ISSN : 0970-5120

THE JOURNAL of the INDIAN ACADEMY of MATHEMATICS

Volume 46 2024 No. 2

INDIAN ACADEMY OF MATHEMATICS

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PUBLISHED BY THE ACADEMY

2024

Advisory Board of Indian Academy of Mathematics 2022-2025

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Journal of Indian Acad. Math. ISSN: 0970-5120
Vol. 46, No. 2 (2024) pp. 75-82

Thomas Koshy

GIBONACCI SUMS INVOLVING TRIANGULAR NUMBERS

Abstract: We explore consequences of a generalization of an infinite gibonacci sum involving triangular numbers.

Keywords: Gibonacci Polynomials, Fibonacci Polynomial, Lucas Polynomial, Gibonacci Sums.

Mathematics Subject Classification (2020) No.: Primary 11B37, 11B39, 11C08.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 3].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $\Delta = \sqrt{x^2+4}$ and $2\alpha = x + \Delta$. With $2\alpha = 1+\sqrt{5}$, have $\alpha^3 = 2 + \sqrt{5}$, $\alpha^6 = 9 + 4\sqrt{5}$, and $2\alpha^{10} = 123 + 55\sqrt{5}$.

Triangular numbers t_n are positive integers that can pictorially be represented by equilateral triangular arrays. They can be defined recursively by $t_n = t_{n-1} + n$, where $t_1 = 1$ and $n \geq 2$ [2]. The first five triangular numbers are 1, 3, 6, 10, and 15.

2. Gibonacci Sums

A generalization of a sum involving gibonacci polynomial squares is studied in [6]. To showcase it, we first let

$$\mu = \begin{cases} 1, & if\ g_n = f_n \\ \Delta^2, & otherwise; \end{cases} \quad \text{and} \quad \nu^* = \begin{cases} 1, & if\ g_n = f_n \\ -1, & othewrwise. \end{cases}$$

With these tools at our fingertips, we now present the cornerstone of our discourse [6].

Theorem 1: *Let k, p, r, and t be positive integers, where* $t \leq 2p$. *Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} \mu\nu^* f_r f_{2pk}}{g_{(2pn+t-p)k}^2 - (-1)^{tk} \mu\nu^* f_{pk}^2} = \frac{g_{tk+r}}{g_{tk}} - \alpha^r. \tag{1}$$

The objective of our discourse is to explore consequences of this theorem when *k, p, r*, and *t* are triangular numbers ≤ 15. To this end, in the interest of brevity and expediency, we use the labels below in the explorations to follow.

A_1	=	10,336;	B_7	=	4,870,800;
A_2	=	11,552;	B_8	=	6,656,320;
A_3	=	11,556;	B_9	=	6,674,746;
A_4	=	13,500;	B_{10}	=	6,677,056;
A_5	=	13,530;	B_{11}	=	13,271,424;

A_6	=	15,125;	B_{12}	=	13,354,112;
A_7	=	18,040;	B_{13}	=	13,358,736;
A_8	=	20,736;	B_{14}	=	14,930,352;
A_9	=	20,801;	B_{15}	=	26,708,224;
A_{10}	=	25,840;	B_{16}	=	29,860,704;
A_{11}	=	27,060;	B_{17}	=	33,385,280;
A_{12}	=	30,258;	B_{18}	=	52,721,408;
A_{13}	=	35,424;	B_{19}	=	59,721,408;
A_{14}	=	60,885;	B_{20}	=	66,747,456;
A_{15}	=	67,650;	B_{21}	=	66,770,568;
B_1	=	103,361;	B_{22}	=	67,186,584;
B_2	=	104,005;	B_{23}	=	149,303,520;
B_3	=	187,575;	C_1	=	276,058,890,624;
B_4	=	248,050;	C_2	=	868,059,063;603;
B_5	=	802,780;	C_3	=	1,380,294,453,120;
B_6	=	1,845,492;	C_4	=	3,474,165,668,824.

