{1.7}$$

where S is Ricci curvature tensor, $L_V g$ denotes the Lie derivative of g along the vector field V, λ and μ are real constants. The data (g, V, λ, μ) which satisfies the equation (1.7) is called an η-Ricci soliton on M. In particular, when $\mu = 0$, the notion of η-Ricci soliton simply reduces to the notion of Ricci soliton. And when $\mu \neq 0$, (g, V, λ, μ) is called proper η-Ricci soliton on M.

In 2018, Siddiqi [34] introduced the notion of conformal η-Ricci soliton as

$$L_V g + 2S + \left[2\lambda - \left(p + \frac{2}{n} \right) \right] g + 2\beta \eta \otimes \eta = 0, \tag{1.8}$$

where $L_V g$ denotes the Lie derivative of g along the vector field V, λ and β are real constants and p is a non-dynamical scalar field.

Definition 1.1: *Let $\mathcal{F}$ and Ω be two tensors of rank 4. A Riemanian manifold (or, pseudo Riemannian manifold) M is said to be Ω-semisymmetric type if $\mathcal{F}(X,Y).\Omega = 0$ for all smooth vector fields X, Y on M, where $\mathcal{F}$ acts on Ω as derivation of tensor algebra.*

In the above definition if we consider $\mathcal{F} = \Omega = R$, then the manifold M is called semi-symmetric [32]. Semi-symmetry and other conditions of semi-symmetry type are studied in detail in [8, 15, 20, 33]. In 2013, Kundu and Shaikh [26] investigated the equivalency of the various geometric structures depending on conditions of semi-symmetry. They have established the following conditions

(i) $E.R = 0$, $E.P = 0$, $E.E = 0$, $E.P = 0$, $E.\mathcal{M} = 0$, $E.W_i = 0$ and $E.\mathcal{W}_i^* = 0$ (for all $i = 1,2,\ldots 9$) are equivalent and named such a class by C_1;

(ii) $R.R = 0$, $R.P = 0$, $R.E = 0$, $R.P = 0$, $R.M = 0$, $R.W_i = 0$ and $R.W_i^* = 0$ (for all $i = 1,2, \ldots\ldots 9$) are equivalent and named such a class by C_2;

(iii) $R.K = 0$ and $R.C = 0$ are equivalent and named such a class by C_3;

(iv) $E.C = 0$ and $E.K = 0$ are equivalent and named such a class by C_4;

where the symbols C, E, P, K, M and Wi stand for conformal curvature tensor [11], concircular curvature tensor [38], projective curvature tensor [38], conharmonic curvature tensor [14], M-projective curvature tensor [22], Wi-curvature tensor [21, 22, 23] and W_i^*-curvature tensor [22], respectively.

$$C(X,Y) = R(X,Y) - \frac{1}{n-2}\left[(X \wedge_g QY) + (QX \wedge_g Y) + \frac{r}{n-1}(X \wedge_g Y)\right], \quad (1.9)$$

$$E(X,Y) = R(X,Y) - \frac{r}{n(n-1)}(X \wedge_g Y), \tag{1.10}$$

$$P(X,Y) = R(X,Y) - \frac{1}{n-1}(X \wedge_g Y), \tag{1.11}$$

$$K(X,Y) = R(X,Y) - \frac{1}{n-2}[(X \wedge_g QY) + (QX \wedge_g Y)], \tag{1.12}$$

$$\mathcal{M}(X,Y) = R(X,Y) - \frac{1}{2(n-1)}[(X \wedge_g QY) + (QX \wedge_g Y)], \tag{1.13}$$

$$\mathcal{W}_0(X,Y) = R(X,Y) - \frac{1}{n-1}(X \wedge_g QY), \tag{1.14}$$

$$\mathcal{W}_0^*(X,Y) = R(X,Y) + \frac{1}{n-1}(X \wedge_g QY), \tag{1.15}$$

$$\mathcal{W}_1(X,Y) = R(X,Y) - \frac{1}{n-1}(X \wedge_S Y), \tag{1.16}$$

$$\mathcal{W}_1^*(X,Y) = R(X,Y) + \frac{1}{n-1}(X \wedge_S Y), \tag{1.17}$$

$$\mathcal{W}_2(X,Y) = R(X,Y) - \frac{1}{n-2}\Big[(QX \wedge_g Y) + (X \wedge_g QY) - (X \wedge_S Y)\Big], \tag{1.18}$$

$$\mathcal{W}_2^*(X,Y) = R(X,Y) + \frac{1}{n-2}\Big[(QX \wedge_g Y) + (X \wedge_g QY) - (X \wedge_S Y)\Big], \tag{1.19}$$

$$\mathcal{W}_3(X,Y) = R(X,Y) - \frac{1}{n-1}(Y \wedge_g QX), \tag{1.20}$$

$$\mathcal{W}_3^*(X,Y) = R(X,Y) + \frac{1}{n-1}(Y \wedge_g QX), \tag{1.21}$$

$$\mathcal{W}_5(X,Y) = R(X,Y) - \frac{1}{n-1}[(X \wedge_g QY) - (X \wedge_S Y)], \tag{1.22}$$

$$\mathcal{W}_5^*(X,Y) = R(X,Y) + \frac{1}{n-1}[(X \wedge_g QY) - (X \wedge_S Y)], \tag{1.23}$$

$$\mathcal{W}_7(X,Y) = R(X,Y) + \frac{1}{n-1}[(QX \wedge_g Y) - (X \wedge_S Y)], \tag{1.24}$$

$$\mathcal{W}_7^*(X,Y) = R(X,Y) - \frac{1}{n-1}[(QX \wedge_g Y) - (X \wedge_S Y)], \tag{1.25}$$

$$\mathcal{W}_4(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[g(X,Z)QY - g(X,Y)QZ], \tag{1.26}$$

$$\mathcal{W}_4^*(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)QY - g(X,Y)QZ], \tag{1.27}$$

$$\mathcal{W}_6(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(X,Z)QY - g(X,Y)QZ], \tag{1.28}$$

$$\mathcal{W}_6^*(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(Y,Z)X - g(X,Y)QZ], \tag{1.29}$$

$$\mathcal{W}_8(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - g(X,Y)Z], \tag{1.30}$$

$$\mathcal{W}_8^*(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(Y,Z)X - g(X,Y)Z], \tag{1.31}$$

$$\mathcal{W}_9(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(X,Y)Z - g(Y,Z)QX], \tag{1.32}$$

$$\mathcal{W}_9^*(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(X,Y)Z - g(Y,Z)QX], \tag{1.33}$$

where

$$(X \wedge_D Y)Z = D(Y,Z)X - D(X,Z)Y .$$

for all $X, Y, Z \in \chi(M)$, where R is the Riemannian curvature tensor of type $(1, 3)$ and r is the scalar curvature.

Definition 1.2: *A para-Sasakian manifold M is called an Einstein manifold if its Ricci tensor is of the form*

$$S(Y,Z) = kg(Y,Z),$$

for all $Y, Z \in \chi(M)$, where k being a scalar.

Definition 1.3: *A para-Sasakian manifold M is called an η-Einstein manifold if its Ricci tensor is of the form*

$$S(Y,Z) = l_1 g(Y,Z) + l_2 \eta(Y)\eta(Z),$$

for all $Y, Z \in \chi(M)$, *where* l_1, l_2 *are scalars.*

Definition 1.4: *A para-Sasakian manifold M is called a generalized* η*-Einstein manifold if its Ricci tensor is of the form*

$$S(Y,Z) = k_1 g(Y,Z) + k_2 \eta(Y)\eta(Z) + k_3 g(Y,\phi Z),$$

for all $Y, Z \in \chi(M)$, *where* k_1, k_2 *and* k_3 *are scalars.*

This paper is structured as follows:

First two sections of the paper has been kept for introduction and preliminaries. In Section-3, we study properties of para-Sasakian manifold with respect to SVK-connection. In Section-4, we introduce conformal Ricci soliton on para-Sasakian manifold with respect to SVK-connection. In Section-5, we study conformal η-Ricci soliton on para-Sasakian manifold with respect to SVK-connection. Section-6 concerns with conformal η-Ricci soliton with respect to SVK-connection on para-Sasakian manifolds of class C_1, C_2, C_3 and C_4.

2. Preliminaries

Let M be an n-dimensional differentiable manifold with structure (ϕ, ξ, η), where η is a 1-form, ξ is the structure vector field, ϕ is a (1, 1)-tensor field satisfying [28]

$$\phi^2(X) = X - \eta(X)\xi, \quad \eta(\xi) = 1 \tag{2.1}$$

$$\phi(\xi) = 0, \eta \circ \phi = 0, \tag{2.2}$$

for all vector field X on M is called almost paracontact manifold. If an almost paracontact manifold M with structure (ϕ, ξ, η) admits a pseudo-Riemannian metric g such that [39]

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

then we say that M is an almost paracontact metric manifold with an almost paracontact metric structure (ϕ, ξ, η, g). From (2.3) one can deduce that

$$g(X,\phi Y) = -g(\phi X,Y), \tag{2.4}$$

$$g(X,\xi) = \eta(\xi). \tag{2.5}$$

An almost paracontact metric structure of M becomes a paracontact metric structure [39] if

$$g(X,\phi Y) = d\eta(X,Y),$$

for all vector fields X, Y on M, where

$$d\eta(X,Y) = \frac{1}{2}\{X\eta(Y) - Y\eta(X) - \eta([X,Y])\}.$$

The manifold M is called a para-Sasakian manifold if

$$(\nabla_X\varphi)Y = -g(X,Y)\xi + \eta(Y)X, \tag{2.6}$$

for any smooth vector fields X, Y on M.

In a para-Sasakian manifold the following relations also hold [39]

$$(\nabla_X\eta)Y = g(X,\phi Y), \nabla_X\xi = -\phi X, \tag{2.7}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.8}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.9}$$

$$R(\xi,X)Y = -g(X,Y)\xi + \eta(Y)X, \tag{2.10}$$

$$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.11}$$

$$R(\xi,X)\xi = X - \eta(X)\xi, \tag{2.12}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{2.13}$$

$$S(\xi,\xi) = -(n-1),\ Q\xi = -(n-1)\xi, \tag{2.14}$$

$$S(\phi X,\ \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y). \tag{2.15}$$

for any smooth vector fields X, Y, Z on M.

3. Schouten-Van Kampen Connection on Para-Sasakian Manifolds

In this section we get the relation between SVK-connection and Levi-Civita connection on para-Sasakian manifold M. Then we obtain Rie-mannian curvature tensor, Ricci curvature tensor, Ricci operator and scalar curvature of M with respect to the SVK-connection. We also establish here the first Bianchi identity with respect to SVK-connection on M.

In view of (1.1), (2.7) and (2.5), we get the expression for SVK-connection in a para-Sasakian manifold M as

$$\overline{\nabla}_X Y = \nabla_X Y + g(X, \phi Y)\xi + \eta(Y)\phi X \ , \tag{3.1}$$

with torsion tensor

$$\overline{T}(X,Y) = 2g(X,\phi Y)\xi + \eta(Y)\phi X - \eta(X)\phi Y \ .$$

On para-Sasakian manifold the connection $\overline{\nabla}$ has the following properties

$$\overline{\nabla}_X \xi = 0, (\overline{\nabla}_X \eta)Y = g(\phi X, Y), \tag{3.2}$$

$$(\overline{\nabla}_X g)(Y,Z) = g(\phi X, Y)\eta(Z) + g(\phi Y, Z)\eta(X). \tag{3.3}$$

for all $X, Y \in \chi(M)$.

Proposition 3.1: *The SVK-connection on a para-Sasakian manifold is non metric compatible connection.*

Proposition 3.2: *The SVK-connection on a para-Sasakian manifold is non symmetric connection.*

Proposition 3.3: *The structure vector field of a para-Sasakian manifold is parallel with respect to SVK-connection.*

Let $\overline{R}$ be the Riemannian curvature tensor with respect to SVK-connection on a para-Sasakian manifold defined as

$$\overline{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z . \tag{3.4}$$

Then using (2.6), (2.7) and (3.1) in (3.4) we get

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &\quad + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y .\end{aligned} \tag{3.5}$$

Writing the equation (3.5) by cyclic permutations of X,Y and Z and using the fact that $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$, we have

$$\overline{R}(X,Y)Z + \overline{R}(Y,Z)X + \overline{R}(Z,X)Y = 0 ,$$

for all $X,Y,Z \in \chi(M)$.

Taking inner product of (3.5) with a vector field U and contracting over X and U we get

$$\overline{S}(Y,Z) = S(Y,Z) + (n-1)\eta(Y)\eta(Z) - \psi g(Y,\phi Z), \tag{3.6}$$

where $\overline{S}$ denotes Ricci curvature tensor with respect to $\overline{\nabla}$ and $\psi = trace(\phi)$.

Proposition 3.4: *The SVK-connection on para-Sasakian manifold satisfies the first Bianchi identity.*

Lemma 3.5: *Let M be an n-dimensional para-Sasakian manifold admitting SVK-connection, then*

$$\overline{R}(X,Y)\xi = 0,\ \ \overline{R}(\xi,Y)Z = 2\left[g(Y,Z)\xi - \eta(Z)Y\right], \tag{3.7}$$

$$\overline{R}(X,\xi)Z = -2[g(X,Z)\xi + \eta(Z)X], \tag{3.8}$$

$$\overline{S}(X,\xi) = 0 = \overline{S}(\xi,Y), \tag{3.9}$$

$$\overline{Q}X = QX + (n-1)\eta(X)\xi + \phi X\psi, \ \ \overline{Q}\xi = 0\,, \tag{3.10}$$

$$\overline{r} = r + (n-1) - \psi^2\,, \tag{3.11}$$

for all $X, Y, Z \in \chi(M)$, *where* $\overline{R}$, $\overline{Q}$ *and* $\overline{r}$ *denote Riemannian curvature tensor, Ricci operator and scalar curvature with respect to* $\overline{\nabla}$, *respectively.*

Remark 3.6: *Eigen value of Ricci operator with respect to SVK-connection corresponding to the eigen vector* ξ *is zero.*

4. Conformal Ricci Soliton on Para-Sasakian Manifold with Respect to SVK-Connection

In this section we find a para-Sasakian manifold M admitting conformal Ricci soliton with respect to SVK-connection in which the potential vector field being pointwise collinear with the structure vector field of M.

Let $V = \alpha\xi$, where α is some non-zero smooth function. Taking covariant derivative of V in the direction of X and using (2.7) we get

$$\nabla_X V = X(\alpha)\xi - \alpha\phi X\,. \tag{4.1}$$

In view of (3.1) and (4.1) we have

$$\overline{\nabla}_X V = X(\alpha)\xi - \alpha\phi X + g(X,\phi V)\xi + \eta(V)\phi X\,. \tag{4.2}$$

Writing equation (1.6) with respect to SVK-connection we have

$$\begin{aligned} 0 &= (\overline{L}_V g)(X,Y) + 2\overline{S}(X,Y) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y) \\ &= g(\overline{\nabla}_X V, Y) + g(X, \overline{\nabla}_Y V) \\ &\quad + 2\overline{S}(X,Y) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y)\,. \end{aligned} \tag{4.3}$$

Using (4.2) in (4.3) we get

$$0 = X(\alpha)\eta(Y) + Y(\alpha)\eta(X) + 2\overline{S}(X,Y) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right]g(X,Y). \tag{4.4}$$

Setting $X = \xi$ and using (3.9) in (4.4) we get

$$0 = \xi(\alpha)\eta(Y) + Y(\alpha) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right]\eta(Y). \tag{4.5}$$

Replacing Y by ξ in (4.5) we obtain

$$\xi(\alpha) = \left[\left(\frac{p}{2} + \frac{1}{n}\right) - \lambda\right]. \tag{4.6}$$

Using (4.6) in (4.5) we get

$$Y(\alpha) = \left[\left(\frac{p}{2} + \frac{1}{n}\right) - \lambda\right]. \tag{4.7}$$

If we consider $Y(\alpha) = 0$, then equation (4.7) yields

$$\lambda = \frac{p}{2} + \frac{1}{n}.$$

Therefore we have the following theorem

Theorem 4.1: *Let $M(\phi, \xi, \eta, g)$ be a para-Sasakian manifold admitting conformal Ricci soliton (g, V, λ) with respect to SVK-connection. If V is pointwise collinear with ξ, then V is a constant multiple of ξ provided $\lambda = \frac{p}{2} + \frac{1}{n}$.*

Now setting $V = \xi$ in (4.3) we have

$$0 = 2\overline{S}(X,Y) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right]g(X,Y). \tag{4.8}$$

Using (3.6) in (4.8) we get

$$S(X,Y) = \left[\left(\frac{p}{2} + \frac{1}{n}\right) - \lambda\right] g(X,Y) - (n-1)\eta(X)\eta(Y) + \psi g(X, \phi Y). \tag{4.9}$$

Corollary 4.2: *If a para-Sasakian manifold M admits conformal Ricci soliton (g, ξ, λ) with respect to SVK-connection, then M is generalized η-Einstein.*

5. Conformal η-Ricci Soliton on Para-Sasakian Manifold with Respect to SVK-Connection

Writing equation (1.6) with respect to SVK-connection we have

$$0 = (\overline{L}_\xi g)(X,Y) + 2\overline{S}(X,Y) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y) + 2\beta\eta(X)\eta(Y). \tag{5.1}$$

Expanding (5.1) we get

$$0 = g(\overline{\nabla}_X \xi, Y) + g(X, \overline{\nabla}_Y \xi) + 2\overline{S}(X,Y) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y) + 2\beta\eta(X)\eta(Y). \tag{5.2}$$

Using (3.2) in (5.2) we obtain

$$0 = 2\overline{S}(X,Y) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y) + 2\beta\eta(X)\eta(Y). \tag{5.3}$$

Setting $X = \xi$ in (5.3) we have

$$\lambda = \left(\frac{p}{2} + \frac{1}{n}\right) - \beta. \tag{5.4}$$

Hence, we have the following theorem

Theorem 5.1: *If an n-dimensional para-Sasakian manifold admits a conformal* η*-Ricci soliton* (g, ξ, λ, β) *with respect to SVK-connection, then the relation between the soliton scalars are given by*

$$\lambda = \left(\frac{p}{2} + \frac{1}{n}\right) - \beta.$$

Using (3.6) in (5.3) we obtain

$$S(X,Y) = \left[\left(\frac{p}{2} + \frac{1}{n}\right) - \lambda\right] g(X,Y) - (n + \beta - 1)\eta(X)\eta(Y) + \psi g(X, \phi Y), \tag{5.5}$$

which shows that M is generalized η-Einstein manifold.

Corollary 5.2: *If an n-dimensional para-Sasakian manifold* M *contains a conformal* η*-Ricci soliton with respect to SVK-connection, then* M *is generalized* η*-Einstein manifold.*

Contracting (5.5) over X and Y we get

$$r = \frac{n}{2}(p - 2) - \lambda n + \beta + \psi^2 + 2. \tag{5.6}$$

Corollary 5.3: *If an n-dimensional para-Sasakian manifold* M *contains a conformal* η*-Ricci soliton with respect to SVK-connection, then the scalar curvature of* M *is given by equation* (5.6).

6. Conformal η-Ricci Soliton with Respect to SVK-Connection on Equivalence Classes C_1, C_2, C_3 and C_4

In this section we consider η-Ricci soliton (g, ξ, λ, β) with respect to SVK-connection on the manifolds belong to the equivalence classes C_1, C_2, C_3 and C_4 and obtain the relation between the soliton constants.

Conformal η-Ricci soliton with respect to $\overline{\nabla}$ on class C_1: The condition that must be satisfied by the Riemannian curvature tensor (R) is

$$(E(\xi, X).R)(Y, Z)V = 0, \tag{6.1}$$

for all $X, Y, Z, V \in \chi(M)$.

Equation (6.1) gives

$$E(\xi, X).R(Y, Z)V = R(E(\xi, X)Y, Z)V + R(Y, E(\xi, X)Z)V + R(Y, Z)E(\xi, X)V. \tag{6.2}$$

Setting $V = \xi$ and using (1.10), (2.9)-(2.11) in (6.2) we get

$$0 = [r + n(n-1)][g(X, Y)Z - g(X, Z)Y] - [r + n(n-1)]R(Y, Z)X. \tag{6.3}$$

Taking an inner product of (6.3) with a vector field U we get

$$0 = [r + n(n-1)]\ [g(X, Y)g(Z, U) - g(X, Z)g(Y, U)] - [r + n(n-1)]g(R(Y, Z)X, U). \tag{6.4}$$

Contracting (6.4) over Z and U we have

$$S(X, Y) = -(n-1)g(X, Y), \tag{6.5}$$

if $r \neq -n(n-1)$.

In view of (5.5) and (6.5) we obtain

$$0 = \left[\left(\frac{p}{2} + \frac{1}{n}\right) - \lambda + n - 1\right]g(X, Y) - (n + \beta - 1)\eta(X)\eta(Y) + \psi g(X, \phi Y). \tag{6.6}$$

Setting $Y = \xi$ in (6.6) we have

$$\lambda = \left(\frac{p}{2} + \frac{1}{n}\right) - \beta. \tag{6.7}$$

Thus, we have the following theorem:

Theorem 6.1: *Let $M(\phi, \xi, \eta, g)$ be an η-dimensional para-Sasakian manifold of class C_1. If M admits a conformal η-Ricci soliton with respect to SVK-connection, then the soliton constants are given by*

$$\lambda = \left(\frac{p}{2} + \frac{1}{n}\right) - \beta,$$

provided $r \neq -n(n-1)$.

Corollary 6.2: *A para-Sasakian manifold of class C_1 is Einstein manifold if $r \neq -n(n-1)$.*

Corollary 6.3: *If a para-Sasakian manifold of class C_1 contains conformal η-Ricci soliton with respect to SVK-connection, then the manifold is generalized η-Einstein, provided $r \neq -n(n-1)$.*

Conformal η-Ricci Soliton with Respect to $\bar{\nabla}$ on Class C_2: The condition that must be satisfied by the Riemannian curvature tensor (R) is

$$(E(\xi, X).R)(Y, Z)V = 0,$$

for all $X, Y, Z, V \in \chi(M)$

$$R(\xi, X).R(Y, Z)V = R(R(\xi, X)Y, Z)V + R(Y, R(\xi, X)Z)V + R(Y, Z)R(\xi, X)V. \tag{6.8}$$

Setting $V = \xi$ and using (2.8)-(2.11) in (6.8) we get

$$0 = [g(X, Y)Z - g(X, Z)Y] - R(Y, Z)X. \tag{6.9}$$

Taking an inner product of (6.9) with a vector field W we get

$$0 = [g(X, Y)g(Z, W) - g(X, Z)g(Y, W)] - g(R(Y, Z)X, W). \tag{6.10}$$

Contracting (6.10) over Z and W we have

$$S(X,Y) = -(n-1)\,g(X,Y), \tag{6.11}$$

In view of (5.5) and (6.11) we obtain

$$\begin{aligned} 0 = \left[\left(\frac{p}{2}+\frac{1}{n}\right) - \lambda + n - 1\right] g(X,Y) \\ -(n+\beta-1)\,\eta(X)\,\eta(Y) + \psi g(X,\phi Y), \end{aligned} \tag{6.12}$$

Setting $Y = \xi$ in (6.12) we have

$$\lambda = \left(\frac{p}{2}+\frac{1}{n}\right) - \beta .$$

This leads to the following theorem:

Theorem 6.4: *Let $M(\phi,\xi,\eta,g)$ be an n-dimensional para-Sasakian manifold of class C_2. If M admits a conformal η-Ricci soliton with respect to SVK-connection, then the soliton constants are given by*

$$\lambda = \left(\frac{p}{2}+\frac{1}{n}\right) - \beta .$$

Corollary 6.5: *A para-Sasakian manifold of class C_2 is always Einstein manifold.*

Conformal η-Ricci Soliton with Respect to $\overline{\nabla}$ on Class C_3: The condition that must be satisfied by conformal curvature tensor (C) is

$$(R(\xi,X).C)(Y,Z)V = 0\,, \tag{6.13}$$

for all $X,Y,Z,V \in \chi(M)$.

Equation (6.13) gives

$$R(\xi,X).C(Y,Z)V = C(R(\xi,X)Y,Z)V + C(Y,R(\xi,X)Z)V + C(Y,Z)R(\xi,X)V. \quad (6.14)$$

Setting $V = \xi$ in (6.14) we have

$$R(\xi,X).C(Y,Z)\xi = C(R(\xi,X)Y,Z)\xi + C(Y,R(\xi,X)Z)\xi + C(Y,Z)R(\xi,X)\xi. \quad (6.15)$$

Using (1.9), (2.9)-(2.11) in (6.14) and taking inner product of (6.15) with a vector field U and then contracting over Z, U we get

$$S(X,Y) = \left[\frac{n+r-1}{n-1}\right]g(X,Y) - \left[\frac{n^2-n+r}{n-1}\right]\eta(X)\eta(Y). \quad (6.16)$$

In consequence of (5.5) and (6.16) we obtain

$$0 = \left[\left(\frac{p}{2}+\frac{1}{n}\right) - \lambda - \frac{n+r-1}{n-1}\right]g(X,Y) + \left[\frac{n^2-n+r}{n-1} - n - \beta + 1\right]\eta(X)\eta(Y) + \psi g(X,\phi Y), \quad (6.17)$$

Setting $Y = \xi$ in (6.17) we have

$$\lambda = \left(\frac{p}{2}+\frac{1}{n}\right) - \beta,$$

which gives the following theorem:

Theorem 6.6: *Let $M(\phi,\xi,\eta,g)$ be an n-dimensional para-Sasakian manifold of class C_3. If M admits a conformal η-Ricci soliton with respect to SVK-connection, then the soliton constants are given by*

$$\lambda = \left(\frac{p}{2}+\frac{1}{n}\right) - \beta.$$

Corollary 6.7: *a para-Sasakian manifold of class* C_3 *is always an* η *-Einstein manifold.*

Conformal η-Ricci soliton with respect to $\bar{\bar{\nabla}}$ on class C_4: The condition that must be satisfied by conformal curvature tensor (C) is

$$(E(\xi, X).C)(Y, Z)V = 0, \tag{6.18}$$

for all $X, Y, Z, V \in \chi(M)$.

Equation (6.13) gives

$$E(\xi, X).C(Y, Z)V = C(E(\xi, X)Y, Z)V + C(Y, E(\xi, X)Z)V + C(Y, Z)E(\xi, X)V. \tag{6.19}$$

Setting $V = \xi$ in (6.14) we have

$$E(\xi, X).C(Y, Z)\xi = C(E(\xi, X)Y, Z)\xi + C(Y, E(\xi, X)Z)\xi + C(Y, Z)E(\xi, X)\xi. \tag{6.20}$$

Using (1.9), (1.10), (2.9)-(2.11) in (6.14) and taking inner product of (6.15) with a vector field U and then contracting over Z, U we get

$$S(X, Y) = \left[\frac{n + r - 1}{n - 1}\right]g(X, Y) - \left[\frac{n^2 - n + r}{n - 1}\right]\eta(X)\eta(Y),$$

if $r \neq -n(n-1)$.

In view of (5.5) and (6.16) we obtain

$$0 = \left[\left(\frac{p}{2} + \frac{1}{n}\right) - \lambda - \frac{n + r - 1}{n - 1}\right]g(X, Y) + \left[\frac{n^2 - n + r}{n - 1} - n - \beta + 1\right]\eta(X)\eta(Y) + \psi g(X, \phi Y), \tag{6.22}$$

Setting $Y = \xi$ in (6.22) we have

$$\lambda = \left(\frac{p}{2} + \frac{1}{n}\right) - \beta,$$

which gives the following theorem:

Theorem 6.8: *Let $M(\phi, \xi, \eta, g)$ be an n-dimensional para-Sasakian manifold of class C_4. If M admits a conformal η-Ricci soliton with respect to SVK-connection, then the soliton constants are given by*

$$\lambda = \left(\frac{p}{2} + \frac{1}{n}\right) - \beta.$$

provided $r \neq -n(n-1)$.

Corollary 6.9: *A para-Sasakian manifold of class C_4 is an η-Einstein manifold if $r \neq -n(n-1)$.*

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BI-CONCAVE AND OZAKI-TYPE BI-CLOSE-TO-CONVEX FUNCTIONS ASSOCIATED WITH MILLER-ROSS TYPE POISSON DISTRIBUTION SUBORDINATE TO INVOLUTION NUMBERS

Abstract: The purpose of this article is to study new subclasses of bi-univalent functions related to the Miller-Ross type Poisson distribution, which is subordinate to the generalized telephone numbers. Here, we introduce two new subclasses of Ozaki-type bi-close-to-convex functions and bi-concave functions. For the functions, In these new classes, we estimate the first two Taylor-Maclaurin coefficients and Fekete-Szeg*ö* problem.

Keywords: Univalent Functions, Bi-Univalent Functions, Bi-Convex Function, Miller-Ross Type Poisson Distribution, Subordination, Fekete-Szeg*ö* Problem.

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1. Introduction

We begin by considering that $\mathfrak{B}$ represents the class of analytic functions defined as

$$\mathcal{T}(z) = z + \sum_{r=2}^{\infty} d_r z^r \quad z \in \mathfrak{O}, \tag{1.1}$$

those are analytic in open unit disk $\mathfrak{O} = \{z : z \in \mathbb{C}, |z| < 1\}$. Let us denote the $\mathcal{G}$ as the family of all analytic and univalent functions in $\mathfrak{O}$.

Koebe one-quarter Theorem [11] states that, the image of $\mathfrak{O}$ under any univalent function $\mathcal{T} \in \mathfrak{S}$ contains the disk of radius 1/4. As a result, every function $\mathcal{T}$ has an inverse $\mathcal{T}^{-1}$ given by

$$\mathcal{T}^{-1}(z) = \mathcal{G}(z) = w - d_2w^2 + (2d_2^2 - d_3)w^3 - (5d_2^3 - 5d_2d_3 + d_4)w^4 + \cdots.$$

A function $\mathcal{T} \in \mathfrak{S}$ is bi-univalent in $\mathfrak{O}$ if both $\mathcal{T}$ and $\mathcal{T}^{-1}$ are univalent in $\mathfrak{O}$. Let us denote Σ as the class of bi-univalent functions.

Assume that g_1 and g_2 are analytic functions that are defined in $\mathfrak{O}$. We say that g_1 is subordinate to g_2 i.e. $g_1(z) \prec g_2(z)$, when we can identify a function w with analytic properties in domain $\mathfrak{O}$, as follows:

$$w(0) = 0, |w(z)| < 1 \text{ and } g_1(z) = g_2(w(z)).$$

In particular, g_2 is univalent in $\mathfrak{O}$ then the below equivalence is obtained.

$$g_1 \prec g_2 \Leftrightarrow g_1(0) = g_2(0) \text{ and } g_1(\mathfrak{O}) \subset g_2(\mathfrak{O}).$$

A function $\mathcal{T} : \mathfrak{S} \to \mathbb{C}$ belongs to the class of concave functions if $\mathcal{T}$ satisfies conditions listed below:

- $\mathcal{T}$ is analytic in $\mathfrak{O}$ and satisfying normalization conditions $\mathcal{T}(0) = \mathcal{T}'(0) - 1 = 0$.
- $\mathcal{T}$ maps $\mathfrak{O}$ conformally onto a set whose complement with respect to $\mathbb{C}$ is convex.
- The opening angle $\mathcal{T}(\mathfrak{O})$ at ∞ is less than or equal to $\pi\vartheta, \vartheta \in (1, 2]$.

The class $\mathcal{CV}(\vartheta)$ represents the class of concave analytic and univalent functions (for details, see [5; 3; 4; 25; 24]) and the functions of this class satisfy below inequality:

$$\Re\left(1+\frac{z\mathcal{T}''(z)}{\mathcal{T}'(z)}\right)<0 \qquad z\in\mathfrak{O}$$

Bhowmik B., Ponnusamy S., Wirths K. [7] established that a function maps $\mathfrak{O}$ onto an angled concave domain $\pi\vartheta$ if and only if

$$\Re\left\{\frac{1}{\vartheta-1}\frac{(\vartheta+1)(1+z)}{2(1-z)}-1-\frac{z\mathcal{T}''(z)}{\mathcal{T}'(z)}\right\}>0\,.$$

Numerous studies on bi-univalent function subclasses may be found in the varied publications [9; 8; 16; 21]. Motivated by works [27; 30; 28; 31; 20; 2], we analyze the novel subclasses of concave and bi-close-to-convex functions.

Let us consider the Miller-Ross function [17] (also see [15; 29]) and is denoted as

$$\mathcal{G}_{\upsilon,\eta}(z)=z^{\upsilon}\sum_{r=0}^{\infty}\frac{(\eta z)^{r}}{\Gamma(r+\upsilon+1)}, \qquad \upsilon,\eta,z\in\mathbb{C}.$$

The two parameter Mittag-Leffler function [32], $\mathcal{E}_{\theta,\mu}(z)$ is given by

$$\mathcal{E}_{\theta,\mu}(z)=\sum_{r=0}^{\infty}\frac{z^{r}}{\Gamma(\theta r+\mu)}, \qquad \theta,\mu,z\in\mathbb{C},\Re\{\theta,\mu\}>0.$$

For $\mu=1$, we have the Mittag-Leffler function [18],

$$\mathcal{E}_{\theta}(z)=\sum_{r=0}^{\infty}\frac{z^{r}}{\Gamma(\theta r+1)}, \qquad \theta,z\in\mathbb{C},\Re\{\theta\}>0.$$

The Miller-Ross function [12] can be represented as

$$G_{\upsilon,\eta}(z)=z^{\upsilon}\mathcal{E}_{1,1+\upsilon}(\eta z).$$

Recently, Şeker *et al*. [26] represented a power series with the corresponding coefficients are the Miller-Ross type Poisson distribution, which is as shown below:

$$\mathcal{Q}_{\upsilon,\eta}^{\sigma}(z) = z\sum_{r=2}^{\infty} \frac{\sigma^{\upsilon}(\eta\sigma)^{r-1}}{\Gamma(r+\upsilon)\mathcal{G}_{\upsilon,\eta}(\sigma)} z^{r}, \qquad z \in \mathfrak{D}, \upsilon > -1, \eta > 0. \tag{1.2}$$

Now, consider the convolution of functions (1.1) and (1.2), an operator $\mathcal{M}_{\upsilon,\eta}^{\sigma} : \mathfrak{B} \to \mathfrak{B}$ written as:

$$\begin{aligned}\mathcal{M}_{\upsilon,\eta}^{\sigma}\mathcal{T}(z) &= \mathcal{Q}_{\upsilon,\eta}^{\sigma}(z) * \mathcal{T}(z) \\ &= z + \sum_{r=2}^{\infty} \frac{\sigma^{\upsilon}(\eta\sigma)^{r-1}}{\Gamma(r+\upsilon)\mathcal{G}_{\upsilon,\eta}(\sigma)} d_r z^{r} \\ &= z + \sum_{r=2}^{\infty} \varphi_r d_r z^{r}, \qquad z \in \mathfrak{D}, \upsilon > -1, \eta > 0.\end{aligned}$$

where, $\varphi_r = \dfrac{\sigma^{\upsilon}(\eta\sigma)^{r-1}}{\Gamma(r+\upsilon)\mathcal{G}_{\upsilon,\eta}(\sigma)}$.

In particular,

$$\varphi_2 = \frac{\sigma^{\upsilon}(\eta\sigma)}{\Gamma(2+\upsilon)\mathcal{G}_{\upsilon,\eta}(\sigma)}, \text{ and } \varphi_3 = \frac{\sigma^{\upsilon}(\eta\sigma)^2}{\Gamma(3+\upsilon)\mathcal{G}_{\upsilon,\eta}(\sigma)}. \tag{1.3}$$

1.1 Involution Numbers: Considering the involution numbers (that are also referred to as telephone numbers (TN)), the recurrence relation is

$$\mathcal{V}(r) = \mathcal{V}(r-1) + (r-1)\mathcal{V}(r-2), \quad r \geq 2,$$

with $\mathcal{V}(0) = \mathcal{V}(1) = 1$.

New generalized telephone numbers (GTNs) were recently identified in 2019 by Bednarz and Wolowiec-Musial [6] and are represented as

$$\mathcal{V}_{\epsilon}(r) = \mathcal{V}_{\epsilon}(r-1) + \epsilon(r-1)\mathcal{V}_{\epsilon}(r-2), \quad r \geq 2, \epsilon \geq 1,$$

with $\mathcal{V}_{\epsilon}(0) = \mathcal{V}_{\epsilon}(1) = 1$.

GTNs are presented in exponential series form by

$$e^{x+\epsilon\frac{x^2}{2}} = \sum_{r=0}^{\infty} \mathcal{V}_\epsilon(r)\frac{x^r}{r!}, \qquad \epsilon \geq 1.$$

Thus, for $\epsilon = 1$, we have TNs $\mathcal{V}(r)$ and for specific values of $r, \mathcal{V}_\epsilon(r)$ is provided as

1. $\mathcal{V}_\epsilon(0) = \mathcal{V}_\epsilon(1) = 1,$
2. $\mathcal{V}_\epsilon(2) = 1 + \epsilon,$
3. $\mathcal{V}_\epsilon(3) = 1 + 3\epsilon,$
4. $\mathcal{V}_\epsilon(4) = 1 + 6\epsilon + 3\epsilon^2,$
5. $\mathcal{V}_\epsilon(5) = 1 + 10\epsilon + 15\epsilon^2.$

Let us consider the function

$$\begin{aligned}\Psi(x) &= e^{x+\epsilon\frac{x^2}{2}}\\ &= \sum_{r=0}^{\infty} \mathcal{V}_\epsilon(r)\frac{x^r}{r!}\\ &= 1 + x + (1+\epsilon)\frac{x^2}{2} + (1+3\epsilon)\frac{x^3}{6} + (1+6\epsilon+3\epsilon^2)\frac{x^4}{24} + \cdots.\end{aligned}$$

For $x \in \mathfrak{O}$. (see also [19; 10])

We introduce two novel subclasses of bi-univalent functions connected with the Poisson distribution of Miller-Ross type that are subordinate to GTNs in our current paper. In addition, we estimate the Fekete-Szego inequality and the Taylor-Maclaurin coefficients $|d2|$, $|d3|$, for the newly defined classes.

2. Ozaki-type Bi-Close-to-Convex Functions

In 1952, Kaplan [13] introduced the class $\mathfrak{K}$ of close-to-convex functions. In 1935, Ozaki [22] had already identified these functions, satisfying the following inequality:

$$\Re\left(1+\frac{z\mathcal{T}'(z)}{\mathcal{T}(z)}\right) > -\frac{1}{2}, \qquad z \in \mathfrak{D}. \tag{2.1}$$

Kaplan [8] states that the function which satisfy inequality (2.1) are close-to-convex functions and which are categorized under class $\mathfrak{S}$. The Ozaki inequality was further generalized by Kargar and Gebadian [14]. (For details see [22; 1]) A function $\mathcal{T} \in \mathfrak{B}$ is locally univalent and is said to be Ozaki close-to-convex function if it satisfy the condition:

$$\Re\left(1+\frac{z\mathcal{T}'(z)}{\mathcal{T}(z)}\right) > \frac{1}{2}-\vartheta, \qquad z \in \mathfrak{D}, \vartheta \in (-\tfrac{1}{2}, \tfrac{1}{2}] .$$

Definition 1: *The class* $OBCV_\Sigma(\vartheta, \sigma, \upsilon, \eta)$ *includes all functions* $\mathcal{T} \in \mathfrak{B}$ *if it satisfies the following subordination conditions*:

$$\frac{2\vartheta-1}{2\vartheta+1}+\frac{2}{2\vartheta+1}\left[\frac{((z\mathcal{M}^{\sigma}_{\upsilon,\eta}\mathcal{T}(z))')'}{(\mathcal{M}^{\sigma}_{\upsilon,\eta}\mathcal{T}(z))'}\right] \prec \Psi(z), \tag{2.2}$$

and

$$\frac{2\vartheta-1}{2\vartheta+1}+\frac{2}{2\vartheta+1}\left[\frac{((w\mathcal{M}^{\sigma}_{\upsilon,\eta}\mathcal{G}(w))')'}{(\mathcal{M}^{\sigma}_{\upsilon,\eta}\mathcal{G}(w))'}\right] \prec \Psi(z), \tag{2.3}$$

where $\mathcal{T}^{-1}(w) = \mathcal{G}(w)$ *and* $\frac{1}{2} \leq \vartheta \leq 1$.

Remark 1: *For* $\vartheta = \frac{1}{2}$ *, the class* $OBCV_\Sigma(\vartheta, \sigma, \upsilon, \eta) = CV_\Sigma(\vartheta)$ *includes all functions* $\mathcal{T} \in \Sigma$ *if*

$$\frac{((z\mathcal{M}^{\sigma}_{\upsilon,\eta}\mathcal{T}(z))')'}{(\mathcal{M}^{\sigma}_{\upsilon,\eta}\mathcal{T}(z))'} \prec \Psi(z), \qquad \text{and} \qquad \frac{((w\mathcal{M}^{\sigma}_{\upsilon,\eta}\mathcal{G}(w))')'}{(\mathcal{M}^{\sigma}_{\upsilon,\eta}\mathcal{G}(w))'} \prec \Psi(z)$$

where $\mathcal{T}^{-1}(w) = \mathcal{G}(w)$.