Case 1: Suppose $p = 1$. Recalling that $t = 1$ and with $k, r \in \{1, 3, 6, 10\}$, the theorem yields [6]:

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}^2 + 1} = -\frac{1}{2} + \frac{\sqrt{5}}{2}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n}^2 - 5} = \frac{1}{2} - \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n+2}^2 + 1} = -1 + \frac{\sqrt{5}}{2}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n+2}^2 - 5} = \frac{1}{4} - \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{6n}^2 + 4} = -\frac{1}{8} + \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6n}^2 - 20} = \frac{1}{32} - \frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n}^2 - 64} = \frac{1}{128} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n}^2 + 320} = -\frac{1}{648} + \frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{20n}^2 - 3,025} = \frac{1}{6,050} - \frac{\sqrt{5}}{A_5}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{20n+1}^2 + A_6} = -\frac{5}{A_{12}} + \frac{\sqrt{5}}{A_{15}}.$$

Case 2: Let $p = 3$. With $k, r \in \{1, 3, 6, 10\}$ and $t \in \{1, 3, 6\}$, the theorem implies [4, 5]:

$$\sum_{n=1}^{\infty} \frac{1}{F_{6n-2}^2 + 4} = -\frac{1}{16} + \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6n-2}^2 - 20} = \frac{1}{16} - \frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{6n}^2 + 4} = -\frac{1}{8} + \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6n}^2 - 20} = \frac{1}{32} - \frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{6n+3}^2 - 4} = \frac{9}{64} - \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6n+3}^2 + 20} = -\frac{1}{36} + \frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{3(6n-2)}^2 + 3,025} = -\frac{1}{2,584} + \frac{\sqrt{5}}{5,168}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(6n-2)}^2 - A_6} = \frac{1}{A_1} - \frac{\sqrt{5}}{A_{10}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{3(6n)}^2 + 3,025} = -\frac{1}{2,312} + \frac{\sqrt{5}}{5,168}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(6n)}^2 - A_6} = \frac{1}{A_2} - \frac{\sqrt{5}}{A_{10}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{3(6n+3)}^2 - 3,025} = \frac{2,889}{B_{10}} - \frac{\sqrt{5}}{5,168}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(6n+3)}^2 + A_6} = -\frac{1}{A_3} + \frac{\sqrt{5}}{A_{10}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{6(6n-2)}^2 - B_{10}} = \frac{1}{B_{11}} - \frac{\sqrt{5}}{B_{16}}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6(6n-2)}^2 + B_{17}} = -\frac{1}{B_{22}} + \frac{\sqrt{5}}{B_{23}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{6(6n)}^2 - B_{10}} = \frac{1}{B_{12}} - \frac{\sqrt{5}}{B_{16}}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6(6n)}^2 + B_{17}} = -\frac{1}{B_{21}} + \frac{\sqrt{5}}{B_{23}};$$

Case 3: Let $p = 6$. With $k \in \{1, 3\}$, $r \in \{1, 3, 6\}$, and $t \in \{1, 3, 6, 10\}$, we get:

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n-5}^2 + 64} = -\frac{1}{288} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n-5}^2 - 320} = \frac{1}{288} - \frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n-3}^2 + 64} = -\frac{1}{144} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n-3}^2 - 320} = \frac{1}{576} - \frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n}^2 - 64} = \frac{1}{128} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n}^2 + 320} = -\frac{1}{648} + \frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n+4}^2 - 64} = \frac{41}{5,280} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n+4}^2 + 320} = -\frac{55}{A_{13}} - \frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{3(12n-5)}^2 + 64} = -\frac{1}{B_{14}} + \frac{\sqrt{5}}{B_{16}}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(12n-5)}^2 - 320} = \frac{1}{B_{18}} - \frac{\sqrt{5}}{B_{23}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{3(12n-3)}^2 + B_{10}} = -\frac{1}{B_{13}} + \frac{\sqrt{5}}{B_{16}}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(12n-3)}^2 - B_{17}} = \frac{1}{B_{21}} - \frac{\sqrt{5}}{B_{23}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{3(12n)}^2 - B_{10}} = \frac{1}{B_{12}} - \frac{\sqrt{5}}{B_{16}}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(12n)}^2 + B_{17}} = -\frac{1}{B_{21}} + \frac{\sqrt{5}}{B_{23}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{3(12n+2)}^2 - B_{10}} = \frac{B_1}{C_3} - \frac{\sqrt{5}}{B_{16}}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(12n+2)}^2 + B_{17}} = -\frac{1}{B_{11}} + \frac{\sqrt{5}}{B_{23}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{3(12n-5)}^2 + B_{10}} = -\frac{1}{B_{14}} + \frac{\sqrt{5}}{B_{16}}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(12n-5)}^2 - B_{17}} = \frac{1}{B_{19}} - \frac{\sqrt{5}}{B_{23}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{3(12n+4)} - B_{10}} = \frac{B_1}{C_3} - \frac{\sqrt{5}}{B_{16}}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{3(12n+4)} + B_{17}} = -\frac{B_2}{C_4} + \frac{\sqrt{5}}{B_{23}}.$$