The following lemma [23] will be necessary for proving the main findings.

Lemma 1: *If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions h, analytic in $\mathfrak{O}$, for which $\Re[h(z)] > 0$ $(z \in \mathfrak{O})$, where*

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots (z \in \mathfrak{O}).$$

Theorem 1: *A function $\mathcal{T} \in \mathfrak{B}$ form (1.1) is in class $OBCV_{\Sigma}(\vartheta, \sigma, \upsilon, \eta)$, then*

$$|d_2| \leq \min\left\{\frac{2\vartheta + 1}{4\varphi_2}, \frac{2\vartheta + 1}{2\sqrt{(2\vartheta + 1)(3\varphi_3 - 2\varphi_2^2) + 2(1 - \epsilon)\varphi_2^2}}\right\}, \tag{2.4}$$

and

$$|d_2| \leq \min\left\{\frac{2\vartheta + 1}{12\varphi_3} + \frac{(2\vartheta + 1)^2}{12\varphi_2^2}, \frac{2\vartheta + 1}{12\varphi_3} + \frac{(2\vartheta + 1)^2}{4[(2\vartheta + 1)(3\varphi_3 - 2\varphi_2^2) + 2(1 - \epsilon)\varphi_2^2}\right\} \tag{2.5}$$

where φ_2 and φ_3 are as given in (1.3).

Proof: Let us consider $s(z)$ and $t(z)$ as

$$s(z) = \frac{1 + l(z)}{1 - l(z)} = 1 + s_1 z + s_2 z^2 + \cdots, \tag{2.6}$$

$$t(w) = \frac{1 + m(w)}{1 - m(w)} = 1 + t_1 w + t_2 w^2 + \cdots, \tag{2.7}$$

or, equivalently,

$$l(z) = \frac{s(z) - 1}{s(z) + 1} = \frac{1}{2}\left[s_1 z + \left(s_2 - \frac{s_1^2}{2}\right)z^2 + \cdots\right], \tag{2.8}$$

and

$$m(w) = \frac{t(w) - 1}{t(w) + 1} = \frac{1}{2}\left[t_1 w + \left(t_2 - \frac{t_1^2}{2}\right)w^2 + \cdots\right], \tag{2.9}$$

Then $s(0) = t(0) = 1$, and $s(z)$ and $t(z)$ are analytic in $\mathfrak{O}$ with a positive real part in $\mathfrak{O}$.

Now consider,

$$\Psi(l(z)) = e^{l(z)+\epsilon\frac{[l(z)]^2}{2}}$$

$$\Psi(l(z)) = 1 + \frac{s_1}{2}z + \left(\frac{s_2}{2} + (1-\epsilon)\frac{s_1^2}{8}\right)z^2 + \left(\frac{s_3}{2} + (\epsilon-1)\frac{s_1 s_2}{4} + (1-3\epsilon)\frac{s_1^3}{48}\right)z^3 + \cdots. \tag{2.10}$$

Similarly,

$$\Psi(m(w)) = 1 + \frac{t_1}{2}w + \left(\frac{t_2}{2} + (1-\epsilon)\frac{t_1^2}{8}\right)w^2 + \left(\frac{t_3}{2} + (\epsilon-1)\frac{t_1 t_2}{4} + (1-3\epsilon)\frac{t_1^3}{48}\right)w^3 + \cdots. \tag{2.11}$$

From (2.2) and (2.3), we have

$$\frac{2\vartheta-1}{2\vartheta+1} + \frac{2}{2\vartheta+1}\left[\frac{((z\mathcal{M}_{\upsilon,\eta}^{\sigma}\mathcal{T}(z))')'}{(\mathcal{M}_{\upsilon,\eta}^{\sigma}\mathcal{T}(z))'}\right] = \Psi(l(z)), \tag{2.12}$$

and

$$\frac{2\vartheta-1}{2\vartheta+1} + \frac{2}{2\vartheta+1}\left[\frac{((w\mathcal{M}_{\upsilon,\eta}^{\sigma}\mathcal{G}(w))')'}{(\mathcal{M}_{\upsilon,\eta}^{\sigma}\mathcal{G}(w))'}\right] = \Psi(m(w)). \tag{2.13}$$

Using (2.10), (2.11) in (2.12), (2.13) and comparing the coefficients, we the following relations

$$\frac{4}{2\vartheta+1}d_2\varphi_2 = \frac{s_1}{2}, \tag{2.14}$$

$$\frac{4}{2\vartheta+1}(3d_3\varphi_3-2d_2^2\varphi_2^2)=\frac{s_2}{2}+(\epsilon-1)\frac{s_1^2}{8}, \tag{2.15}$$

$$-\left(\frac{4}{2\vartheta+1}\right)d_2\varphi_2=\frac{t_1}{2}, \tag{2.16}$$

$$\frac{4}{2\vartheta+1}(3[2d_2^2-d_3]\varphi_3-2d_2^2\varphi_2^2)=\frac{s_2}{2}+(\epsilon-1)\frac{s_1^2}{8}. \tag{2.17}$$

From (2.14) and (2.16), it follows that

$$s_1=-t_1. \tag{2.18}$$

Squaring and adding (2.14) and (2.16), we obtain that

$$\frac{128}{(2\vartheta+1)^2}d_2^2\varphi_2^2=s_1^2+t_1^2. \tag{2.19}$$

Now, applying Lemma 1 to (2.19), we get

$$|d_2|\leq\frac{2\vartheta+1}{4\varphi_2^2}. \tag{2.20}$$

Adding (2.15) and (2.17), we can find out that

$$\frac{4}{2\vartheta+1}[6\varphi_3d_2^2-4\varphi_2^2d_2^2]=\frac{s_2+t_2}{2}+(\epsilon-1)\left(\frac{s_1^2+t_1^2}{8}\right). \tag{2.21}$$

If we use (2.19) in (2.21), then we have

$$d_2^2=\frac{(s_2+t_2)(2\vartheta+1)^2}{16[(2\vartheta+1)(3\varphi_3-2\varphi_2^2)+2(1-\epsilon)\varphi_2^2]}. \tag{2.22}$$

Employing Lemma 1, we obtain

$$|d_2| \leq \frac{2\vartheta + 1}{2\sqrt{(2\vartheta + 1)(3\varphi_3 - 2\varphi_2^2) + 2(1 - \epsilon)\varphi_2^2}}.$$

Subtracting (2.17) from (2.15) and using (2.18), it follows that

$$\frac{24}{2\vartheta + 1}(d_3 - d_2^2)\varphi_3 = \frac{s_2 - t_2}{2},$$

i.e.,

$$d_3 = \frac{(s_2 - t_2)(2\vartheta + 1)}{48\varphi_3} + d_2^2. \tag{2.23}$$

Substituting the value of d_2^2 from (2.19) in (2.23), we obtain

$$d_3 = \frac{(s_2 - t_2)(2\vartheta + 1)}{48\varphi_3} + \frac{(s_1^2 - t_1^2)(2\vartheta + 1)^2}{128\varphi_2^2}. \tag{2.24}$$

Applying Lemma 1, we obtain

$$|d_3| \leq \frac{2\vartheta + 1}{12\varphi_3} + \frac{(2\vartheta + 1)^2}{12\varphi_2^2}.$$

Using (2.22) in (2.23), we obtain

$$d_3 = \frac{(s_2 + t_2)(2\vartheta + 1)}{48\varphi_3} + \frac{(s_2 + t_2)(2\vartheta + 1)^2}{16[(2\vartheta + 1)(3\varphi_3 - 2\varphi_2^2) + 2(1 - \epsilon)\varphi_2^2]}. \tag{2.25}$$

Using Lemma (1), it follows that

$$|d_3| \leq \frac{2\vartheta + 1}{12\varphi_3} + \frac{(2\vartheta + 1)^2}{4[(2\vartheta + 1)(3\varphi_3 - 2\varphi_2^2) + 2(1 - \epsilon)\varphi_2^2]}.$$

□

Lemma 2 [13]: *Let* $b_1, b_2 \in \mathbb{R}$ *and* $c_1, c_2 \in \mathbb{C}$. *Suppose that* $|c_1|, |c_2| < \xi$, *then*

$$|(b_1 + b_2)c_1 + (b_1 - b_2)c_2| \leq \begin{cases} 2|b_1|\xi, & |b_1| \geq |b_2| \\ 2|b_2|\xi, & |b_1| \leq |b_2|. \end{cases}$$

Theorem 2: *A function* $\mathcal{T} \in \mathfrak{B}$ *form* (1.1) *is in class* $OBCV_\Sigma(\vartheta, \sigma, \upsilon, \eta)$ *and* $\tau \in \mathbb{R}$, *then*

$$|d_3 - \tau d_2^2| \leq \begin{cases} \frac{2\vartheta+1}{12\varphi_3}, & 0 \leq \chi(\vartheta, \tau, \epsilon) \leq \frac{2\vartheta+1}{48\varphi_3} \\ 4|\chi(\vartheta, \tau, \epsilon)|, & \chi(\vartheta, \tau, \epsilon) \geq \frac{2\vartheta+1}{48\varphi_3}. \end{cases}$$

Proof: From (2.22) and (2.23), we have

$$d_3 - \tau d_2^2 = (1-\tau)d_2^2 + \frac{(s_2 - t_2)(2\vartheta + 1)}{48\varphi_3}$$

$$= \frac{(s_2 + t_2)(1-\tau)(2\vartheta+1)^2}{16[(2\vartheta+1)(3\varphi_3 - 2\varphi_2^2) + (1-\epsilon)\varphi_2^2]} + \frac{(s_2 - t_2)(2\vartheta+1)}{48\varphi_3}$$

$$= \left(\chi(\vartheta, \tau, \epsilon) + \frac{2\vartheta+1}{48\varphi_3}\right)s_2 + \left(\chi(\vartheta, \tau, \epsilon) - \frac{2\vartheta+1}{48\varphi_3}\right)t_2,$$

where

$$\chi(\vartheta, \tau, \epsilon) = \frac{(1-\tau)(2\vartheta+1)^2}{16[(2\vartheta+1)(3\varphi_3 - 2\varphi_2^2) + (1-\epsilon)\varphi_2^2]}$$

Applying Lemma 2, we deduce that

$$|d_3 - \tau d_2^2| \leq \begin{cases} \frac{2\vartheta+1}{12\varphi_3}, & 0 \leq \chi(\vartheta,\tau,\epsilon) \leq \frac{2\vartheta+1}{48\varphi_3} \\ 4|\chi(\vartheta,\tau,\epsilon)|, & \chi(\vartheta,\tau,\epsilon) \geq \frac{2\vartheta+1}{48\varphi_3}. \end{cases} \qquad \square$$

Corollary 1: *A function* $\mathcal{T} \in \mathfrak{B}$ *form* (1.1) *is in class* $OBCV_\Sigma(\vartheta, \sigma, \upsilon, \eta)$, *then*

$$|d_3 - d_2^2| \leq \frac{2\vartheta + 1}{12\varphi_3}.$$

3. Bi-concave Functions

Definition 2: *A function* $\mathcal{T} \in \mathfrak{B}$ *is belongs to the class* $BCV_\Sigma(\vartheta, \sigma, \upsilon, \eta)$ *if it satisfies the following subordination conditions*:

$$\frac{2}{\varrho - 1}\left[\frac{(1+\varrho)(1+z)}{2(1-z)} - 1 - \frac{z(\mathcal{M}_\sigma^{\upsilon,\eta}\mathcal{T}(z))''}{(\mathcal{M}_\sigma^{\upsilon,\eta}\mathcal{T}(z))'}\right] \prec \Psi(z), \tag{3.1}$$

and

$$\frac{2}{\varrho - 1}\left[\frac{(1+\varrho)(1+w)}{2(1-w)} - 1 - \frac{w(\mathcal{M}_\sigma^{\upsilon,\eta}\mathcal{T}(w))''}{(\mathcal{M}_\sigma^{\upsilon,\eta}\mathcal{T}(w))'}\right] \prec \Psi(w) \tag{3.2}$$

where $\mathcal{T}^{-1}(w) = \mathcal{G}(w)$ *and* $1 < \varrho \leq 2$.

Theorem 3: *A function* $\mathcal{T} \in \mathfrak{B}$ *form* (1.1) *is in class* $BCV_\Sigma(\varrho, \sigma, \upsilon, \eta)$, *then*

$$|d_2| \leq \min\left\{\sqrt{\frac{9\varrho^2 + 6\varrho + 1}{16\varphi_2^2}},\right.$$

$$\left.\left(\frac{1}{(\varrho-1)(2\varphi_2^2 - 3\varphi_3) + 2(1-\epsilon)}\left[\frac{3\varrho^2 - 2\varrho - 1}{4} + (\epsilon - 1)\frac{\varrho 2 + \varrho}{\varphi_2^2}\right]\right)^{\frac{1}{2}}\right\}, \tag{3.3}$$

and

$$|d_3| \leq \min\left\{\frac{9\varrho^2 + 6\varrho + 1}{16\varphi_2^2} + \frac{\varrho - 1}{12\varphi_3},\right.$$

$$\left.\frac{1}{(\varrho - 1)(2\varphi_2^2 - 3\varphi_3) + 2(1 - \epsilon)}\left[\frac{3\varrho^2 - 2\varrho - 1}{4} + (\epsilon - 1)\frac{\varrho 2 + \varrho}{\varphi_2^2}\right] + \frac{\varrho - 1}{12\varphi_3}\right\}, \quad (3.4)$$

where $1 < \varrho \leq 2$ *and* φ_2 *and* φ_3 *are as given in* (1.3).

Proof: From (3.1) and (3.2), we can write

$$\frac{2}{\varrho - 1}\left[\frac{(1 + \varrho)(1 + z)}{2(1 - z)} - 1 - \frac{z(\mathcal{M}_\sigma^{\upsilon,\eta}\mathcal{T}(z))''}{(\mathcal{M}_\sigma^{\upsilon,\eta}\mathcal{T}(z))'}\right] \prec \Psi(l(z)), \quad (3.5)$$

and

$$\frac{2}{\varrho - 1}\left[\frac{(1 + \varrho)(1 + w)}{2(1 - w)} - 1 - \frac{w(\mathcal{M}_\sigma^{\upsilon,\eta}\mathcal{G}(w))''}{(\mathcal{M}_\sigma^{\upsilon,\eta}\mathcal{G}(w))'}\right] \ll \Psi(m(w)). \quad (3.6)$$

Using (2.10), (2.11) in (3.5), (3.6) respectively and equating the coefficients, we obtain

$$\frac{2}{\varrho - 1}[(1 + \varrho) - 2\varphi_2 d_2] = \frac{s_1}{2}, \quad (3.7)$$

$$\frac{2}{\varrho - 1}[(1 + \varrho) + 4\varphi_2^2 d_2^2 - 6\varphi_3 d_3] = \frac{s_2}{2} + (\epsilon - 1)\frac{s_1^2}{8}, \quad (3.8)$$

$$-\frac{2}{\varrho - 1}[(1 + \varrho) - 2\varphi_2 d_2] = \frac{t_1}{2}, \quad (3.9)$$

$$\frac{2}{\varrho - 1}[(1 + \varrho) + 4\varphi_2^2 d_2^2 - 6\varphi_3(2d_2^2 - d_3)] = \frac{t_2}{2} + (\epsilon - 1)\frac{t_1^2}{8}. \quad (3.10)$$

From (3.7) and (3.9), it follows that

$$s_1 = t_1. \quad (3.11)$$

From (3.7) and (3.9), we can write

$$d_2 = \frac{1+\varrho}{2\varphi_2} - \frac{(\varrho-1)s_1}{8\varphi_2}, \tag{3.12}$$

$$d_2 = \frac{1+\varrho}{2\varphi_2} + \frac{(\varrho-1)t_1}{8\varphi_2}. \tag{3.13}$$

Squaring and adding (3.12) and (3.13), we obtain

$$d_2^2 = \frac{(1+\varrho)^2}{4\varphi_2^2} + \frac{(s_1^2+t_1^2)(\varrho-1)^2}{128\varphi_2^2} - \frac{(\varrho^2-1)(s_1-t_1)}{16\varphi_2^2}, \tag{3.14}$$

i.e.,

$$\frac{(s_1^2+t_1^2)(\varrho-1)^2}{64\varphi_2^2} = 2d_2^2 - \frac{(1+\varrho)^2}{2\varphi_2^2} + \frac{(\varrho^2-1)(s_1-t_1)}{8\varphi_2^2}. \tag{3.15}$$

Applying Lemma 1 to (3.14), we have

$$|d_2|^2 \leq \frac{9\varrho^2+6\varrho+1}{16\varphi_2^2}.$$

Adding (3.8) and (3.10), we obtain

$$\frac{2}{\varrho-1}\{2(1+\varrho)+8\varphi_2^2 d_2^2 - 12\varphi_3 d_2^2\} = \frac{(s_2+t_2)}{2} + (\epsilon-1)\left(\frac{s_1^2+t_1^2}{8}\right). \tag{3.16}$$

Implies that,

$$(2\varphi_2^2 - 3\varphi_3)d_2^2 = \frac{(\varrho-1)(s_2+t_2)}{16} + \frac{(\epsilon-1)(\varrho-1)(s_1^2+t_1^2)}{64} - \frac{(\varrho+1)}{2}.$$

Multiplying both sides by $(\varrho \quad 1)$, thus, we get

$$(\varrho-1)(2\varphi_2^2 - 3\varphi_3)d_2^2 = \frac{(\varrho-1)^2(s_2+t_2)}{16} + \frac{(\epsilon-1)(\varrho-1)^2(s_1^2+t_1^2)}{64} - \frac{(\varrho^2+1)}{2}. \tag{3.17}$$

Using (3.15) in (3.17), we obtain

$$(\varrho - 1)(2\varphi_2^2 - 3\varphi_3)d_2^2 + 2(1-\epsilon)d_2^2 = \frac{(\varrho - 1)^2(s_2 + t_2)}{16} - (\epsilon - 1)\frac{(1+\varrho)^2}{2\varphi_2^2} + (\epsilon - 1)\frac{(\varrho^2 - 1)}{8\varphi_2^2}(s_1 - t_1) - \left(\frac{(\varrho^2 - 1)}{2}\right). \quad (3.18)$$

Thus, we have

$$d_2^2 = \frac{1}{[(\varrho - 1)(2\varphi_2^2 - 3\varphi_3) + 2(1-\epsilon)]}\left[\frac{(\varrho - 1)^2(s_2 + t_2)}{16} - (\epsilon - 1)\frac{(1+\varrho)^2}{2\varphi_2^2} + (\epsilon - 1)\frac{(\varrho^2 - 1)}{8\varphi_2^2}(s_1 - t_1) - \left(\frac{(\varrho^2 - 1)}{2}\right)\right]. \quad (3.19)$$

Applying Lemma 1 to (3.19), we obtain

$$|d_2^2|^2 \le \frac{1}{(\varrho - 1)(2\varphi_2^2 - 3\varphi_3) + 2(1-\epsilon)}\left[\frac{3\varrho^2 - 2\varrho - 1}{4} + (\epsilon - 1)\frac{\varrho^2 + \varrho}{\varphi_2^2}\right].$$

Now, subtracting (3.10) from (3.8) and using (3.11), we get

$$d_2^2 - d_3 = \frac{(s_2 - t_2)(\varrho - 1)}{48\varphi_3}. \quad (3.20)$$

Using (3.14) in (3.20), we find that

$$d_3 = \frac{(1+\varrho)^2}{4\varphi_2^2} + \frac{(s_1^2 + t_1^2)(\varrho - 1)^2}{128\varphi_2^2} - \frac{(\varrho^2 - 1)(s_1 - t_1)}{16\varphi_2^2} - \frac{(s_2 - t_2)(\varrho - 1)}{48\varphi_3}. \quad (3.21)$$

According to Lemma 1, we deduce that

$$|d_3| \le \frac{9\varrho^2 + 6\varrho + 1}{16\varphi_2^2} + \frac{\varrho - 1}{12\varphi_3}.$$

Next we use the value of d_2^2 form (3.19) in (3.20), we obtain

$$d_3 = \frac{1}{[(\varrho-1)(2\varphi_2^2-3\varphi_3)+2(1-\epsilon)]}\left[\frac{(\varrho-1)^2(s_2+t_2)}{16}\right.$$

$$\left.-(\epsilon-1)\frac{(1+\varrho)^2}{2\varphi_2^2}+(\epsilon-1)\frac{(\varrho^2-1)}{8\varphi_2^2}(s_1-t_1)-\left(\frac{(\varrho^2-1)}{2}\right)\right]-\frac{(s_2-t_2)(\varrho-1)}{48\varphi_3}. \tag{3.22}$$

Using Lemma 1, we get

$$d_3 \leq \frac{1}{[(\varrho-1)(2\varphi_2^2-3\varphi_3)+2(1-\epsilon)]}\left[\frac{3\varrho^2-2\varrho-1}{4}+(\epsilon-1)\frac{\varrho^2+\varrho}{\varphi_2^2}\right]+\frac{\varrho-1}{12\varphi_3}$$

□

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EDGE ODD GRACEFUL LABELING OF SOME SNAKE GRAPHS

Abstract: An edge odd graceful labeling of graph G is a bijection f from the edges of the graph to $\{1, 3, \ldots, 2q-1\}$ such that, when each vertex is assigned the sum of all the edges incident to it $\bmod 2q$ the resulting vertex labels are distinct. A graph is called an edge odd graceful graph as it admits an edge odd graceful labeling. It was intoduced by Solairaju and Chithra in 2008. In this research paper, Edge odd graceful labeling of some snake graphs such as double alternate triangular snake $DA(T_n)$, double alternate quadrilateral snake $DA(Q_n)$ and alternate pentagonal snake $A(PS_n)$ have been discussed.

Keywords: Edge Odd Graceful Labeling, Edge Odd Graceful Graph, Snake Graphs.

Mathematics Subject Classification (2000) No.: 05C78.

1. Introduction

In this research article, all graphs $G = (V(G), E(G))$ are finite, simple, connected and undirected. Here $V(G)$ be the vertex set and $E(G)$ be the edge set of a graph. Graph labeling is an assignment of integers to edges or vertices or both, subject to certain conditions. For an extensive survey on graph labeling and bibliographic references, we refer to Gallian [2]. A graceful labeling of a graph G, which was introduced by Rosa [6] in 1967, is a injection f from the vertices of the graph to the set $\{1, 2, \ldots, q\}$ such that the induced function f^* from the set of edges to the set $\{0, 1, 2, \ldots, q\}$ defined as $f^*(e = uv) = | f(u) - f(v) |$, is bijective. Soleha *et al*. [10] have proved that the alternate triangular snake and alternate quadrilateral snake graphs are edge odd graceful.

Definition 1.1 [9]: *A function* f *is called an edge odd graceful labeling of a graph* G *if* $f : E(G) \to \{1, 3, \ldots, 2q - 1\}$ *is bijective and the induced function* $f^* : V(G) \to \{0, 1, 2, \ldots, 2q - 1\}$, *defined as* $f^*(u) = \sum_{uv \in E(G)} f(uv)$ $(\bmod 2q)$ *is injective.*

A graph which admits an edge odd graceful labeling is called an edge odd graceful graph.

Definition 1.2 [1]: *An alternate triangular snake* $A(T_n)$ *is obtained from a path* P_n *with vertices* $u_1, u_2, \ldots, u_n$ *by joining* u_i *and* $u_i + 1$ *(alternatively) to new vertex* v_i, *where* $1 \leqslant i \leqslant n - 1$ *for even* n *and for* $1 \leqslant i \leqslant n - 2$ *for odd* n.

That is every alternate edge of a path P_n *is replaced by* C_3.

Definition 1.3 [1]: *A double alternate triangular snake* $DA(T_n)$ *is obtained from a path* P_n *with vertices* $u_1, u_2, \ldots, u_n$ *by joining* u_i *and* u_{i+1} *(alternatively) to two new vertices* v_i *and* w_i, *where* $1 \leqslant i \leqslant n - 1$ *for even* n *and for* $1 \leqslant i \leqslant n - 2$ *for odd* n.

In other words, the double alternate triangular snake $DA(T_n)$ *consists of two alternate triangular snakes that have a common path.*

Definition 1.4 [1]: *An alternate quadrilateral snake* $A(Q_n)$ *is obtained from a path* P_n *with vertices* $u_1, u_2, \ldots, u_n$ *by joining* u_i *and* u_{i+1}

(alternatively) to two new vertices v_i *and* w_i, *respectively and then joining* v_i, *and* w_i *where* $1 \leqslant i \leqslant n-1$ *for even* n *and for* $1 \leqslant i \leqslant n-2$ *for odd* n.

That is every alternate edge of a path P_n *is replaced by* C_4.

Definition 1.5 [1]: *The double alternate quadrilateral snake* $DA(Q_n)$ *obtained from a path* P_n *with vertices* $u_1, u_2, \ldots, u_n$ *by joining* u_i *and* $u_i + 1$ *(alternatively) to two new vertices* v_i, w_i *and* v_{i+1}, w_{i+1} *respectively and then joining* v_i, v_{i+1} *and* w_i, w_{i+1}, *where* $1 \leqslant i \leqslant n-1$ *for even* n *and for* $1 \leqslant i \leqslant n-2$ *for odd* n.

In other words, the double alternate quadrilateral snake graph $DA(Q_n)$ *consists of two alternate quadrilateral snakes that have a common path.*

Definition 1.6 [8]: *An alternate pentagonal snake* $A(PS_n)$ *is obtained from a path* P_n *with vertices* $u_1, u_2, \ldots, u_n$ *by joining* u_i *and* $u_i + 1$ *(alternatively) to new vertices* v_i *and* w_i *respectively and then joining* v_i *and* w_i *to the new vertex* x_i, *where* $1 \leqslant i \leqslant n-1$ *for even* n *and for* $1 \leqslant i \leqslant n-2$ *for odd* n.

That is, every alternate edge of path P_n *is replaced by a cycle* C_5.

2. Main Results

Theorem 2.1: *The double alternate triangular snake* $DA(T_n)$ *is an edge odd graceful graph for all* $n \geqslant 2$.

Proof: Let G be a double alternate triangular snake $DA(T_n)$ which is obtained from a path P_n with vertices $u_1, u_2, \ldots, u_n$ by joining u_i and $u_i + 1$ (alternatively) to two new vertices v_j and w_j, where $1 \leqslant i \leqslant n-1$ for even n, $1 \leqslant i \leqslant n-2$ for odd n and $1 \leqslant j \leqslant \lfloor \frac{n}{2} \rfloor$.

Therefore

$V(G) = \{u_i, v_j, w_j \ / \ 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant \lfloor \frac{n}{2} \rfloor\}$

$E(G) = \{u_i u_{i+1} / 1 \leqslant i \leqslant n-1\} \cup \{u_{2i-1} v_i / 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor\} \cup \{u_{2i} v_i / 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor\}$

$\cup \{u_{2i-1} w_i / 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor\} \cup \{u_{2i} w_i / 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor\}.$

Here note that

$$|V(G)| = \begin{cases} 2n, & \text{if } n \equiv 0 \pmod 2 \\ 2n-1, & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

$$|E(G)| = \begin{cases} 3n-1, & \text{if } n \equiv 0 \pmod 2 \\ 3n-3, & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

Case 1: $n \equiv 0, 2 \pmod 4$

Subcase 1: $n = 2$

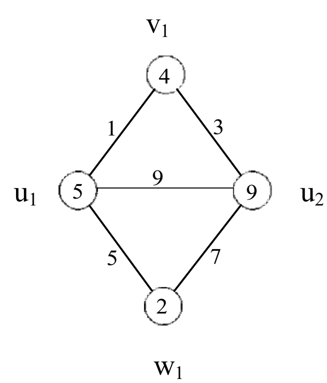

Figure 1: Edge odd graceful labeling of double alternate triangular snake $DA(T_2)$

Here Figure 1 shows that the double alternate triangular snake $DA(T_2)$ is an edge odd graceful graph.

Subcase 2: $n \equiv 0, 2 \pmod 4$ and $n \neq 2$

Define edge labeling $f : E(G) \to \{1, 3, 5, \ldots, 6n-3\}$ is as follows:

$f(u_i u_{i+1}) = 4n - 2i - 1;\ 1 \leqslant i \leqslant n - 1$

$f(u_{2i-1} v_i) = 4i - 3;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(u_{2i} v_i) = 4i - 1;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(u_{2i-1} w_i) = 6n - 4i + 1;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(u_{2i} w_i) = 6n - 4i - 1;\ 1 \leqslant i \leqslant \frac{n}{2}$

The corresponding labels of vertices u_i and $v_i, i = 1, 2, 3 \ldots \mod (6n - 2)$ are

$f^*(u_1) = f(u_1 u_2) + f(u_1 v_1) + f(u_1 w_1) = 4n - 3;$

$$f^*(u_i) = f(u_{i-1} u_i) + f(u_i u_{i+1}) + f(u_i v_{\lceil \frac{i}{2} \rceil}) + f(u_i w_{\lceil \frac{i}{2} \rceil})$$
$$= 2n - 4i + 2;\ 2 \leqslant i \leqslant n - 1$$

$f^*(u_n) = f(u_{n-1} u_n) + f(u_n v_{\frac{n}{2}}) + f(u_n w_{\frac{n}{2}}) = 2n + 1;$

$f^*(v_i) = f(u_{2i-1} v_i) + f(u_{2i} v_i) = 8i - 4;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f^*(w_i) = f(u_{2i-1} w_i) + f(u_{2i} w_i) = 6n - 8i + 2;\ 1 \leqslant i \leqslant \frac{n}{2}$

The labels of edges are in the set $\{1, 3, 5, \ldots, 6n - 3\}$. Then the labels of vertices are in the set

$$\{4, 12, \ldots, 4n - 4\} \cup \{4n - 3\} \cup \{2n - 6, 2n - 10, \ldots, 4n + 4\} \cup \{2n + 1\}$$
$$\cup \{6n - 6, 6n - 14, \ldots, 2n + 2\}.$$

Here $\forall i \neq j,\ f(v_i) \neq f(v_j)$.

Therefore, the induced function $f^* : V(G) \to \{0, 1, 2, \ldots, 2q - 1\}$, defined as $f^*(u) = \sum_{uv \in E(G)} f(uv) \mod (6n - 2)$ is injective.

Case 2: $n \equiv 1 \pmod 4$

Define edge labeling $f : E(G) \to \{1, 3, 5, \dots, 6n - 7\}$ is as follows:

$f(u_i u_{i+1}) = 4n + 2i - 3;\ 1 \leqslant i \leqslant n - 2$

$f(u_{n-1} u_n) = 2n - 1;$

$f(u_{2i-1} v_i) = 4i - 3;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(u_{2i} v_i) = 4i - 1;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(u_{2i-1} w_i) = 4n - 4i + 1;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(u_{2i} w_i) = 4n - 4i - 1;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

The corresponding labels of vertices u_i and v_i, $i = 1, 2, 3 \dots \bmod (6n - 6)$ are

$f^*(u_1) = f(u_1 u_2) + f(u_1 v_1) + f(u_1 w_1) = 8n - 3;$

$f^*(u_i) = f(u_{i-1} u_i) + f(u_i u_{i+1}) + f(u_i v_{\lceil \frac{i}{2} \rceil}) + f(u_i w_{\lceil \frac{i}{2} \rceil})$

$= 6n + 4i - 4;\ 2 \leqslant i \leqslant n - 2$

$f^*(u_{n-1}) = f(u_{n-2} u_{n-1}) + f(u_{n-1} u_n) + f(u_{n-1} v_{\lfloor \frac{n}{2} \rfloor}) + f(u_n w_{\lfloor \frac{n}{2} \rfloor}) = 6n - 4$

$f^*(u_n) = f(u_{n-1} u_n) = 2n - 1;$

$f^*(v_i) = f(u_{2i-1} v_i) + f(u_{2i} v_i) = 8i - 4;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f^*(w_i) = f(u_{2i-1} wi) + f(u_{2i} w_i) = 2n - 8i + 6;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

The labels of edges are in the set $\{1, 3, 5, \dots, 6n - 7\}$. Then the labels of vertices are in the set $\{4, 12, \dots, 4n - 8\} \cup \{2n + 3\} \cup \{6n + 4, 6n + 8, \dots, 4n - 6\} \cup \{2n - 1\} \cup \{2n - 2, 2n - 10, \dots, 4n + 4\}$.

Here $\forall i \neq j,\ f(v_i) \neq f(v_j)$.

Therefore, the induced function $f^* : V(G) \to \{0, 1, 2, \ldots, 2q - 1\}$, defined as $f^*(u) = \sum_{uv \in E}(G) f(uv) \bmod (6n - 6)$ is injective.

Case 3: $n \equiv 3 \pmod 4$

Define edge labeling $f : E(G) \to \{1, 3, 5, \ldots, 6n - 7\}$ is as follows:

$f(u_i u_{i+1}) = 2n + 2i - 3;\ 1 \leqslant i \leqslant n - 1$

$f(u_{2i-1} v_i) = 4i - 3;\ ;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(u_{2i} v_i) = 4i - 1;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(u_{2i-1} w_i) = 4n + 4i - 7;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(u_{2i} w_i) = 4n + 4i - 5;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

The corresponding labels of vertices u_i and v_i, $i = 1, 2, 3 \ldots \bmod (6n - 6)$ are

$f^*(u_1) = f(u_1 u_2) + f(u_1 v_1) + f(u_1 w_1) = 6n - 3;$

$$f^*(u_i) = f(u_{i-1} u_i) + f(u_i u_{i+1}) + f(u_i v_{\lceil \frac{i}{2} \rceil}) + f(u_i w_{\lceil \frac{i}{2} \rceil})$$
$$= 2n + 8i - 8;\ 2 \leqslant i \leqslant n - 1$$

$f^*(u_n) = f(u_{n-1} u_n) = 4n - 5;$

$f^*(v_i) = f(u_{2i-1} v_i) + f(u_{2i} v_i) = 8i - 4;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f^*(w_i) = f(u_{2i-1} w_i) + f(u_{2i}\, w_i) = ;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

The labels of edges are in the set $\{1, 3, 5, \ldots, 6n - 7\}$. Then the labels of vertices are in the set $\{4, 12, \ldots, 4n - 4\} \cup \{6n - 3\} \cup \{2n + 8, 2n + 16, \ldots, 4n - 10\} \cup \{4n - 5\} \cup \{2n + 2, 2n + 10, \ldots, 6n - 10\}$. Here $\forall i \neq j,\ f(v_i) \neq f(v_j)$.

Therefore, the induced function $f^* : V(G) \to \{0, 1, 2, \ldots, 2q - 1\}$, defined as $f^*(u) = \sum_{uv \in E(G)} f(uv) \bmod (6n - 6)$ is injective.

Example 2.2: *The edge odd graceful labeling of double alternate triangular snake* $DA(T_8)$, $DA(T_9)$ *and* $DA(T_{11})$ *is shown in* Figure 2, 3, *and* 4.

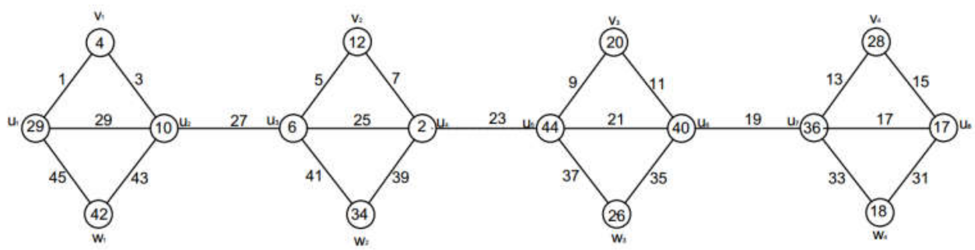

Figure 2: The edge odd graceful labeling of double alternate triangular snake $DA(T_8)$

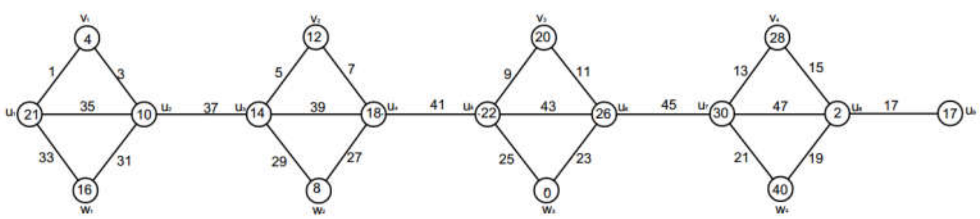

Figure 3: The edge odd graceful labeling of double alternate triangular snake $DA(T_9)$

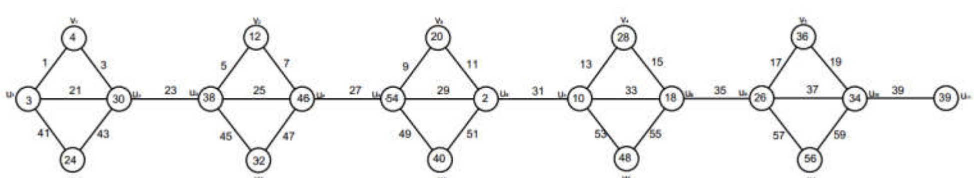

Figure 4: The edge odd graceful labeling of double alternate triangular snake $DA(T_{11})$.

In each possibility the graph under consideration satisfies the vertex conditions and edge conditions for an edge odd graceful labeling. Hence, the double alternate triangular snake $DA(T_n)$ is an edge odd graceful graph for all $n \geqslant 2$. $\square$

Theorem 2.3: *The double alternate quadrilateral snake* $DA(Q_n)$ *is an edge odd graceful graph for all* $n \geqslant 2$.

Proof: Let G be a double alternate quadrilateral snake $DA(Q_n)$ which is obtained from a path P_n with vertices $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to two new vertices v_i, w_i and v_{i+1}, w_{i+1} respectively and then joining v_i, v_{i+1} and w_i, w_{i+1}, where $1 \leqslant i \leqslant n-1$ for even n and for $1 \leqslant i \leqslant n-2$ for odd n.

Therefore,

$$V(G) = \{u_i, v_i, w_i / 1 \leqslant i \leqslant n\}$$

$$E(G) = \{u_i u_{i+1} / 1 \leqslant i \leqslant n-1\} \cup \{v_{2i-1} v_{2i} / 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor\}$$
$$\cup \{w_{2i-1} w_{2i} / 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor\} \cup \{u_i v_i / 1 \leqslant i \leqslant n\} \cup \{v_i w_i / 1 \leqslant i \leqslant n\}.$$

Here note that

$$|V(G)| = \begin{cases} 3n, & \text{if } n \equiv 0 \pmod 2 \\ 3n-2, & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

$$|E(G)| = \begin{cases} 4n-1, & \text{if } n \equiv 0 \pmod 2 \\ 4n-4, & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

Case 1: $n \equiv 0 \pmod 2$

Subcase 1: $n = 2$

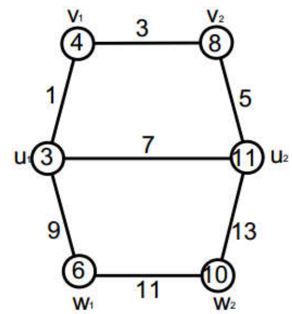

Figure 5: Edge odd graceful labeling of double alternate quadrilateral snake graph $DA(Q_2)$

Here Figure 5 shows that double alternate quadrilateral snake $DA(Q_2)$ is an edge odd graceful graph.

Subcase 2: $n \equiv 0 \ (\bmod 2)$ and $n \neq 2$

Define edge labeling $f : E(G) \rightarrow \{1, 3, 5, \ldots 8n - 3\}$ is as follows:

$f(u_i u_{i+1}) = 3n + 2i - 1; \ 1 \leqslant i \leqslant n - 1$

$f(u_{2i-1} v_{2i-1}) = 6i - 5; \ 1 \leqslant i \leqslant \frac{n}{2}$

$f(u_{2i} v_{2i}) = 6i - 1; \ 1 \leqslant i \leqslant \frac{n}{2}$

$f(v_i v_{i+1}) = 6i - 3; \ 1 \leqslant i \leqslant \frac{n}{2}$

$f(u_{2i-1} w_{2i-1}) = 8n - 6i + 3; \ 1 \leqslant i \leqslant \frac{n}{2}$

$f(u_{2i} w_{2i}) = 8n - 6i - 1; \ 2 \leqslant i \leqslant \frac{n}{2}$

$f(w_i w_{i+1}) = 8n - 6i + 1; \ 1 \leqslant i \leqslant \frac{n}{2}$

The corresponding labels of vertices u_i and v_i, $i = 1, 2, 3 \ldots \bmod (8n - 2)$ are

$f^*(u_1) = f(u_1 u_2) + f(u_1 v_1) + f(u_1 w_1) = 3n + 1;$

$f^*(u_i) = f(u_{i-1}u_i) + f(u_iu_{i+1}) + f(u_iv_i) + f(u_iw_i) = 6n + 4i - 4;\ 2 \leqslant i \leqslant n-1$

$f^*(u_n) = f(u_{n-1}u_n) + f(u_nv_n) + f(u_nw_n) = 5n - 3$

$f^*(v_{2i-1}) = f(u_{2i-1}v_{2i-1}) + f(v_{2i-1}v_{2i}) = 12i - 8;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f^*(v_{2i}) = f(v_{2i-1}v_{2i}) + f(v_{2i}u_{2i}) = 12i - 4;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f^*(w_{2i-1}) = f(u_{2i-1}w_{2i-1}) + f(w_{2i-1}w_{2i}) = 8n - 12i + 6;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f^*(w_{2i}) = f(w_{2i-1}w_{2i}) + f(w_{2i}u_{2i}) = 8n - 12i + 2;\ 1 \leqslant i \leqslant \frac{n}{2}$

The labels of edges are in the set $\{1, 3, 5, \ldots, 8n-3\}$. Then the labels of vertices are in the set $\{4, 16, \ldots, 6n-8\} \cup \{8, 20, \ldots, 6n-4\} \cup \{3n+1\} \cup \{6n+4, 6n+8, \ldots, 2n-6\} \cup \{5n-3\} \cup \{8n-6, 8n-18, \ldots, 2n+6\} \cup \{8n-10, 8n-22, \ldots, 2n+2\}$. Here $\forall i \neq j,\ f(v_i) \neq f(v_j)$.

Therefore, the induced function $f^* : V(G) \to \{0, 1, 2, \ldots, 2q-1\}$, defined as $f^*(u) = \sum_{uv \in E(G)} f(uv) \bmod (8n-2)$ is injective.

Case 2: $n \equiv 1 \pmod 2$

Define edge labeling $f : E(G) \to \{1, 3, 5, \ldots 8n-9\}$ is as follows:

$f(u_iu_{i+1}) = 6n + 2i - 7;\ 1 \leqslant i \leqslant n-1$

$f(u_{2i-1}v_{2i-1}) = 4i - 3;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(u_{2i}v_{2i}) = 4n + 4i - 7;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(v_iv_{i+1}) = 2n + 4i - 5;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(u_{2i-1}w_{2i-1}) = 4n + 4i - 5;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(u_{2i}w_{2i}) = 4i - 1;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(w_i w_{i+1}) = 2n + 4i - 3;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

The corresponding labels of vertices u_i and $v_i,\ i = 1, 2, 3 \ldots \mod (8n - 8)$ are

$f^*(u_1) = f(u_1u_2) + f(u_1v_1) + f(u_1w_1) = 2n + 3$;

$$f^*(u_{2i-1}) = f(u_{2i-2}u_{2i-1}) + f(u_{2i-1}u_{2i}) + f(u_{2i-1}v_{2i-1}) + f(u_{2i-1}w_{2i-1}),$$
$$= 8n + 16i - 20;\ 2 \leqslant i \leqslant \lfloor \tfrac{n-2}{2} \rfloor$$

$$f^*(u_{2i}) = f(u_{2i-1}u_{2i}) + f(u_{2i}u_{2i+1}) + f(u_{2i}v_{2i}) + f(u_{2i}w_{2i})$$
$$= 8n + 16i - 16;\ 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor$$

$f^*(un) = f(u_{n-1}u_n) = 8n - 9$

$f^*(v_{2i-1}) = f(u_{2i-1}v_{2i-1}) + f(v_{2i-1}v_{2i}) = 2n + 8i - 8;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f^*(v_{2i}) = f(u_{2i}v_{2i}) + f(v_{2i-\ 1}v_{2i}) = 6n + 8i - 12;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f^*(w_{2i-1}) = f(u_{2i-1}w_{2i-1}) + f(w_{2i-1}w_{2i}) = 6n + 8i - 8;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f^*(w_{2i}) = f(u_{2i}w_{2i}) + f(w_{2i-1}w_{2i}) = 2n + 8i - 4;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

The labels of edges are in the set $\{1, 3, 5, \ldots, 8n - 9\}$. Then the labels of vertices are in the set $\{2n,\ 2n + 8, \ldots, 6n - 12\} \cup \{6n - 4, 6n + 4, \ldots, 2n - 8\}$ $\cup\ \{2n + 3\} \cup \{8, 24, \ldots, 8n - 16\} \cup \{2n + 3\} \cup \{20, 36, , \ldots, 8n - 20\} \cup \{8n - 9\}$ $\cup\ \{6n, 6n + 8, \ldots, 2n - 4\} \cup \{2n + 4, 2n + 12, \ldots, 6n - 8\}$.

Here $\forall i \neq j,\ f(v_i) \neq f(v_j)$.

Therefore, the induced function $f^* : V(G) \to \{0, 1, 2, \ldots, 2q - 1\}$, defined $f^*(u) = \sum_{uv \in E(G)} f(uv) \mod (8n - 8)$ is injective.

Example 2.4: *The edge odd graceful labeling of the double alternate quadrilateral snake* $DA(Q_6)$ *and* $DA(Q_7)$ *is shown in the following* Figure 6 *and* 7.

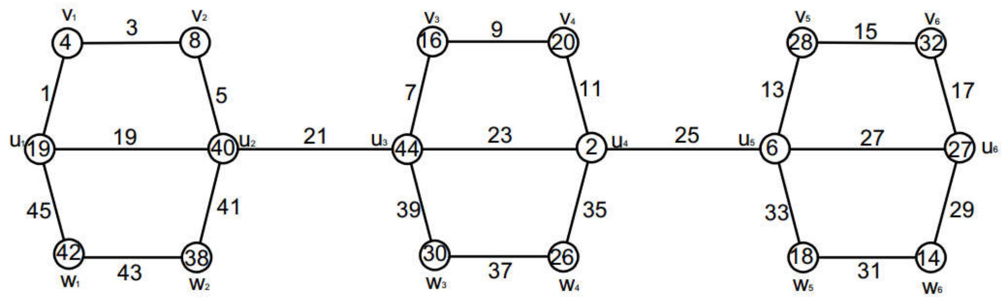

Figure 6: The edge odd graceful labeling of double alternate quadrilateral snake $DA(Q6)$

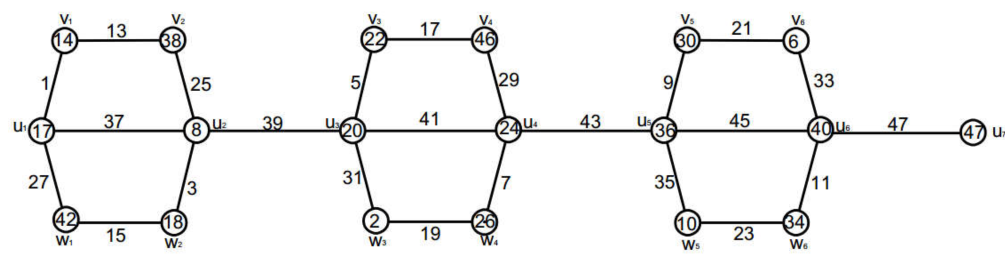

Figure 7: The edge odd graceful labeling of double alternate quadrilateral snake $DA(Q_7)$.

In each possibility the graph under consideration satisfies the vertex conditions and edge conditions for an edge odd graceful labeling. Hence, the double alternate quadrilateral snake $DA(Q_n)$ is an edge odd graceful graph for all $n \geqslant 2$. □

Theorem 2.5: *The alternate pentagonal snake* $A(PS_n)$ *is an edge odd graceful graph for all* $n \geqslant 2$.

Proof: Let G be a alternate pentagonal snake $A(PS_n)$ which is obtained from a path P_n with vertices $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to new vertices v_j and w_j respectively and then joining v_j and w_j to the new vertex x_j, where $1 \leqslant i \leqslant n-1$ for even $n, 1 \leqslant i \leqslant n-2$ for odd n and $1 \leqslant j \leqslant \lfloor \frac{n}{2} \rfloor$.

Therefore,

$V(G) = \{u_i, v_j, w_j, x_j / 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant \lfloor \frac{n}{2} \rfloor\}$

$$E(G) = \{u_i u_{i+1} / 1 \leqslant i \leqslant n-1\} \cup \{u_{2i-1} v_i / 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor\} \cup \{v_i x_i / 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor\}$$
$$\cup \{x_i w_i / 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor\} \cup \{w_i u_{2i} / 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor\}.$$

Here note that

$$|V(G)| = \begin{cases} \frac{5n}{2}, & \text{if } n \equiv 0 \pmod 2 \\ \lfloor \frac{5n}{2} \rfloor + 1, & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

$$|E(G)| = \begin{cases} 3n-1, & \text{if } n \equiv 0 \pmod 2 \\ 3n-3, & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

Case 1: $n \equiv 0 \pmod 2$

Subcase 1: $n \equiv 0 \pmod 6$

Define edge labeling $f : E(G) \to \{1, 3, 5, \ldots, 6n-2\}$ is as follows:

$f(u_i u_{i+1}) = 4n + 2i - 1;\ 1 \leqslant i \leqslant n-1$

$f(u_{2i-1} v_i) = 2n + 4i - 3;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(v_i x_i) = 4i - 3;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(w_i x_i) = 2n + 4i - 1;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(u_{2i} w_i) = 4i - 1;\ 1 \leqslant i \leqslant \frac{n}{2}$

The corresponding labels of vertices u_i and v_i, $i = 1, 2, 3 \ldots \mod (6n - 2)$ are

$f^*(u_1) = f(u_1u_2) + f(u_1v_1) = 6n + 2$

$f^*(u_{2i}) = f(u_{2i-1}u_{2i}) + f(u_{2i}u_{2i+1}) + f(u_{2i}w_i) = 2n + 12i - 3;\ 1 \leqslant i \leqslant \frac{n-2}{2}$

$f^*(u_{2i-1}) = f(u_{2i-1}u_{2i}) + f(u_{2i}u_{2i+1}) + f(u_{2i-1}v_i) = 4n + 12i - 9;\ 2 \leqslant i \leqslant \frac{n}{2}$

$f^*(u_n) = f(u_{n-1}u_n) + f(w_{n/2}u_n) = 8n - 4$;

$f^*(v_i) = f(u_{2i-1}v_i) + f(v_ix_i) = 2n + 8i - 6;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f^*(x_i) = f(v_ix_i) + f(x_iw_i) = 2n + 8i - 4;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f^*(w_i) = f(x_iw_i) + f(w_iu_{2i}) = 2n + 8i - 2;\ 1 \leqslant i \leqslant \frac{n}{2}$

The labels of edges are in the set $\{1, 3, 5, \ldots, 6n - 3\}$.Then the labels of vertices are in the set $\{6n + 2\} \cup \{2n + 9, 2n + 21, \ldots, 2n - 13\} \cup \{4n + 15, 4n + 27, \ldots, 4n - 7\} \cup \{2n - 2\} \cup \{2n + 2, 2n + 10, \ldots, 6n - 6\} \cup \{2n + 4, 2n + 12, \ldots, 6n - 4\} \cup \{2n + 6, 2n + 14, \ldots, 6n - 2\}$.

Here $\forall i \neq j,\ f(v_i) \neq f(v_j)$.

Therefore, the induced function $f^* : V(G) \to \{0, 1, 2, \ldots, 2q - 1\}$, defined as $f^*(u) = \sum_{uv \in E(G)} f(uv) \mod (6n - 2)$ is injective.

Subcase 2: $n \equiv 2, 4 \pmod 6$

Define edge labeling $f : E(G) \to \{1, 3, 5, \ldots, 6n - 2\}$ is as follows:

$f(u_iu_{i+1}) = 4n + 2i - 1;\ 1 \leqslant i \leqslant n - 1$

$f(u_{2i-1}v_i) = 8i - 7;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(v_ix_i) = 8i - 5;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(w_i x_i) = 8i - 3;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(u_{2i} w_i) = 8i - 1;\ 1 \leqslant i \leqslant \frac{n}{2}$

The corresponding labels of vertices v_i and $v_i, i = 1, 2, 3 \ldots \mod (6n - 2)$ are

$f^*(u_1) = f(u_1 u_2) + f(u_1 v_1) = 4n + 2\,;$

$f^*(u_{2i}) = f(u_{2i} u_{2i+1}) + f(u_{2i-1} u_{2i}) + f(u_{2i} w_i) = 2n + 16i - 3;\ 1 \leqslant i \leqslant \frac{n-2}{2}$

$f^*(u_{2i-1}) = f(u_{2i-1} u_{2i}) + f(u_{2i} u_{2i+1}) + f(u_{2i-1} v_i) = 2n + 16i - 13;\ 2 \leqslant i \leqslant \frac{n}{2}$

$f^*(u_n) = f(u_{n-1} u_n) + f(w_{n/2} u_n) = 4n - 2\,;$

$f^*(v_i) = f(u_{2i-1} v_i) + f(v_i x_i) = 16i - 12;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f^*(x_i) = f(v_i x_i) + f(x_i w_i) = 16i - 8;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f^*(w_i) = f(x_i w_i) + f(w_i u_{2i}) = 16i - 4;\ 1 \leqslant i \leqslant \frac{n}{2}$

The labels of edges are in the set $\{1, 3, 5, \ldots, 6n - 3\}$. Then the labels of vertices are in the set $\{4n + 2\} \cup \{2n + 13, 2n + 29, \ldots, 4n - 17\}$ $\cup \{2n + 19, 2n + 35, \ldots, 4n - 11\} \cup \{8, 24, \ldots, 2n - 6\} \cup \{12, 28, \ldots, 2n - 2\}$.

Here $\forall i \neq j,\ f(v_i) \neq f(v_j)$.

Therefore, the induced function $f^* : V(G) \to \{0, 1, 2, \ldots, 2q - 1\}$, defined as $f^*(u) = \sum_{uv \in E(G)} f(uv) \mod (6n - 2)$ is injective.

Case 2: $n \equiv 1 \pmod 2$

Here note that

$$|V(G)| = 5\lfloor \tfrac{n}{2} \rfloor + 1$$

$$|E(G)| = 3n - 3$$

Define edge labeling $f : E(G) \to \{1, 3, 5, \ldots 6n - 7\}$ is as follows:

$$f(u_i u_{i+1}) = 6n - 2i - 5;\ 1 \leqslant i \leqslant n - 1$$

$$f(u_{2i-1} v_i) = 4i - 3;\ 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor$$

$$f(v_i x_i) = 2n + 4i - 5;\ 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor$$

$$f(w_i x_i) = 4i - 1;\ 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor$$

$$f(u_{2i} w_i) = 2n + 4i - 3;\ 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor$$

The corresponding labels of vertices u_i and v_i, $i = 1, 2, 3 \ldots \mod (6n - 6)$ are

$$f^*(u_1) = f(u_1 u_2) + f(u_1 v_1) = 6n - 6;$$

$$f^*(u_{2i}) = f(u_{2i} u_{2i+1}) + f(u_{2i-1} u_{2i}) + f(u_{2i} w_i) = 2n - 4i + 1;\ 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor$$

$$f^*(u_{2i-1}) = f(u_{2i-1} u_{2i}) + f(u_{2i} u_{2i+1}) + f(u_{2i-1} v_i) = 6n - 4i - 1;\ 2 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor$$

$$f^*(u_n) = f(u_{n-1} u_n) = 4n - 3;$$

$$f^*(v_i) = f(u_{2i-1} v_i) + f(v_i x_i) = 2n + 8i - 8;\ 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor$$

$$f^*(x_i) = f(v_i x_i) + f(x_i w_i) = 2n + 8i - 6;\ 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor$$

$$f^*(w_i) = f(x_i w_i) + f(w_i u_{2i}) = 2n + 8i - 4;\ 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor$$

The labels of edges are in the set $\{1, 3, 5, \ldots, 6n - 7\}$.

Then the labels of vertices are in the set $\{6n - 6\} \cup \{2n - 3, 2n - 7, \ldots, 6n - 3\}$ $\cup \{6n - 9, 6n - 13, \ldots, 4n + 1\} \cup \{4n - 3\} \cup \{2n, 2n + 8, \ldots, 6n - 12\}$ $\cup \{2n + 2, 2n + 10, \ldots, 6n - 10\} \cup \{2n + 4, 2n + 12, \ldots, 6n - 8\}$.

Here $\forall i \neq j,\ f(v_i) \neq f(v_j)$.

Therefore, the induced function $f^* : V(G) \to \{0, 1, 2, \ldots, 2q - 1\}$, defined as $f^*(u) = \sum_{uv \in E(G)} f(uv) \bmod (6n - 6)$ is injective.

Example 2.6: *The edge odd graceful labeling of the alternate pentagonal snake* $A(PS_6)$, $A(PS_{10})$ *and* $A(PS_9)$ *is shown in* Figure 8, 9 *and* 10.

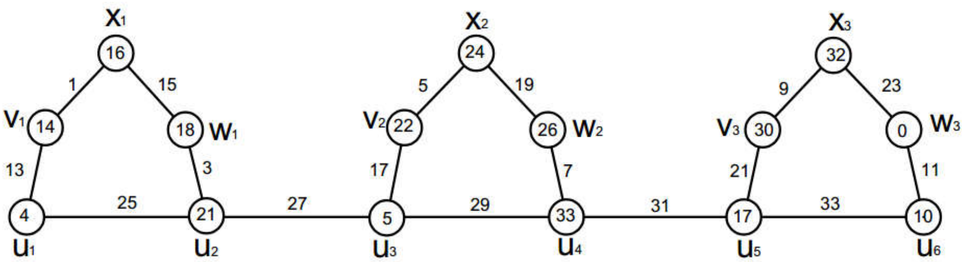

Figure 8: The edge odd graceful labeling of alternate pentagonal snake $A(PS_6)$.

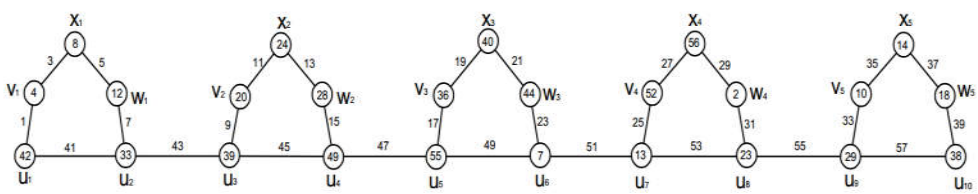

Figure 9: The edge odd graceful labeling of alternate pentagonal snake $A(PS_{10})$.

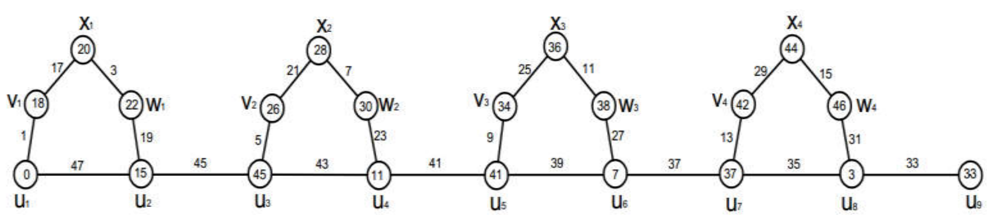

Figure 10: The edge odd graceful labeling of alternate pentagonal snake graph $A(PS_9)$

In each possibility the graph under consideration satisfies the vertex conditions and edge conditions for an edge odd graceful labeling. Hence, the alternate pentagonal snake $A(PS_n)$ is an edge odd graceful graph for all $n \geqslant 2$. □

3 Conclusion

In this paper, it is proved that double alternate triangular snake $DA(T_n)$, double alternate quadrilateral snake $DA(Q_n)$ and alternate pentagonal snake $A(PS_n)$ are edge odd graceful graphs. To derive new families of graphs that admit edge odd graceful labeling is an open area of research.

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ANTIMAGIC LABELING OF LINE GRAPH OF SOME GRAPHS

Abstract: Motivated from the study of magic square, Hartsfield and Ringel defined antimagic labeling as a bijection $f : E(G) \to \{1, 2, 3, \ldots, |E(G)|\}$ such that $\forall u, v \in V(G), uf \neq v$, sum of $f(e)$ for all e incident to u is different from sum of $f(e)$ for all e incident to v. In this paper, we discussed antimagic labeling of the line graph of armed crown, double comb, ladder, wheel and tadpole.

Keywords: Graph Labeling, Antimagic Labeling, Graph Operation, Line Graph.

Mathematics Subject Classification (2020) No.: 05C78, 05C76.

1. Introduction

All the graphs considered in this paper are simple, finite, connected and undirected. A graph $G = (V(G), E(G))$ with q edges is said to be antimagic, if there exist a bijective labeling *f* from edge set of G to $1, 2, 3, \ldots, q$ such that the sums of the labels of the edges incident to each vertex is distinct. Hartsfield and Ringel [12] in 'Pearls in graph theory' introduced antimagic labeling and conjecture that 'every connected graph different from K_2 is antimagic'.

Many authors have tried to attack antimagic conjecture, Alon *et al.* [1] have derived conditions on degree of a vertices for graph to be antimagic. Arumugam *et al.* [2] have shown that various pyramid graphs are antimagic graphs. Cheng [6] has proved that Cartesian products of two or more regular graphs are antimagic. Joseph and Kureethara[15] have investigated that Cartesian product of wheel graph and path graph is antimagic. Bača *et al.* [3] as well as Wang *et al* .[24] have discussed antimagic labeling for some join graphs. Latchoumanane and Varadhan [17] have studied antimagicness for tensor product of wheel and star. Lozano *et al.* [18] have proved antimagic labeling of caterpillars. Sethuraman and Shermily [21] have verified binomial tree and Fibonacci tree are antimagic. Barasara and Prajapati [4, 5] have obtained antimagic labeling of some degree splitting graphs as well as for some snake graphs. Although researchers applied various techniques, still antimagic conjecture remains open.

A detailed survey on antimagic labeling can be found in Jin and Tu [14]. While survey on graph labeling is carried out by Gallian [10].

In this paper, we study antimagic labeling in the context of line graph operation.

2. Preliminaries

Definition 2.1 ([7]): *The line graph $L(G)$ of a graph G is the graph whose vertex set is $E(G)$ and two vertices are adjacent in $L(G)$ whenever they are incident in G.*

Definition 2.2 ([23]): *The armed crown AC_n is a graph in which path P_2 is attached at each vertex of cycle C_n by an edge.*

Definition 2.3 ([19]): *The* Cartesian product of graphs G_1 *and* G_2 *denoted by* $G_1 \square G_2$ *is the graph with vertex set* $V(G_1) \times V(G_2) = \{(u, v)/u \in V(G_1) \text{ and } v \in V(G_2)\}$ *and* (u, v) *is adjacent to* (u', v') *if and only if either* $u = u'$ *and* $vv' \in E(G_2)$ *or* $v = v'$ *and* $uu' \in E(G_1)$.

Definition 2.4 ([11]): *The ladder graph L_n is defined as $L_n = P_n \square K_2$.*

Definition 2.5 ([9]): *Let G and H be two graphs. The corona product of G and H, denoted by $G \odot H$, is obtained by taking one copy of G and $|V(G)|$ copies of H, and by joining each vertex of the i^{th} copies of H to the i^{th} vertex of G, for $1, 2, 3, \ldots, |V(G)|$.*

Definition 2.6 ([13]): *Let P_n be a path graph with n vertices. The double comb graph is defined as $P_n \odot 2K_1$.*

Definition 2.7 ([22]): *The graph obtained by joining cycle C_n to a path P_m with an edge is called tadpole graph. It is denoted by $T(n, m)$.*

Preposition 2.1 ([1]): *If G has $n \geqslant 4$ vertices and $\Delta(G) \geqslant n - 2$ then G is antimagic.*

Preposition 2.2 (Exercise in [12]): *The cycle C_n is antimagic.*

3. Main Results

Theorem 3.1: *The armed crown graph AC_n is an antimagic graph.*

Proof: Let AC_n be an armed crown graph with $V(AC_n) = \{v_i, v_i', v_i'' / i = 1, 2, \ldots, n\}$ and $E(AC_n) = \{v_i v_{i+1} / i = 1, 2, \ldots, n-1\} \cup \{v_1 v_n\} \cup \{v_i v_i' / i = 1, 2, \ldots, n\} \cup \{v_i' v_i'' / i = 1, 2, \ldots, n\}$.

Then $|V(AC_n)| = 3n$ and $|E(AC_n)| = 3n$.

We define $f : E(AC_n) \to \{1, 2, \ldots, 3n\}$ as follows.

$f(v_1 v_n) = 2n + 1$,

$f(v_i v_{i+1}) = 3n - i + 1$; For $1 \leqslant i \leqslant n - 1$,

$f(v_i v_i') = n + i;$ For $1 \leqslant i \leqslant n$,

$f(v_i' v_i'') = i;$ For $1 \leqslant i \leqslant n$.

Above define edge labeling function will generate distinct vertex labels for all the vertices of AC_n. Thus, f is an antimagic labeling.

Hence, the armed crown graph AC_n is an antimagic graph. □

Illustration 3.1: *The graph AC_8 and its antimagic labeling is shown in* Figure 1.

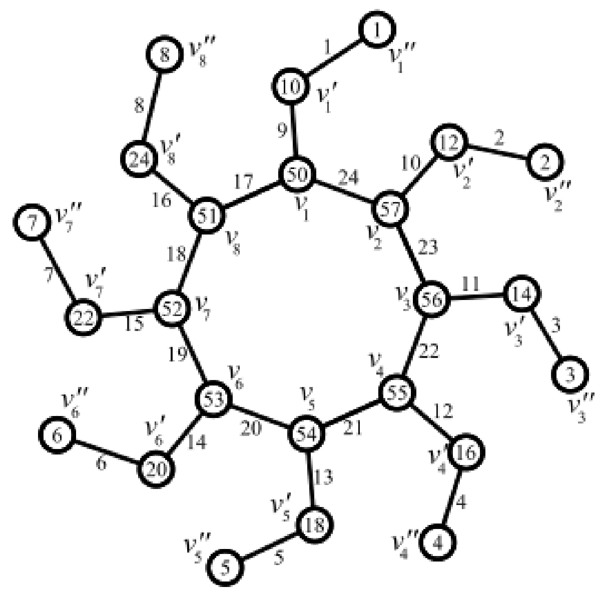

Figure 1: AC_8 and its antimagic labeling.

Theorem 3.2: *The graph $L(AC_n)$ is an antimagic graph.*

Proof: Let AC_n be an armed crown graph with $V(AC_n) = \{v_i, v_i', v_i'' / i = 1, 2, \ldots, n\}$ and $E(AC_n) = \{e_i = v_i v_{i+1} / i = 1, 2, \ldots, n-1\} \cup \{e_n = v_1 v_n\} \cup \{e_i' = v_i v_i' / i = 1, 2, \ldots, n\} \cup \{e_i'' = v_i' v_i'' / i = 1, 2, \ldots, n\}$.To construct $L(AC_n)$, let the vertices corresponding to e_i be x_i, e_i' be x_i' and e_i'' be x_i'' for each i. Then $|V(L(AC_n)| = 3n$ and $|E(L(AC_n)| = 4n$.

We define $f : E(L(AC_n)) \to \{1, 2, \dots, 4n\}$ as follows.

$$f(x_i'x_i'') = i; \qquad \text{For } 1 \leqslant i \leqslant n,$$

$$f(x_n x_1') = n + 1,$$

$$f(x_i x_{i+1}') = n + 1 + i; \qquad \text{For } 1 \leqslant i \leqslant n - 1,$$

$$f(x_i x_i') = 3n + 1 - i; \qquad \text{For } 1 \leqslant i \leqslant n,$$

$$f(x_i x_{i+1}) = 3n + 1 + i; \qquad \text{For } 1 \leqslant i \leqslant n - 1,$$

$$f(x_1 x_n) = 3n + 1.$$

Above define edge labeling function will generate distinct vertex labels for all the vertices of $L(AC_n)$. Thus, f is an antimagic labeling.

Hence, the graph $L(AC_n)$ is an antimagic graph. □

Illustration 3.2: *The graph* $L(AC_6)$ *and its antimagic labeling is shown in* Figure 2.

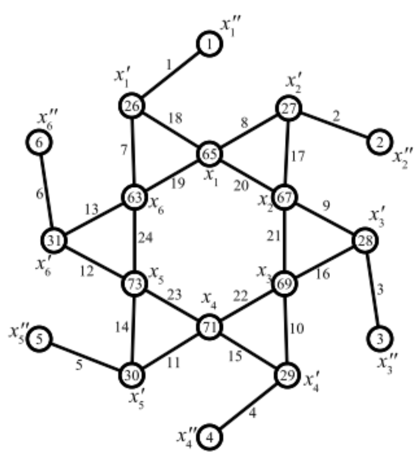

Figure 2: $L(AC_6)$ and its antimagic labeling.

Theorem 3.3: *The graph $L(P_n \odot 2K_1)$ is an antimagic graph.*

Proof: Let $P_n \odot 2K_1$ be a double comb graph with $V(P_n \odot 2K_1) = \{v_i, v_i', v_i'' / i = 1, 2, \dots, n\}$ and $E(P_n \odot 2K_1) = \{e_i = v_i v_{i+1} / i = 1, 2, \dots, n-1\} \cup \{e_i' = v_i v_i' / i = 1, 2, \dots, n\} \cup \{e_i'' = v_i v_i'' / i = 1, 2, \dots, n\}$. To construct $L(P_n \odot 2K_1)$, let the vertices corresponding to e_i be x_i, e_i' be x_i' and e_i'' be x_i'' for each i. Then $|V(L(P_n \odot 2K_1)| = 3n - 1$ and $|E(L(P_n \odot 2K_1)| = 6n - 6$.

We define $f : E(P_n \odot 2K1) \to \{1, 2, \dots, 6n - 6\}$ as per following two cases.

Case 1: For $n = 2$.

The graph $L(P_2 \odot 2K_1)$ has 5 vertices and $\Delta(L(P_2 \odot 2K_1)) = 4$. Thus, by Preposition 2.1, $L(P_2 \odot 2K_1)$ is an antimagic graph.

Case 2: For $n \geqslant 3$.

$f(x_1' x_1'') = 1$,

$f(x_n' x_n'') = 2$,

$f(x_i x_i') = 2i + 1$; For $1 \leqslant i \leqslant n - 1$,

$f(x_{i-1} x_i') = 2i$; For $2 \leqslant i \leqslant n$,

$f(x_i x_i'') = 4n - 2i$; For $1 \leqslant i \leqslant n - 1$,

$f(x_i x_{i+1}'') = 4n - 1 - 2i$; For $1 \leqslant i \leqslant n - 1$,

$f(x_i x_{i+1}) = 5n - 4 + i$; For $1 \leqslant i \leqslant n - 2$,

$f(x_i' x_i'') = 4n - 3 + i$; For $2 \leqslant i \leqslant n - 1$.

Above define edge labeling function will generate distinct vertex labels for all the vertices of $L(P_n \odot 2K_1)$. Thus, f is an antimagic labeling.

Hence, the graph $L(P_n \odot 2K_1)$ is an antimagic graph. □

Illustration 3.3: *The graph* $L(P_7 \odot 2K_1)$ *and its antimagic labeling is shown in* Figure 3.

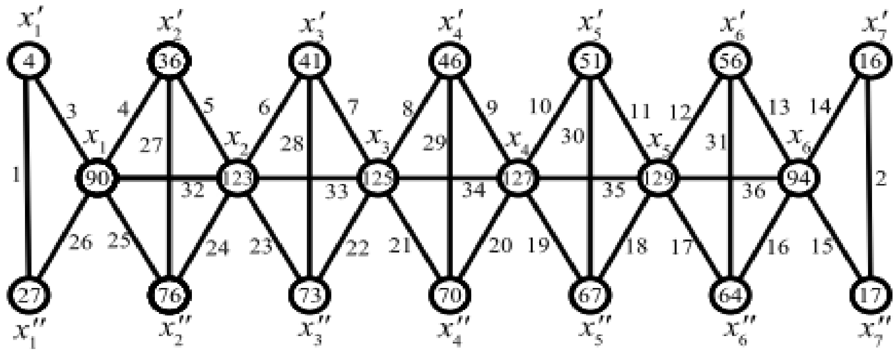

Figure 3: $L(P_7 \odot 2K_1)$ and its antimagic labeling.

Theorem 3.4: *The graph* $L(L_n)$ *is an antimagic graph.*

Proof: Let L_n be a ladder graph with $V(L_n) = \{v_i, v_i' / i = 1, 2, \dots, n\}$ and $E(L_n) = \{e_i' = v_i v_{i+1} / i = 1, 2, \dots, n-1\} \cup \{e_i = v_i v_i' / i = 1, 2, \dots, n\}$ $\cup \{e_i'' = v_i' v_{i+1}' / i = 1, 2, \dots, n-1\}$. To construct $L(L_n)$, let the vertices corresponding to e_i be x_i, e_i' be x_i' and e_i'' be x_i'' for each i.

Then $|V(L(L_n)| = 3n - 2$ and $E(L(L_n)| = 6n - 8$.

We define $f : E(L(Ln)) \to \{1, 2, \dots, 6n - 8\}$ as per following four cases.

Case 1: For $n = 2$.

The graph $L(L_2)$ is also known as cycle C_4. Thus, by Preposition 2.2, $L(L_2)$ is an antimagic graph.

Case 2: For $n = 4$.

The antimagic labeling of graph $L(L_4)$ is demonstrated in following Figure 4.

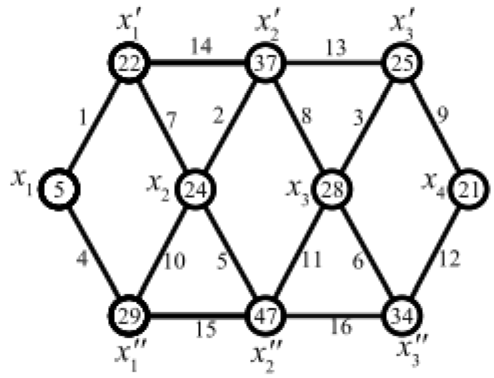

Figure 4: $L(L_4)$ and its antimagic labeling.

Case 3: For $n \equiv 0, 1, 3 (\text{mod } 4)$ and $n \neq 4$.

$f(x_i x_i') = i$; For $1 \leqslant i \leqslant n-1$,

$f(x_{i+1} x_i') = n - 1 + i$; For $1 \leqslant i \leqslant n-1$,

$f(x_i x_i'') = 2(n-1) + i$; For $1 \leqslant i \leqslant n-1$,

$f(x_{i+1} x_i'') = 3(n-1) + i$; For $1 \leqslant i \leqslant n-1$,

$f(x_i' x_{i+1}') = 4(n-1) + i$; For $1 \leqslant i \leqslant n-2$,

$f(x_i'' x_{i+1}'') = 5(n-1) - 1 + i$; For $1 \leqslant i \leqslant n-2$.

Case 4: For $n \equiv 2 (\text{mod } 4)$ and $n \neq 2$.

$f(x_i x_i') = i$; For $1 \leqslant i \leqslant n-1$,

$f(x_{i+1} x_i') = 2(n-1) + i$; For $1 \leqslant i \leqslant n-1$,

$f(x_i x_i'') = n - 1 + i$; For $1 \leqslant i \leqslant n-1$,

$f(x_{i+1}x_i'') = 3(n-1) + i;$ For $1 \leqslant i \leqslant n-1$,

$f(x_i'x_{i+1}') = 4(n-1) + i;$ For $1 \leqslant i \leqslant n-2$,

$f(x_i''x_{i+1}'') = 5(n-1) - 1 + i;$ For $1 \leqslant i \leqslant n-2$.

Above define edge labeling function will generate distinct vertex labels for all the vertices of $L(L_n)$. Thus, f is an antimagic labeling.

Hence, the graph $L(L_n)$ is an antimagic graph. □

Illustration 3.4: *The graph* $L(L_6)$ *and its antimagic labeling is shown in* Figure 5.

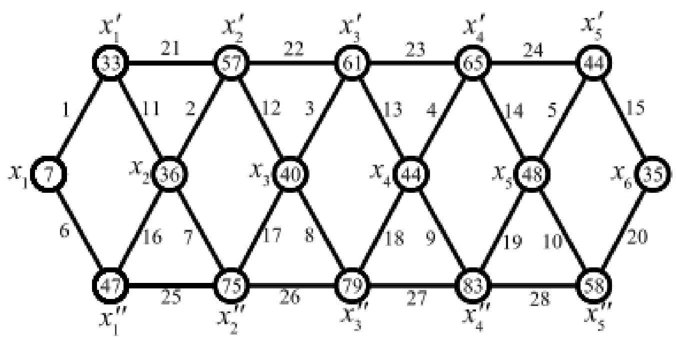

Figure 5: $L(L_6)$ and its antimagic labeling.

Theorem 3.5: *The graph* $L(W_n)$ *is an antimagic graph.*

Proof: Let W_n be a wheel graph with $V(W_n) = \{v, v_i / i = 1, 2, \ldots, n\}$ and $E(W_n) = \{e_i = v_iv_{i+1} / i = 1, 2, \ldots, n-1\} \cup \{e_n = v_1v_n\} \cup \{e_i' = vv_i / i = 1, 2, \ldots, n\}$.

To construct $L(W_n)$, let the vertices corresponding to e_i be x_i and e_i' be x_i' for each i.

Then $|V(L(W_n)| = 2n$ and $|E(L(W_n)| = \frac{n^2 + 5n}{2}$.

We define $f : E(L(W_n)) \to \left\{1, 2, \ldots, \frac{n^2 + 5n}{2}\right\}$ as follows.

$$f(x_i x_{i+1}) = i; \qquad \text{For } 1 \leqslant i \leqslant n-1,$$

$$f(x_n x_1) = n,$$

$$f(x_i x_i') = n + i; \qquad \text{For } 1 \leqslant i \leqslant n,$$

$$f(x_i' x_{i+1}) = 3n + 1 - i; \qquad \text{For } 1 \leqslant i \leqslant n-1,$$

$$f(x_n' x_1) = 2n + 1,$$

$$f(x_i' x_{i+j}') = i(n-1) + 2n + j + 1; \qquad \text{For } \begin{cases} 1 \leqslant i \leqslant 2, \\ 1 \leqslant j \leqslant n - i, \end{cases}$$

$$f(x_i' x_{i+j}') = i(n-1) + 2n + j + 1 - \frac{(i-1)(i-2)}{2}; \quad \text{For } \begin{cases} 3 \leqslant i \leqslant n-1, \\ 1 \leqslant j \leqslant n - i, \end{cases}$$

Above define edge labeling function will generate distinct vertex labels for all the vertices of $L(W_n)$.

Thus, f is an antimagic labeling.

Hence, the graph $L(W_n)$ is an antimagic graph. □

Illustration 3.5: *The graph* $L(W_5)$ *and its antimagic labeling is shown in* Figure 6.

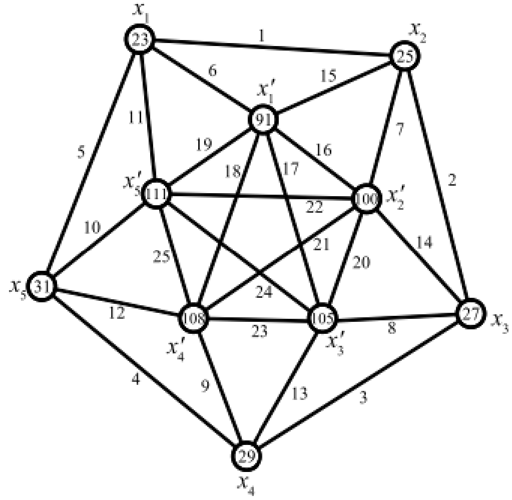

Figure 6: $L(W_5)$ and its antimagic labeling.

Theorem 3.6: *The graph $L(T(n, m))$ is an antimagic graph.*

Proof: Let $T(n, m)$ be a tadpole with $V(T(n, m)) = \{v_1, v_2, \dots, v_{m+n}\}$ and $E(T(n, m)) = \{e_i = v_i v_{i+1} / i = 1, 2, \dots, n + m\} \cup \{e_{n+m} = v_{n+m} v_{m+1}\}$. To construct $L(T(n, m))$, let the vertices corresponding to e_i be x_i for each i.

Then $|V(L(T(n, m))| = n + m$ and $|E(L(T(n, m))| = n + m + 1$.

We define $f : E(L(T(n, m)) \to \{1, 2, \dots, n + m + 1\}$ as per following seven cases.

Case 1: For $n = 3$ and $m \geqslant n - 1$.

$f(x_i x_{i+1}) = i;$ For $1 \leqslant i \leqslant n + m - 1$,

$f(x_{n+m} x_m) = n + m$,

$f(x_{m+1} x_{n+m}) = n + m + 1$.

Case 2: For $n \geqslant 4$ and ($m \geqslant n - 1$ or $m = n - 3$).

$f(x_i x_{i+1}) = i;$ For $1 \leqslant i \leqslant n + m - 1$,

$f(x_{n+m} x_m) = n + m$,

$f(x_{m+1} x_{n+m}) = n + m + 1$.

Case 3: For ($n = 3$ or $n = 4$) and $m = n - 2$.

$f(x_i x_{i+1}) = i;$ For $1 \leqslant i \leqslant n + m - 1$,

$f(x_{n+m} x_m) = n + m + 1$,

$f(x_{m+1} x_{n+m}) = n + m$.

Case 4: For odd $n \geqslant 5$ and $m = 1$.

$f(x_i x_{i+1}) = n - 2i + 2;$ For $2 \leqslant i \leqslant \frac{n + m}{2}$

$f(x_i x_{i+1}) = 2i - (n + m);$ For $\frac{n + m}{2} + 1 \leqslant i \leqslant n + m - 1$

$f(x_1x_2) = n + m,$

$f(x_1x_{n+m}) = n + m + 1,$

$f(x_2x_{n+m}) = n + m - 1.$

Case 5: For even $n \geqslant 6$ and $m = 1$.