Case 4: Next, we let $p = 10$. Then $t \le 20$. With $k = 1 = r$ and $t \in \{1, 3, 6, 10, 15\}$, the theorem yields:

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{20n-9} + 3,025} = -\frac{1}{A_5} + \frac{\sqrt{5}}{A_5}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{20n-9} - A_6} = \frac{1}{A_5} - \frac{\sqrt{5}}{A_{15}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{20n-7} + 3,025} = -\frac{1}{6,765} + \frac{\sqrt{5}}{A_5}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{20n-7} - A_6} = \frac{1}{A_{11}} - \frac{\sqrt{5}}{A_{15}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{20n-4} - 3,025} = \frac{3}{A_7} - \frac{\sqrt{5}}{A_5}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{20n-4} + A_6} = -\frac{2}{A_{14}} + \frac{\sqrt{5}}{A_{15}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{20n} - 3,025} = \frac{1}{B_4} - \frac{\sqrt{5}}{A_5}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{20n} + A_6} = -\frac{1}{A_{12}} + \frac{\sqrt{5}}{A_{15}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{20n+5} + 3,025} = -\frac{31}{B_3} + \frac{\sqrt{5}}{A_5}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{20n+5} - A_6} = \frac{61}{B_6} - \frac{\sqrt{5}}{A_{15}}.$$

We now encourage gibonacci enthusiasts to compute the sums when $k = 3$ and $t \in \{1, 3, 6, 10, 15\}$.

Case 5: Finally, we let $p = 15$, and restrict $k = 1 = r$ and $t \in \{1, 3, 6, 10, 15\}$. With the labels

$$D1 = 372,100; \quad E1 = 3,720,992;$$
$$D2 = 744,200; \quad E2 = 3,744,180;$$
$$D3 = 832,040; \quad E3 = 40,936,368,$$

we then have:

$$\sum_{n=1}^{\infty} \frac{1}{F_{30n-14}^2 + D_1} = -\frac{1}{2D_3} + \frac{\sqrt{5}}{2D_3}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{30n-14}^2 - 5D_1} = \frac{1}{2D_3} - \frac{\sqrt{5}}{10D_3};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{30n-12}^2 + D_1} = -\frac{1}{D_3} + \frac{\sqrt{5}}{2D_3}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{30n-12}^2 - 5D_1} = \frac{1}{4D_3} - \frac{\sqrt{5}}{10D_3};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{30n-8}^2 - D_1} = \frac{9}{8D_3} - \frac{\sqrt{5}}{2D_3}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{30n-8}^2 + 5D_1} = -\frac{1}{E_2} + \frac{\sqrt{5}}{10D_3};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{30n-5}^2 - D_1} = \frac{123}{11D_3} - \frac{\sqrt{5}}{2D_3}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{30n-5}^2 + 5D_1} = -\frac{1}{E_3} + \frac{\sqrt{5}}{10D_3};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{30n}^2 + D_1} = -\frac{1}{D_2} + \frac{\sqrt{5}}{2D_3}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{30n}^2 - 5D_1} = -\frac{1}{E_1} + \frac{\sqrt{5}}{10D_3}.$$

Again, we encourage the gibonacci enthusiasts to compute the sums with $k \in \{3, 6\}$.

Acknowledgment

The author is grateful to Z. Gao for a careful reading of the article and his computational assistance.