$f(x_ix_{i+1}) = n - 2i + 2;$ For $2 \leqslant i \leqslant \frac{n+m-1}{2}$

$f(x_ix_{i+1}) = 2i - (n + m);$ For $\frac{n+m+1}{2} \leqslant i \leqslant n + m - 1,$

$f(x_1x_2) = n + m,$

$f(x_1x_{n+m}) = n + m + 1,$

$f(x_2x_{n+m}) = n + m - 1.$

Case 6: For $n \geqslant 5$ and $(1 < m \leqslant n - 4$ or $m = n - 2)$ and $n + m$ is even.

$f(x_ix_{i+1}) = 2i;$ For $1 \leqslant i \leqslant \frac{n+m}{2}$,

$f(x_ix_{i+1}) = 2i - (n + m) - 1;$ For $\frac{n+m}{2} + 1 \leqslant i \leqslant n + m - 1$

$f(x_{n+m}x_m) = n + m - 1,$

$f(x_{m+1}x_{n+m}) = n + m + 1.$

Case 7: For $n \geqslant 6$ and $(1 < m \leqslant n - 4$ or $m = n - 2)$ and $n + m$ is odd.

$f(x_ix_{i+1}) = 2i;$ For $1 \leqslant i \leqslant \frac{n+m+1}{2}$

$f(x_ix_{i+1}) = 2i - (n + m) - 2;$ For $\frac{n+m+1}{2} + 1 \leqslant i \leqslant n + m - 1$

$f(x_{n+m}x_m) = n + m - 2,$

$f(x_{m+1}x_{n+m}) = n + m.$

Above define edge labeling function will generate distinct vertex labels for all the vertices of $L(T(n,m))$. Thus, f is an antimagic labeling.

Hence, the graph $L(T(n,m))$ is an antimagic graph. □

Illustration 3.6: *The graph* $L(T(5,5))$ *and its antimagic labeling is shown in* Figure 7.

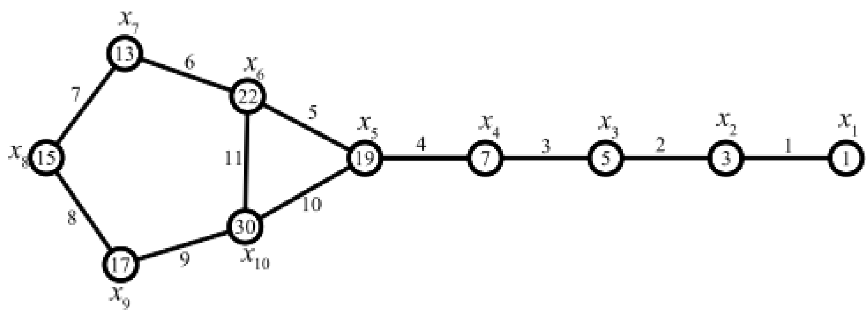

Figure 7: $L(T(5,5))$ and its antimagic labeling.

4. Applications of Antimagic Labeling

Labeled graph has many applications in computer science, applied sciences, social sciences and cryptography. Development of encryption and decryption algorithm using antimagic labeling was studied by Krishnaa [16], Femina and Xavier [8] and Selvakumar and Gupta [20].

5. Conclusions

It is quite difficult to verify that the given connected graph admits antimagic labeling. Many authors [3, 4, 6, 15, 17, 24] have studied antimagic labeling for various graph operations.

While in this paper, antimagic labeling for the line graph of armed crown, double comb, ladder, wheel and tadpole is verified.

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VECTOR BASIS S-CORDIAL LABELING OF FRIENDSHIP GRAPH, FAN GRAPH, AND LILLY GRAPH

Abstract: Let G be a (p,q) graph. Let V be an inner product space with basis S. Let $\varphi : V(G) \to S$ be a map. For each xy assign the label $< x, y >$, where $< x, y >$ denotes the inner product of x and y. We say that φ is a vector basis S-cordial labeling if $| \varphi_x - \varphi_y | \leq 1$ and $| \gamma_i - \gamma_j | \leq 1$ where φ_x denotes the number of vertices labeled with the vector x and γ_i denotes the number of edges labeled with the scalar i. A graph with a vector basis S-cordial labeling is called a vector basis S-cordial graph. In this paper, we investigate the vector basis S-cordial labeling of certain graphs like friendship graph, fan graph, lilly graph, bistar graph, crown graph and armed crown graph where S = {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)} is a basis in R^4 .

Keywords: Friendship Graph, Fan Graph, Lilly Graph, Bistar Graph, and Crown Graph.

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1. Introduction

In this paper, we consider only finite, simple and undirected graph

$G = (V(G), E(G))$ where $V(G)$ and $E(G)$ respectively, denote the vertex set and edge set of G. Note that $p = |V(G)|$ and $q = |E(G)|$ denote the number of vertices and edges of G respectively. The idea of graph labeling was first introduced by Rosa in 1967 [16]. Vertex odd graceful labeling has studied in [5]. Baskar Babujee and Shobana [3] have examined the prime and prime cordial labeling for some special graphs. Radio geometric mean labeling of some star like graphs have investigated in [8]. Parmar [18] proved that for the wheel, fan and friendship graphs are edge vertex prime.

The join $G_1 + G_2$ [6] of two graphs G_1 and G_2 is defined as the graph whose vertex set is $V(G_1) + V(G_2)$ and the edge set consists of these edges which are in G_1 and in G_2 and the edges contained by joining each vertex of G_1 to each vertex of G_2. The fan graph F_n [18] is a graph that is constructed by joining all vertices of a path P_n to a further vertex, called center. That is, $F_n = K_1 + P_n$. Amutha and Uma Devi [1] have explored the super graceful labeling for some families of fan graphs. Barasara [2] proved that the comb is an edge and total edge product cordial. For a dynamic survey on graph labeling, we refer to Gallian [6].

The friendship graph $C_3(n)$ [6] can be constructed by joining n copies of the cycle graph C_3 with a common vertex, which becomes a universal vertex for the graph. The corona $G_1 \odot G_2$ [6] of two graphs G_1 and G_2 is obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and joining each vertex of the i^{th} copy of G_2 to the i^{th} vertex of G_1.

The concept of cordial labeling was first introduced by I. Cahit [4]. Mitra and Bhoumik [11] have introduced the tribonacci cordial labeling of graphs. Parthiban and Sharma proved that the Lilly graph is a prime cordial graph in [13]. The Lilly graph $I_n, n \geq 2$ [13] can be constructed by two star graphs $2K_{1,n}, n \geq 2$ joining two paths $2P_n, n \geq 2$ with sharing a common vertex. That is, $I_n = 2K_{1,n} + 2P_n$. For the terminologies and different notations of graph theory, we refer the book of Harary [7] and of algebra; we refer the book of Herstein [9]. Sum divisor cordial labeling of theta graph was examined by Sugumaran and Rajesh in [17]. Difference cordial labeling for plus and hanging pyramid graphs have studied in [12]. Prajapati and A. Vantiya have proved that the triangular snake, double triangular snake, quadrilateral snake, double quadrilateral snake are SD-prime cordial in [14]. Kaneria *et al*. [10] have investigated the balanced mean cordial labeling and graph operations.

We have introduced new labeling called vector basis S-cordial labeling in [15] and investigated the vector basis vector basis {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial labeling behavior of some standard graphs like path, cycle, comb, star and complete graph. In this paper, we investigate the vector basis {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial labeling of certain graphs like friendship graph, fan graph, lilly graph, bistar graph, crown graph and armed crown graph.

2. Vector basis S-cordial labeling

Definition 2.1: Let G be a (p, q) graph. Let V be an inner product space with basis S. Let $\varphi : V(G) \to S$ be a map. For each xy assign the label $< x, y >$, where $< x, y >$ denotes the inner product of x and y. We say that φ is a vector basis S-cordial labeling if $| \varphi_x - \varphi_y | \le 1$ and $| \gamma_i - \gamma_j | \le 1$ where φ_x denotes the number of vertices labeled with the vector x and γ_i denotes the number of edges labeled with the scalar i. A graph with a vector basis S-cordial labeling is called a vector basis S- cordial graph.

Theorem 2.2: [15] The set S = {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)} is a basis for R^4 over R.

Theorem 2.3: [15] A graph G is vector basis {(1,0),(0,1)}-cordial if and only if G is a cordial graph.

Theorem 2.4: [15] The path P_n is a vector basis {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial graph for all $n \ge 1$.

Theorem 2.5: [15] The cycle C_n is a vector basis {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial if and only if $n \equiv 1, 2, 3 \pmod 4$.

In this paper, we consider the inner product space Rn and the standard inner product $< x, y > = x_1y_1 + x_2y_2 + \cdots + x_ny_n$ where $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, $x_i, y_i \in R$.

3. Main Results

In this section, we consider the basis S = {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}.

Theorem 3.1: The friendship graph $C_3(n)$ is a vector basis {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial if and only if $n \equiv 0,1,2 \pmod 4$.

Proof: The friendship graph $C_3(n)$ is a planar, undirected graph with $2n+1$ vertices and $3n$ edges. Let $V(C_3(n)) = \{u, u_i \mid 1 \le i \le 2n\}$ and $E(C_3(n)) = \{uu_{2i-1}, uu_{2i}, u_{2i-1}u_{2i} \mid 1 \le i \le n\}$ respectively be the vertex and edge sets of $C_3(n)$.Then $|V(C_3(n))| = p = 2n+1$ and $|E(C_3(n))| = q = 3n$. There are four case arises.

Case (i): $n \equiv 0 \pmod 4$

Let $n = 4k$. Then, $p = 2n+1 = 8k+1$. Next, we assign the vector (1,1,1,1) to the vertex u. Assign the vector (1,1,1,1) to the vertices $u_1, u_2, \dots, u_{2k}$. We assign the vector (1,1,1,0) to the next vertices $u_{2k+1}, u_{2k+2}, \dots, u_{4k}$. Then assign the vector (1,1,0,0) to the next vertices $u_{4k+1}, u_{4k+2}, \dots, u_{6k}$. Also assign the vector (1,0,0,0) to the remaining vertices $u_{6k+1}, u_{6k+2}, \dots, u_{8k}$.

Case (ii): $n \equiv 1 \pmod 4$

Let $n = 4k+1$. Then, $p = 8k+3$. Now, we assign the vector (1,1,1,1) to the vertex u. So assign the vector (1,1,1,1) to the vertices $u_1, u_2, \dots, u_{2k}$. Assign the vector (1,1,1,0) to the next vertices $u_{2k+1}, u_{2k+2}, \dots, u_{4k}$. We assign the vector (1,1,0,0) to the next vertices $u_{4k+1}, u_{4k+2}, \dots, u_{6k}$. Further, assign the vector (1,1,1,0) to the vertex u_{6k+1}. Assign the vector (1,0,0,0) to the vertex u_{6k+2}. Then assign the vector (1,1,0,0) to the vertex u_{6k+3}. Finally, assign the vector (1,0,0,0) to the remaining $2k-1$ vertices $u_{6k+4}, u_{6k+5}, \dots, u_{8k+3}$.

Case (iii): $n \equiv 2 \pmod 4$

Let $n = 4k+2$. Then, $p = 8k+5$. Also, we assign the vector (1,1,1,1) to the vertex u. Assign the vector (1,1,1,1) to the vertices $u_1, u_2, \dots, u_{2k+1}$. We assign the vector (1,1,1,0) to the next vertices u_{2k+2}, $u_{2k+3}, \dots, u_{4k+2}$. Then assign the vector (1,1,0,0) to the vertices $u_{4k+3}, u_{4k+4}, \dots, u_{6k+3}$. Moreover, assign the vector (1,0,0,0) to the remaining $2k$ vertices $u_{6k+4}, u_{6k+5}, \dots, u_{8k+4}$.

Case (iv): $n \equiv 3 \ (mod\ 4)$

Let $n = 4k+3$. Then $p = 8k+7$ and $q = 12k+9$. If we assign vector (1,1,1,1) to the vertex u and we have to assign (1,1,1,1) to the $2k+1$ vertices, then $\gamma_4 = 2k+1+\frac{2k+1}{2} < 2k+1+k+1 < 3k+2$, a contradiction. But $\gamma_4 = 3k+1$ or $\gamma_4 = k+1$ according as the vertex u receive the vector (1,1,1,1) or not. This is a contradiction since the size of $C_3(n)$ is $12k+9$.

Clearly the above labeling pattern provides a vector basis {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial labeling for the friendship graph $C_3(n)$ if $n \equiv 0,1,2 \pmod 4$.

Theorem 3.2: The fan graph F_n is a vector basis {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial if and only if $n \equiv 0 \pmod 4$.

Proof: Let $V(F_n) = \{u, u_i \mid 1 \le i \le n\}$ and $E(F_n) = \{uu_i \mid 1 \le i \le n\} \cup \{u_i u_{i+1} \mid 1 \le i \le n-1\}$ respectively be the vertex and edge sets of F_n. Then $|V(F_n)| = p = n+1$ and $E|(F_n)| = q = 2n-1$. There are four cases arises.

Case (i): $n \equiv 0 \pmod 4$

Let $n = 4k$. Then, $p = 4k+1$. Next, we assign the vector (1,1,1,1) to the vertex u. Assign the vector (1,1,1,1) to the first k vertices $u_1, u_2, \dots, u_k$. Then, assign the vector (1,1,1,0) to the next k vertices $u_{k+1}, u_{k+2}, \ldots, u_{2k}$. Also, assign the vector (1,1,0,0) to the next k vertices $u_{2k+1}, u_{2k+2}, \ldots, u_{3k}$. Moreover, assign the vector (1,0,0,0) to the remaining k vertices $u_{3k+1}, u_{3k+2}, \ldots, u_{4k}$.

Case (ii): $n \equiv 1 \pmod 4$

Let $n = 4k+1$. Then $p = n+1 = 4k+2 = (k+1)+(k+1)+k+k$ and $q = 2n-1 = 8k+1 = (2k+1)+2k+2k+2k$. Clearly, $\gamma_4 = 2k-1$ or $\gamma_4 = k$ according as the vertex u receive the vector (1,1,1,1) or not. This is a contradiction since the size of F_n is $8k+1$.

Case (iii): $n \equiv 2 \pmod 4$

Let $n = 4k+2$. Then $p = 4k+3 = (k+1)+(k+1)+(k+1)+k$ and $q = 8k+3 = (2k+1)+(2k+1)+(2k+1)+2k$. Thus, $\gamma_4 = 2k-1$ or $\gamma_4 = k$ according as the vertex u receive the vector (1,1,1,1) or not. We get a contradiction since the size of F_n is $8k+3$.

Case (iv): $n \equiv 3 \pmod 4$

Let $n = 4k+3$. Then $p = 4k+4 = (k+1)+(k+1)+(k+1)+(k+1)$ and $q = 8k+5 = (2k+2)+(2k+1)+(2k+1)+(2k+1)$. Hence, $\gamma_4 = 2k-1$ or $\gamma_4 = k$ according as the vertex u receive the vector (1,1,1,1) or not. We get a contradiction since the size of F_n is $8k+5$.

Clearly the above labeling method provides a vector basis {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial labeling for the fan graph F_n if $n \equiv 0 \pmod 4$.

Example 3.3: The following Figure 1 illustrates the vector basis {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial labeling fan graph F_4.

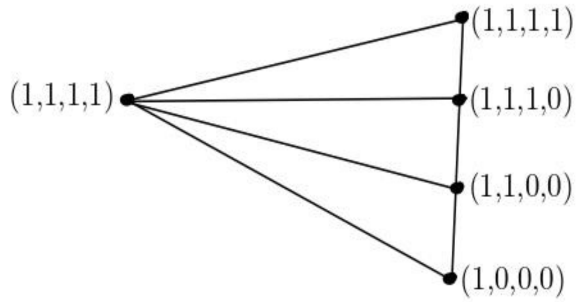

Figure 1
Vector basis $\{$(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)$\}$-cordial labeling of F_4.

Theorem 3.4: The Lilly graph I_n is a vector basis $\{$(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)$\}$-cordial graph for all $n \geq 2$.

Proof: Consider the Lilly graph I_n, $n \geq 2$. Let $V(I_n) = \{u, u_i, v_i \mid 1 \leq i \leq n-1\} \cup \{x_i, y_i \mid 1 \leq i \leq n\}$ and $E(I_n) = \{ux_i, uy_i \mid 1 \leq i \leq n\} \cup \{uv_1, uu_1, u_iu_{i+1},, v_iv_{i+1} \mid 1 \leq i \leq n-2\}$ respectively be the vertex and edge sets of I_n. Then $|V(I_n)| = p = 4n-1$ and $|E(I_n)| = q = 4n-2$.

First, we assign the vector (1,1,1,1) to the vertex u. Assign the vector (1,1,1,1) to the vertices $u_1, u_2, \dots, u_{n-1}$. Then, assign the vector (1,1,1,0) to the vertices $v_1, v_2, \dots, v_{n-1}$. Also, assign the vector (1,1,0,0) to the vertices $x_1, x_2, \dots, x_n$. Moreover, assign the vector (1,0,0,0) to the vertices $y_1, y_2, \dots, y_n$.

Hence, the above labeling technique provides a vector basis $\{$(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)$\}$- cordial labeling for the Lilly graph I_n.

Example 3.5: The following Figure 2 illustrates the vector basis $\{$(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)$\}$-cordial labeling of Lilly graph I_4.

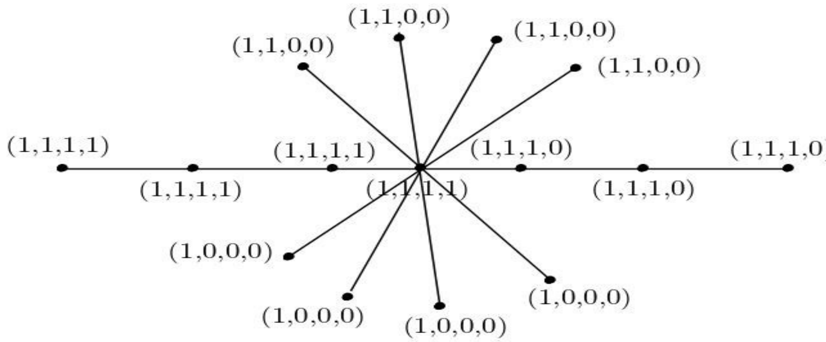

Figure 2
Vector basis {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial labeling of I_4 .

Theorem 3.6: The crown graph $C_n \odot K_1$ is a vector basis {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial if and only if n is odd.

Proof: Consider the crown graph $C_n \odot K_1$. Let C_n be the cycle $u_1u_2 \ldots u_nu_1$. Let $V(C_n \odot K_1) = V(C_n) \cup \{v_i \mid 1 \le i \le n\}$ and $E(C_n \odot K_1) = E(C) \cup \{u_iv_i \mid 1 \le i \le n\}$ respectively be the vertex and edge sets of $C_n \odot K_1$. Then $|V(C_n \odot K_1)| = p = 2n$ and $|E(C_n \odot K_1)| = q = 2n$. We have considered the two cases.

Case (i): $p \equiv 0 \ (mod\ 4)$

Let $p = 4k$. To get the edge label 4, the vector (1,1,1,1) should be assigned to the consecutive vertices of $C_n \odot K_1$. As the size of $C_n \odot K_1$ is $2n = p = 4k$, the maximum edges with label 4 is $k-1$, a contradiction arises.

Case (ii): $p \equiv 2 \ (mod\ 4)$

Let $p = 4k+2$. Then, assign the vector in the following order $u_1, u_2, \ldots u_n, v_1, v_2, \ldots v_n$. We assign the vector (1,1,1,1) to the first $k+1$ vertices $u_1, u_2, \ldots, u_{k+1}$. Also, assign the vector (1,1,1,0) to the next k vertices

$u_{k+2}, u_{k+3}, \ldots, u_n$. We assign the vector (1,1,0,0) to the $k+1$ vertices $v_1, v_2, \ldots, v_{k+1}$. Moreover, assign the vector (1,0,0,0) to the next k vertices $v_{k+2}, v_{k+3}, \ldots, v_n$.

Thus, the above labeling technique provides a vector basis {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial labeling for the crown graph $C_n \odot K_1$.

Theorem 3.7: The armed crown graph AC_n is a vector basis {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial if and only if $n \equiv 1, 2, 3 \pmod 4$.

Proof: The armed crown graph AC_n is the graph obtained from the cycle $u_1 u_2 \ldots u_n u_1$ with $V(ACn) = V(C_n) \cup \{v_i, w_i \mid 1 \le i \le n\}$ and $E(AC_n) + E(C_n) \cup \{u_i v_i, v_i w_i \mid 1 \le i \le n\}$ respectively be the vertex and edge sets of AC_n. Then $\left|V(AC_n)\right| = p = 3n$ and $\left|E(AC_n)\right| = q = 3n$. We have considered the four cases.

Case (i): $p \equiv 0 \pmod 4$

Let $p = 4k$. To get the edge label 4, the vector (1,1,1,1) should be assigned to the consecutive vertices of the graph. AC_n As the size of AC_n is $3n = p = 4k$, the maximum edges with label 4 is $k - 1$, a contradiction.

Case (ii): $p \equiv 1 \pmod 4$

Let $p = 4k + 1$. Then, assign the vector in the following order $u_1, u_2, \ldots u_n$, $v_1, w_1, v_2, w_2, \ldots v_n, w_n$. We assign the vector (1,1,1,1) to the first $k+1$ vertices. Next, assign the vector (1,1,1,0) to the next k vertices. Also assign the vector (1,1,0,0) to the next k vertices. Further, assign the vector (1,0,0,0) to the remaining k vertices.

Case (iii): $p \equiv 2 \pmod 4$

Let $p = 4k + 2$. Then, assign the vector in the following order $u_1, u_2, \ldots u_n$,

$v_1, w_1, v_2, w_2, \dots v_n, w_n$. Now, assign the vector (1,1,1,1) to the first $k+1$ vertices. So assign the vector (1,1,1,0) to the next k vertices. Next, assign the vector (1,1,0,0) to the next $k+1$ vertices. Finally, assign the vector (1,0,0,0) to the remaining k vertices.

Case (iv): $p \equiv 3 \pmod 4$

Let $p = 4k + 3$. Then, assign the vector in the following order $u_1, u_2, \dots u_n$, $v_1, w_1, v_2, w_2, \dots v_n, w_n$. Also, assign the vector (1,1,1,1) to the first $k+1$ vertices. Assign the vector (1,1,1,0) to the next k vertices. Then, assign the vector (1,1,0,0) to the next $k+1$ vertices. Moreover, assign the vector (1,0,0,0) to the remaining $k+1$ vertices.

Therefore, the above labeling method provides a vector basis {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial labeling for the armed crown graph AC_n.

Theorem 3.8: The bistar graph $B_{n,n}$ is a vector basis {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial graph for all n.

Proof: Let $V(B_{n,n}) = \{u, u, u_i, v_i \mid 1 \le i \le n\}$ and $E(B_{n,n}) = \{uv, uu_i, vv_i \mid 1 \le i \le n\}$ respectively be the vertex and edge sets of $B_{n,n}$. Note that $\left|V(Bn, n)\right| = p = 2n + 2$ and $\left|E(B_{n,n})\right| = q = 2n + 1$. We have considered the two cases.

Case (i): $p \equiv 0 \pmod 4$

Let $p = 4k$. Next, we assign the vector (1,1,1,1) to the vertices u and v. Assign the vector (1,1,1,1) to the vertices $u_1, u_2, \dots, u_{k-2}$. Then, assign the vector (1,1,1,0) to the vertices $u_{k-1}, u_k, \dots, u_{2k-2}$. Also, assign the vector (1,1,0,0) to the next k vertices $v_1, v_2, \dots, v_k$. Moreover, assign the vector (1,0,0,0) to the remaining k vertices.

Case (ii): $p \equiv 2 \pmod 4$

Let $p = 4k + 2$. Now, we assign the vector (1,1,1,1) to the vertices u and v. Assign the vector (1,1,1,1) to the vertices $u_1, u_2, \dots, u_{k-1}$. Next, assign the vector (1,1,1,0) to the vertices $u_k, u_{k+1}, \dots, u_{2k-1}$. So, assign the vector (1,1,0,0) to the next k vertices $v_1, v_2, \dots, v_k$. Further, assign the vector (1,0,0,0) to the remaining vertices.

Clearly the above labeling method provides a vector basis {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial labeling for the bistar graph $B_{n,n}$.

4. Conclusion

Vector basis {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial labeling behavior of certain standard graphs like friendship graph, fan graph, lilly graph, bistar graph, crown graph and armed crown graph have been investigated in this paper. The investigation of different kinds of families of graphs for existence of vector basis S-cordial labeling is an open problem.

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A FAMILY OF GENERALIZED GIBONACCI SUMS: GRAPH-THEORETIC CONFIRMATION

Abstract: We confirm a generalized sum of a family of gibonacci polynomial squares using graph-theoretic techniques, and its graph-theoretic and Pell consequences.

Keywords: Generalized Gibonacci Polynomials, Pell Polynomials, Fibonacci Polynomials, Lucas Polynomials.

Mathematics Subject Classification (2020) No.: Primary 11B37, 11B39, 11C08.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*.

They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively [2, 6].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n; $b_n = p_n$ or q_n; $\Delta = \sqrt{x^2 + 4}$ and $2\alpha = x + \Delta$ [6, 7].

1.1 Fundamental Gibonacci Identities: Gibonacci polynomials satisfy the following properties [2, 3, 4, 5, 6, 7]:

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1}f_k^2, & \text{if } g_n = f_n \\ (-1)^{n+k}\Delta^2 f_k^2, & \text{otherwise;} \end{cases} \tag{1}$$

$$g_{n+k+r}g_{n-k} - g_{n+k}g_{n-k+r} = \begin{cases} (-1)^{n+k+1}f_r f_{2k}, & \text{if } g_n = f_n \\ (-1)^{n+k}\Delta^2 f_r f_{2k}, & \text{otherwise;} \end{cases} \tag{2}$$

$$g_{n+k+r}g_{n-k} + g_{n+k}g_{n-k+r} = \begin{cases} \frac{1}{\Delta^2}\left[2l_{2n+r} - (-1)^{n+k}l_{2k}l_r\right], & \text{if } g_n = f_n \\ 2l_{2n+r} + (-1)^{n+k}l_{2k}l_r, & \text{otherwise,} \end{cases} \tag{3}$$

where k and r are positive integers. These properties can be confirmed using Binet-like formulas.

Consequently, we have

$$g_{n+k+r}^2 g_{n-k}^2 - g_{n+k}^2 g_{n-k+r}^2 = \begin{cases} \frac{(-1)^{n+k+1}}{\Delta^2}\left[2l_{2n+r} - (-1)^{n+k}l_{2k}l_r\right] f_{2k}f_r, & \text{if } g_n = f_n \\ (-1)^{n+k}\Delta^2\left[2l_{2n+r} + (-1)^{n+k}l_{2k}l_r\right]f_{2k}f_r, & \text{otherwise.} \end{cases} \tag{4}$$

Again, in the interest of brevity and convenience, we now let

$$A = 2l_{2(2pn+t-p)k+r} - (-1)^{tk}l_{2pk}l_r; \text{ and } B = 2l_{2(2pn+t-p)k+r} + (-1)^{tk}l_{2pk}l_r.$$

It follows identities (1) and (4) that

$$g_{(2pn+t)k}g_{(2pn+t-2p)k} - g^2_{(2pn+t-p)k} = \begin{cases}(-1)_{tk+1}f^2_{pk}, & \text{if } g_n = f_n\\ (-1)^{tk}\Delta^2 f^2_{pk}, & \text{otherwise;}\end{cases} \tag{5}$$

$$g^2_{(2\ pn+t)k+r}g^2_{(2\ pn+t-2p)k} - g^2_{(2pn+t)k}g^2_{(2pn+t-2p)k+r} = \begin{cases}\frac{(-1)^{tk+1}}{\Delta^2}Af_{2pk}f_r, & \text{if } g_n = f_n\\ (-1)^{tk}\Delta^2 Bf_{2pk}f_r, & \text{otherwise,}\end{cases} \tag{6}$$

respectively, where $k, p, r,$ and t are positive integers and $t \leq 2p$ [6].

2. A Telescoping Gibonacci Sum

Using recursion, we established the following telescoping sum in [6]. In the interest of brevity, we omit its proof here.

Lemma 1: *Let k, p, r, t , and λ be positive integers, where $t \leq 2p$. Then*

$$\sum_{n-1}^{\infty}\left[\frac{g^\lambda_{(2pn+t-2p)k+r}}{g^\lambda_{(2pn+t-2p)k}} - \frac{g^\lambda_{(2pn+t)k+r}}{g^\lambda_{(2pn+t)k}}\right] = \frac{g^\lambda_{tk+r}}{g^\lambda_{tk}} - \alpha^{\lambda r}. \tag{7}$$

3. A Family of Gibonacci Sums

This lemma, coupled with identities (5) and (6), played a major role in the development of the following theorem. To present it in a concise fashion, we now let:

$$\mu = \begin{cases}1, & if\ g_n = f_n\\ \Delta^2, & otherwise;\end{cases} \qquad \mu^* = \begin{cases}\frac{1}{\Delta^2}, & if\ g_n = f_n\\ \Delta^2, & otherwise;\end{cases}$$

and $$\nu = \begin{cases}-1, & if\ g_n = f_n\\ 1, & otherwise.\end{cases}$$

These tools served as building blocks in the development of the theorem [6].

Theorem 1: *Let* $k, p, r,$ *and* t *be positive integers, where* $t \le 2p$. *Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} \mu^* [2l_{2(2pn+t-p)k+r} + (-1)^{tk} \nu l_{2pk} l_r] f_{2pk} f_r}{[g^2_{(pn+t-p)k} + (-1)^{tk} \mu\nu f^2_{pk}]^2} = \frac{g^2_{tk+r}}{g^2_{tk}} - \alpha^{2r}. \qquad (8)$$

The objective of our discourse is to confirm this result using graph-theoretic techniques. To this end, we now present the needed tools.

4. Graph-Theoretic Tools

Consider the Fibonacci digraph in Figure 1 with vertices v_1 and v_2, where a weight is assigned to each edge [2, 5]. It follows from its *weighted adjacency matrix* $Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ that

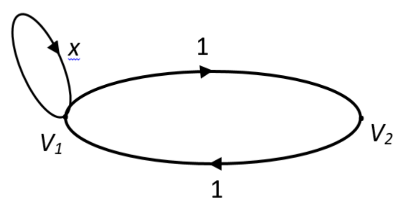

Figure 1: Weighted Fibonacci Digraph

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $n \ge 1$ [2, 3, 4, 5]. We extend the exponent n to 0, which is consistent with the *Cassini-like formula* $f_{n+1} f_{n-1} - f_n^2 = (-1)^n$, where $f_{-1} = 1$ [2, 5].

A *walk* from vertex v_i to vertex v_j is a sequence $v_i - e_i - v_{i+1} - \cdots - v_{j-1} - e_{j-1} - v_j$ of vertices v_k and edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The walk is *closed* if $v_i = v_j$; and *open*, otherwise. The *length* of a walk is the number of edges in the walk. The *weight* of a walk is the product of the weights of the edges along the walk.

The ijth entry of Q^n gives the sum of the weights of all walks of length n from v_i to v_j in the weighted digraph, where $1 \le i, j \le n$ [2, 3, 4]. Consequently, the sum of the weights of closed walks of length n originating at v_1 in the digraph is f_{n+1} and that of those originating at v_2 is f_{n-1}. So, the sum of the weights of all closed walks of length n in the digraph is $f_{n+1} + f_{n-1} = l_n$ [2, 5].

Let A and B denote sets of walks of varying lengths originating at a vertex v. Then the sum of the weights of the elements (a, b) in the product set $A \times B$ is defined as the product of the sums of weights from each component [3, 4]. This definition can be extended to any finite number of component sets. In particular, let $A, B, C,$ and D denote the sets of walks of varying lengths originating at a vertex v, respectively. Then the sum of the weights of the elements (a, b, c, d) in the product set $A \times B \times C \times D$ is the product of the sums of weights from each component [3, 4].

We now make an interesting observation. Let $A = \{u\}$ and $B = \{v\}$, where u denotes the closed walk $v_1 - v_1$ and v denotes the closed walk $v_1 - v_2 - v_1$. The weight of the element (u, u) in $A \times A$ is x^2, and that in $B \times B$ is 1. Consequently, the sum w of the weights of the elements in $C^* = (A \times A) \cup (B \times B) \cup (B \times B) \cup (B \times B) \cup (B \times B)$ is given by $w = x^2 + 4 = \Delta^2$.

These tools play a major role in the discourse. With them at our finger tips, we are now ready for our pursuit of the graph-theoretic confirmation.

5. Graph-Theoretic Confirmation

Let T_n^* denote the set of closed walks of length n in the digraph originating at v_1, and U_n^* the set of all closed walks of the same length n in the digraph. Correspondingly, let T_n denote the sum of the weights of all elements in T_n^*, and U_n that of those in U_n^*. Clearly, $T_n = f_{n+1}$ and $U_n = f_{n+1} + f_{n-1} = l_n$ [2, 5]. With this brief background, we now begin the proof of the gibonacci sum (8) in two cases, where $k, p, r, t \ge 1$ and $t \le 2p$.

Proof: Case 1: Suppose $g_n = f_n$. The sum of the weights of the elements in the product set $T^*_{2pk-1} \times T^*_{r-1}$ is $T_{2pk-1}T_{r-1} = f_{2pk}f_r$; the sum of those in $T^*_{(2pn+t-p)k-1} \times T^*_{(2pn+t-p)k-1}$ is $T^2_{(2pn+t-p)k-1} = f^2_{(2pn+t-p)k}$; and that of those in $T^*_{pk-1} \times T^*_{pk-1}$ is $T^2_{pk-1} = f^2_{pk}$.

We now let

$$S_n = \frac{(-1)^{tk+1}[2U_{2(2pn+t-p)k+r} - (-1)^{tk}U_{2pk}U_r]T_{2pk-1}T_{r-1}}{w[T^2_{(2pn+t-p)k} - (-1)^{tk}T^2_{pk-1}]^2}$$

$$= \frac{(-1)^{tk+1}[2l_{2(2pn+t-p)k+r} - (-1)^{tk}l_{2pk}l_r]f_{2pk}f_r}{\Delta^2[f^2_{(2pn+t-p)k} - (-1)^{tk}f^2_{pk}]^2}$$

With identities (3) and (4), and Lemma 1, this yields

$$\frac{(-1)^{tk+1}[Af_{2pk}f_r}{\Delta^2[f^2_{(2pn+t-p)k} - (-1)^{tk}f^2_{pk}]^2} = \frac{f^2_{(2pn+t)k+r}f^2_{(2pn+t-2p)k} - f^2_{(2pn+t)k}f^2_{(2pn+t-2p)k+r}}{f^2_{(2pn+t)k}f^2_{(2pn+t-2p)k}}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{tk}[Af_{2pk}f_r}{\Delta^2[f^2_{(2pn+t-p)k} - (-1)^{tk}f^2_{pk}]^2} = \sum_{n=1}^{\infty}\left[\frac{f^2_{(2pn+t-2p)k+r}}{f^2_{(2pn+t-2p)k}} - \frac{f^2_{(2pn+t)k+r}}{f^2_{(2pn+t)k}}\right]$$

$$= \frac{f^2_{tk+r}}{f^2_{tk}} - \alpha^{2r}. \tag{9}$$

We now turn to the next case.

Case 2: Let $g_n = l_n$. Recall that the sum w of the weights of the elements in C^* is given by $w = x^2 + 4 = \Delta^2$, and that of the elements in the product set $C^* \times T^*_{2k-1} \times T^*_{r-1}$ is given by $wT_{2pk-1}T_{r-1} = \Delta^2 f_{2pk}f_r$. The sum of the weights of the elements in the product set $U^*_{(2pn+t-p)k} \times U^*_{(2pn+t-p)k}$ is $U^2_{(2pn+t-p)k} = l^2_{(2pn+t-p)k}$; and that of those in $T^*_{2pk-1} \times T^*_{pk-1}$ is $T^2_{2pk-1} = f^2_{2pk}$.

As above, we now let

$$S_n = \frac{(-1)^{tk+1} w[2U_{2(2pn+t-p)k+r} + (-1)^{tk} U_{2pk} U_r] T_{2pk-1} T_{r-1}}{[U^2_{(2pn+t-p)k} + (-1)^{tk} w T^2_{pk-1}]^2}$$

$$= \frac{(-1)^{tk+1} \Delta^2 [2l_{2(2pn+t-p)k+r} + (-1)^{tk} l_{2pk} l_r] f_{2pk} f_r}{[l^2_{(2pn+t-p)k} + (-1)^{tk} \Delta^2 f^2_{pk}]^2}.$$

It then follows by identities (3) and (4), and Lemma 1 that

$$\frac{(-1)^{tk+1} \Delta^2 B f_{2pk} f_r}{[l^2_{(2pn+t-p)k} + (-1)^{tk} \Delta^2 f^2_{pk}]^2} = \frac{l^2_{(2pn+t)k} l^2_{(2pn+t-2p)k+r} - l^2_{(2pn+t)k+r} l^2_{(2pn+t-2p)k}}{l^2_{(2pn+t)k} l^2_{(2pn+t-2p)k}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} \Delta^2 B f_{2pk} f_r}{[l^2_{(2pn+t-p)k} - (-1)^{tk} \Delta^2 f^2_{pk}]^2} = \sum_{n=1}^{\infty} \left[\frac{l^2_{(2pn+t-2p)k+r}}{l^2_{(2pn+t-2p)k}} - \frac{l^2_{(2pn+t)k+r}}{l^2_{(2pn+t)k}} \right]$$

$$= \frac{l^2_{tk+r}}{l^2_{tk}} - \alpha^{2r}. \tag{10}$$

This equation, coupled with equation (9), yields Theorem 1, as desired. □

Interestingly, equation (9) can be rewritten in terms of graph-theoretic tools.

To realize this goal, we define $T_0 = 1$, $U_0 = 2$; $H_n = T_n$ or U_n;

$$\mu = \begin{cases} 1, & if\ H_n = T_n \\ w, & otherwise; \end{cases} \qquad \mu^* = \begin{cases} \frac{1}{w}, & if\ H_n = T_n \\ w, & otherwise; \end{cases}$$

$$\nu = \begin{cases} -1, & if\ H_n = T_n \\ 1, & otherwise; \end{cases} \qquad \nu^* = \begin{cases} 1 & if\ H_n = T_n \\ -1 & otherwise; \end{cases}$$

$$\nu' = \begin{cases} -1, & if\ H_n = T_n \\ 0, & otherwise. \end{cases}$$

With these new tools, and integers $k, p, r,$ and t as before, we now present the graph-theoretic version of equation (8):

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk}\mu^*\nu^*[2U_{2(2pn+t-p)k+r} + (-1)^{tk}\nu U_{2pk}U_r]T_{2pk-1}T_{r-1}}{[H^2_{(2pn+t-p)k-\nu} + (-1)^{tk}\mu\nu T^2_{pk-1}]^2} = \frac{H^2_{tk+r+\nu'}}{H^2_{tk+\nu'}} - \alpha^{2r}. \tag{11}$$

Next, we turn to the Pell implications of the graph-theoretic techniques.

6. Pell Consequence

With the gibonacci-Pell relationship $b_n(x) = g_n(2x)$, we can construct the graph-theoretic proof of the Pell version of Theorem 1 independently by changing the weight of the loop at v_1 from x to $2x$. We encourage gibonacci enthusiasts to explore this path.

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INVESTIGATING THE EXISTENCE OF THE EXPONENTIAL DIOPHANTINE RECTANGLES OVER STAR AND PRONIC NUMBERS

Abstract: This paper is focused on collecting a different sort of rectangle called Exponential Diophantine Rectangles over Star and Pronic Numbers. We demonstrated that there is only one Exponential Diophantine Rectangle over the *Star numbers* and no Exponential Diophantine rectangles over the Pronic numbers. Python programming is provided for the existence of such rectangles.

Keywords: Binomial Expansion, Catalan's Conjecture, Diophantine Equation, Exponential Diophantine Equation, Star numbers, Pronic numbers, Rectangles.

Mathematics Subject Classification (2020) No.: 11A07, 11D61, 11D72.

1. Introduction

An exponential Diophantine equation is a special type of Diophantine equation where the variables exist in exponents. Many authors solved the different forms of exponential Diophantine equations. In particular, William Sobredo Gayo, Jr. and Jerico Bravo Bacani [12] solved the exponential Diophantine equation of the form $M_p^x + (M_q + 1)^y = z^2$ and Mahalakshmi, M. *et al*. [5], [6], and [7] solved various Diophantine equations to collect various geometrical shapes, including peble triangles and almost equilateral triangles.

We define and collect the exponential Diophantine rectangle over Special numbers (ED Rectangles over Special numbers), inspired by the above. In this paper we deal only with two types of special numbers especially star and Pronic numbers. After the introduction basis preliminaries provided. In section (3), the definition of ED Rectangles over Star numbers (S_m) and the lemmas needed for the main theorem, while subsections establish theorems for the existence of solutions to the Exponential Diophantie equations. Python code is displayed for the existence of such rectangles. Section (3) provides the definition theorems and Python programming for the existence of exponential Diopahantine Rectangles over Pronic numbers.

2. Preliminaries

This section contains basic definitions and lemmas required for this article.

Lemma 2.1 (*Catalan's Conjecture*)**:** $(3, 2, 2, 3)$ *is the unique solution for the exponential Diophantine equation* $a^x - b^y = 1$, *where* $a, b, x, y \in \mathbb{Z}$ *such that* $\min\{a, b, x, y\} \geq 2$.

Definition 2.1 (*Binomial Expansion*)**:** *For* $x \in \mathbb{Z}$ *and* $n \in \mathbb{N}$, *the expansion for* $(1 + x)^n$ *is* $1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$ *and for* $(1 - x)^n$ *is* $1 - nx + \frac{n(n-1)}{2!}x^2 - \ldots$.

In general, $(x + y)^n = \sum_{k=0}^{n} nC_k x^{n-k} y^k = \sum_{k=0}^{n} nC_k x^k y^{n-k}$.

Definition 2.2: *The* m^{th} *Star number* (S_m) *is given by* $S_m = 6m^2 - 6m + 1$, *for* $m \in \mathbb{N}$.

Example 1: $S_1 = 1$, $S_{10} = 541$.

Definition 2.3: *The* m^{th} *Pronic numbers* (P_m) *are of the form* $P_m = m^2 + m$, *for all* $m \in \mathbb{N}$:

Definition 2.4: *An Exponential Diophantine rectangle is defined as a rectangle with the length* (l) *and breadth* (b) *as* $(l, b) = (rp + (r - 1)(x + y), q(r - 1) + (r + 2)(x + y))$ *where* $p, q, r \in \mathbb{N}$ *and* $x; y$ *are non-negative integers such that*

$$p^x \pm q^x = r^y \tag{1}$$

3. Exponential Diophantine Rectangles over Star Numbers

This section defines Exponential Diophantine Rectangles over S_m, provides some lemmas for solving exponential Diophantine equations, and is divided into two subsections that examine some theorems. The existence of Exponential Diophantine Rectangles over S_m is proved by python programming with certain limits.

Definition 3.1 ($Exponential\ Diophantine\ Rectangles\ over S_m$): *An Exponential Diophantine rectangle over star numbers* (S_m) *is defined as a rectangle with the length* (l) *and breadth* (b) *as* $(l, b) = ((m+1)S_{m+1} + m(x+y),\ mS_m + (m+1)(x+y))$ *where* $m \in \mathbb{N}$ *and* x, y *are non-negative integers such that*

$$S_{m+1}^x + S_m^x = (m+1)^y \tag{2}$$

or

$$S_{m+1}^x - S_m^x = (m+1)^y \tag{3}$$

Notation: Exponential Diophantine Rectangles over star numbers - ED Rectangles over S_m.

Lemma 3.1: *The inequality* $(1+n)^x > 2$ *holds for all* $n, x > 1$.

Proof: Let us show this by using induction hypothesis on n. Now, for $n = 2$, the inequality becomes $3^x > 2$. This is true for $x > 1$. Now, assume that the inequality $(1+n)^y > 2$ holds for $n = k$. That is, $(k+1)^y > 2$ for $k > 1$. We have to show that the given inequality holds for $n = k+1$. We know that $(k+2) > (k+1)$ implies $(k+2)^y > 2$, for all $k, y > 1$. □

Lemma 3.2: *The equation* $b^4 - 6b^2 + 4b^3 + 4b - 1 = 0$ *has no positive integer solution for* $b > 1$.

Proof: The equation $b^4 - 6b^2 + 4b^3 + 4b - 1 \equiv 2, 3 \pmod 4$, which is an absurd one. □

Lemma 3.3: *If* $y > 2$, *the inequality* $(1 + n)^y > 4n$ *holds for all* $n > 1$.

Lemma 3.4: *The equation* $m^{12} + 12m^{11} + 66m^{10} + 220m^9 + 495m^8 + 792m^7 + 924m^6 + 792m^5 + 495m^4 + 220m^3 + 66m^2 + 1 = 0$ *has no solution* $\forall m \in \mathbb{N}$.

Proof: *The equation* $m^{12} + 12m^{11} + 66m^{10} + 220m^9 + 495m^8 + 792m^7 + 924m^6 + 792m^5 + 495m^4 + 220m^3 + 66m^2 + 1 \quad 0 \pmod{10}$

Hence it has no solution. □

3.1 The exponential Diophantine equation $S^x_{m+1} + S^x_m = (m + 1)^y$:

The existence of the solution to the equation $S^x_{m+1} + S^x_m = (m + 1)^y$ is discussed here.

Theorem 3.1: *The exponential Diophantine equation* $S^x_{m+1} + S^x_m = (m + 1)^y$ *has only one solution* $(x, y, m) = (0, 1, 1)$ *for all* $x, y \in \mathbb{Z}^+ \cup \{0\}$ *and* $m \in \mathbb{N}$.

Proof: Consider the exponential Diophantine equation $S^x_{m+1} + S^x_m = (m + 1)^y$ and solve this in various possibilities.

Possibility 1: For $x = 0$ and $y = 0$, the equation (2) has no solution.

Possibility 2: If $x = 0$ and $y = 1$, then we obtain $m = 1$ from the exponential Diophantine equation.

Possibility 3: If $x = 1$ and $y = 0$, the equation becomes $12m^2 = -1$ which has no solution as $m \in \mathbb{N}$.

Possibility 4: For $x = y = 1$, it reduces to a quadratic equation $12m^2 - m - 1 = 0$. On solving this we obtain $m = \frac{8}{24} \notin \mathbb{N}$.

Possibility 5: When $x = 0$ and $y > 1$, the equation becomes $(m + 1)^y = 2$. This is not possible by lemma (3.3).

Possibility 6: For $x > 1$ and $y = 0$, the equation becomes $(6m^2 + 6m + 1)^x + (6m^2 - 6m + 1)^x = 1$. Here $S_m^x + S_{x+1}^m \equiv 0 \pmod 2$ which is an absurd one.

Possibility 7: For $x = 1$ and $y > 1$, we get $12m^2 + 2 = (m + 1)^y$. By using definition (2.1) and equating the coefficient of m^2, we get the value $y = 4$. Putting the value of y in $12m^2 + 2 = (m + 1)^y$ implies $(m + 1)^4 = 12m^2 + 2$. By lemma (3.2), it has no solutions.

Possibility 8: Now $x > 1$ and $y = 1$, the equation changes into $S_m^x + S_{m+1}^x - m - 1 = 0$. For $m = 1$, it reduces to $13^x = 1$, which is an impossible one as $x > 1$. Now, $m > 1$ for, $S_m + S_{m+1} < m + 1 \Rightarrow 12m^2 - m + 1 < 0$. This is not possible as $m > 1$.

Possibility 9: Here $x, y > 1$ and $m = 1$, the equation has no solution by lemma (2.1). Now for $x, y, m > 1$, this possibility fails by using definition (2.1).

Hence, there is only one solution for the Diophantine equation $S_{m+1}^x + S_m^x = (m + 1)^y$ (i.e.,$(x, y, m) = (0, 1, 1)$) . □

3.2 The exponential Diophantine equation $S_{m+1}^{x} - S_{m}^{x} = (m+1)^{y}$: This subsection provides the theorem for determining the solution for the equation $S_{m+1}^{x} + S_{m}^{x} = (m+1)^{y}$.

Theorem 3.2: *No integral solution exists for the exponential Diophantine equation* $S_{m+1}^{x} - S_{m}^{x} = (m+1)^{y}$ $\forall x,\ y \in \mathbb{Z}+ \cup \{0\}$ *and* $m \in \mathbb{N}$.

Proof: Consider the exponential Diophantine equation $S_{m+1}^{x} + S_{m}^{x} = (m+1)^{y}$ and solve this in various possibilities.

Possibility 1: For $x = 0$ and $y = 0$, this possibility fails.

Possibility 2: $x = 0$ and $y = 1$, the equation becomes $m = -1$, which contradicts.

Possibility 3: Here $x = y = 1$, then on solving the above equation, the value of m obtained as $\frac{1}{11} \notin \mathbb{N}$.

Possibility 4: For $x = 1$ and $y = 0$, we obtain $m \notin \mathbb{N}$.

Possibility 5: Now $x = 0$ and $y > 1$, then from the equation we obtain $(m+1)^{y} = 0$, as $m \geq 1$. This possibility fails.

Possibility 6: When $x > 1$ and $y = 0$, the equation becomes $S_{m+1}^{x} - S_{m=1}^{x}$. But $S_{m+1}^{x} - S_{m}^{x} \equiv 1 (\mathrm{mod}\ 2)$. This is not possible.

Possibility 7: For $x = 1$ and $y > 1$, now by the definition (2.1) we get $y = 12$. If $y = 12$, then this possibility fails by lemma (3.4).

Possibility 8: When $y = 1$ and $x > 1$, the equation reduces into $m + 1 > S_{m+1} - S_{m}$ which implies $11m < 1$.

Possibility 9: $x > 1$ and $y > 1$, This possibility also fails by Binomial Expansion.

Therefore the given equation has no solution. □

Theorem 3.3: $(27, 1)$ *is the only one ED Rectangle over* S_m .

Proof: By the theorem (3.1) and (3.2), there exists only one (x, y, z) and so there only one ED Rectangle exists over S_m . □

3.3 Python Programing for Existence of ED Rectangles over S_m **:** In this section, we provided the python programming for the existence and non existence of the ED Rectangles over S_m

```
# ED rectangle over Star number
 import math
 def rectangle ():
  print ('x\ty\tn\tSm\tSn\t(l,b)')
    for x in range (0, m + 1) :
     for y in range (0, m + 1) :
      for n in range (1, m + 1) :
      Sm = 6* n **2 + 6* n + 1
      Sn = 6* n **2 - 6* n + 1
      l = Sm *(n + 1) + n*(x + y)
      b = n*Sn + (n -1) *(x + y)
      if (Sm)**x + ( Sn)**x == ( n + 1) **y :
      print (x,'\t', y,'\t', n,'\t', Sm ,'\t', Sn ,'\t' ,(l, b))
m = int ( input (" Enter the maximum range :"))
# m is the maximum range
rectangle ()
```

Coding 1: Calculating the solution for $S_m^x + S_{m+1}^x = (m+1)^y$

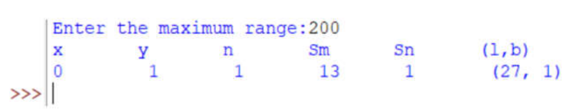

Figure 1: Output: Coding 1

```
#  ED rectangle over Star number
import math
def rectangle ():
 print ('x\ty\tn\tSm\tSn\t(l,b)')
  for x in range (0,m + 1) :
   for y in range (0,m + 1) :
    for n in range (1,m + 1) :
     Sm = 6* n **2 + 6* n + 1
     Sn = 6* n **2 - 6* n + 1
     l = Sm *(n + 1) + n*(x + y)
     b = n*Sn + (n -1) *(x + y)
     if (Sm)**x - ( Sn)**x == ( n + 1) **y :
     print (x,'\t',y,'\t',n,'\t', Sm ,'\t',Sn ,'\t' ,(l,b))
m = int ( input (" Enter the maximum range :"))
#m is the maximum range
rectangle ()
```

Coding 2: Calculating the solution for $S_m^x - S_{m+1}^x = (m+1)^y$

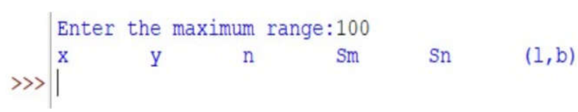

Figure 2: Output: Coding 2

4. Exponential Diophantine Rectangles over Pronic Numbers

This section includes defintion of Exponential Diophantine Rectangles over P_m and also contains three subsections. First two subsections provide the theorems for solving the exponential Diopahntine equations. In the final subsection the python programming is provided for the existence of Exponential Diophantine Rectangles over P_m within a specific limit.

Definition 4.1: *An Exponential Diophantine rectangle over Pronic numbers* P_m *is defined as a rectangle with the length* (l) *and breadth* (b) *as* $(l,b) = (P_{m+1}(m+1) + m(x+y),\ \ mP_m + (m-1)(x+y))$ *where* $m \in \mathbb{N}$ *and* x, y *are non-negative integers such that*

$$P^x_{m+1} + P^x_m = (m+1)^y \tag{4}$$

or

$$P^x_{m+1} - P^x_m = (m+1)^y \tag{5}$$

Notation - Exponential Diophantine Rectangles over Pronic numbers- ED Rectangles over P_m .

4.1 The exponential Diophantine equation $P^x_{m+1} + P^x_m = (m+1)^y$ **:** This subsections contains the theorem for finding the solution for the exponential Diophantine equation $P^x_{m+1} + P^x_m = (m+1)^y$.

Theorem 4.1: *The only solution for the Exponential Diophantine equation* $P^x_{m+1} + P^x_m = (m+1)^y$ *are* $(x, y, m) \in \{(0, 1, 1)\}$ *with* $m \in \mathbb{N}$ *and* $x, y \in \mathbb{Z}^+ \cup \{0\}$.

Proof: Consider the exponential Diophantine equation $P^x_{m+1} + P^x_m = (m+1)^y$ and solve this in various possibilities.

Possibility 1: Whenever $x = y = 0$, there is no possibility.

Possibility 2: Here $x = 0$ and $y = 1$, we obtain $m = 1$.

Possibility 3: Suppose $x = 1$ and $y = 0$, the equation 4 reduced to $12m^2 = 1$ which is an impossible one.

Possibility 4: Now $x = y = 1$, the equation (4) reduces to a quadratic equation $2m^2 + 3m + 1 = 0$ and it is not solvabe over $\mathbb{N}$.

Possibility 5: $x = 0$ and $y > 1$ the equation (4) reduces $(m+1)^y = 2$. For $m = 1$, it becomes $y = 1$ which is contradiction to our assumption. For $m > 1, (m+1)^y$ is always greater than 2. This possibility fails.

Possibility 6: For $y = 0$ and $x > 1$, the equation (4) reduces to $P_{m=1}^x + P_m^x = 1$. Thus, $P_{m+1}^x + P_m^x \equiv 0 \pmod{2}$. We get a contradiction.

Possibility 7: Whenever $x = 1$ and $y > 1$, the equation reduced into $2m^2 + 4m + 2 = (m + 1)^y$. By using Binomial Expansion we get $y = 2$. Now for $y = 2$, the equation reduces into a quadratic equation $2m + 1 = 0$ and it is not solvable.

Possibility 8: Here $y = 1$ and $x > 1$, we have $m + 1 > P_{m+1} + P_m$ implies $1 > m^2 + 3m + 2$. This is an impossible one.

Possibility 9: When $x, y > 1$ Now for $m = 1$, we obtain an impossible one. If $m > 1$ and by using the definition 2.1, the possibility fails.

$\therefore$ The given equation has only one solution. □

4.2 The exponential Diophantine equation $P_{m+1}^x - P_m^x = (m + 1)^y$: This subsection discusses about the solution for the equation $P_{m+1}^x - P_m^x = (m + 1)^y$.

Theorem 4.2: *There is no solution exists for the Exponential Diophantine equation $P_{m+1}^x - P_m^x = (m + 1)^y$ with $m \in \mathbb{N}$ and $x, y \in \mathbb{Z}^+ \cup \{0\}$.*

Proof: Consider the equation $P_{m+1}^x - P_m^x = (m + 1)^y$ and deal with different possibilities.

Possibility 1: Whenever $x = y = 0$, this is one fails.

Possibility 2: Suppose $x = 0$ and $y = 1$, we obtain $m = -1$. This is a contradiction.

Possibility 3: If $x = 1$ and $y = 0$, then $m \notin \mathbb{N}$.

Possibility 4: For $x = y = 1$, we obtain $m = -1$.

Possibility 5: When $x = 0$ and $y > 1$, the equation becomes $(m+1)^y = 0$. This is impossible.

Possibility 6: Now $y = 0$ and $x > 1$, we deal with two possibilities. For $m = 1$, we obtain a contradiction. For $m > 1$, then by lemma (2.1) P_{m+1} and P_m are obtained as 3 and 2 respectively, which contradicts.

Possibility 7: For $x = 1$ and $y > 1$, by using Definition (2.1) we obtain $y = 2$. This is an absurd one.

Possibility 8: Here $x > 1$ and $y = 1$, the equation becomes $m + 1 > 2m + 2$ (not possible).

Possibility 9: $x, y > 1$, When $m = 1$, we get an impossible one and $m > 1$ this possibility fails (by Definition 2.1)

Hence, the Diophantine equation $P_{m+1}^x - P_m^x = (m+1)^y$ has no solution. □

Theorem 4.3: *There is no ED Rectangles over Pronic numbers exists.*

Proof: By the above two theorems (4.1) and (4.2), we get the side of ED rectangle over P_m is (13, 0), this is not possible and conclude that there is no ED rectangles over Pronic numbers.

4.3 Python Programing for Existence of ED Rectangles over P_m: The Python programming for the existence and nonexistence of the exponential Diophantine equation solution is provided in this part; however, the ED rectangle over P_m, does not exist.

```
#ED rectangle over Pm
import math
def rectangle ():
  print ('x\ty\tm\tPm\tPn\t(l,b)')
   for x in range (0, n + 1) :
    for y in range (0, n + 1) :
```

```
  for m in range (1, n + 1) :
  Pm = (m + 1) *(m + 2)
  Pn = m*(m - 1)
  l = Pm *(m + 1) +m*(x + y)
  b = m*Pn + (m - 1) *(x + y)
  if (Pm)**x + ( Pn)**x ==( m + 1) **y :
    print (x,'\t',y,'\t',m,'\t', Pm ,'\t',Pn ,'\t' ,(l,b))
n = int ( input (" Enter the maximum range :"))
#n is the maximum range
rectangle ()
```

Coding 3: Calculating the solution for $P_{m+1}^{x} + P_{m}^{x} = (m+1)^{y}$

```
Enter the maximum range:200
x       y       m       Pm      Pn      (l,b)
0       1       1       6       0       (13, 0)
>>>
```

Figure 3: Output: Coding 3

```
#ED rectangle over Pm
import math
def rectangle ():
 print ('x\ty\tm\tPm\tPn\t(l,b)')
  for x in range (0,n +1) :
   for y in range (0,n +1) :
    for m in range (1,n +1) :
     Pm =(m +1) *(m +2)
     Pn=m*(m -1)
     l=Pm *(m +1) +m*(x+y)
     b=m*Pn +(m -1) *(x+y)
     if (Pm)**x -( Pn)**x == ( m +1) **y :
       print (x,'\t',y,'\t',m,'\t', Pm ,'\t',Pn ,'\t' ,(l,b))
n = int ( input (" Enter the maximum range :"))
#n is the maximum range
rectangle ()
```

Coding 4: Calculating the solution for $P_{m+1}^{x} - P_{m}^{x} = (m+1)^{y}$

```
Enter the maximum range:100
x       y       m       Pm      Pn      (l,b)
>>>
```

Figure 4: Output: Coding 4

Since the sides of the rectangles are positive, but we have $l = 13$ and $b = 0$ and therefore does not exists ED rectangle over P_m .

5. Conclusion

Finally we infer that there exists only one ED Rectangle over Star numbers and there is no ED Rectangles over Pronic numbers. In the future, this could be employed in cryptographic concepts like it helps to develop efficient algorithms. Additionally, it can be applied to furnishings design (making tables, chairs, etc.), building construction, graphic design (such as creating logos), etc. One can also work on these topics using various types of equations.

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DUALITY CRITERIA INVOLVING (p, φ ,d)-INVEXITY AND (p, φ, d) -PSEUODINVEXITY IN INTERVAL -VALUED MULTIOBJECTIVE OPTIMIZATION PROBLEMS

Abstract: In this present research work, study of duality associated with a special class of multiobjective optimization that include the interval valued components is delt. We define (ρ,φ,d)-Invexity and (ρ,φ,d-Pseuodinvexity, which are connected with an interval valued multiple integral functional. For such class of variational problems, we write dual problem associated with primal problem. We prove weak, strong and converse duality theorems for this type of variational problems. A brief comparison with existed methods have been done to show the importance of this research work. Additionally, numerical examples have been displayed at the appropriate places to support the results which shows the significance of our Study.

Keywords: Multiobjective Optimization, (ρ, φ, d)-Invexity and (ρ, φ, d)-Pseuodinvexity, Duality.

Mathematics Subject Classification (2010) No.: 58E17, 65K05, 90C46, 90C29, 26B25, 49K20, 49N15.

1. Introduction and Literature Review

Mathematical optimization problems involving multiple objective functions that must be optimized simultaneously fall under the purview of multi-objective optimization, also known as Pareto optimization or multi-objective programming, vector optimization, multicriteria optimization, or multiattribute optimization. Several scientific domains, such as engineering, economics, and logistics, have used multiobjective vector optimization to make optimal judgments when there are trade-offs between two or more competing objectives. Multi-objective optimization issues with two or three objectives include things like optimizing performance while limiting fuel consumption and vehicle emissions, and minimizing cost while maximizing comfort while purchasing an automobile. There may be more than three objectives in practical tasks.

The application of duality theory to more general classes of functions has grown as a result of its success in mathematical programming. Kumar et al. [1] have considered multiobjective semi-infinite variational problem (MSVP) and generalised the concept of inveity. Kumar *et al*. [2] defined certain conditions on the functionals of multi-objective fractional variational problem in order that it becomes F-Kuhn Tucker pseudo invex or F-Fritz John pseudo invex. Bhardwaj and Ram [3] established the relationships between a class of interval-valued vector optimization problems and interval-valued vector variational-like inequality problems of both Stampacchia and Minty kinds in terms of convexificators.

Upadhyay *et al*. [4] dealt with a certain class of multiobjective semi-infinite programming problems with switching constraints (in short, MSIPSC) in the framework of Hadamard manifolds. Sahay and Bhatia [5] introduced new classes of higher order generalized strong invex functions under non-differentiable settings.

Soni *et al*. [6] discussed optimization problems with multiobjective functions and their applications in engineering field.

Zalmai [7] established global semiparametric sufficient efficiency results under various generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity assumptions for a multiobjective fractional subset programming problem. Hachimi and Aghezzaf [8] generalized a fairly large number of sufficient optimality conditions and duality results previously obtained for multiobjective variational problems. Treanţă [9] introduced necessary efficiency conditions for a class of multi-time vector fractional variational problems with nonlinear equality and inequality constraints involving higher-order partial derivatives. Treanţă [10] introduced a generalised condition on the functionals involved in a multidimensional vector control problem and prove that a (strongly) *b*-V-KT-pseudoinvex multidimensional control problem is characterized so that all Kuhn-Tucker points are efficient solutions. Kim [11] formulated duality for nondifferentiable multiobjective variational problems and established the weak, strong, and converse duality theorems under generalized (F, ρ)-convexity assumptions. Gulati *et al*. [12] obtained Fritz John and Kuhn-Tucker type necessary optimality conditions for a Pareto optimal (efficient) solution of a multiobjective control problem are by first reducing the multiobjective control problem to a system of single objective control problems, and then using already established optimality conditions. Nahak and Nanda [13] presented the sufficient optimality criteria for a class of multiobjective variational control problems under the V-invexity assumption. They also proved duality results under a variety of V-invexity assumptions.

Antczak and Jiménez [14] generalized the notion of B-(p, r)-invexity and proved sufficient optimality conditions under the assumptions that the functions constituting them are B-(p, r)-invex. Antczak [15] extended the notions of (Φ, ρ)-invexity and generalized (Φ, ρ)-invexity to the continuous case and we use these concepts to establish sufficient optimality conditions for the considered class of nonconvex multiobjective variational control problems and established several mixed duality results are under (Φ, ρ)-invexity. Khazafi *et al*. [16] introduced the classes of (B, ρ)-type I and generalized (B, ρ)-type I, and derived various sufficient optimality conditions and mixed type duality results for multiobjective control problems under (B, ρ)-type I and generalized (B, ρ)-type I assumptions. Zhang *et al*. [17] extended the vector-valued G-invex functions to multiobjective variational control problems, by using this concept, a number of sufficient optimality results and Mond-Weir type duality results were obtained for multiobjective variational control programming problem. Treanţă and Arana [18] defined a Kuhn-Tucker (KT)-pseudoinvex multidimensional control problem and introduced a new condition on the functions, which were involved in a multidimensional control problem proved that a KT-pseudoinvex multidimensional control problem is characterized such that a KT point is an optimal solution. Mititelu [19] established necessary conditions for normal efficient solutions of a class of multiobjective fractional variational problem (MFP) with nonlinear equality and inequality constraints using a parametric approach to relate efficient solutions of fractional problems and a non-fractional problem and established the sufficiency of these conditions for efficiency solutions in problem (MFP) using the (ρ, b)-quasiinvexity notion.

Mititelu and Treanţă [20] formulated and proved necessary and sufficient optimality conditions in multiobjective control problems which involve multiple integral and under (ρ, b)-quasiinvexity assumptions, sufficient efficiency conditions for a feasible solution were derived. Treanţă and Mititelu [21] introduced several results of duality for a class of multiobjective fractional control problems involving multiple integrals and under (ρ, b)-quasiinvexity assumptions, they formulated and prove weak, strong and converse duality results. Treanţă [22] formulated and proved efficiency conditions for the considered uncertain variational control problem and established sufficiency of Karush-Kuhn-Tucker conditions under some invexity and (ρ, b)-quasiinvexity assumptions of the involved functionals. Treanţă [23] formulated and proved weak, strong, and converse duality results for the considered class of variational control problems by using the new notion of (ρ,ψ,d)-quasiinvexity associated with an interval-valued multiple-integral functional. Treanţă [24] investigated some connections between an LU-optimal solution of a variational control problem governed by interval-valued multiple integral functional and a saddle-point associated with an LU-Lagrange functional corresponding to a modified interval-valued variational control problem.

In contrast to earlier studies, the current work addresses the duality study related to a novel class of multiobjective optimization problems that involve interval-valued ratio vector components. When taken into account simultaneously, these three emphasized components are completely novel in the relevant literature. Additionally, numerical example is given to show how useful the conclusions drawn in the study are.

The following table compares our study with the available literature in this field

Research Article	**Mutliobjective Optimization**	**Invexity and Pseudoinvexity**	**Inverval Valued Components**	**Duality Criteria**
Kumar *et al*. [1]	Yes	Generalised Invexity	No	No
Bhardwaj *et al*. [3]	No	Generalised Approximate Invexity	Yes	No
Upadhyay *et al*. [4]	Yes	No	No	Yes
Hachimi and Aghezzaf [8]	Yes	No	No	Yes
Kim [11]	Yes	No	No	Yes
Gulati *et al*. [12]	Yes	No	No	Yes
Nahak and Nanda [13]	Yes	V-Invexity	No	Yes
Antczak and Jiménez [14]	Yes	B-(p, r)-Invexity	No	Yes
Antczak [15]	Yes	No	No	Yes
Khazafi *et al*. [16]	Yes	Yes	No	No
Mititelu [19]	Yes	No	No	No
Treanţă and Mititelu [21]	Yes	(ρ, b)-Q uasiinvexity	No	Yes
Treanţă [23]	Yes	(ρ, ψ, d)-Quasiinvexity	Yes	Yes
Treanţă [24]	Yes	(p, b, d)-Invexity	Yes	No
Our Proposed Paper	**Yes**	**Both (ρ,φ,d)-Invexity and (ρ, φ,d)-Pseuodinvexity**	**Yes**	**Yes**

In the field of multiobjective optimization, somewhere invexity or pseudoinvexity were discussed, somewhere mutliobjective optimization with interval valued components were discussed, while somewhere duality results were discussed. To the best of our knowledge, all four components simultaneously with (ρ,φ,d)-Invexity and (ρ, φ, d)-Pseudinvexity were not discussed, so there was a research gap in this field.

The structure of the paper is as follows: The problem formulation, preliminary mathematical tools, and notations are included in Section 2 of this article. The key findings are presented in Section 3 of this document. Results for Mond-Weir weak, strong, and converse duality are developed and demonstrated for the recently introduced category of multiobjective optimization problems. The paper is finally concluded in Section 4.

2. The formulation of Problem and Notations

This part presents the definitions, notations, and preliminary findings that will be utilized in the follow-up. Given this, we take into account:

Let us assume **Ω** be a compact domain which is a subset of Euclidean space $\mathbb{R}^m$ and a point in this compact domain **Ω** is represented by $t = (t^\alpha)$ where $\alpha = 1, 2 \ldots \ldots m$.

Now, following continuous differentiable functions are defined

$X = (X_\alpha^i) : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^{mn}$ where $i = 1, 2 \ldots .n$ and $\alpha = 1, 2 \ldots \ldots m$

$Y = (Y_1, Y_2 \ldots \ldots Y_q) = (Y_\beta) : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^q$ where $\beta = 1, 2 \ldots .q$

It is assumed that the functions that are continuously differentiable

$X_\alpha = (X_\alpha^i) : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^{mn}$ where $i = 1, 2 \ldots .n$ and $\alpha = 1, 2 \ldots \ldots m$

Satisfy the complete integrability conditions (closeness conditions)

$$D_\tau X_\alpha^i = D_\alpha X_\tau^i \text{ where } \alpha \neq \tau, \quad \alpha, \tau = 1, 2 \ldots \ldots m \text{ and } i = 1, 2 \ldots \ldots n$$

Where D_τ represent the total derivative operator.

If we consider any two vectors $d = (d_1, d_2.....d_s)$ and $e = (e_1, e_2.....e_s)$ in $\mathbb{R}^s$, then following partial ordering is used

$$d = e \Leftrightarrow d_r = e_r, \qquad d \leq e \Leftrightarrow d_r \leq e_r,$$

$$d < e \Leftrightarrow d_r < e_r, \qquad d \preceq e \Leftrightarrow d_r \leq e_r, d_r \neq e_r, r = 1, 2.........s$$

Now let us assume that K is the set of all closed and bounded real intervals, we represent a closed and bounded interval by $F = [f^L, f^U]$, where f^L and f^U are the lower and upper bounds of F, respectively. The interval operations covered in this paper can be carried out in the following ways:

(1) $F = G \Rightarrow f^L = g^L$ and $f^U = g^U$;

(2) if $f^L = f^U = f$ then $F = [f, f] = f$;

(3) $F + G = [f^l + g^L, f^U + g^U]$;

(4) $-F = -[f^L, f^U] = [-f^L, -f^U]$;

(5) For any $h \in R, h + F = \{h + f^L, h + f^U]$;

(6) For any $h \in R$ and $h \geq 0, hF = [hf^L, hf^U]$;

(7) For any $h \in R$ and $h < 0, hF = [hf^U, hf^L]$;

(8) $F - G = [f^L - g^L, f^U - g^U]$;

(9) $F/G = [f^L/g^L, f^U/g^U]$, where $g^L, g^U > 0$.

Now we have some following definitions

Definition 1: If F and G are two closed and bounded real intervals, i.e. $F, G \in K$, then we have

$$F \leq G \Leftrightarrow f^L \leq g^U \text{ and } f^U \leq g^U$$

Definition 2: If F and G are two closed and bounded real intervals, i.e. $F, G \in K$, then we have

$$F < G \Leftrightarrow f^L < g^U \text{ and } f^U < g^U$$

Definition 3: Interval valued Functions

If we define a function f from $\Omega \times \mathbb{R}^n \times \mathbb{R}^k$ to K, i.e. $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \to K$ such that

$$f(t, b(t), c(t)) = [f^L(t, b(t), c(t)), f^U(t, b(t), c(t))], \text{ where } t \in \Omega$$

Where both $f^L(t, b(t), c(t))$ and $f^U(t, b(t), c(t))$ are real valued functions and the condition $f^L(t, b(t), c(t)) \leq f^U(t, b(t), c(t)) \forall t \in \Omega$ is satisfied , then f is said to be an interval valued function.

The following (per Mititelu and Treantă [19], and Treantă [21]) was used to formulate and demonstrate the primary findings of this work, now we are going to introduce (ρ, φ, d)-Invexity and (ρ, φ, d)-Pseudoinvexity with the help of functional which is interval valued multiple integral.

For this first we consider an interval-valued function which is continuously differentiable

$$h : \Omega \times R^n \times R^{mn} \times R^k \to K \text{ such that}$$

$$h = h(t, b(t), b_\alpha(t), c(t)) = [h^L(t, b(t), b_\alpha(t), c(t)), h^U(t, b(t), b_\alpha(t), c(t))]$$

Where $b_\alpha(t)$ represents partial derivative of $b(t)$ with respect to t^α i.e. $b_\alpha(t) = \dfrac{\partial b}{\partial t^\alpha}(t)$.

Now for any $b \in B$ and $c \in C$, we define following interval-valued multiple integral functional:

$H : B \times C \to K$ such that

$$H(b, c) = \int_\Omega h(t, b(t), b_\alpha(t), c(t))dt$$
$$= [\int_\Omega h^L(t, b(t), b_\alpha(t), c(t))dt, \int_\Omega h^U(t, b(t), b_\alpha(t), c(t))dt]$$

If ρ is a real number and $\varphi : B \times C \times B \times C \to [0, \infty)$ be a positive functional and $(d(b, c), (b^0, c^0))$ is a real valued function defined on $(B \times C)^2$.

Definition 4: (ρ, φ,d)-Invexity and (ρ, φ,d)-Pseudoinvexity

(i) Now if there exists a functional such that

$$\xi : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \text{ such that}$$

$\xi = \xi(t, b(t), c(t), b^0(t), c^0(t)) = (\xi_i(t, b(t), c(t), b^0(t), c^0(t)))$ where $i = 1, 2 \ldots . n$, of the C^1 class functional with $\xi(t, b(t), c(t), b^0(t), c^0(t)) = 0,\ \forall t \in \Omega,\ \xi|_{\partial\Omega} = 0$, and another functional such that

$$\eta : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^K \quad \text{such that}$$

$\eta = \eta(t, b(t), c(t), b^0(t), c^0(t)) = (\eta_j(t, b(t), c(t), b^0(t), c^0(t)))$ where $j = 1, 2 \ldots . k$, of the C^0 class function with $\eta(t, b(t), c(t), b^0(t), c^0(t)) = 0,\ \forall t \in \Omega,\ \eta|_{\partial\Omega} = 0$ such that for each $(b, c) \in B \times C$:

$$\begin{aligned} H(b, c) \le H(b^0, c^0) \\ \Rightarrow\ & \varphi(b, c, b^0, c^0)\int_\Omega [h_b^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_b^U(t, b^0(t), b_\alpha^0(t), c^0(t))]\xi dt \\ & +\varphi(b, c, b^0, c^0)\int_\Omega [h_{b_\alpha}^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_{b_\alpha}^U(t, b^0(t), b_\alpha^0(t), c^0(t))]D_\alpha\xi dt \\ & +\varphi(b, c, b^0, c^0)\int_\Omega [h_c^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_c^U(t, b^0(t), b_\alpha^0(t), c^0(t))]\eta dt \\ & +\rho\varphi(b, c, b^0, c^0)d^2((b, c), (b^0, c^0)) \le 0 \end{aligned}$$

Or in other words

$$\begin{aligned} & \varphi(b, c, b^0, c^0)\int_\Omega [h_b^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_b^U(t, b^0(t), b_\alpha^0(t), c^0(t))]\xi dt \\ & +\varphi(b, c, b^0, c^0)\int_\Omega [h_{b_\alpha}^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_{b_\alpha}^U(t, b^0(t), b_\alpha^0(t), c^0(t))]D_\alpha\xi dt \\ & +\varphi(b, c, b^0, c^0)\int_\Omega [h_c^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_c^U(t, b^0(t), b_\alpha^0(t), c^0(t))]\eta dt \\ & +\rho\varphi(b, c, b^0, c^0)d^2((b, c), (b^0, c^0)) > 0 \Rightarrow H(b, c) > H(b^0, c^0) \end{aligned}$$

In this case H is called as (ρ, φ, d)-Invex at point $(b^0, c^0) \epsilon B \times C$ with respect to ξ and η.

(ii) Now if there exists a functional such that

$$\xi : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \quad \text{such that}$$

$\xi = \xi(t, b(t), c(t), b^0(t), c^0(t)) = (\xi_i(t, b(t), c(t), b^0(t), c^0(t)))$ where $i = 1, 2 \ldots .n$, of the C^1 class functional with $\xi(t, b(t), c(t), b^0(t), c^0(t)) = 0$, $\forall t \in \Omega$, $\xi|_{\partial\Omega} = 0$, and another functional such that

$$\eta : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^K \quad \text{such that}$$

$\eta = \eta(t, b(t), c(t), b^0(t), c^0(t)) = (\eta_j(t, b(t), c(t), b^0(t), c^0(t)))$ where $j = 1, 2 \ldots .k$, of the C^0 class function with $\eta(t, b(t), c(t), b^0(t), c^0(t)) = 0$, $\forall t \in \Omega$, $\eta|_{\partial\Omega} = 0$ such that for each $(b, c) \neq (b^0, c^0) \in B \times C$:

$$\begin{aligned}
&H(b, c) < H(b^0, c^0) \\
&\quad \Rightarrow \varphi(b, c, b^0, c^0)\int_\Omega [h_b^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_b^U(t, b^0(t), b_\alpha^0(t), c^0(t))]\xi dt \\
&\quad +\varphi(b, c, b^0, c^0)\int_\Omega [h_{b_\alpha}^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_{b_\alpha}^U(t, b^0(t), b_\alpha^0(t), c^0(t))]D_\alpha\xi dt \\
&\quad +\varphi(b, c, b^0, c^0)\int_\Omega [h_c^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_c^U(t, b^0(t), b_\alpha^0(t), c^0(t))]\eta dt \\
&\quad +\rho\varphi(b, c, b^0, c^0)d^2((b, c), (b^0, c^0)) < 0
\end{aligned}$$

Or in other words

$$\begin{aligned}
&\varphi(b, c, b^0, c^0)\int_\Omega [h_b^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_b^U(t, b^0(t), b_\alpha^0(t), c^0(t))]\xi dt \\
&+\varphi(b, c, b^0, c^0)\int_\Omega [h_{b_\alpha}^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_{b_\alpha}^U(t, b^0(t), b_\alpha^0(t), c^0(t))]D_\alpha\xi dt \\
&+\varphi(b, c, b^0, c^0)\int_\Omega [h_c^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_c^U(t, b^0(t), b_\alpha^0(t), c^0(t))]\eta dt \\
&+\rho\varphi(b, c, b^0, c^0)d^2((b, c), (b^0, c^0)) \geq 0 \Rightarrow H(b, c) \geq H(b^0, c^0)
\end{aligned}$$

In this case H is called as (ρ, φ, d)-pseudoinvex at point $(b^0, c^0)\epsilon B \times C$ with respect to ξ and η.

Definition 5: Now if we consider a vector valued continuously differentiable function h such that

$$h : \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \times \mathbb{R}^k \times \to \mathbb{K}^p \quad \text{such that}$$

$$h = h(t, b(t), b_\alpha(t), c(t)) \quad \text{where} \quad r = 1, 2 \ldots\ldots p$$

$$= ([h_1^L(t, b(t), b_\alpha(t), c(t)), h_1^U(t, b(t), b_\alpha(t), c(t))] \ldots\ldots\ldots\ldots\ldots [h_p^L(t, b(t), b_\alpha(t), c(t)), h_p^U(t, b(t), b_\alpha(t), c(t))])$$

Now we define vector multiple integral functional H with the help of above continuously differentiable function

$$H : B \times C \to K^p \text{ such that}$$

$$H(b, c) = \int_\Omega h(t, b(t), b_\alpha(t), c(t))dt$$

$$\left(\left[\int_\Omega h_1^L(t, b(t), b_\alpha(t), c(t))dt, \int_\Omega h_1^U(t, b(t), b_\alpha(t), c(t))dt\right], \ldots \left[\int_\Omega h_p^L(t, b(t), b_\alpha(t), c(t))dt, \int_\Omega h_p^U(t, b(t), b_\alpha(t), c(t))dt\right]\right)$$

Now this vector valued multiple integral functional H is said to be (ρ, φ,d)-Invex or (ρ, φ,d)-Pseudoinvex at point $(b^0, c^0)\epsilon B \times C$ with respect to ξ and η if each of the interval valued component of the vector is (ρ, φ,d)-Invex or (ρ, φ, d)-Pseudoinvex respectively at point $(b^0, c^0)\epsilon B \times C$ with respect to ξ and η.

Now consider a vector valued continuous differentiable function g such that

$$g = (g_1, g_2 \ldots\ldots g_p) \quad \text{where} \quad g_r : \Omega \times R^n \times R^k \times K^p, \qquad r = 1, 2 \ldots\ldots p$$

We may now design a new class of multiobjective variational control problems with interval-valued components that we refer to as Primal Problems

(abbreviated PP for short)

$$\min_{(b,c)}\left\{G(b,c) = \left(\int_{\Omega} g_1(t,b(t),c(t)dt, \ldots\ldots\int_{\Omega} g_p(t,b(t),c(t)dt\right)\right\}$$

subject to

$$\frac{\partial b^i}{\partial t^\alpha}(t) = X^i_\alpha(t,b(t),c(t),\ i = 1,2\ldots.n \text{ and } \alpha = 1,2\ldots\ldots.m \text{ and } t \in \Omega \quad (1)$$

$$Y(t,b(t),c(t)) \leq 0, \quad t \in \Omega \quad (2)$$

$$b(t)|_{\partial\Omega} = \psi(t) = given \quad (3)$$

Now for $r = 1,2\ldots\ldots.p$ we have

$$\int_{\Omega} g_r(t,b(t),c(t)dt = [\int_{\Omega} g^L_r(t,b(t),c(t)dt, g^U_r(t,b(t),c(t)dt]$$

or $$G_r(b,c) = [G^L_r(b,c), G^U_r(b,c)]$$

The set of all feasible solutions in primal problem is defined by

$D = \{(b,c)|\ b \in B$ and $c \in C\}$ satisfying equations (1), (2) and (3).