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(*Received*, May 14, 2024)

Journal of Indian Acad. Math.
Vol. 46, No. 2 (2024) pp. 83-99

ISSN: 0970-5120

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HYPERBOLIC HYBRID VALUED PROBABILISTIC MEASURES UNDER THE FLAVOUR OF KOLMOGOROV'S AXIOMS

Abstract: The main aim of this paper is to introduce a notion of a hyperbolic hybrid valued probabilistic measurable space to generalize 'Kolmogorov's system of axioms'. The probability which we define here may take values $\mathbf{e}_+$ and $\mathbf{e}_-$ for the certain event other than **1** which is the key difference from the probability in $\mathbb{R}$, where $\mathbf{e}_+$ and $\mathbf{e}_-$ are very special kind of zero divisors in the ring of hybrid numbers. In this work we also prove the usual properties of probability theory like extended addition theorem, Boole's inequality, continuity theorem, Bonferroni's inequalities etc. by this new measure.

Keywords and Phrases: Kolmogorov's Axioms, Measurable Spaces, Hybrid Numbers, Idempotent Representation, Hyperbolic Hybrid Valued Probabilistic Measure.

Mathematics Subject Classification (2020) No.: 60A05, 60A10

1. Introduction, Notations and Definitions

A probabilistic measurable space $(\Omega, \Sigma, P_{\mathbb{R}})$ is a triplet formed by a set Ω which has no structure but represents the sample space of a random experiment, a σ-field of subsets Σ of Ω and a measure $P_{\mathbb{R}}$ on the measurable space (Ω, Σ) which satisfies $P_{\mathbb{R}}(\Omega) = 1$. In 1933 Russian mathematician Andrei Nikolaevitch Kolmogorov introduced an alternative approach in formalizing the probability by three axioms known as Kolmogorov's axioms which stated as follows:

Axiom-I: The probability of an event is always a non-negative real number.

Axiom-II: The probability of that at least one of the elementary events in the entire sample space is 1.

Axiom-III: Probability of countable union events is same as the countable sum of the probability of each events.

Kolmogorov's empowered us to study the probability theory in the context of measure theory. We can derive many important results from these axioms for an example the law of large numbers. More precisely, we can say a probability space is a measurable space with total mass equal to 1 and a random variable which is a real valued measurable function [6], [8], [10], [7]. Kolmogorov introduced the set Ω as the space of elementary events.

In the last century, a lot of researcher's works with some two-dimensional systems like complex, dual and hyperbolic number systems which have the most significant roles in algebraic, geometric, physics, engineering, etc. The geometry of the Euclidean plane, the Minkowski plane and the Gallian plane can be described with the help of complex numbers

$$\mathbb{C} = \{a + \mathbf{i}b : a, \; b \in \mathbb{R}, \mathbf{i}^2 = -1\},$$

dual numbers

$$\mathbb{D} = \{a + \varepsilon b : a, b \in \mathbb{R}, \; \varepsilon^2 = 0\}$$

and hyperbolic numbers

$$\mathbb{H} = \{a + \mathbf{h}b : a, b \in \mathbb{R}, \mathbf{h}^2 = 1\}.$$

We know that the complex numbers, dual numbers and hyperbolic numbers can be described as the quotient of the polynomial ring $\mathbb{R}[x]$ by the ideal generated by the polynomials $x^2 + 1$, x^2 and $x^2 - 1$ respectively, i.e.,

$$\mathbb{C} = \mathbb{R}[x]/ < x^2 + 1 >, \quad \mathbb{D} = \mathbb{R}[x]/ < x^2 >, \quad \mathbb{H} = \mathbb{R}[x]/ < x^2 - 1 >$$

For a piece of detailed information about complex, dual and hyperbolic numbers one can read some outstanding articles and books [1], [5], [11], [2], [3], [4]. Now any combination of the complex, dual and hyperbolic numbers satisfying the relation $\mathbf{ih} = -\mathbf{hi} = \mathbf{i} + \varepsilon$ forms a new non-commutative ring [9] and opens a new direction of thoughts, which we discuss in the following section.