Definition 6: A feasible solution $(b^0, c^0) \in D$ in primal problem is said to be an LU-optimal solution if there does not exist any $(b,c) \in D$ such that $G(b,c) < G(b^0,c^0)$.

Constrained by certain qualification assumptions, if $(b^0, c^0) \in D$ is an LU-Optimal solution of the variational control, then Treantă [21] and Mititelu and Treantă [19] can be considered. According to this there exists piecewise smooth functions θ, μ and λ, with $\theta(t) = (\theta^L(t), \theta^U(t)), \mu(t) = (\mu^\beta(t))$ and $\lambda(t) = \lambda^\alpha_i(t)$ such that

$$\theta^r_l \frac{\partial g^l_r}{\partial b^i}(t,b^0(t),c^0(t)) + \lambda^\alpha_i(t)\frac{\partial X^i_\alpha}{\partial b^i}(t,b^0(t),c^0(t))$$

$$+ \ \mu^{\beta}(t)\frac{\partial Y_{\beta}}{\partial b^{i}}(t, b^{0}(t), c^{0}(t)) + \frac{\partial \lambda_{i}^{\alpha}}{\partial t^{\alpha}}(t) = 0. \tag{4}$$

Where $i = 1, 2\ldots\ldots\ldots n$ and $l = L, U$.

$$\theta_{l}^{r}\frac{\partial g_{r}^{l}}{\partial c^{j}}(t, b^{0}(t), c^{0}(t)) + \lambda_{i}^{\alpha}(t)\frac{\partial X_{\alpha}^{i}}{\partial c^{j}}(t, b^{0}(t), c^{0}(t))$$

$$+ \ \mu^{\beta}(t)\frac{\partial Y_{\beta}}{\partial c^{j}}(t, b^{0}(t), c^{0}(t)) = 0. \tag{5}$$

Where $j = 1, 2\ldots\ldots\ldots k$ and $l = L, U$.

And $\mu^{\beta}(t)Y_{\beta}(t, b^{0}(t), c^{0}(t)) = 0$ (no summation) $\theta(t), \mu(t) \geq 0$ (6)

for all $t \in \Omega$ except at the point of discontinuities.

Definition 7: For the primal problem an LU-Optimal solution $(b^{0}, c^{0}) \in D$ is called an normal LU-optimal solution if above necessary LU-optimality conditions in equation (4) to (6) are satisfied.

3. Dual problem associated with Primal problem

Suppose that the set $P = \{1, 2\ldots\ldots q\}$ is partitioned into the set $\{P_1, P_2, \ldots\ldots\ldots P_s\}$, where $s < q$. Using the same notations as in Section 2, we relate the next multiobjective variational control problem with interval-valued vector components, known as the Dual Problem (DP), to the above primal problem for $(a, u) \in B \times C$:

$$\min_{(a,u)} \left\{G(a, u) = \left(\int_{\Omega} g_1(t, a(t), u(t)dt, \ldots\ldots\ldots \int_{\Omega} g_p(t, a(t), u(t)dt\right)\right\}$$

subject to

$$\theta_{l}^{r}\frac{\partial g_{r}^{l}}{\partial a^{i}}(t, a(t), u(t)) + \lambda_{i}^{\alpha}(t)\frac{\partial X_{\alpha}^{i}}{\partial a^{i}}(t, a(t), u(t))$$

$$+ \ \mu^{\beta}(t)\frac{\partial Y_{\beta}}{\partial a^{i}}(t, a(t), u(t)) + \frac{\partial \lambda_{i}^{\alpha}}{\partial t^{\alpha}}(t) = 0. \tag{7}$$

Where $i = 1, 2 \ldots\ldots\ldots n$ and $l = L, U$

$$\theta_l^r \frac{\partial g_r^l}{\partial u^j}(t, a(t), u(t)) + \lambda_i^\alpha(t) \frac{\partial X_\alpha^i}{\partial u^j}(t, a(t), u(t)) + \mu^\beta(t) \frac{\partial Y_\beta}{\partial u^j}(t, a(t), u(t)) = 0. \tag{8}$$

Where $j = 1, 2 \ldots\ldots\ldots k$ and $l = L, U$.

$$\lambda_i^\alpha(t) \left[X_\alpha^i(t, a(t), u(t)0 - \frac{\partial b^i}{\partial t^\alpha}(t) \right] \geq 0. \tag{9}$$

And $\mu^{P_\theta}(t) Y_{P_\theta}(t, a(t), u(t)0 = 0$ where $\theta = 1, 2 \ldots\ldots\ldots s$ (10)

Where $\theta = (\theta_l^r) \geq 0, \mu(t) = (\mu^\beta(t)) \geq 0$, $a(t)|_{\partial\Omega} = \psi(t) = given$ $l = L, U$. (11)

And the expression is $\mu^{P_\theta}(t) Y_{P_\theta}(t, a(t), u(t))$ is

$$\mu^{P_\theta}(t) Y_{P_\theta}(t, a(t), u(t)) = \sum_{\beta \epsilon P_\theta} \mu^\beta(t) Y_\beta(t, a(t), u(t))$$

In this section, we prove that, under (ρ, φ, d)-Invexity hypotheses, the multiobjective optimization problems with interval-valued components, Primal Problem and Dual Problem, are a Mond-Weir dual pair. Moreover, keep in mind that γ is the collection of all feasible solutions related to dual problem.

Now we formulate and establish the initial duality result, which is also known as weak duality.

Weak Duality theorem-For any multiobjective variational problem with interval-valued components (Primal Problem), let $(b, c) \in D$ be a feasible solution; similarly, let $(a, u, \theta, \lambda, \mu) \in \gamma$ be a feasible solution for the multiobjective variational problem with interval-valued components (Dual Problem). Furthermore, keep in mind that the following prerequisites are satisfied:

(i) For each r, the functional

$\mathcal{g}_{r,l}^{a,u}(b, c) = \int_{\Omega} g_r(t, b(t), c(t)dt \quad r = 1, 2......p$ and $l = L, U$ is $(\rho^1$, φ, d)-Invex at (a, u) with regard to ξ and η or in other words, each interval-valued multiple-integral functional

$\mathcal{g}_{r}^{a,u}(b, c) = [\mathcal{g}_{r,L}^{a,u}(b, c), \mathcal{g}_{r,U}^{a,u}(b, c)]$, $r = 1, 2......p$ is $(\rho^1$, φ, d)-Invex at (a, u) with regard to ξ and η.

(ii) The functional $X(b, c) = \int_{\Omega} \lambda_i^{\alpha}(t)\left[X_{\alpha}^i(t, b(t), c(t), -\frac{\partial b^i}{\partial t^{\alpha}}(t)\right]dt$ is $(\rho^2$,φ, d)-Invex at (a, u) with regard to ξ and η.

(iii) Each functional

$Y_{\theta}(b, c) = \int_{\Omega} \mu^{Q\theta}(t) Y_{Q\theta}(t, b(t), c(t)dt \quad \theta = 1, 2.......s$ is $(\rho_{\theta}^3, \varphi, d)$-Invex at (a, u) with respect to ξ and η.

(iv) With regard to ξ *and* η, at least one of the functionals provided in (i) to (iii) is $(\rho$, φ, d)-Pseudoinvex at (a, u), where $\rho = \rho_r^1, \rho^2$ and ρ_{θ}^3.

(v) For the given

$\theta_l^r \rho_r^1 + \rho^2 + \sum_{\theta=1}^{s} \rho_{\theta}^3 \geq 0$ where ρ_r^1, ρ^2 and $\rho_{\theta}^3 \in \mathbb{R}$.

Then, supremum of dual problem is less than or equal to the infimum of primal problem.

Proof: The values of primal problem at $(b, c) \in D$ and dual problem at $(a, u, \theta, \lambda, \mu) \in \gamma$ are denoted by $\delta(b, c)$ and $\pi(a, u, \theta, \lambda, \mu)$ respectively. Contrast to the result, if possible, suppose that $\delta(b, c) \leq \pi(a, u, \theta, \lambda, \mu)$.

Now, take into consideration the following non-empty set for $r = 1, 2......p, l = L, U$ and $\theta = 1, 2, ...s$:

$S = \{(b, c) \in B \times C | \mathscr{g}_{r,l}^{a,u}(b, c) \leq \mathscr{g}_{r,l}^{b,c}(a, u), X(b, c) \leq X(a, u), Y_\theta(b, c) \leq Y_\theta(a, u)\}$

Now by using above (i) for $(b, c) \in S$ and $r = 1, 2 \ldots\ldots p$ and $l = L, U$, we have

$$\mathscr{g}_{r,l}^{a,u}(b, c) \leq \mathscr{g}_{r,l}^{b,c}(a, u) \Rightarrow$$

$$\varphi(b, c, a, u)\int_\Omega^\cdot (g_r^l)_a(t, a(t), u(t)\xi dt + \varphi(b, c, a, u)\int_\Omega^\cdot (g_r^l)_u(t, a(t), u(t)\eta dt$$

$$\leq -\rho_r^1\varphi(b, c, a, u)d^2((b, c)(a, u))$$

Now we multiply this by $\theta = \theta_l^r \geq 0$ where $l = L, U$ and take summation over $r = 1, 2 \ldots\ldots p$, we get the following

$$\varphi(b, c, a, u)\int_\Omega^\cdot \theta_l^r(g_r^l)_a(t, a(t), u(t)\xi dt + \varphi(b, c, a, u)\int_\Omega^\cdot \theta_l^r(g_r^l)_u(t, a(t), u(t)\eta dt$$

$$\leq -\theta_l^r\rho_r^1\varphi(b, c, a, u)d^2((b, c)(a, u)). \tag{12}$$

Now since for each $(b, c) \in S$, the inequality $X(b, c) \leq X(a, u)$ satisfies, now according to (ii), we have the following

$$\varphi(b, c, a, u)\int_\Omega^\cdot [\lambda_i^\alpha(t)(X_\alpha^i)_a(t, a(t), u(t))\xi - \lambda^\alpha(t)D_\alpha\xi + \lambda_i^\alpha(t)(X_\alpha^i)_u(t, a(t), u(t))\eta]dt$$
$$\leq -\rho^2\varphi(b, c, a, u)d^2((b, c)(a, u)). \tag{13}$$

Similarly, for each $(b, c) \in S$, the inequality $Y_\theta(b, c) \leq Y_\theta(a, u)$ for $\theta = 1, 2 \ldots\ldots s$ exists, now using (iii) , we have the following

$$\varphi(b, c, a, u)\int_\Omega^\cdot [\mu^{Q\theta}(t)(Y_{Q_\theta})_a(t, a(t), u(t))\xi + \mu^{Q\theta}(t)(Y_{Q_\theta})_u(t, a(t), u(t))\eta]dt$$
$$\leq -\rho_\theta^3\varphi(b, c, a, u)d^2((b, c)(a, u)).$$

Now, taking the summation over $\theta = 1, 2 \ldots\ldots s$, we have

$$\varphi(b, c, a, u)\int_{\Omega}^{\cdot}[\mu^{\beta}(t)(Y_{\beta})_{a}(t, a(t), u(t))\xi + \mu^{\beta}(t)(Y_{\beta})_{u}(t, a(t), u(t))\eta]dt$$

$$\leq -\sum_{\theta=1}^{s}\rho_{\theta}^{3}\varphi(b, c, a, u)d^{2}((b, c)(a, u)) \tag{14}$$

Now, adding equations (12),(13) and (14) and taking condition (iv) under consideration, we have

$$\varphi(b, c, a, u)\int_{\Omega}^{\cdot}\theta_{l}^{r}(g_{r}^{l})_{a}(t, a(t), u(t))\xi dt$$

$$+\varphi(b, c, a, u)\int_{\Omega}^{\cdot}[\lambda_{i}^{\alpha}(t)(X_{\alpha}^{i})_{a}(t, a(t), u(t)) + \mu^{\beta}(t)(Y_{\beta})_{a}(t, a(t), u(t))]dt$$

$$+\varphi(b, c, a, u)\int_{\Omega}^{\cdot}\theta_{l}^{r}(g_{r}^{l})_{u}(t, a(t), u(t))\eta dt$$

$$+\varphi(b, c, a, u)\int_{\Omega}^{\cdot}[\lambda_{i}^{\alpha}(t)(X_{\alpha}^{i})_{u}(t, a(t), u(t)) + \mu^{\beta}(t)(Y_{\beta})_{u}(t, a(t), u(t))]\eta dt$$

$$-\varphi(b, c, a, u)\int_{\Omega}^{\cdot}[\lambda^{\alpha}(t)D_{\alpha}\xi]dt < -\left(\theta_{l}^{r}\rho_{r}^{1} + \rho^{2} + \sum_{\theta=1}^{s}\rho_{\theta}^{3}\right)\varphi(b, c, a, u)d^{2}((b, c), (a, u))$$

Where $l = L, U$.

Since, $\varphi(b, c, a, u) > 0$, using this, we have the following

$$\int_{\Omega}^{\cdot}\theta_{l}^{r}(g_{r}^{l})_{q}(t, a(t), u(t))\xi dt$$

$$+\int_{\Omega}^{\cdot}[\lambda_{i}^{\alpha}(t)(X_{\alpha}^{i})_{a}(t, a(t), u(t)) + \mu^{\beta}(t)(Y_{\beta})_{a}(t, a(t), u(t))]\xi dt$$

$$+\int_{\Omega}^{\cdot}\theta_{l}^{r}(g_{r}^{l})_{u}(t, a(t), u(t))\eta dt$$

$$+\int_{\Omega}^{\cdot}[\lambda_{i}^{\alpha}(t)(X_{\alpha}^{i})_{u}(t, a(t), u(t)) + \mu^{\beta}(t)(Y_{\beta})_{u}(t, a(t), u(t))]\eta dt$$

$$-\int_{\Omega}^{\cdot}[\lambda^{\alpha}(t)D_{\alpha}\xi]dt < -\left(\theta_{l}^{r}\rho_{r}^{1} + \rho^{2} + \sum_{\theta=1}^{s}\rho_{\theta}^{3}\right)\varphi(b, c, a, u)d^{2}((b, c), (a, u)).$$

Where $l = L, U$.

Now using the constraints (7) and (8) of dual problem, we have

$$-\int_{\Omega} [\xi D_{\alpha}\lambda^{\alpha}(t)dt - \int_{\Omega} [\lambda^{\alpha}(t)D_{\alpha}\xi]dt + 0$$

$$< -\left(\theta_l^r \rho_r^1 + \rho^2 + \sum_{\theta=1}^{s} \rho_{\theta}^3 \right) \varphi(b, c, a, u) d^2((b, c), (a, u))$$

Where $l = L, U$.

By direct formula of derivative, we know that

$$D_{\alpha}[\xi\lambda^{\alpha}(t)] = \lambda^{\alpha}(t)D_{\alpha}\xi + \xi D_{\alpha}\lambda^{\alpha}(t)$$

$$\xi D_{\alpha}\lambda^{\alpha}(t) = D_{\alpha}[\xi\lambda^{\alpha}(t)] - \lambda^{\alpha}(t)D_{\alpha}\xi$$

Now applying integral over the region Ω , we have

$$\int_{\Omega} \xi D_{\alpha}\lambda^{\alpha}(t)dt = \int_{\Omega} D_{\alpha}[\xi\lambda^{\alpha}(t)]dt - \int_{\Omega} [\lambda^{\alpha}(t)D_{\alpha}\xi]dt$$

Using the condition $\xi|_{\partial\Omega} = 0$ and applying the flow-divergence formula, we get

$$\int_{\Omega} D_{\alpha}[\xi\lambda^{\alpha}(t)dt = \int_{\partial\Omega} [\xi\lambda^{\alpha}(t)\vec{n}\, d\sigma = 0$$

Where $\vec{n} = (n)_{\alpha}$ where $\alpha = 1, 2 \ldots\ldots m$, is the unit normal vector to the hyper surface $\partial\Omega$, now it follows that

$$\int_{\Omega} \xi D_{\alpha}\lambda^{\alpha}(t)dt = \int_{\Omega} [\lambda^{\alpha}(t)D_{\alpha}\xi]dt$$

or

$$-\int_{\Omega} \xi D_{\alpha}\lambda^{\alpha}(t)dt - \int_{\Omega} [\lambda^{\alpha}(t)D_{\alpha}\xi]dt = 0.$$

Therefore, we have

$$0 < -\left(\theta_l^r \rho_r^1 + \rho^2 + \sum_{\theta=1}^{s} \rho_{\theta}^3 \right) \varphi(b, c, a, u) d^2((b, c), (a, u)).$$

Where $l = L, U$.

Now applying the condition (v) and $d^2((b, c), (a, u)) \geq 0$, we get a contradiction. Therefore, supremum of dual problem is less than or equal to the infimum of primal problem.

The following outcome proves a strong duality between the two multiobjective optimization problems with interval-valued components under consideration.

Strong Duality theorem-If we consider the same (ρ, φ, d)-Invexity hypotheses mentioned in above weak duality theorem, if $(b^0, c^0) \in D$ is a normal LU-optimal solution of the given primal problem, then $\exists\ \theta^0, \mu^0(t)$ and $\lambda^0(t)$ such that $(b^0, c^0, \theta^0, \lambda^0, \mu^0) \in \gamma$ is an LU-optimal solution of the dual problem, and the values of corresponding objective functions are equal.

Proof: Consider that $(b^0, c^0) \in D$ is a normal LU-optimal solution of the primal problem, the necessary LU-optimality conditions mentioned in equations (4) to (6) involve that $\exists\ \theta^0, \mu^0(t)$ and $\lambda^0(t)$ such that $(b^0, c^0, \theta^0, \lambda^0, \mu^0) \in \gamma$ is an feasible solution for dual problem.

$$\frac{\partial b^{0i}}{\partial t^\alpha}(t) = X^i_\alpha(t, b^0(t), c^0(t)) \text{ for } i = 1, 2, \ldots\ldots n,\ \alpha = 1, 2\ldots\ldots m\ \ t \in \Omega$$

Now by equation (6)

$$\mu^\beta(t) Y_\beta(t, b^0, (t), c^0(t)0 = 0,\ \text{(summation is taken over } \beta\text{)} \quad \text{and } t \in \Omega.$$

Therefore, the value of objective function of dual problem has the same value of objective function of primal problem. Hence by weak duality theorem $(b^0, c^0, \theta^0, \lambda^0, \mu^0) \in \gamma$ is an LU-optimal solution of dual problem.

A converse duality conclusion related to considered multiobjective optimization problems with interval-valued components is formulated in the following theorem.

Converse Duality theorem-Assume that the LU-optimal solution of dual problem is $(b^0, c^0, \theta^0, \lambda^0, \mu^0) \in \gamma$. Furthermore, presumptively the following circumstances hold true:

(i) $(\overline{b}, \overline{c}) \in D$ is a normal LU-optimal solution of the given primal problem.

(ii) For $(b^0, c^0, \theta^0, \lambda^0, \mu^0)$, the hypotheses of weak duality theorem are met.

Consequently, the corresponding objective values are equal and $(\overline{b}, \overline{c}) = (b^0, c^0)$.

Proof: In contrast to the outcome, let's assume that $(\overline{b}, \overline{c}) \neq (b^0, c^0)$ and that (b^0, c^0) is not a normal LU-optimal solution of primal problem. According to Treantă and Mititelu and Treantă, since $(\overline{b}, \overline{c}) \in D$ is a normal LU-optimal solution of primal problem, there exist $\overline{\theta}, \overline{u}(t)$ and $\overline{\lambda}(t)$, satisfying equations (4) to (6) and definition of normal LU-optimal solution. Consequently

$$\overline{\lambda}_i^{\alpha}(t)\left[X_{\alpha}^i(t, \overline{b}(t), \overline{c}(t) - \frac{\partial \overline{b}^i}{\partial t^{\alpha}}(t)\right] \geq 0,$$

$$\overline{\mu}^{Q\theta}(t) Y_{Q\theta}(t, \overline{b}(t), \overline{c}(t) \geq 0, \quad \theta = 1, 2\ldots\ldots s$$

where $(\overline{b}, \overline{c}, \overline{\theta}, \overline{\lambda}, \overline{\mu}) \in \gamma$ as a result. Additionally, $\delta(\overline{b}, \overline{c}) = (\overline{b}, \overline{c}, \overline{\theta}, \overline{\lambda}, \overline{\mu}) \in \gamma$ is present. We obtain $\delta(\overline{b}, \overline{c}) \geq \pi(b^0, c^0, \theta^0, \lambda^0, \mu^0)$ in accordance with weak duality theorem, or $\pi(\overline{b}, \overline{c}, \overline{\theta}, \overline{\lambda}, \overline{\mu}) \geq \pi(b^0, c^0, \theta^0, \lambda^0, \mu^0)$. The maximal LU-optimality of $(b^0, c^0, \theta^0, \lambda^0, \mu^0)$ is in conflict with this. As a result, the corresponding objective values are identical and $(\overline{b}, \overline{c}) = (b^0, c^0)$.

Illustrative instance: The following two-dimensional interval-valued variational control problem is taken into consideration:

$$\min_{(b,c)} \int_{\Omega(0.3)} g(t, b(t), c(t), dt$$

$$= \left[\int_{\Omega(0.3)} (c^2(t) - 8c(t) + 16) dt^1 dt^2, \int_{\Omega(0.3)} (c^2(t) dt^1 dt^2\right],$$

Subject to

$$\frac{\partial b}{\partial t^1}(t) = \frac{\partial b}{\partial t^2}(t) = 3 - c(t) \quad \text{where } t = (t^1, t^2) \in \Omega_{(0.3)}$$

$$81 - b^2(t) \leq 0 \quad \text{where } t = (t^1, t^2) \in \Omega_{(0.3)}$$

$$b(0) = b(0, 0) = 6, \qquad b(3) = b(3, 3) = 8$$

where $t_0 = (t_0^1, t_0^2) = (0, 0)$ and $t_1 = (t_1^1, t_1^2) = 33$ in $\mathbb{R}^2$ are the diagonally opposed points that fix the square $b : \Omega_{(0.3)} \to \mathbb{R}.$, $c : \Omega_{(0.3)} \to \left[-\frac{8}{3}, \frac{8}{3}\right]$ and $\Omega_{(t_0^1, t_0^2)} = \Omega_{(0.3)}$.

Furthermore, we consider that in the examined variational control problem in which affine state functions are the only ones that interest us. It is possible to demonstrate by direct computation that the feasible point

$$b^0(t) = \frac{1}{3}(t^1 + t^2) + 6, \qquad c^0(t) = \frac{8}{3}, \qquad t = (t^1, t^2) \in \Omega_{(0.3)}$$

is a normal LU-optimal solution with $\lambda = (\lambda^1, \bar{\lambda}^2) = (1, \frac{5}{3}), \theta = (\theta^L, \theta^U) = (1, 1)$ and $\mu = 0$ for the optimization problem under consideration. Moreover, the $(\rho, 1, 0)$-invexity (with $\rho \in \mathbb{R}$) of the functionals involved (refer to weak duality theorem) at (b^0, c^0) with regard to ξ and η may be easily verified as follows: Given by $\xi, \eta : \Omega_{(0.3)} \times (\mathbb{R} \times \mathbb{R})^2 \to \mathbb{R}$

$$\xi(t, b(t), c(t), b^0(t), c^0(t) = \begin{cases} b(t) - b^0(t), & t \in \operatorname{int}(\Omega_{(0.3)}) \\ 0, & t \in \partial\Omega_{(0.3)} \end{cases}$$

$$\eta(t, b(t), c(t), b^0(t), c^0(t) = \begin{cases} c(t) - c^0(t), & t \in \operatorname{int}(\Omega_{(0.3)}) \\ 0, & t \in \partial\Omega_{(0.3)} \end{cases}$$

where $\mathrm{int}(\Omega_{(0.3)})$ and $\partial(\Omega_{(0.3)})$ represent interior region and boundary of $\Omega_{(0.3)}$ respectively.

Therefore, by strong duality theorem $\left(\frac{1}{3}(t^1 + t^2) + 6, \frac{8}{3}, (1,1), (1, \frac{5}{3}), 0\right)$ will be an LU-optimal solution for the dual problem mentioned below

$$\max_{(a,u)}\int_{\Omega_{(0.3)}}^{\cdot} g(t, a(t), u(t), dt$$

$$= \left[\int_{\Omega_{(0.3)}}^{\cdot} (u^2(t) - 8u(t) + 16)dt^1dt^2, \int_{\Omega_{(0.3)}}^{\cdot} u^2(t)dt^1dt^2\right]$$

subject to

$$-2\mu(t)a(t) + \frac{\partial\lambda^1}{\partial t^1}(t) = \frac{\partial\lambda^2}{\partial t^2}(t) = 0 \qquad \text{where } t = (t^1, t^2) \in \Omega_{(0.3)}$$

$$2\theta^L u(t) - 8\theta^L + 2\theta^U u(t) - \lambda^1(t) - \lambda^2(t) = 0 \qquad \text{where } t = (t^1, t^2) \in \Omega_{(0.3)}$$

$$\lambda^1(t)\left(3 - u(t) - \frac{\partial a}{\partial t^1}(t)\right) + \lambda^2(t)\left(3 - u(t) - \frac{\partial b}{\partial t^2}(t)\right) \geq 0$$

$$\text{where } t = (t^1, t^2) \in \Omega_{(0.3)}$$

$$\mu(t)(81 - a^2(t)) \geq 0, \qquad \text{where } t = (t^1, t^2) \in \Omega_{(0.3)}$$

$$\theta = \theta^L, \theta^U \geq [0, 0], \quad \mu(t) \geq 0, \quad a(0) = a(0,0) = 6, \quad b(3) = b(3,3) = 8.$$

and the values of objective of both primal and dual problem are equal.

4. Conclusions

In this paper we have formulated and proved Mond-Weir weak, strong, and converse duality theorems for a completely new concept of multiobjective optimization problems having interval-valued components, based on the completely new notion of (ρ, φ, d)-Invexity and (ρ, φ, d)-Pseudoinvexity related with an interval-valued multiple-integral functional. Considering the relevance of interval analysis and duality theory to optimization and control, this work constitutes a significant contribution for applied sciences researchers and engineers.

Future Scope

This paper can be extended from numerous points of view for additional exploration. In this paper we have studied for one parameter t, which can be generalised for two or three parameters. On the other hand, here, we have studied Both (ρ, φ, d)-Invexity and (ρ, φ, d)-Pseudinvexity for multiobjective optimization, which can also be studied for fractional programming or Inverse optimization. So, this study has great future scope.

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Conflict of Interest

All authors declare that they have no conflicts of interest.

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INVARIANT ANALYSIS OF HEAT GENERATION AND THERMAL RADIATION EFFECTS ON MHD NON-NEWTONIAN POWER-LAW NANOFLUID OVER LINEARLY STRETCHING SURFACE WITH CONVECTIVE BOUNDARY CONDITIONS

Abstract: This study examines the effects of thermal radiation and heat generation along a stretching surface. The power-law non-Newtonian model under the influence of Brownian motion and thermophoresis for nanofluids is analysed for determining their effects on various parameters of nanofluid like temperature, velocity etc. The uniform magnetic field and boundary conditions for convective mode are also considered for nanofluid flow. The objective of similarity invariants is to convert non-linear partial differential equations into ordinary differential equations invariantly. The numerical results of the investigation for the impacts of various parameters on skin friction coefficients, Nusselt-Sherwood numbers are determined. The behaviour of different physical factors on skin friction coefficients in x and y directions, on the local Nusselt number, and on the Sherwood number is analysed. An increment in the power-law index increases the Nusselt number. The results of the experiment indicates that an increase in the heat generation parameter will result in a drop in the Nusselt number and an increase in the Sherwood number. Sherwood number will decrease and Nusselt number will increase with an increase in thermal radiation parameter.

Keywords: MHD Nanofluid; Non-Newtonian Power-law Model; Heat Generation and Radiation; Similarity Invariants; Convective Boundary Condition.

Mathematics Subject Classification No.: 35Q35, 76M60, 58J70, 35G60.

1. Introduction

The dispersion of nanoparticles in a base fluid, such as water, ethylene glycol, or oil, is known as nanofluid. It was introduced and studied by Choi(1995). In his experimental research, he also noticed that heat transfer was enhanced in nanofluid compared to regular fluids. There are many attractive applications of nanofluid like coolants, brake fluid, gear lubrication in automobile industries. It is useful in solar devices, as delivery of cancer drugs in the medical field, and coolants in electronic devices. So, it is an essential to study the influence of different physical factors and various physical situations on nanofluid flow.

Tesfaye *et al*. (2020) analysed the erratic flow of Williamson nanofluid over a stretched sheet under the influence of a magnetic field, heat radiation, and chemical reaction. Kalidas *et al*. (2018) examined heat generation/absorption effects for Oldroyd-B type nanofluid, two-dimensional flow over a permeable stretching surface under the effect of magnetic field and slip velocity. Umadevi and Nithyadevi (2016) investigated two-dimensional nanofluid flows under uniform heat generation or absorption with a uniform magnetic field for different thermal boundaries. Bilal *et al*. (2018) examined the impact of the various physical factors for three-dimensional Maxwell nanofluid MHD flow passing through a bidirectional stretching surface under nonlinear thermal radiation. Hayat *et al*. (2017) addressed three-dimensional Maxwell MHD nanofluid flow under the influence of heat generation-absorption and thermal radiation on a stretching surface. Burger's nano-liquid flow over a stretching sheet was studied by Ganesh *et al*. (2018) with the impact of non-linear radiation and non-uniform heat generation and absorption. The thermal radiation effects on the MHD stagnation point, the two-dimensional flow of a non-Newtonian Williamson fluid, over a stretching plate, were examined by Hasmawani *et al*. (2019) by applying similarity transformations. The two-dimensional flow of Maxwell nanofluid on a linearly stretching surface under heat generation and absorption impacts was investigated by Awais *et al*. (2015). The two-dimensional flow passing over an exponentially stretching sheet of MHD Casson fluid was studied with internal heat generation by Animasaun *et al*. (2016).

Waqas *et al*. (2017) modelled and analysed Oldroyd-B nano-liquid two-dimensional flow over a moving sheet with heat generation and absorption effects using the Homotopy analysis method. The MHD nanofluid three- dimensional flow over a shrinking sheet under viscous dissipation and heat generation and absorption with entropy generation was examined by Hiranmoy *et al*. (2019). The solution for unsteady, two-dimensional nanofluid flow over a stretching surface was studied numerically by utilising the fourth-fifth order RKF technique under the influence of radiation, thermophoresis, and heat generation and absorption by Pandey and Manoj (2018). Ahmed *et al*. (2019) examined MHD Maxwell nanofluids flow over a stretching surface under the influence of heat generation-absorption and non-linear thermal radiation in the porous medium by applying similarity variables and the shooting technique. Kalpna and Sumit (2017) investigated two- dimensional (MHD) Jeffrey nanofluid flow in the presence of thermal radiation, heat generation/absorption, and viscous dissipation over an impermeable surface by assuming similarity transformations and applying the Homotopy analysis method. Makinde (2011) introduced similarity variables and used the fourth-order Runge-Kutta method and the shooting method to examine the impacts of internal heat generation on two-dimensional boundary layer flow on a vertical plate with a convective surface boundary condition. Lalrinpuia and Surender (2019) used the homotopy analysis approach to assess MHD nanofluid flow in a saturated porous medium, in an inclined channel with a heat source/sink, accounting for hydrodynamic slip and convection at the boundary. Khan et al. (2014) analysed the impacts of heat generation/ absorption on the 3-D flow of an Oldroyd-B nanofluid over a sheet stretching in both x and y directions. They applied similarity transformations.

The influence of heat generation, radiation, and viscous dissipation on the flow of MHD nanofluid over a sheet stretched exponentially in a porous medium was studied by Thiagarajan and Dinesh Kumar (2019). The MHD-Carreau nanofluid flow over a radially stretched sheet under the influence of chemical reaction, nonlinear thermal radiation, and heat generation/absorption was examined by Dianchen *et al*. (2018). The second grade Cattaneo-Christov two-dimensional fluid flow caused by a linear stretched Riga plate was studied under the impact of heat generation/absorption by Aisha *et al*. (2018). Abdul Khan *et al*. (2018) analysed Williamson nanofluid flow in three dimensions across a linear porous stretching surface for the impact of thermal radiation. Sulochana *et al*. (2016) investigated Newtonian and non-Newtonian, 3-D magnetohydrodynamic fluid flow across a stretched sheet. Chuo-Jeng and Kuo-Ann (2021) examined the effects of zero nanoparticle flux, internal heat generation, nonlinear radiation, and changing viscosity on free convection on a non-Newtonian power-law nanofluid flowing via a vertical truncated cone embedded in a fluid-saturated porous medium. Considering thermal radiation and heat absorption/generation, Mabood *et al*. (2020) investigated MHD Oldroyd-B two-dimensional, thermal stratified flow across an inclined linearly stretched sheet. Recently, Newtonian and various non-Newtonian fluid models like Sisko, Powell-Eyring, Power-Law Model, Prandtl-Eyring were analysed using invariant analysis via the group- theoretic technique by deriving dependent and independent invariants. (Patel *et al*. 2015, Shukla *et al*. 2017, 2018, 2020). Impact of heat generation/absorption in the context of nonlinear thermal radiation on magnetohydrodynamic stagnation-point two-dimensional Newtonian nanofluid flow across a convective stretching surface were examined by Feroz *et al*. (2018). Shukla *et al*. (2020) analysed flow over linearly stretching surface for 3-D Power-low nanofluid.

Due to the significance role of heat generation and thermal radiation on nanofluid flow, we have extended the work done by Shukla *et al*. (2020) and considered the influence of heat generation and thermal radiation. In this paper, we have studied a power-law fluid flow in three dimensions on a linearly stretched sheet. A survey of the literature shows that most studies have focused on flows in two directions, X and Y. The scenario is more real in three dimensions, X, Y, and Z. We have also examined the effects of thermophoresis, magnetic field, and Brownian motion on heat generation and thermal radiation. The convective boundary conditions have been considered for the analysis of the present non-Newtonian fluid flow model. Various parameters like Nusselt number, skin friction coefficients, and Sherwood number have been considered for analysing the flow. Similarity-dependent and independent invariants have been used with the aim of transforming the nonlinear PDEs into ODEs invariantly.

2. Governing Equation of the Boundary Value Problem

Here, we have considered the three-dimensional power-law nano non-Newtonian fluid model. The flow is incompressible, steady, laminar over a linearly stretching sheet with the velocity $u_w = ax$ and $v_w = by$ in X and Y-direction respectively. Here, the stretched sheet is exposed to a homogeneous magnetic field B that is directed in the surface's normal direction. The conditions of convective boundaries are considered for the flow analysis. Heat generation/absorption impacts, as well as the impact of thermal radiation, are also considered in the heat transfer study.

We have taken the following parameters for the flow analysis.

T_∞ - Temperature at Infinite distance from the sheet's surface

C_∞ - Concentration at Infinite distance from the sheet's surface

h_f - Heat transfer coefficient

h_s - Convective mass transfer coefficient

Convective heat transfer mode is used to heat or cool the sheet's surface by maintaining a hot fluid temperature T_f and a convective concentration of fluid C_f.

We have used the following boundary value flow governing equations.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\lambda}{\rho}\frac{\partial}{\partial z}\left(-\frac{\partial u}{\partial z}\right)^n - \frac{\sigma B^2}{\rho}u \tag{2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{\lambda}{\rho}\frac{\partial}{\partial z}\left[\left(-\frac{\partial u}{\partial z}\right)^{n-1}\frac{\partial v}{\partial z}\right] - \frac{\sigma B^2}{\rho}v \tag{3}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \alpha\frac{\partial^2 T}{\partial z^2} + \tau\left[D_B\left(\frac{\partial T}{\partial z}\frac{\partial C}{\partial z}\right) + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial z}\right)^2\right] + \frac{Q_0}{\rho c_p}(T - T_\infty) - \frac{1}{\rho c_p}\frac{\partial q_r}{\partial z} \tag{4}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = D_B\left(\frac{\partial^2 C}{\partial z^2}\right) + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial z^2} \tag{5}$$

Boundary values for convective mode are given by

$$u = u_w = ax,\ v = v_w = by,\ w = 0, -k\frac{\partial T}{\partial z} = h_f(T_f - T), -D_B\frac{\partial C}{\partial z} = h_s(C_f - C)$$

at $z = 0, u = 0, v = 0, w = 0, T = T_\infty, C = C_\infty$ at $z = \infty$ (6)

Where,

u - Velocity in the x direction, v -Velocity in the y direction, w -Velocity in the z direction

T - Fluid temperature, C -Fluid concentration, ρ-Fluid density, τ-Heat capacitance ratio

D_T - Thermophoresis diffusion coefficient, D_B -Brownian diffusion coefficient, n : flow index

λ (> 0) - Rheological constant, σ-electrical conductivity of the fluid, α-thermal diffusivity

Q_0 : coefficient of internal heat generation, q_r -radiative heat flux.

q_r is defined as $q_r = \dfrac{-4\sigma^* {T_\infty}^3}{3k^*} \dfrac{\partial T^4}{\partial z}$

Where σ^* - the Stefan-Boltzmann constant, k^* - absorption coefficient.

Now, expanding T^4 about T_∞ and neglecting higher terms, we get following expression:

$$T^4 = {T_\infty}^4 + 4{T_\infty}^3 T - 4{T_\infty}^3 T_\infty$$

$$\frac{\partial q_r}{\partial z} = \frac{\partial}{\partial z}\left(\frac{-4\sigma^* {T_\infty}^3}{3k^*} \frac{\partial T^4}{\partial z} \right)$$

$$\frac{\partial q_r}{\partial z} = \frac{-16\sigma^* {T_\infty}^3}{3k^*} \frac{\partial^2 T}{\partial z^2} \tag{7}$$

By putting $\dfrac{\partial q_r}{\partial z}$ in equation (4), we get the following equation.