Definition 1.1: [9] *The set of hybrid numbers, denoted by* $\mathbb{K}$, *contains complex, dual and hyperbolic numbers as well as any combination of these three types of numbers with* $\mathbf{ih} = -\mathbf{hi} = \mathbf{i} + \varepsilon$ *i.e.,*

$$\mathbb{K} = \{Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} : a, b, c, d \in \mathbb{R}, \mathbf{i}^2 = -1, \ \varepsilon^2 = 0, \mathbf{h}^2 = 1 , \\ \mathbf{ih} = -\mathbf{hi} = \mathbf{i} + \varepsilon\}$$

The geometry corresponding to the hybrid numbers is called **Hybridian plane geometry**. This plane is a two-dimensional subspace of $\mathbb{R}^4$. The real part $'a'$ of the hybrid number $Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ is called **scalar part** and is denoted by $S(Z)$ whereas the remaining part $b\mathbf{i} + c\varepsilon + d\mathbf{h}$ is called **vector part** and is denoted by $\mathbf{V(Z)}$.

Definition 1.2: [9] (***Conjugate of a hybrid number***) *The conjugate of a hybrid number* $Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ *is denoted by* $\overline{Z}$ *and is defined by*

$$\overline{Z} = S(Z) - V(Z) = a - b\mathbf{i} - c\varepsilon - d\mathbf{h} .$$

This conjugation of a hybrid number is additive, involutive and multiplicative operation on $\mathbb{K}$. i.e., for any two hybrid numbers Z_1 and Z_2

a. $\overline{Z_1 + Z_2} = \overline{Z_1} + \overline{Z_2}$

b. $\overline{(\overline{Z_1})} = Z_1$

c. $\overline{Z_1 . Z_2} = \overline{Z_1} . \overline{Z_2}$

Definition 1.3: [9] (***Character of a hybrid number***) *The real number* $C(Z) = Z\overline{Z} = a^2 + (b - c)^2 - c^2 - d^2$ *is called the character of the hybrid number Z.*

Since, $C(Z) \in \mathbb{R}$, so depending on the value of $C(Z)$ a hybrid number can be categorized into three parts, spacelike, timelike, or lightlike according as the character is negative, positive or zero.

Also, the real number $\sqrt{|C(Z)|}$ is called the **norm** of the hybrid number Z and is denoted by $\|Z\|$.

Definition 1.4: [9] *The inverse of the hybrid number* $Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$, $\|Z\| \neq 0$ *is defined as*

$$Z^{-1} = \frac{\bar{Z}}{C(Z)}.$$

Therefore, we can conclude that a lightlike hybrid number never possesses an inverse. So if we consider a set of zero divisors $\mathfrak{Z}$ of the ring of hybrid numbers then it contains all non-zero lightlike hybrid numbers.

Definition 1.5: [9] (***Hybrid vector***) *For the hybrid number* $Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$, *the vector* $\varepsilon_Z = ((b - c), (c, d)$ *is called the hybrid vector of* Z .

Definition 1.6: [9] (***Type of a hybrid number***) *The real number* $\mathcal{T}_\varepsilon(Z) = -(b - c)^2 + c^2 + d^2$ *is called the type of the hybrid number Z.*

Depending on the real value of the type of a hybrid number, it is classified as complike (elliptic), hiperlike (hyperbolic) or duallike (parabolic) for $\mathcal{T}_\varepsilon(Z) < 0$, $\mathcal{T}_\varepsilon(Z) > 0$ or $\mathcal{T}_\varepsilon(Z) = 0$ respectively. Also, the real number, $\sqrt{|\mathcal{T}_\varepsilon(Z)|}$ is called the norm of the hybrid vector of Z and is denoted by $\mathcal{N}(Z)$.

1.1 Idempotent Representation of hyperbolic Hybrid Numbers: Let us denote the set of hyperbolic hybrid numbers as $\mathcal{P}$. Every hyperbolic hybrid number $Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ can be written [9] as

$$Z = s + \mathbf{H}v$$

where

$$\mathbf{H} = \frac{b\mathbf{i} + c\varepsilon + d\mathbf{h}}{\mathcal{N}(Z)} \quad \text{and} \quad \mathbf{H}^2 = 1\,.$$

Now, let us consider two numbers $\mathbf{e}_+ = \frac{1+\mathbf{H}}{2}$ and $\mathbf{e}_- = \frac{1-\mathbf{H}}{2}$. These two numbers are satisfying the equalities, $\|\mathbf{e}_+\| = \|\mathbf{e}_-\| = \mathbf{e}_+\mathbf{e}_- = 0$, $\mathbf{e}_+^2 = \mathbf{e}_+$ and $\mathbf{e}_-^2 = \mathbf{e}_-$ and is called idempotent elements. Since, $\|\mathbf{e}_+\| = \|\mathbf{e}_-\| = 0$, so $\mathbf{e}_+, \mathbf{e}_- \in \mathfrak{Z}$ and both are mutually complementary idempotent elements. Thus, the two principal ideals $\mathcal{P}_{\mathbf{e}+} := \mathbf{e}_+.\mathcal{P}$ and $\mathcal{P}_{\mathbf{e}-} := \mathbf{e}_-.\mathcal{P}$ in the ring $\mathcal{P}$ have the following properties

$$\mathcal{P}_{\mathbf{e}+} \cap \mathcal{P}_{\mathbf{e}-} = \{0\} \quad \text{and} \quad \mathcal{P} = \mathcal{P}_{\mathbf{e}+} \cup \mathcal{P}_{\mathbf{e}-} \tag{1.1}$$

and property 1.1 is known as the idempotent decomposition of $\mathcal{P}$. With the help of these idempotent elements $\mathbf{e}_+$ and $\mathbf{e}_-$, every hyperbolic hybrid number can be uniquely expressed as their linear combination

$$Z = (s+v)\mathbf{e}_+ + (s-v)\mathbf{e}_- = z_+\mathbf{e}_+ + z_-\mathbf{e}_- \tag{1.2}$$

where $z_+ = s + v \in \mathbb{R}$ and $z_- = s - v \in \mathbb{R}$ and this representation is called the idempotent representation of a hyperbolic hybrid number.

Lemma 1.1: The *sets* $\mathcal{P}_{\mathbf{e}+} = \{r_1\mathbf{e}_+ : r_1 \in \mathbb{R}\} = \mathbb{R}\mathbf{e}_+$ *and* $\mathcal{P}_{\mathbf{e}-} = \{r_2\mathbf{e}_- : r_2 \in \mathbb{R}\} = \mathbb{R}\mathbf{e}_-$ *satisfy the following properties*

a. $\mu \in \mathcal{P}\mathbf{e}_+ \Longleftrightarrow \mu\mathbf{e}_+ = \mu$;

b. $\nu \in \mathcal{P}_{\mathbf{e}-} \Longleftrightarrow \nu\mathbf{e}_- = \nu$.

Lemma 1.2: *Let us consider a set* $\mathcal{P}^+ = \{z_+\mathbf{e}_+ + z_-\mathbf{e}_- : z_+, z_- \geq 0\}$ *and a relation on* $\mathcal{P}$ *in such a way that for any two hyperbolic hybrid numbers* ζ_1 *and* ζ_2, $\zeta_1 \quad \zeta_2$ *if and only if* $\zeta_2 - \zeta_1 \in \mathcal{P}^+$.

Clearly this relation is a reflexive, antisymmetric and transitive relation and hence it is a poset (partial order relation) on $\mathcal{P}$.

The poset have the following properties:

For $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathcal{P}$ and $\xi \in \mathcal{P}^+$,

a. If $\zeta_1 \preceq \zeta_2$ and $\xi\zeta_1 \preceq \xi\zeta_2$.

b. If $\zeta_1 \preceq \zeta_2$ and $\zeta_3 \preceq \zeta_4$ then $\zeta_1 + \zeta_3 \preceq \zeta_2 + \zeta_4$.

c. If $\zeta_1 \preceq \zeta_2$ then $-\zeta_2 \preceq -\zeta_1$.

Before proving our main results, we have to go through following definition.