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \alpha \frac{\partial^2 T}{\partial z^2} + \tau \left[DB \left(\frac{\partial T}{\partial z} \frac{\partial C}{\partial z} \right) + \left(\frac{D_T}{T_\infty} \frac{\partial T}{\partial z} \right)^2 \right]$$

$$+ \frac{Q_0}{\rho c_p} (T - T_\infty) + \frac{1}{\rho c_p} \frac{-16\sigma^* {T_\infty}^3}{3k^*} \frac{\partial^2 T}{\partial z^2} \tag{8}$$

3. Invariance Analysis by Generalized Group Theoretic Method

We have used the following dependent and independent absolute invariants to convert governing partial differential equations into ordinary differential equations invariantly. (Shukla *et al*. 2020)

$$\begin{cases} \eta = d_1 z(x)^{\frac{1-n}{1+n}} \\ H_1(\eta) = \frac{u}{d_2 x} \\ H_2(\eta) = \frac{v}{d_3 y} \\ H_3(\eta) = \frac{W}{d_4 (x)^{\frac{n-1}{1+n}}} \\ H_4(\eta) = \theta = \frac{T - T_\infty}{T_f - T_\infty} \\ H_5(\eta) = \phi = \frac{C - C_\infty}{C_f - C_\infty} \end{cases}$$

We have assumed the following values for the coefficients and parameters.

$$d^1 = \left(\frac{a^{2-n}}{\frac{\lambda}{\rho}} \right)^{\frac{1}{n+1}}, d_2 = a \quad d_3 = b, \quad d_4 = -a \left(\frac{a^{n-2}}{\frac{\rho}{\lambda}} \right)^{\frac{1}{n+1}}$$

$$pr = \frac{\rho c_p u_w x}{k} (Re) \frac{-2}{n+1}, \quad Re = \frac{(u_w)^{2-n} {}_x n \rho}{\lambda}, \; M \; \frac{\sigma B^2}{\rho}$$

$$Bi_1 = \frac{h_f}{k} x (Re) \frac{-1}{n+1}, \quad N_b = \tau D_B \frac{(C_f - C_\infty)}{\alpha}, \quad R_d = \frac{-16 \sigma T_\infty{}^3}{3k^*}, \quad \lambda_1 = \frac{Q_0}{\rho \alpha c_p} \tag{10}$$

Where Re-local Reynolds number, pr-generalised Prandtl number, Le-the Lewis number, N_b-Brownian motion parameter, N_t-thermophoresis parameter Bi_1 and Bi_2-generalised Biot number. We have taken skin-friction coefficients C_{fx} and C_{fy} along the x- and y-axes, the Nusselt number, and the Sherwood number for analysing the fluid flow. We have used the following equations for above parameters:

$$(Re)^{\frac{1}{n+1}} C_{fx} = -(H_1'(0))^n \tag{11}$$

$$(Re)^{\frac{1}{n+1}} C_{fy} = -\frac{a v_w}{b u_w} (H_1'(0))^{n-1} H_2'(0) \tag{12}$$

$$(Re)^{\frac{1}{n+1}} Nu_x = -(1 + R_d) H_4'(0) \tag{13}$$

$$(Re)^{\frac{1}{n+1}} Sh_x = -H_5'(0) \tag{14}$$

Differentiating absolute invariants of equation (9) with respect to similarity independent variable η and applying on governing equations (1 to 5, 8), we obtain following equations.

$$aH_1 + bH_2 - aH_3^1 + \frac{1-n}{1+n} a\eta H_1' = 0 \tag{15}$$

$$a(H_1)^2 - aH_1'H_3 + \frac{1-n}{1+n} a\eta H_1'H_1 - na(-H_1')^{n-1} H_1'' + MH_1 = 0 \tag{16}$$

$$b(H_1)^2 - aH_2'H_3 + \frac{1-n}{1+n} a\eta H_2'H_1 + a(n-1)(-H_1')^{n-2} H_2'H_1''' \\ -a(-H_1')^{n-1} H_2'' + MH_2 = 0 \tag{17}$$

$$prH_4'H_3 - pr\frac{1-n}{1+n} \eta H_1H_4' + H_4'' - R_dH_4'' + \lambda_1 H_4 + N_bH_4'H_5' + N_t(H_4')^2 = 0 \tag{18}$$

$$H_5'' + \frac{N_t}{N_b} H_4'' + prLeH_5'H_3' - prLe\frac{1-n}{1+n} \eta H_1H_5' = 0 \tag{19}$$

Similarly, we have obtained the following equations of boundary conditions from equation (6).

$$At\eta = 0,\ H_1 = 1,\ H_2 = 1,\ H_3 = 0,\ H_4' = -Bi_1(1 - H_4),\ H_5' = -Bi_2(1 - H_5), \\ At\eta = \infty,\ H_1 = H_2 = H_3 = H_4 = H_5 = 0. \tag{20}$$

The following equations are obtained from equations (15-19)

$$H_1 = \mathcal{G}'_1, H_2 = \mathcal{G}'_2, H_3 = \frac{2n}{1+n}\mathcal{G}_1 + \frac{b}{a}\mathcal{G}_2 + \frac{1-n}{1+n}\eta\mathcal{G}'_1 \tag{21}$$

$$a(\mathcal{G}'_1)^2 - b\mathcal{G}''_1\ \mathcal{G}_2 - \tfrac{2n}{1+n}\, a\mathcal{G}''_1\mathcal{G}_1 - na(-\mathcal{G}''_1)^{n-1}\mathcal{G}'''_1 + M\mathcal{G}'_1 = 0, \tag{22}$$

$$b(\mathcal{G}'_2)^2 - b\mathcal{G}''_2\ \mathcal{G}_2 - \tfrac{2n}{1+n}\, a\mathcal{G}''_2\mathcal{G}_1 - a(n-1)(-\mathcal{G}''_1)^{n-2}\mathcal{G}''_2\mathcal{G}'''_1 - a(-\mathcal{G}''_1)^{n-1}\mathcal{G}'''_2 + M\mathcal{G}_2 = 0, \tag{23}$$

$$H''_4 + N_b H'_4 H'_5 + N_t (H'_4)^2 + \frac{b}{a}\, prH'_4\mathcal{G}_2 + \frac{2n}{1+n}\, pr\mathcal{G}_1(\eta)H'_4 - R_a H'_4 + \lambda_1 H_4 \tag{24}$$

$$H''_5 + \frac{N_t}{N_b} H''_4 + \frac{b}{a}\, Le\,pr\,H'_5\mathcal{G}_2 + \frac{2n}{1+n}\, pr\,Le\,\mathcal{G}_1 H'_5 = 0 \tag{25}$$

4. Numerical Solution

We have transformed the aforementioned system of equations into a system of first order differential equations in order to use Bvp4c - MATLAB software.

By replacing functions $\mathcal{G}_1, \mathcal{G}'_1, \mathcal{G}''_1, \mathcal{G}_2, \mathcal{G}'_2, \mathcal{G}''_2, H_4, H'_4, H_5, H'_5$ by y_i, for $i = 1, 2, \dots, 10$ respectively, we get the following equations.

$$y'_1 = y_2 \tag{26}$$

$$y'_2 = y_3 \tag{27}$$

$$y'_3 = \frac{(a(y_2)^2 - by_3y_4 - \tfrac{2n}{1+n}\, ay_1y_3 + My_2)}{na(-y_3)^{n-1}} \tag{28}$$

$$y'_4 = y_5 \tag{29}$$

$$y'_5 = y_6 \tag{30}$$

$$y'_6 = \frac{b(y_5)^2 - by_4y_6 - \tfrac{2n}{1+n}\, ay_1y_6 - a(n-1)(-y_3)^{n-2}y'_3y_6 + My_5}{a(-y_3)^{n-1}} \tag{31}$$

$$y'_7 = y_8 \tag{32}$$

$$y'_8 = \frac{-N_b y_8 y_{10} - N_t (y_8)^2 - \frac{b}{a} pr y_4 y_8 - \frac{2n}{1+n} pr y_1 y_8 - \lambda y_7}{1 - R_d} \tag{33}$$

$$y'_9 = y_{10} \tag{34}$$

$$y'_{10} = -\frac{N_t}{N_b} y'_8 - \frac{b}{a} Lepr y_4 y_{10} - \frac{2n}{1+n} prLe y_1 y_{10} \tag{35}$$

$$\eta = 0 \Rightarrow y_1 = y_4 = 0, y_2 = y_5 = 1$$

$$y_8 = -Bi_1(1 - y_7(0)), y_{10} = -Bi_2(1 - y_9(0))$$

$$\eta = \infty \Rightarrow y_1 = 0, y_4 = 0, y_7 = 0, y_9 = 0 \tag{36}$$

We have obtained the following equations from equations (11-14)

$$(Re)^{\frac{1}{n+1}} C_{fx} = -(H'_1(0))^n = -(\mathcal{G}''_1(0))^n = -(y_3(0))n \tag{37}$$

$$(Re)^{\frac{1}{n+1}} C_{fy} = -\frac{av_w}{bu_w}(H'_1(0))^{n-1} H'_2(0) = -\frac{av_w}{bu_w}(\mathcal{G}''_1(0))^{n-1} \mathcal{G}''_2(0)$$

$$\frac{u_w}{v_w}(Re)^{\frac{1}{n+1}} C_{fy} = -\frac{a}{b}(y_3(0))^{n-1} y_6(0) \tag{38}$$

$$(Re)^{-\frac{1}{n+1}} Nu_x = -(1 + R_d) H'_4(0) = -(1 + R_d) y_8(0) \tag{39}$$

$$(Re)^{-\frac{1}{n+1}} Sh_x = -H'_5(0) = -y_{10}(0) \tag{40}$$

6. Results and Discussion

We have used MATLAB bvp4c solver for analysing fluid flow problem. Tables 1 and 2 show the values for Skin friction coefficients, Nusselt number, and Sherwood number.

Table 1: Skin friction coefficient values for various parameters in the x and y directions

n	a	b	pr	N_t	N_b	M	R_d	Le	λ	Cf_x	Cf_y
1	1	1	1	0.1	0.1	0.5	0	0.2	0	1.7538893508	1.7538893508
1	1	1	1	0.2	0.1	0.5	0	0.2	0	1.7538897120	1.7538897120
1	1	1	1	0.3	0.1	0.5	0	0.2	0	1.7538902961	1.7538902961
1	1	1	1	0.1	0.1	0.5	0	0.2	0	1.7538893508	1.7538893508
1	1	1	1	0.1	0.2	0.5	0	0.2	0	1.7538895038	1.7538895038
1	1	1	1	0.1	0.3	0.5	0	0.2	0	1.7538896175	1.7538896175
1	1	1	1	0.1	0.1	0.5	0	0.2	0	1.7538893508	1.7538893508
1	1	1	1	0.1	0.1	1.2	0	0.2	0	1.9058126194	1.9058126194
1	1	1	1	0.1	0.1	1.5	0	0.2	0	1.9704591416	1.9704591416
1	1	1	1	0.1	0.1	0.5	0	0.2	0.2	1.7538900002	1.7538900002
1	1	1	1	0.1	0.1	0.5	0.1	0.2	0.2	1.7538901651	1.7538901651
1	1	1	1	0.1	0.1	0.5	0.2	0.2	0.2	1.7538907395	1.7538907395
1	1	1	1	0.1	0.1	0.5	0	0	0.2	1.7538899995	1.7538899995
1	1	1	1	0.1	0.1	0.5	0	0.1	0.2	1.7538899984	1.7538899984
1	1	1	1	0.1	0.1	0.5	0	0.2	0.2	1.7538900002	1.7538900002
1	10	2	2	0.1	0.1	0.5	0.1	0.2	0.2	1.5511878626	1.8971685570
1	10	4	2	0.1	0.1	0.5	0.1	0.2	0.2	2.8042554030	1.3992670114
1	10	6	2	0.1	0.1	0.5	0.1	0.2	0.2	3.1138892553	1.1463275550

From Table 1, it is observed that the value of skin friction coefficient in X and Y direction both enhances with rising values of the thermophoresis parameter N_t as well as thermal radiation R_d. The reason behind it is that if the thermophoresis parameter is increasing the temperature and concentration, differences between the surface of the semi-infinite vertical plate and the ambient fluid are increasing and hence accelerates the heat transfer rate. Table 1 shows the skin friction coefficient for various values of the Brownian motion parameter N_b.

The skin friction coefficient in both directions is seen to grow with increasing values of the Brownian motion parameter N_b and opposite behaviour observed for Lewis number Le. The skin friction coefficient increases as the magnetic field parameter M increases because it reflects an increase in surface velocity gradients. A similar phenomenon is noticed in Table 1. Effect of stretching ratio parameter significantly affects skin friction coefficient. An increase in parameter b, the skin friction coefficient in the X direction rises, whereas the Y direction exhibits the opposite behaviour.

Table 2: Sherwood number and Nusselt number Values for different parameters

n	b	n	N_t	N_b	M	pr	Le	λ	Rd	Sh_x	Nu_x
1	1	1	0.1	0.1	0.5	1	0.2	0	0	0.1055262378	0.3824750511
1	1	1	0.2	0.1	0.5	1	0.2	0	0	-0.0451858252	0.3801468924
1	1	1	0.3	0.1	0.5	1	0.2	0	0	-0.1918276471	0.3777829817
1	1	1	0.1	0.1	0.5	1	0.2	0	0	0.1055262378	0.3824750511
1	1	1	0.1	0.2	0.5	1	0.2	0	0	0.1838693036	0.3801079022
1	1	1	0.1	0.3	0.5	1	0.2	0	0	0.2099947839	0.3777045682
1	1	1	0.1	0.1	0.5	1	0.2	0	0	0.1055262378	0.3824750511
1	1	1	0.1	0.1	1.2	1	0.2	0	0	0.1066120068	0.3811253420
1	1	1	0.1	0.1	1.5	1	0.2	0	0	0.1070520017	0.3803950922
1	1	1	0.1	0.1	0.5	0.7	0.2	0	0	0.6808097648	-0.0591201311
1	1	1	0.1	0.1	0.5	1.2	0.2	0	0	0.4517559332	0.0767238041
1	1	1	0.1	0.1	0.5	1.7	0.2	0	0	0.1075016163	0.4341594390
1	1	1	0.1	0.1	0.5	1	0	0.2	0	0.0640340920	0.3645373365
1	1	1	0.1	0.1	0.5	1	0.1	0.2	0	0.0926736299	0.3644023170
1	1	1	0.1	0.1	0.5	1	0.2	0	0	0.1055262378	0.3824750511
1	1	1	0.1	0.1	0.5	1	0.2	0.2	0	0.1210982576	0.3642872395
1	1	1	0.1	0.1	0.5	2	0.2	0.2	0.1	0.1154854564	0.4960100946
1	1	1.2	0.1	0.1	0.5	2	0.2	0.2	0.1	0.1151724519	0.4966261688
1	1	1.3	0.1	0.1	0.5	2	0.2	0.2	0.1	0.1150959844	0.4968748668
1	1	1.4	0.1	0.1	0.5	2	0.2	0.2	0.1	0.1150477775	0.4970963117

Table 2 indicates the effect of various parameters on the Sherwood number and Nusselt number. Growing thermophoresis parameter values are accompanied by decreasing Sharwood and Nusselt numbers. Table 2 demonstrates that when the Brownian motion parameter increases, the rate of heat transmission slows down, resulting in a fall in the Nusselt number and an observed increase in the Sherwood number. It is observed that the Nusselt number decreases and the Sherwood number increases with an acceleration of the magnetic parameter .

The Lorentz force is increased when the magnetic parameter increases, slowing down fluid motion and lowering the rate of heat flux in the process. By increasing the value of the Lewis number, nanoparticle volume fraction distribution decreases, because of reduction in mass diffusion. This, in turn, increases the Sherwood number, with the opposite effect being seen on the Nusselt number. Based on the table's numerical values, it can be determined that as the radiation parameter is raised, the Sherwood number falls and the Nusselt number rises. An analogous result was noted with the Prandtl number. The Sherwood number rises, the heat generation parameter lambda increases, and the Nusselt number decreases. The Sherwood number decreases as n (the power-law index) increases, but the Nusselt number increases.

Figures 1 and 2 depict, how the Lewis number changes the Nusselt and Sherwood numbers in response to thermophoresis and thermal radiation, respectively. The Nusselt number decreases as the thermophoresis parameter and Lewis number grow, while inverse patterns are seen as the thermal radiation parameter increases. As thermophoresis and Lewis numbers rise, Sherwood number tends to increase; conversely, as the thermal radiation parameter increases, it tends to decrease.

Figures 3 and 4 show the impact of the heat source/sink parameter under the influence of thermal radiation and thermophoresis parameter on the Nusselt number and Sherwood number. Figures 5 and 6 demonstrate the influence of the Brownian motion parameter, the thermophoresis parameter, and the thermal radiation parameter on the Sherwood number and Nusselt number respectively. Sherwood number decreases as thermal radiation parameter value increases. Nusselt number increasing as a result of the thermal radiation parameter increasing.

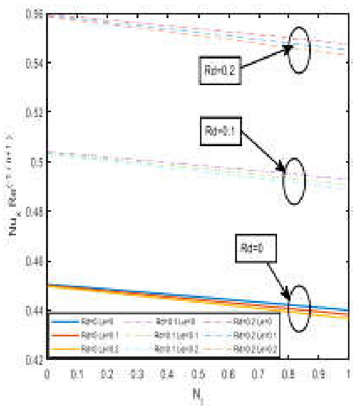

Figure 1 Impact of thermal radiation and thermophoresis on the Nusselt number under the impact of Lewis number

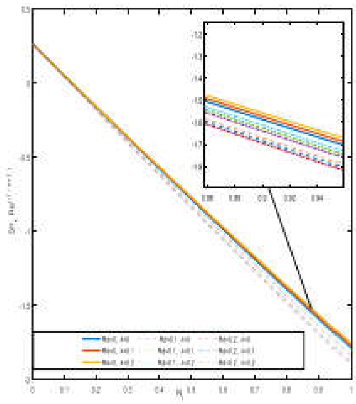

Figure 2 ***Influence of Lewis number on the Sherwood number under the impact of thermophoresis and thermal radiation***

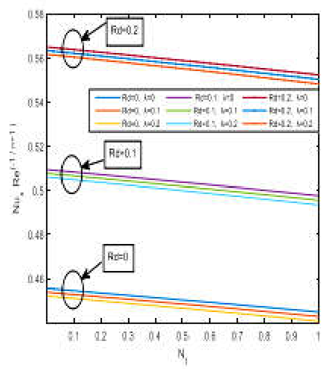

Figure 3 Impact on the Nusselt number of heat generation parameter with the impact of thermophoresis and thermal radiation

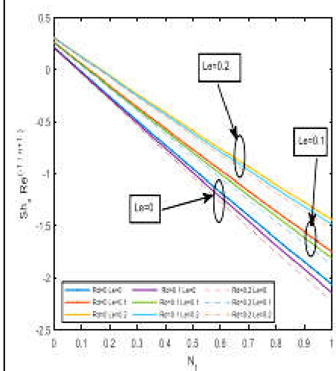

Figure 4 Impact parameter on the Sherwood number of heat generation parameter with the impact of thermophoresis and thermal radiation

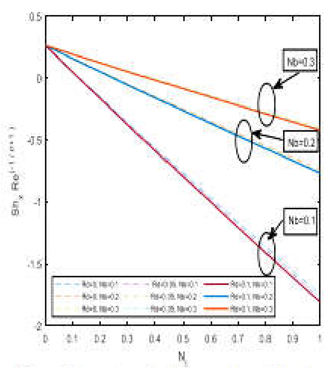

Figure 5 Impact on the Sherwood number of thermophoresis parameter, thermal radiation parameter and Brownian motion parameter

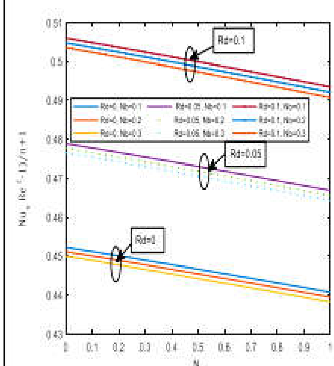

Figure 6 Impact on the Nusselt number of thermophoresis parameter , thermal radiation parameter and Brownian motion parameter

5. Conclusion

We have used both similarity dependent and independent invariants to get a similarity solution for the boundary value problem associated with power-law nanofluid flow. The power-law nanofluid problem's governing equations have been converted into ordinary differential equations with the help of invariants. The numerical solutions of derived ordinary differential equations are utilized by using MATLAB bvp4c software to find the effects of various parameters like Nusselt number, Sharwood number and Skin friction coefficients on fluid flow. The following are the findings of the analysis of the fluid flow using invariants.

- The findings indicate that an increase in the Lewis number Le results in a drop in the coefficient of skin friction in the x and y directions, an increase in the Sharwood number, and a decrease in the Nusselt number.
- An increase in the magnetic parameter M causes the skin friction coefficient to increase in both the x and y directions, the Nusselt number decreases, and the Sharwood number increases.
- A rise in the power-law index , a fall in the Sherwood number, and an increase in the Nusselt number.
- The Sherwood and Nusselt numbers decrease with an increase in the thermophoresis parameter.
- As the radiation parameter increases, the Nusselt number rises while the Sherwood number reduces.

Conflict of Interest Statement

We (authors) do not have any conflict of interest (financial or academic) for this work.

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EXACT SOLUTION FOR FLOW THROUGH POROUS MEDIUM OF A ROTATING VARIABLY INCLINED MHD FLUID BY MAGNETOGRAPH TRANSFORMATION

Abstract. An analytical study of the motion of a steady, homogenous, incompressible, plane rotating MHD fluid flow through a porous medium for exact solutions is carried out. The velocity vector of the fluid particle is thought to be variably inclined to the magnetic field vector at every point. The flow of fluid is governed by non-linear partial differential equations. These governing equations are converted into a system of linear partial differential equations by means of transformation technique known as magnetograph transformation. The two components of the magnetic field in the physical plane and two independent variables are switched around using the magnetograph transformation. Further, the flow equations have been derived using the Legendre transform of the magnetic flux function. Finally, several examples have been used to apply and illustrate the developed theory and exact solutions have been determined. The expressions for the components of velocity vector, components of magnetic field vector, magnetic lines and pressure distribution are obtained and analyzed graphically.

Keywords: MHD, Exact Solution, Magnetograph Transformation, Magnetic Flux Function, Legendre Transform Function, Porous Medium.

Mathematics Subject Classification (2020) No.: 35F05, 35Q30, 35Q35.

1. Introduction

The governing equations for the flow of non-Newtonian fluids give rise to systems of non-linear partial differential equations; these equations have no general solution. The several approaches used to solve these equations and their applications have received excellent coverage from Ames [1]. Hodograph transformations, as employed by Martin [2] in fluid mechanics, are a class of transformations that change variables from the physical plane to the velocity plane.

The magnetograph transformation- a method for accurately solving non-linear partial differential equations- which govern the steady flow of a homogeneous, incompressible, viscous fluid with finite electrical conductivity in a porous medium in a rotating reference frame-is the subject of the current study. It is common practice to solve non-linear partial differential equations using transformation techniques. The magnetograph is a curve formed by the extremities of the magnetic field vectors when they are extended from a given point. An equivalent linear system is produced by using the magnetograph transformation to switch the roles of the independent and dependent variables. In other words, the transformations that are used to switch the roles of the two independent variables in the physical plane and the two components of the magnetic field are known as magnetograph transformations.

The governing non-linear equations are transformed into a linear form that may be solved by using the magnetograph transformation. Using magnetograph transformation, several researchers have studied MHD fluid flow and discovered precise answers. In order to investigate orthogonal MHD flow, S. N. Singh [3] invented and used magnetograph transformation. Researchers Venkateshappa, Siddabasappa, and Rudraswamy [26] as well as C. S. Bagewadi and Siddabasappa [4], looked on rotating MHD ow that was variably inclined in the magnetograph plane. Exact solutions were found by M. Kumar and S. Sil [5] after studying aligned MHD flow in the magnetograph plane.

The study of fluid flow in a rotating frame is important for many technical applications that are directly affected by the coriolis force created by the earth's rotation. Examples of these applications include spin coating, the creation and use of computer disks, rotational viscometers, centrifugal machinery, the pumping of liquid metals at high melting points, the growth of crystals from molten silicon, turbo-machinery etc. The coriolis force is shown to have a significant impact when compared to the viscous and inertial forces in the equations of motion.

The coriolis force has a major impact on the hydromagnetic flow in the liquid core of the earth, which is essential to the mean geomagnetic field [6]. Because of its role in solar physics and its relationship to the formation of sunspots and the solar cycle, the theory of rotating fluid is also significant. Several studies with rotating fluid have been carried out [9, 11, 10, 12, 7, 8, 13, 26]. Many works have been conducted on various types of flows for both non-MHD and MHD.

In the study of soil percolation in hydrology, the petroleum industry, agricultural engineering, and many other significant fields, the flow of a viscous fluid through a porous material is crucial. Numerous authors [17, 19, 14, 20, 21, 23, 22, 16, 15, 24, 25, 18, 28, 29] have investigated fluid flows across porous media and discovered an exact solution.

The objective of this research is to analyze the motion of a rotating, steady, homogenous, incompressible, variably inclined MHD plane flow through a porous medium in order to obtain exact solutions. The fluid flow equation is described by nonlinear partial differential equations. The magnetograph transformation helps the nonlinear partial differential equations turn into a system of linear partial differential equations. Two independent variables and the two components of the magnetic field in the physical plane have been swapped out using the magnetograph transformation. Moreover, the magnetic flux function's Legendre transform function has been utilized to illustrate the flow equations. Finally, a few examples have been used to clarify the proposed theory and exact solutions have been found.

The expressions for the pressure distribution, magnetic lines, velocity vector components and magnetic field vector components are obtained and graphically examined. We first consider the appropriate steady flow equations in a rotating frame of reference, which includes coriolis force and centrifugal force with non-uniform angular velocity. Using a Legendre transform of the magnetic flux function and rewriting all of the equations in terms of this transformed function, the exact solutions are found by switching the dependent and independent variables in the magnetograph plane. Examples are considered to point out the usefulness of the method. The geometry of streamlines and magnetic lines are discussed. The general solution for angular velocity is also found with the variation of pressure and angular velocity is discussed by plotting various graphs for some different form of suitable examples.

2. Basic Equations

The fundamental equations that regulate the steady flow of a homogeneous incompressible viscous fluid with finite electrical conductivity in a porous medium in the presence of a magnetic field in a rotating reference frame are

$$\nabla \cdot \mathbf{V} = 0 \ , (\mathit{Continuity\ equation}) \tag{1}$$

$$\rho((\mathbf{V}\cdot\nabla) + 2\mathbf{\Omega} \times \mathbf{V} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})) = -\nabla P + \eta\nabla^2\mathbf{V} + \mu\mathbf{Q} \times \mathbf{H} - \frac{\eta}{\kappa}\mathbf{V} \, , (\mathit{Momentum\ Equation}) \tag{2}$$

$$\nabla \times (\mathbf{V} \times \mathbf{H}) = \nabla \times (\gamma_{\mathbf{H}} \nabla \times \mathbf{H}), \quad (Diffusion\ equation) \tag{3}$$

$$\nabla \cdot \mathbf{H} = 0, \qquad (Solenoidal\ equation) \tag{4}$$

where $\mathbf{V}$ = velocity field vector, P = fluid pressure, $\mathbf{H}$ = magnetic field vector, $\mathbf{Q}$ = current density, μ = magnetic permeability, σ = electrical conductivity of the fluid, ρ = the constant fluid field density, $\mathbf{\Omega}$ = angular velocity, η = coefficient of viscosity, κ = permeability of the medium, $\mathbf{r}$ = radius vector and $\gamma\mathbf{H}$ = magnetic viscosity, $\mathbf{\Omega} \times (\mathbf{\Omega} \times \tilde{\mathbf{r}})$ = centripital acceleration, $2\mathbf{\Omega} \times \mathbf{V}$ = coriolis acceleration.

On introducing the function

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \qquad (vorticity\ function) \tag{5}$$

$$Q = \frac{\partial \tilde{H}_2}{\partial x} - \frac{\partial \tilde{H}_1}{\partial y}, \qquad (Current\ density\ function) \tag{6}$$

$$B = \frac{1}{2}\rho V^2 + P' + \frac{1}{2}|\mathbf{\Omega} \times \mathbf{r}|^2, \quad (Bernoulli\ function) \tag{7}$$

where $V^2 = \tilde{u}^2 + \tilde{v}^2$, P' is the reduced pressure and $P' = P - \frac{1}{2}|\mathbf{\Omega} \times \mathbf{r}|^2$ and the last term being the centrifugal contribution of the pressure. The above system reduces to

$$\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} = 0, \tag{8}$$

$$\frac{\partial B}{\partial x} + \eta \frac{\partial \omega}{\partial y} - 2\rho\tilde{v}\Omega - \rho\omega\tilde{v} + \eta\tilde{H}_2 Q + \frac{\eta}{\kappa}\tilde{u} = 0, \tag{9}$$

$$\frac{\partial B}{\partial y} - \eta \frac{\partial \omega}{\partial y} - 2\rho\tilde{v}\Omega - \rho\omega\tilde{v} + \eta\tilde{H}_1 Q + \frac{\eta}{\kappa}\tilde{v} = 0, \tag{10}$$

$$\tilde{u}\tilde{H}_2 - \tilde{v}\tilde{H}_1 = \gamma_H \frac{\partial \tilde{H}_2}{\partial x} - \frac{\partial \tilde{H}_1}{\partial y} + c, \tag{11}$$

$$\frac{\partial \tilde{H}_2}{\partial x} - \frac{\partial \tilde{H}_1}{\partial y} = 0, \tag{12}$$

$$Q(x, y) = \frac{\partial \tilde{H}_2}{\partial x} - \frac{\partial \tilde{H}_1}{\partial y}, \tag{13}$$

$$\omega(x, y) = \frac{\partial \tilde{v}}{\partial x} - \frac{\partial \tilde{u}}{\partial y}, \tag{14}$$

of seven partial differential equations in eight unknown functions $\tilde{u}, \tilde{v}, \tilde{H}_1, \tilde{H}_2$, Ω, ω, Q and B which are functions of (x, y). In addition, c is an arbitrary integration constant that may be found using the diffusion equation (3). Martin [2] has successfully employed a first-order system similar to this one to investigate viscous non-MHD flows.

Let $\alpha = \alpha(x, y)$ be the variable angle such that $\alpha(x, y) \neq 0$ for every (x, y) in the region of flow. Equation (11) yields

$$\tilde{u}\tilde{H}_2 - \tilde{v}\tilde{H}_1 = UH \sin\alpha = c + \gamma HQ, \tag{15}$$

$$\tilde{u}\tilde{H}_1 + \tilde{v}\tilde{H}_2 = UH \cos\alpha = (c + \gamma_H Q)\cot\alpha, \tag{16}$$

where $H = \sqrt{(\tilde{H}_1{}^2 + \tilde{H}_2{}^2)}$. Considering these as two linear algebraic equations in the unknown's u and v, we solve (15) and (16) in terms of $\tilde{H}_1, \tilde{H}_2$, and α.

$$\tilde{u} = (c + \gamma_H Q)\left(\frac{\tilde{H}_2 + \tilde{H}_1 \cot\alpha}{\tilde{H}_1^2 + \tilde{H}_2^2}\right), \tag{17}$$

$$\tilde{v} = (c + \gamma_H Q)\left(\frac{\tilde{H}_2 \cot\alpha - \tilde{H}_1}{\tilde{H}_1^2 + \tilde{H}_2^2}\right), \tag{18}$$

we can eliminate $\tilde{u}$ and v from the system (8)-(14) by using equations (17) and (18) and then obtaining a system of equations to be solved for $\tilde{H}_1, \tilde{H}_2, \Omega, \omega, B, Q$ and α as functions of x and y, this approach leads to the study of system (8)-(14) in the magnetograph plane. By using (17)-(18) and removing u and v from the system of (8)-(14) we get the system of six partial differential equations as under,

$$\frac{\partial \tilde{H}_1}{\partial x} + \frac{\partial \tilde{H}_2}{\partial y} = 0\,, \tag{19}$$

$$\eta\,\frac{\partial \Omega}{\partial x} - \rho(\omega + 2\Omega)\left[(c + \gamma_H Q)\left(\frac{\tilde{H}_1 \cot\alpha + \tilde{H}_2}{\tilde{H}_1^2 + \tilde{H}_2^2}\right)\right]$$

$$+\eta Q \tilde{H}_2 + \frac{\eta}{\kappa}(c + \gamma_H Q)\left(\frac{\tilde{H}_2 + \tilde{H}_1 \cot\alpha}{\tilde{H}_1^2 + \tilde{H}_2^2}\right) = -\frac{\partial B}{\partial x}\,, \tag{20}$$

$$\eta\,\frac{\partial \Omega}{\partial y} - \rho(\omega + 2\Omega)\left[(c + \gamma_H Q)\left(\frac{\tilde{H}_2 + \tilde{H}_1 \cot\alpha}{\tilde{H}_1^2 + \tilde{H}_2^2}\right)\right]$$

$$+\eta Q \tilde{H}_1 + \frac{\eta}{\kappa}(c + \gamma_H Q)\left(\frac{\tilde{H}_2 \cot\alpha - \tilde{H}_1}{\tilde{H}_1^2 + \tilde{H}_2^2}\right) = \frac{\partial B}{\partial y}\,, \tag{21}$$

$$(c + \gamma_H Q)\left[\tilde{H}_1 \frac{\partial \cot\alpha}{\partial x} + \tilde{H}_2 \frac{\partial \cot\alpha}{\partial y}\right.$$

$$+\left(\frac{\partial \tilde{H}_2}{\partial x} - \frac{\partial \tilde{H}_1}{\partial y}\right)\left(\frac{\tilde{H}_1^2 - \tilde{H}_2^2 - 2\tilde{H}_1\tilde{H}_2 \cot\alpha}{\tilde{H}_1^2 + \tilde{H}_2^2}\right)$$

$$\left.+\left(\frac{\partial \tilde{H}_2}{\partial y} - \frac{\partial \tilde{H}_1}{\partial x}\right)\left(\frac{\tilde{H}_1^2 \cot\alpha - \tilde{H}_2^2 \cot\alpha + 2\tilde{H}_1\tilde{H}_2 \cot\alpha}{\tilde{H}_1^2 + \tilde{H}_2^2}\right)\right]$$

$$+\gamma_H\left[(\tilde{H}_2 + \tilde{H}_1 \cot\alpha)\frac{\partial Q}{\partial x} + (\tilde{H}_2 + \tilde{H}_1 \cot\alpha)\frac{\partial Q}{\partial y}\right] = 0\,, \tag{22}$$

$$\frac{\partial \tilde{H}_2}{\partial x} - \frac{\partial \tilde{H}_1}{\partial y} = Q\,, \tag{23}$$

$$\frac{\partial\left[(c + \gamma_H Q)\left(\frac{\tilde{H}_2 \cot\alpha - \tilde{H}_1}{\tilde{H}_1^2 + \tilde{H}_2^2}\right)\right]}{\partial x} - \frac{\partial\left[(c + \gamma_H Q)\left(\frac{\tilde{H}_2 + \tilde{H}_1 \cot\alpha}{\tilde{H}_1^2 + \tilde{H}_2^2}\right)\right]}{\partial y} = \omega, \tag{24}$$

3. Magnetograph transformations

As mentioned in the equations of flow $\tilde{H}_1 = \tilde{H}_1(x, y)$, $\tilde{H}_2 = \tilde{H}_2(x, y)$ the Jacobian

$$J(x, y) = \frac{\partial \tilde{H}_2}{\partial x}\frac{\partial \tilde{H}_2}{\partial y} - \frac{\partial \tilde{H}_1}{\partial y}\frac{\partial \tilde{H}_2}{\partial x} \neq 0 \tag{25}$$

Let x and y be functions of $\tilde{H}_1$ and $\tilde{H}_2$, that is, $x = x(\tilde{H}_1, \tilde{H}^2)$, $y = y(\tilde{H}_1, \tilde{H}^2)$.

Given these assumptions, we may have the following relations:

$$\frac{\partial \tilde{H}_1}{\partial x} = J\frac{\partial y}{\partial \tilde{H}_2}, \frac{\partial \tilde{H}_2}{\partial x} = -J\frac{\partial y}{\partial \tilde{H}_1}, \frac{\partial \tilde{H}_1}{\partial y} = -J\frac{\partial x}{\partial \tilde{H}_2}, \frac{\partial \tilde{H}_2}{\partial y} = J\frac{\partial y}{\partial \tilde{H}_1} \tag{26}$$

Further,

$$J(x, y) = \frac{\partial(\tilde{H}_1, \tilde{H}_2)}{\partial(x, y)} = \left[\frac{\partial(x, y)}{\partial(\tilde{H}_1, \tilde{H}_2)}\right]^{-1} = j(\tilde{H}_1, \tilde{H}_2),$$

$$\frac{\partial f}{\partial x} = j\frac{\partial(f, y)}{\partial(\tilde{H}_1, \tilde{H}_2)}, \frac{\partial f}{\partial y} = j\frac{\partial(x, f)}{\partial(\tilde{H}_1, \tilde{H}_2)}, \tag{27}$$

where $f(\tilde{H}_1, \tilde{H}_2)$ is transformed function of continuously differentiable function of f in the $\tilde{H}_1\tilde{H}_2$-plane.

4. Flow Equations in Magnetograph Plane

Applying the aforementioned transformation relations to the system of equations (19)-(24) in the magnetograph plane, or $(\tilde{H}_1, \tilde{H}_2)$ plane, for the first order partial derivatives results in the transformed system of partial differential equations being

$$\frac{\partial x}{\partial \tilde{H}_1} + \frac{\partial y}{\partial \tilde{H}_2} = 0\,, \tag{28}$$

$$\eta j \frac{\partial(x,\omega)}{\partial(\tilde{H}_1,\tilde{H}_2)} + \frac{\partial y}{\partial \tilde{H}_2} - \rho(\omega + 2\Omega)\left[(c + \gamma_H Q)\left(\frac{\tilde{H}_2 \cot\alpha - \tilde{H}_1}{\tilde{H}_1^2 + \tilde{H}_2^2}\right)\right]$$

$$+\eta Q\tilde{H}_2 + \frac{\eta}{\kappa}(c + \gamma_H Q)\left(\frac{\tilde{H}_2 + \tilde{H}_1 \cot\alpha}{\tilde{H}_1^2 + \tilde{H}_2^2}\right) = -j\frac{\partial(B,y)}{\partial(\tilde{H}_1,\tilde{H}_2)}\,, \tag{29}$$

$$\eta j \frac{\partial(\omega,y)}{\partial(\tilde{H}_1,\tilde{H}_2)} - \rho(\omega + 2\Omega)\left[(c + \gamma_H Q)\left(\frac{\tilde{H}_2 + \tilde{H}_2 \cot\alpha}{\tilde{H}_1^2 + \tilde{H}_2^2}\right)\right]$$

$$+\eta Q\tilde{H}_1 - \frac{\eta}{\kappa}(c + \gamma_H Q)\left(\frac{\tilde{H}_2 \cot\alpha - \tilde{H}_1}{\tilde{H}_1^2 + \tilde{H}_2^2}\right) = \frac{\partial(x,B)}{\partial(\tilde{H}_1,\tilde{H}_2)}\,, \tag{30}$$

$$j\left(\frac{\partial x}{\partial \tilde{H}_2} - \frac{\partial y}{\partial \tilde{H}_1}\right) = Q\,, \tag{31}$$

$$j\left[\frac{\partial\left(\left[(c + \gamma_H Q)\left(\frac{\tilde{H}_2 \cot\alpha - \tilde{H}_1}{\tilde{H}_1^2 + \tilde{H}_2^2}\right)\right], y\right)}{\partial(\tilde{H}_1,\tilde{H}_2)} - \frac{\partial\left(x, \left[(c + \gamma_H Q)\left(\frac{\tilde{H}_2 + \tilde{H}_1 \cot\alpha}{\tilde{H}_1^2 + \tilde{H}_2^2}\right)\right]\right)}{\partial(\tilde{H}_1,\tilde{H}_2)}\right] = \omega\,, \tag{32}$$

$$\frac{\partial x}{\partial \tilde{H}_1}\left[(c + \gamma_H Q)\left\{H_1^2 \cot\alpha - H_2^2 \cot\alpha + 2\tilde{H}_1\tilde{H}_2 \cot + \tilde{H}_2 H^2 \frac{\partial \cot\alpha}{\partial \tilde{H}_1}\right\}\right.$$

$$\left. + H^2\gamma_H \frac{\partial Q}{\partial \tilde{H}_1}(\tilde{H}_2 \cot\alpha - \tilde{H}_1)\right]$$

$$+\frac{\partial x}{\partial \tilde{H}_2}\left[(c + \gamma_H Q)\left\{\tilde{H}_1^2 - \tilde{H}_2^2 - 2\tilde{H}_1\tilde{H}_2 \cot + \tilde{H}_2 H^2 \frac{\partial \cot\alpha}{\partial \tilde{H}_1}\right\}\right.$$

$$+H^2\gamma_H \frac{\partial Q}{\partial \tilde{H}_1}(\tilde{H}_1 - \tilde{H}_2 \cot\alpha)\Bigg]$$

$$+\frac{\partial y}{\partial \tilde{H}_2}\Bigg[(c + \gamma_H Q)\left\{\tilde{H}_2^2 - \tilde{H}_1^2 + 2\tilde{H}_1\tilde{H}_2 \cot\alpha + \tilde{H}_1 H^2 \frac{\partial \cot\alpha}{\partial \tilde{H}_2}\right\}$$

$$+H^2\gamma_H \frac{\partial Q}{\partial \tilde{H}_2}(\tilde{H}_1 \cot\alpha + \tilde{H}_2)\Bigg]$$

$$+\frac{\partial y}{\partial \tilde{H}_1}\Bigg[(c + \gamma_H Q)\left\{H_2^2 \cot\alpha - H_1^2 \cot\alpha + 2\tilde{H}_1\tilde{H}_2 + \tilde{H}_1 H^2 \frac{\partial \cot\alpha}{\partial \tilde{H}_1}\right\}$$

$$+H^2\gamma_H \frac{\partial Q}{\partial \tilde{H}_1}(\tilde{H}_1 \cot\alpha - \tilde{H}_2)\Bigg] = 0. \tag{33}$$

5. Legendre Transform of Magnetic Flux Function

The solenoidal equation (19) verified the existence of the magnetic flux function $\phi(x, y)$ and is such that

$$d\phi = -\tilde{H}_2 dx + \tilde{H}_1 dy \quad \text{or} \quad \frac{\partial \phi}{\partial x} = -\tilde{H}_2, \frac{\partial \phi}{\partial y} = \tilde{H}_1, \tag{34}$$

Similarly, for the magnetic flux function $\phi(x, y)$, equation (28) verified the existence of the function $L(\tilde{H}_1, \tilde{H}_2)$, also known as Legendre's transform function. It is such that

$$dL = -y d\tilde{H}_1 + x d\tilde{H}_2 \quad \text{or} \quad \frac{\partial L}{\partial \tilde{H}_1} = -y, \frac{\partial L}{\partial \tilde{H}_2} = x, \tag{35}$$

and these two equation are connected by $L(\tilde{H}_1, \tilde{H}_2) = \tilde{H}_2 x - \tilde{H}_1 y + \phi(x, y)$.