Definition 1.7: *Consider the set* $\mathcal{P} \cup \{0, 1\}$ *as* $\mathfrak{H}$ *and let* (Ω, Σ) *be a measurable space, a function* $P_{\mathfrak{H}} : \Sigma \mapsto \mathfrak{H}$ *with the properties*:

1. *for any event* $A \in \Sigma$, $P_{\mathfrak{H}}(A) \succeq 0$,

2. *for the certain event* Ω, $P_{\mathfrak{H}}(\Omega) = \mathfrak{p}$, *where* $\mathfrak{p}$ *is either* 1 *or* $\mathbf{e}_+$ *or* $\mathbf{e}_-$, *and*

3. *for a given sequence* $\{A_n\} \subset \Sigma$ *of pairwise disjoint events,*

$$P_{\mathfrak{H}}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P_{\mathfrak{H}}(A_n)$$

is called a $\mathfrak{H}$*-valued probabilistic measure, or a* $\mathfrak{H}$*-valued probability, on the* σ*-algebra of events* Σ *and the triplet* $(\Omega, \Sigma, P_{\mathfrak{H}})$ *is called a* $\mathfrak{H}$*-probabilistic space.*

Since, $\mathcal{P}_{\mathfrak{H}}(A) \in \mathfrak{H}$. So, it can be expressed as

$$\mathcal{P}_{\mathfrak{H}}(A) = \mathcal{P}_1(A)\mathbf{e}_+ + \mathcal{P}_2(A)\mathbf{e}_-.$$

Now property 1. of $\mathfrak{H}$-valued probabilistic measure implies that

$$P_1(A) \geq 0 \quad \text{and} \quad P_2(A) \geq 0, \; \forall A \in \Sigma.$$

From property 2, we get that

$$P_{\mathfrak{H}}(\Omega) = \mathfrak{p} = P_1(\Omega)\mathbf{e}_+ + P_2(\Omega)\mathbf{e}_-,$$

i.e.,

(I) If $\mathfrak{p} = 1$ then $P_1(\Omega) = 1, P_2(\Omega) = 1$.

(II) If $\mathfrak{p} = \mathbf{e}_+$ then $P_1(\Omega) = 1, \; P_2(\Omega) = 0$.

(III) If $\mathfrak{p} = \mathbf{e}_-$ then $P_1(\Omega) = 0, \; P_2(\Omega) = 1$.

The property 3. of $P_{\mathfrak{H}}$ implies

$$P_{\mathfrak{i}}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P_{\mathfrak{i}}(A_n) \quad \text{for } i = 1 \text{ and } 2.$$

Therefore one can say that, in general, the $\mathfrak{H}$-valued probabilis- tic measure is equivalent if we consider a pair of unrelated usual $\mathbb{R}$-valued measures on the same measurable space.

Here we can observe that P_1 is a probabilistic measure in the cases (I) and (II); P_2 is a probabilistic measure in the cases (I) and (III) whereas P_2 and P_1 are trivial measures for the cases (II) and (III) respectively. The cases (II) and (III) can be seen as two different embedding of the $\mathbb{R}$-valued probabilistic measures into our new concept of $\mathfrak{H}$-valued probabilistic measures: we identify such real-valued measures with $\mathfrak{H}$-probabilistic measures which takes as its values only zero divisors.

Now we are in a position to prove our main results.

2. Main Results

In this section, we have discussed the usual properties of $\mathfrak{H}$-valued

Probabilistic Measures.

Theorem 2.1: *Let* $(\Omega, \Sigma, P_{\mathfrak{H}})$ *be a* $\mathfrak{H}$*-probabilistic space. Then*

(i) *the* $\mathfrak{H}$*-valued probability of the null event and a complementary event of* $A \in \Sigma$ *are respectively,* 0 *and* $\mathfrak{p} - P_{\mathfrak{H}}(A)$.

(ii) *if* $A_1, A_2, \ldots, A_n \in \Sigma$ *are* n *events, then*

$$P_{\mathfrak{H}}\left(\bigcup_{i=1}^{n} A_n\right) = \sum_{i=1}^{n} P_{\mathfrak{H}}(A_i) - \sum_{1 \le i < j \le n} P_{\mathfrak{H}}(A_i \cap A_j)$$

$$+ \sum_{1 \le i < j < k \le n} P_{\mathfrak{H}}(A_i \cap A_j \cap A_k) \tag{2.1}$$

$$- \ldots + (-1)^{n-1} P_{\mathfrak{H}}(A_1 \cap A_2 \cap \ldots . \cap A_n).$$

(iii) *if* $A,\ B \in \Sigma$ *with* $A \subset B$ *then*

$$P_{\mathfrak{H}}(A) \preceq P_{\mathfrak{H}}(B).$$

i.e., $P_{\mathfrak{H}}(A)$ *and* $P_{\mathfrak{H}}(B)$ *are comparable with respect to the partial order* .