Introducing $L(\tilde{H}_1, \tilde{H}_2)$ into the system of equations (28)-(33) it follows that equation (28) is identically satisfied with *j* given by (27) and the system is substituted by

$$\eta j \frac{\partial(\frac{\partial L}{\partial \tilde{H}_2}, \omega)}{\partial(\tilde{H}_1, \tilde{H}_2)} + \frac{\partial y}{\partial \tilde{H}_2} - \rho(\omega + 2\Omega)\left[(c + \gamma_H Q)\left(\frac{\tilde{H}_2 \cot\alpha - \tilde{H}_1}{\tilde{H}_1^2 + \tilde{H}_2^2}\right)\right]$$

$$+\eta J\tilde{H}_2 + \frac{\eta}{\kappa}(c + \gamma_H)\frac{\tilde{H}_2 + \tilde{H}_1 \cot\alpha}{\tilde{H}_1^2 + \tilde{H}_2^2} = j\frac{\partial B, \frac{\partial L}{\partial \tilde{H}_1}}{\partial(\tilde{H}_1, \tilde{H}_2)} \tag{36}$$

$$\eta j\frac{\partial(\omega, \frac{\partial L}{\partial \tilde{H}_1})}{\partial(\tilde{H}_1, \tilde{H}_2)} - \rho(\omega + 2\Omega)\left[(c + \gamma_H Q)\left(\frac{\tilde{H}_2 + \tilde{H}_1 \cot\alpha}{\tilde{H}_1^2 + \tilde{H}_2^2}\right)\right]$$

$$+\eta Q\tilde{H}_1 - \frac{\eta}{\kappa}(c + \gamma_H Q)\left(\frac{\tilde{H}_2 \cot\alpha - \tilde{H}_1}{\tilde{H}_1^2 + \tilde{H}_2^2}\right) = \frac{\partial(\frac{\partial L}{\partial \tilde{H}_2}, B)}{\partial(\tilde{H}_1, \tilde{H}_2)} \tag{37}$$

$$j\left[\frac{\partial\left(\left[(c + \gamma_H Q)\left(\frac{\tilde{H}_2 \cot\alpha - \tilde{H}_1}{\tilde{H}_1^2 + \tilde{H}_2^2}\right)\right], \frac{\partial L}{\partial \tilde{H}_1}\right)}{\partial(\tilde{H}_1, \tilde{H}_2)} - \frac{\partial\left(\frac{\partial L}{\partial \tilde{H}_2}, \left[(c + \gamma_H Q)\left(\frac{\tilde{H}_2 + \tilde{H}_1 \cot\alpha}{\tilde{H}_1^2 + \tilde{H}_2^2}\right)\right]\right)}{\partial(\tilde{H}_1, \tilde{H}_2)}\right] = \omega, \tag{38}$$

$$j\left(\frac{\partial^2 L}{\partial \tilde{H}_2^2} - \frac{\partial^2 L}{\partial \tilde{H}_1^2}\right) = Q, \tag{39}$$

$$\begin{aligned}
&\frac{\partial^2 L}{\partial \tilde{H}_2^2}(c + \gamma_H Q)\left\{\tilde{H}_2^2 - \tilde{H}_1^2 + 2\tilde{H}_1\tilde{H}_2 \cot + \tilde{H}_2 H^2 \frac{\partial \cot\alpha}{\partial \tilde{H}_1}\right\}\\
&\quad + H^2\gamma_H \frac{\partial Q}{\partial \tilde{H}_1}(\tilde{H}_1 - \tilde{H}_2 \cot\alpha)\Big]\\
&\quad - \frac{\partial^2 L}{\partial \tilde{H}_1^2}(c + \gamma_H Q)\left\{\tilde{H}_1^2 - \tilde{H}_2^2 + 2\tilde{H}_1\tilde{H}_2 \cot + \tilde{H}_2 H^2 \frac{\partial \cot\alpha}{\partial \tilde{H}_1}\right\}\\
&\quad + H^2\gamma_H \frac{\partial Q}{\partial \tilde{H}_1}(\tilde{H}_2 \cot\alpha) + \tilde{H}_1\Big]\\
&\quad + \frac{\partial^2 y}{\partial \tilde{H}_1 \partial \tilde{H}_2}(c + \gamma_H Q)H^2\left(\frac{\partial \cot\alpha}{\partial \tilde{H}_2} - \tilde{H}_1 \frac{\partial \cot\alpha}{\partial \tilde{H}_1}\right)\\
&\quad + H^2\gamma_H\left\{\frac{\partial Q}{\partial \tilde{H}_2}(\tilde{H}_2 \cot\alpha - \tilde{H}_1) - \frac{\partial Q}{\partial \tilde{H}_1}(\tilde{H}_1 \cot\alpha + \tilde{H}_2)\right\}\\
&\quad + (c + \gamma_H Q)(2\tilde{H}_1^2 \cot\alpha - \tilde{H}_2^2 \cot\alpha + 4\tilde{H}_1\tilde{H}_2 + \tilde{H}_1 H^2)\Big] = 0
\end{aligned} \tag{40}$$

$$j = \left[\frac{\partial^2 L}{\partial \tilde{H}_1^2} \frac{\partial^2 L}{\partial \tilde{H}_2{}^2} - \left(\frac{\partial^2 L}{\partial \tilde{H}_1 \partial \tilde{H}_2} \right)^2 \right]^{-1}, \tag{41}$$

for the seven functions $L(\tilde{H}_1\tilde{H}_2)$, $B(\tilde{H}_1\tilde{H}_2)$, $\omega(\tilde{H}_1\tilde{H}_2)$, $j(\tilde{H}_1\tilde{H}_2)$, $\alpha(\tilde{H}_1\tilde{H}_2)$, $J(\tilde{H}_1\tilde{H}_2)$ and $\Omega(\tilde{H}_1\tilde{H}_2)$.

Introducing polar co-ordinates $(H, \theta)\tilde{H}_1 = H \cos\theta$ and $\tilde{H}_2 = H \sin\theta$

$$\frac{\partial(F, G)}{\partial(\tilde{H}_1, \tilde{H}_2)} = \frac{1}{H} \frac{\partial(F^*, G^*)}{\partial(\tilde{H}_1, \tilde{H}_2)},$$

$$\frac{\partial}{\partial \tilde{H}_1} = \cos\theta \frac{\partial}{\partial H} - \frac{\sin\theta}{H} \frac{\partial}{\partial \theta},$$

$$\frac{\partial}{\partial \tilde{H}_2} = \sin\theta \frac{\partial}{\partial H} + \frac{\cos\theta}{q} \frac{\partial}{\partial \theta}$$

where $F(\tilde{H}_1, \tilde{H}_2) = F^*(H, \theta)$; $G(\tilde{H}_1\tilde{H}_2) = G^*(H, \theta)$ are continuously differentiable functions in (H, θ) coordinates, the equation (40) takes the form

$$\frac{\partial^2 L^*}{\partial H_2^2} \left[(c + \gamma_H J) H \frac{\partial \cot \alpha^*}{\partial \theta} + \gamma_H \cot \alpha^* \frac{\partial Q}{\partial \theta} \right]$$

$$+ \left(\frac{1}{H^2} \frac{\partial 2L^*}{\partial \theta^2} + \frac{1}{H^2} \frac{\partial L^*}{\partial \theta} \right) \left[H\gamma_H \frac{\partial j}{\partial H} - (c + \gamma_H Q) \right]$$

$$+ \left(\frac{1}{H} \frac{\partial^2 L^*}{\partial H \partial \theta} - \frac{1}{H^*} \frac{\partial L^*}{\partial \theta} \right) \left[(c + \gamma_H Q) \left(2 \cot \alpha^* - H \frac{\partial \cot \alpha^*}{\partial H} \right) - \cot \alpha^* H \gamma_H \frac{\partial Q^*}{\partial \theta} \right] = 0 . \tag{42}$$

6. Applications

Example 1: Let

$$L(\tilde{H}_1\tilde{H}_2) = N_1 \tan^{-1}\left(\frac{\tilde{H}_2}{\tilde{H}_1}\right) + N_2, \alpha(\tilde{H}_1, \tilde{H}_2) = \cot^{-1}(N_3 H_1^2 + N_3 H_2^2) \tag{43}$$

form a set of solution of the partial differential equation (40) when $N_1 \neq 0, N_2$ and N_3 are arbitrary constants. As N_3 is arbitrary, there are two cases of the solution which may considered by (43).

(i) If $N_3 \neq 0$ i.e., variably inclined flows and

(ii) If $N_3 = 0$ i.e., crossed flows.

When (i) $N_3 \neq 0$.

Using (43) in (35) we have

$$\tilde{H}_1(x, y) = \frac{N_1 x}{r^2}; \ \tilde{H}_2(x, y) = \frac{N_1 y}{r^2}, \quad r^2 = x^2 + y^2. \tag{44}$$

This represents radial flow and magnetic field profile is thus the arc of a rectangular hyperbola, using (44) we obtain

$$\tilde{u} = \frac{c}{N_1 r^2}(yr^2 + N_3 N_1^2 x), \ \tilde{v} = \frac{c}{N_1 r^2}(N_3 N_1^2 y - xr^2),$$

$$\omega(x, y) = \frac{-2c}{N_1}, \qquad Q = 0, \qquad \alpha(x, y) = \cot^{-1}\left(\frac{N_1^2 N_3}{r^2}\right) \tag{45}$$

With the help of (45) and integrability condition on B i.e.,

$$\frac{\partial^2 B}{\partial x \partial y} = \frac{\partial^2 B}{\partial y \partial x}$$

from equations (9) and (10) we get angular velocity

$$\left[y(x^2 + y^2) + N_3 N_1^2 x\right]\frac{\partial \Omega}{\partial x} - \left[x(x^2 + y^2) - N_3 N_1^2 y\right]\frac{\partial \Omega}{\partial y} - \frac{\eta}{k\rho}(x^2 + y^2) = 0 \tag{46}$$

The Lagrange form of solution of this equation is

$$\Omega = -N_3 N_1^2 \tan^{-1}\frac{y}{x} + C\,, \quad \text{where} \quad N_1^2 N_3 = \frac{\eta}{\rho k}\,, \tag{47}$$

the streamlines are given by $(x^2 + y^2) + N_1 N_3 \tan^{-1}\frac{y}{x} = constant$, the magnetic flux function is

$$\tan^{-1}\frac{y}{x} = constant$$

and from (9) and (10) we have

$$B(x, y) = \left(4\rho c^2 N_3 + \frac{\eta c}{kN_1} + \frac{2\eta c}{\kappa} N_3 N_1 - \frac{\eta c y}{kN_1}\right)\tan^{-1}\frac{y}{x} + \frac{\rho c^2}{N_1^2}(x^2 + y^2)$$

$$+\frac{\eta c x^2}{kN_1} + \frac{\eta c}{\kappa} + N_3 N_1\left(\tan^{-1}\frac{y}{x}\right)^2$$

$$+\frac{\rho c^2}{N_1^2}(x^2 + y^2)\tan^{-1}\frac{y}{x} - \frac{\eta c}{\kappa} N_3 N_1 \ln(x^2 + y^2) + constant, \tag{48}$$

and hence the pressure

$$P(x, y) = B - \frac{1}{2}\rho V^2,$$

is

$$P(x, y) = \left(4\rho c^2 N_3 + \frac{\eta c}{kN_1} + \frac{2\eta c}{\kappa} N_3 N_1 - \frac{\eta c y}{kN_1}\right)\tan^{-1}\frac{y}{x} + \frac{\rho c^2}{N_1^2}(x^2 + y^2)P(x, y)$$

$$+\frac{\eta c x^2}{kN_1} + \frac{\eta c}{\kappa} + N_3 N_1\left(\tan^{-1}\frac{y}{x}\right)^2 + \frac{\rho c^2}{N_1^2}(x^2 + y^2)\left(\tan^{-1}\frac{y}{x}\right) - \frac{\eta c}{\kappa} N_3 N_1 \ln(x^2 + y^2)$$

$$-\frac{1}{2}\frac{\rho c^2}{N_1^2}(x^2 + y^2) - \frac{1}{2}\frac{\rho c^2 N_3 N_1}{2(x^2 + y^2)} + constant \tag{49}$$

and

(ii) for $N_3 = 0$ i.e., crossed flows, the value of u, v, α calculate similarly by putting $N_3 = 0$ in equation (6.1).

By putting $N_3 = 0$ in equation (6.4) we get

$$\Omega = C_1 - \frac{\eta}{\rho k} \ln y$$

Again, B and P can be calculated by putting $N_3 = 0$ in equations (6.6) and (6.7) respectively.

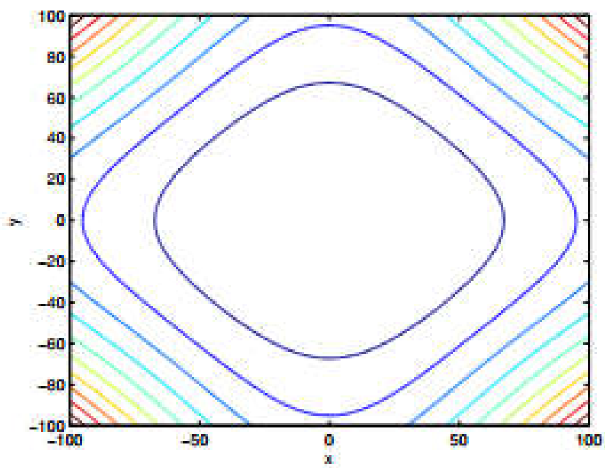

Figure 1: Streamlines for example 1

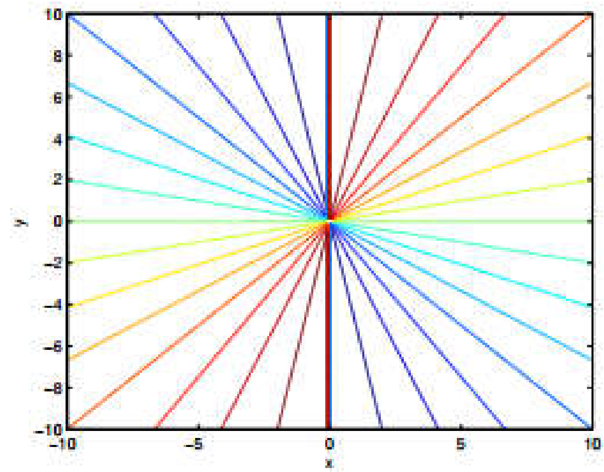

Figure 2: Magnetic lines for example 1

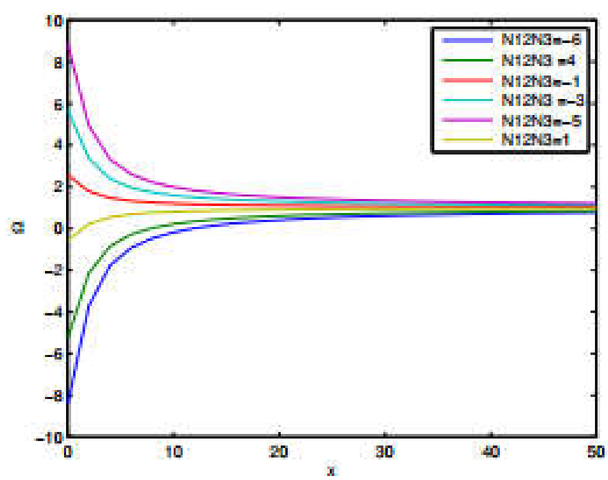

Figure 3: Variation of angular velocity versus x for example 1

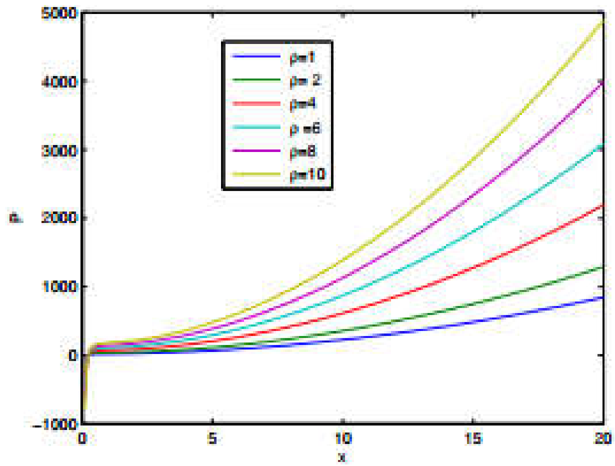

Figure 4: Variation of pressure versus x at $y = 2$ for density variation example 1

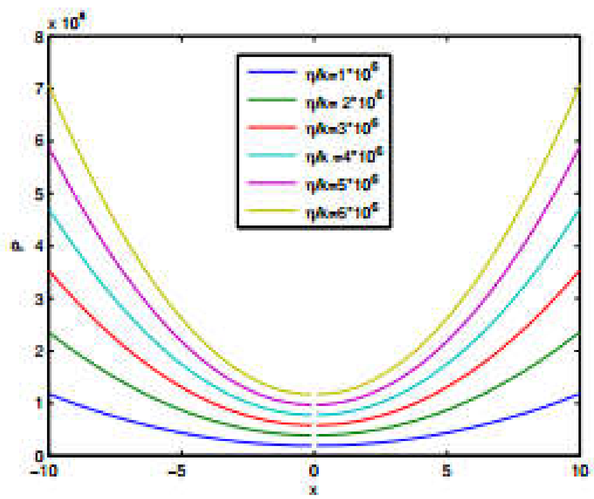

Figure 5: Variation of pressure versus x at $y = 2$ for porosity variation for example1

Example 2: Another solution of equation (5.7) is

$$L(\tilde{H}_1, \tilde{H}_2) = M_1(H_2^2 + H_1^2) + M_2, \ \ \alpha(\tilde{H}_1\tilde{H}_2) = \cot^{-1}(M_3H_1^2 + M_3H_2^2 + M_4) \tag{50}$$

Where $M_1 \neq 0, M_2, M_3$ and M_4 are arbitrary constants.

We have

$$\frac{dL}{d\tilde{H}_1} = -y, \frac{dL}{d\tilde{H}_2} = x,$$

We examine the case where M_3 and M_4 are arbitrary constants. When flows are variably inclined, $M_3 \neq 0$, i.e. The resulting flows are crossed if $M_3 = M_4 = 0$ and constantly inclined if $M_3 = 0, M_4 \neq 0$. Now consider the case when $M_3 \neq 0, M_4 \neq 0$. Using (49) in (35) we obtain

$$x = 2M_1\tilde{H}_2, \; y = -2M_1\tilde{H}_1$$

and therefore

$$\tilde{H}_1 = \frac{-y}{2M_1}, \;\; \tilde{H}_2 = \frac{-x}{2M_2}, \tag{51}$$

This indicates that the radial distance from the central axis directly affects the magnetic field $H = \dfrac{-r}{2M_1}$.

The changing angle between the velocity and magnetic fields in the physical plane is given by

$$\alpha(x, y) = \cot^{-1}\left[\frac{M_3 r^2}{4M_1^2} + M_4\right]$$

and hence vorticity, current density and velocity components are given by

$$\omega = \left(\frac{M_3}{M_1}c + \frac{\gamma_H}{M_1}\right), \;\; Q = \frac{1}{M_1}$$

$$\tilde{u} = \left(c + \frac{\gamma_H}{M_1}\right)\left[\frac{2M_1(x - M_4 y)}{(x^2 + y^2)} - \frac{M_3 y}{2M_1}\right];$$

$$\tilde{v} = \left(c + \frac{\gamma_H}{M_1}\right) \left[\frac{2M_1(x - M_4 y)}{(x^2 + y^2)} - \frac{M_3 y}{2M_1}\right]. \tag{52}$$

It is to be noted that velocity of the fluid is infinite when $r = 0$ i.e., when $H = 0$. And fluid velocity is zero when the radial distance is infinite and so the velocity of the fluid decreases as the r-increases. From (48) and integrability condition on B equations (9) and (10) yields the angular velocity Ω as

$$[M_3 x(x^2 + y^2) + 4M_1^2(y + M_4 x)] \frac{\partial \Omega}{\partial y} + [4M_1^2(x - M_4 y - M_3 x)(x^2 + y^2)] \frac{\partial \Omega}{\partial x}$$
$$+ \frac{\eta M_3}{k\rho}(x^2 + y^2) = 0. \tag{53}$$

The solution to this problem in Lagrange form is

$$\Omega = 4M_1^2 \tan^{-1} \frac{y}{x} - 2M_1^2 M_4 \ln(x^2 + y^2) - (x^2 + y^2) \frac{M_3 + 2}{2},$$

where
$$\frac{4M_1^2}{M_3} = \frac{\eta}{\rho k}, \tag{54}$$

The streamlines are provided by

$$8M_1^2 \tan^{-1} \frac{y}{x} - 8M_1^2 M_4 \ln(x^2 + y^2) + M_3(x^2 + y^2) = constant,$$

the magnetic flux function is

$$x^2 + y^2 = constant$$

and (54) yield the energy function B as

$$B(x, y) = \rho\omega\left(c + \frac{\gamma_H}{M_1}\right)\left[-4M_1 \tan^{-1} \frac{y}{x} + 2M_1 M_4 \ln(x^2 + y^2) + \frac{M_3}{2M_1}(x^2 + y^2)\right]$$

$$+2\rho\left(c+\frac{\gamma_H}{M_1}\right)\left[8\frac{M_1^3}{M_3}\left(\tan^{-1}\frac{y}{x}\right)^2-8\frac{M_1^3M_4}{M_3}\tan^{-1}\frac{y}{x}\ln(x^2+y^2)+2M_1xy\right.$$

$$\left.-2M_1(x^2+y^2)\tan^{-1}\frac{y}{x}+2\frac{M_1^3M_4}{M_3}(\ln(x^2+y^2))^2+M_1M_4(x^2+y^2)\right]$$

$$-\frac{\eta}{4M_1^2}(x^2+y^2)+\frac{\eta M_3}{kM_1}c+\frac{\gamma_H}{M_1}xy+constant. \tag{55}$$

And pressure is

$$P(x,y)=\rho\omega\left(c+\frac{\gamma_H}{M_1}\right)\left[-4M_1\tan^{-1}\frac{y}{x}+2M_1M_4\ln(x^2+y^2)+\frac{M_3}{2M_1}(x^2+y^2)\right]$$

$$+2\rho\left(c+\frac{\gamma_H}{M_1}\right)\left[8\frac{M_1^3}{M_3}\left(\tan^{-1}\frac{y}{x}\right)^2-8\frac{M_1^3M_4}{M_3}\tan^{-1}\frac{y}{x}\ln(x^2+y^2)+2M_1xy\right.$$

$$\left.-2M_1(x^2+y^2)\tan^{-1}\frac{y}{x}+2\frac{M_1^3M_4}{M_3}(\ln(x^2+y^2))^2+M_1M_4(x^2+y^2)\right]$$

$$-\frac{\eta}{4M_1^2}(x^2+y^2)+\frac{\eta M_3}{kM_1}\left(c+\frac{\gamma_H}{M_1}\right)xy$$

$$-\frac{1}{2}\rho\left(c+\frac{\gamma_H}{M_1}\right)^2\left[\frac{4M(1+M_4^2)}{(x^2+y^2)}+\frac{M_3}{4M_2}(x^2+y^2)+2M_3M_4\right]+P_0.$$

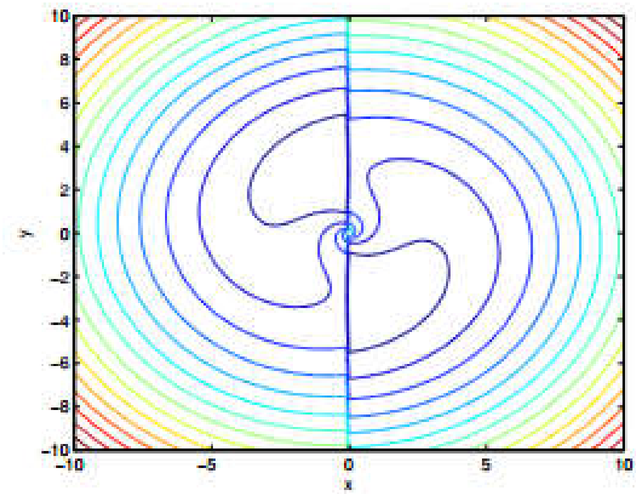

Figure 6: Streamlines for example 2

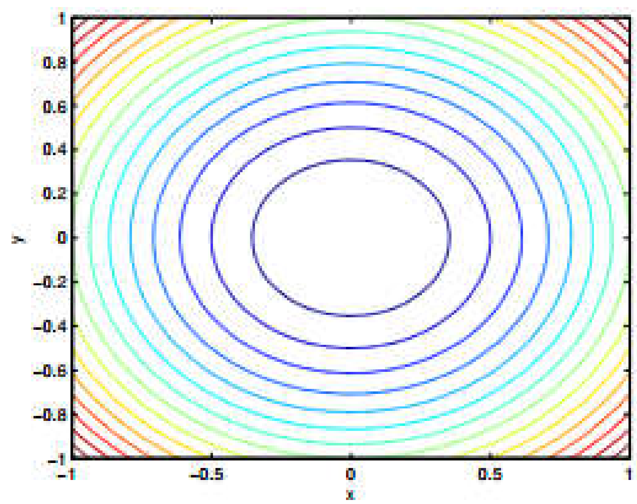

Figure 7: Magnetic lines for example 2

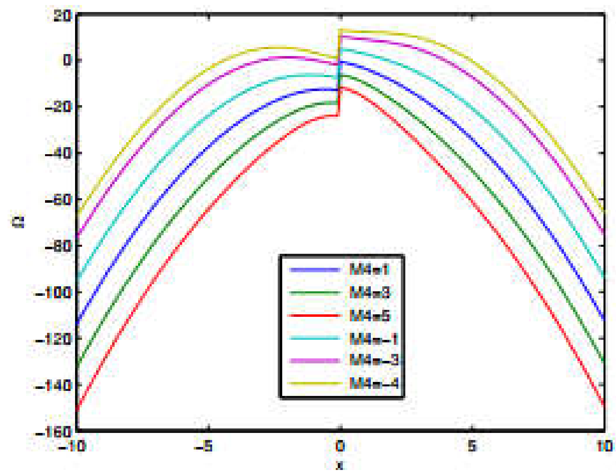

Figure 8: Variation of angular velocity versus x for example 2

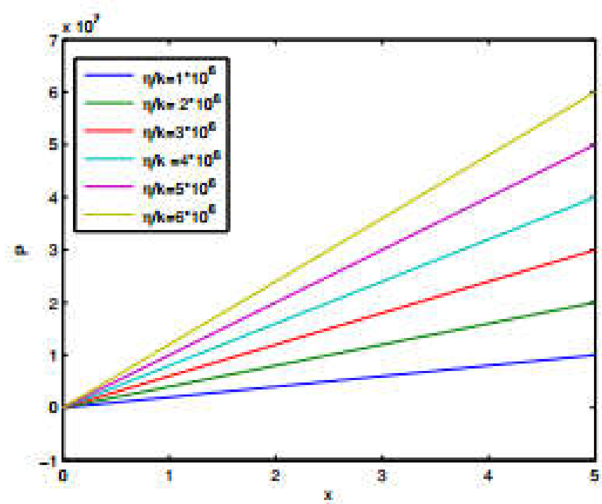

Figure 9: Variation of pressure versus x at $y = 2$ for $\frac{\eta}{\kappa}$ variation example 2

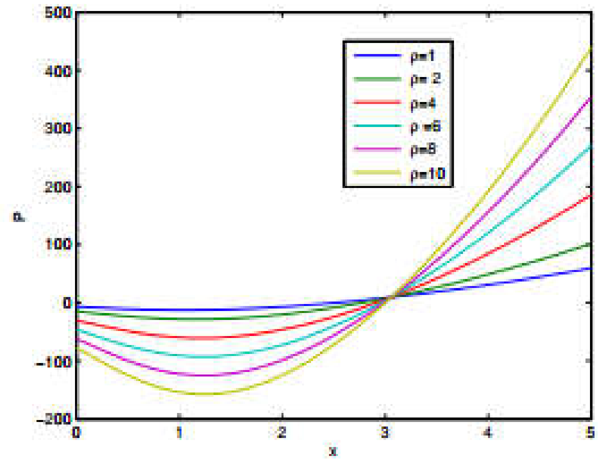

Figure 10: Variation of pressure versus x at $y = 2$ for $\kappa\,\eta$ variation example 2

Example 3: Consider

$$L^*(H,\theta) = A\theta + B\ln H + D \tag{57}$$

In (H,θ) coordinates, where D is an arbitrary constant and A and B are real values that are not zero. Applying this in (42) we have

$$B\frac{\partial}{\partial\theta}\cot\alpha - AH\frac{\partial}{\partial H}\cot\alpha + 2A\cot\alpha + 2B = 0$$

This has solution

$$\cot\alpha = -\frac{B}{A} + M_1H_2, M_1 = arbitrary\ constant$$

and

$$L^*(H,\theta) = A\theta + B\ln H + D,$$

and

$$\alpha^* = \cot^{-1}\left(-\frac{B}{A} + M_1H^2\right),$$

forms a solution set of the partial differential equation (42). If $M_1 = 0$ the flows are constantly inclined with

$$\alpha^* = \cot^{-1}\left(-\frac{B}{A}\right), \tag{58}$$

and when $M_1 \neq 0$, the flows are variably inclined, we have

$$\tilde{H}_1(x, y) = \frac{Ax - By}{r^2}; \quad \tilde{H}_2(x, y) = \frac{Bx + Ay}{r^2}, \quad r^2 = x^2 + y^2. \tag{59}$$

$$\tilde{u} = c\left\{\frac{y}{A} + \frac{(M_1(Ax - By))}{r^2}\right\}, \quad \tilde{v} = c\left\{-\frac{x}{A} + \frac{(M_1(Ay + Bx))}{r^2}\right\},$$

$$\omega(x, y) = \frac{-2c}{A}, \qquad Q = 0.$$

Now integrability condition for B yields

$$\{y(x^2 + y^2) + M_1 A(Ax - By)\}\frac{\partial\Omega}{\partial x} - \{AM_1(Bx + Ay) - x(x^2 + y^2)\}\frac{\partial\Omega}{\partial y} - \frac{\eta}{k\rho}(x^2 + y^2) = 0. \tag{60}$$

The Lagrange form of solution of this equation is

$$\Omega = M_1 A^2 \tan^{-1}\frac{y}{x} + M_1 AB\ln(x^2 + y^2), \text{ where } A^2 M_1 = \frac{\eta}{\rho k}. \tag{61}$$

the streamlines are given by

$$M_1 A^2 \tan^{-1}\frac{y}{x} + M_1 AB\ln(x^2 + y^2) + (x^2 + y^2) = constant$$

and the magnetic flux function is

$$A\tan^{-1}\frac{y}{x} + B\ln r = constant.$$

Now equation (9), (10) and Ω gives us the energy function

$$B(x, y) = \rho \frac{c^2}{A}\left\{-(x^2 + y^2) + \frac{M_1}{2}(A + B)\ln(x^2 + y^2) + M_1 \tan^{-1}\frac{x}{y}\right\}$$

$$+2\rho c\left\{\frac{2}{3}M_1Axy + \frac{2}{3}M_1A(x^2 + y^2)\tan^{-1}\frac{y}{x} + \frac{m_1^2A^2}{2}(1 + A)\left(\tan^{-1}\frac{y}{x}\right)\right.$$

$$-\left(\frac{M_1B}{2} + \frac{M_1^2A^2B}{2}\right)\ln(x^2 + y^2)\tan^{-1}\frac{y}{x}$$

$$\left.-\frac{M_1B}{2}(x^2 + y^2)\ln(x^2 + y^2) + \frac{M_1B}{2}(x^2 + y^2) + \frac{M_1^2A^2B}{2}(\ln(x^2 + y^2))^2\right\}$$

$$+\frac{\eta c}{\kappa}\left(-2\frac{xy}{A} + 2M_1B\tan^{-1}\frac{x}{y}\right) + P. \tag{62}$$

And hence, the pressure function is

$$P(x, y) = \rho \frac{c^2}{A}\left\{-(x^2 + y^2) + \frac{M_1}{2}(A + B)\ln(x^2 + y^2) + M_1 \tan^{-1}\frac{x}{y}\right\}$$

$$+2\rho c\left\{\frac{2}{3}M_1Axy + \frac{2}{3}M_1A(x^2 + y^2)\tan^{-1}\frac{y}{x} + \frac{m_1^2A^2}{2}(1 + A)\left(\tan^{-1}\frac{y}{x}\right)\right.$$

$$-\left(\frac{M_1B}{2} + \frac{M_1^2A^2B}{2}\right)\ln(x^2 + y^2)\tan^{-1}\frac{y}{x}$$

$$\left.-\frac{M_1B}{2}(x^2 + y^2)\ln(x^2 + y^2) + \frac{M_1B}{2}(x^2 + y^2) + \frac{M_1^2A^2B}{2}(\ln(x^2 + y^2))^2\right\}$$

$$+\frac{\eta c}{\kappa}\left(-2\frac{xy}{A} + 2M_1B\tan^{-1}\frac{x}{y}\right) - \frac{1}{2}\rho c\left[\frac{(x^2 + y^2)}{A^2} + M_1^2\frac{A^2 + B^2}{(x^2 + y^2)} - 2\frac{M_1B}{A^2}\right] + P_0. \tag{63}$$

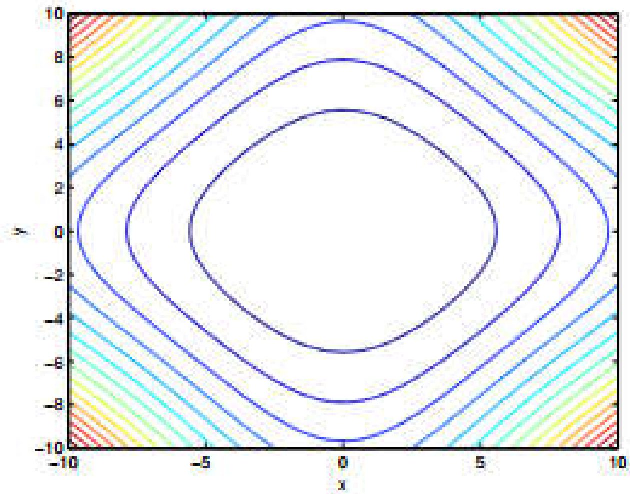

Figure 11: Streamline for example 3

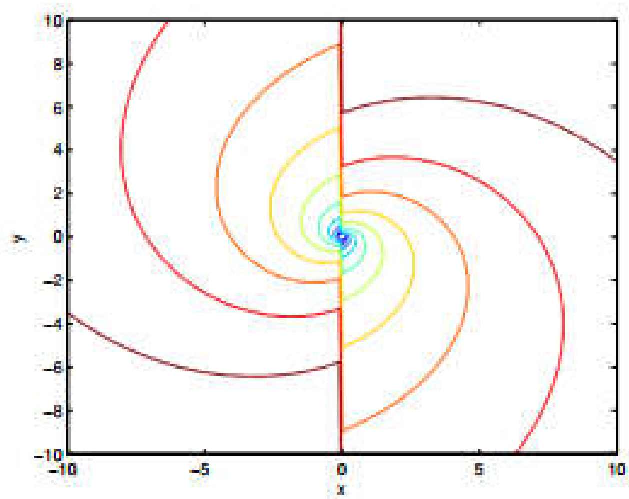

Figure 12: Magnetic line for example 3

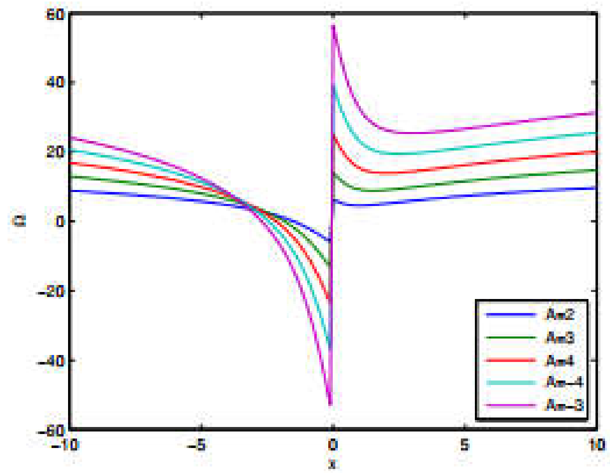

Figure 13: Variation of angular velocity verses x for example 3

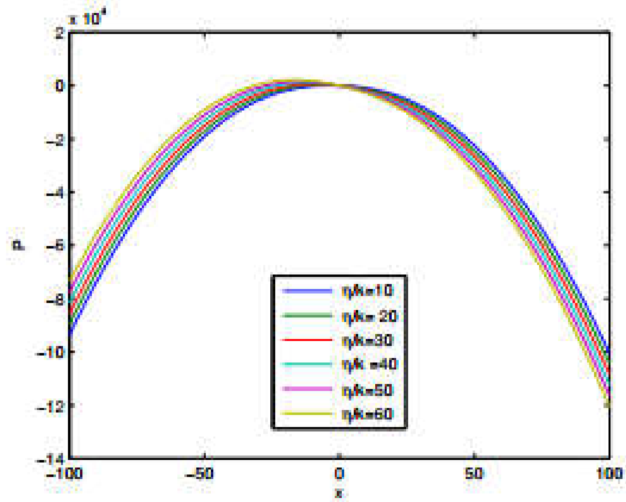

Figure 14: Variation of pressure verses x at $y = 2$ for $\dfrac{\eta}{\kappa}$ variation for example 3

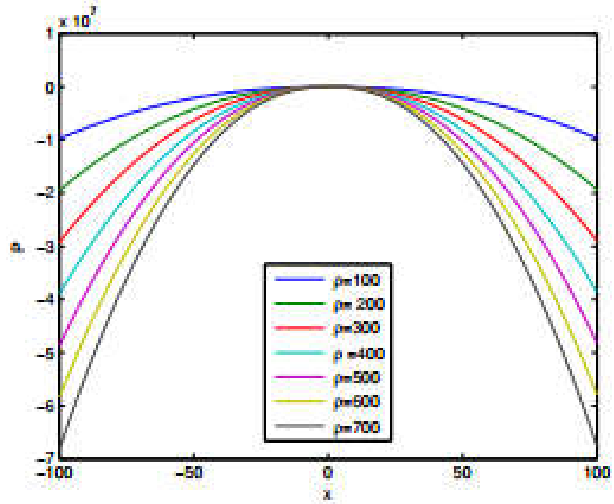

Figure 15: Variation of pressure verses x at $y = 2$ for fluid density variation for example 3

7. Conclusion

In this work, an approach has been carried out where magnetograph transformation method has been applied for the exact solution of the equations governing the flow of a homogeneous, incompressible viscous fluid through porous media of a variably inclined rotating MHD with finite electrical conductivity. We have utilized magnetograph transformation in this problem to reformulate the governing non-linear equation into linear once. Three different forms of Legendre transform function of the magnetic flux function have been considered as examples to illustrate the technique of solving for the exact solution. The expressions for streamlines, magnetic lines, angular velocity and pressure distribution are found out in each case. The main results are listed below:

- In example 1 the streamlines are given by $(x^2 + y^2) + N_1N_3 \tan^{-1}\frac{y}{x} = constant$ and magnetic lines are given by $\tan^{-1}\frac{y}{x} = constant$.

- In example 2 streamlines and magnetic lines are given by $8M_1^2 \tan^{-1}\frac{y}{x} - 8M_1^2M_4 \ln(x^2 + y^2) + M_3(x^2 + y^2) = constant$ and $x^2 + y^2 = constant$ respectively.

- In example 3, the streamlines are given by $M_1 A^2 \tan^{-1} \frac{y}{x} + M_1 AB \ln(x^2 + y^2) + (x^2 + y^2) = constant$ and magnetic lines are given by $A \tan^{-1} \frac{y}{x} + B \ln r = constant$.

- Also for example 1 components of velocity are independent of permeability of porous medium and angular velocity of rotating frame. The vorticity function is constant and current density is zero. Pressure depends on angular velocity, permeability of the medium and the fluid density.

- In example 2 for the form of Legendre transform function we find that the magnetic field varies with the radial distance from central axis. Current density function is constant and verticity function containing magnetic viscosity term is also a constant. The components of velocity depends on magnetic viscosity and current density function. Also, velocity of the fluid decreases with radial distance. Magnetic viscosity, current function, angular velocity, permeability of the medium and fluid density affects the pressure function.

- For the form of Legendre transform function considered in example 3 verticity function is constant, components of velocity does not involve permeability of medium and angular velocity. Pressure depends on angular velocity, permeability of medium and fluid density.

- Angular velocity depends on permeability of porous medium for all examples.

- In example 1 angular velocity for positive $N_1^2 N_3$ decreases with x and for negative increases x becoming almost constant beyond $x = 7$ for both cases. For the form of Legendre transform function represents radial flow and magnetic field profile is arc of a rectangular hyperbola.

- In example 2 angular velocity is found to increase with x in the beginning and shoots up at $x = 100$ and decreases afterward in Figure 4.

- In example 3 angular velocity is found to decrease with x in the beginning shoot up at $x = 0$ and shows varying trend there afterwards (Figure 8). In example 1 (Figure 2) pressure increases at constant $\frac{\eta}{\kappa}$ for different fluid of different densities. For different $\frac{\eta}{\kappa}$ values at constant fluid density ρ the pressure shows parabolic variations with x and is almost symmetric about $x = 100$.

- In examples 2 (Figure 5) Pressure varies linearly with x for different values of $\frac{\eta}{\kappa}$ at constant fluid density. For fluid of different densities at constant $\frac{\eta}{\kappa}$ pressure declines initially and increases rapidly with large x values.
- In example 3 pressure has a inverted parabolic variation (Figure 8 and 9) with x for different $\frac{\eta}{\kappa}$ at constant density ρ as well as different fluid density at constant $\frac{\eta}{\kappa}$ which are symmetric about $x = 0$.

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