(iv) *given* n *events* $A_1, A_2, \ldots, A_n$ *there follows*:

$$P_{\mathfrak{H}}\left(\bigcup_{i=1}^{n} A_i\right) \preceq \sum_{i=1}^{n} P_{\mathfrak{H}}(A_i). \tag{2.2}$$

(v) *for any* n *events* $A_1, A_2, \ldots, A_n$ *the following relation holds*

$$P_{\mathfrak{H}}\left(\bigcup_{i=1}^{n} A_i\right) \succeq \sum_{i=1}^{n} P_{\mathfrak{H}}(A_i) - \sum_{1 \le i < j \le n} P_{\mathfrak{H}}(A_i \cap A_j) \tag{2.3}$$

(vi) ***Continuity theorem of the*** $\mathfrak{H}$***-probability:***

if $A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$ *and* $A := A_1 \cup A_2 \cup \ldots \cup A_n \cup \ldots$ *then*

$$\lim_{n\to\infty} P_{\mathfrak{H}}(A_n) = P_{\mathfrak{H}}(A) = P_{\mathfrak{H}}\left(\bigcup_{n=1}^{\infty} A_n\right).$$

(vii) *for any two events* A *and* B,

$$P_{\mathfrak{H}}(A^C \cap B) = P(B) - P(A \cap B)$$

Proof: (i) We know that $A \cup A^C = \Omega,\ A \cap A^C = \varnothing$ and so that

$$P_{\mathfrak{H}}(A) + P_{\mathfrak{H}}(A^C) = P_{\mathfrak{H}}(\Omega) = \mathfrak{p}$$

$$P_{\mathfrak{H}}(A^C) = \mathfrak{p} - P_{\mathfrak{H}}(A)\ .$$

Now, $P_{\mathfrak{H}}(\varnothing) = P_{\mathfrak{H}}\,(\Omega^C) = \mathfrak{p} - P(\Omega) = 0\,.$

Hence, it follows the results.

(ii) By the help of mathematical induction we can prove the theorem. Since for $n = 2$, $A_1 \cup A_2 = A_1 \cup (A_1{}^C \cap A_2)$ and also $A_2 = (A_1 \cap A_2) \cup (A_1{}^C \cap A_2)$, then

$$P_{\mathfrak{H}}(A_1 \cup A_2) = P_{\mathfrak{H}}(A_1) + P_{\mathfrak{H}}(A_1{}^C \cap A_2)$$

$$= P_{\mathfrak{H}}(A_1) + P_{\mathfrak{H}}(A_2) - P_{\mathfrak{H}}(A_1 \cap A_2).$$

Therefore, the equation (2.1) holds for $n = 2$.

Now, suppose that the equation (2.1) is true for $n = r$.

i.e,.

$$P_{\mathfrak{H}}\left(\bigcup_{i=1}^{r} A_i\right) = \sum_{i=1}^{r} P_{\mathfrak{H}}(A_i) - \sum_{1\le i<j\le r} P_{\mathfrak{H}}(A_i \cap A_j)$$

$$+ \sum_{1\le i<j<k\le r} P_{\mathfrak{H}}(A_i \cap A_j \cap A_k) - \ldots + (-1)^{r-1} P_{\mathfrak{H}}(A_1 \cap A_2 \cap \ldots. \cap A_r)\,.$$

Then, for $n = r + 1$

$$P_{\mathfrak{H}}(A_1 \cup \ldots \cup A_r \cup A_{r+1}) = P_{\mathfrak{H}}\left(\left[\bigcup_{i=1}^{r} A_i\right] \cup A_{r+1}\right)$$

$$= P_{\mathfrak{H}}\left(\bigcup_{i=1}^{r} A_i\right) + P_{\mathfrak{H}}(A_{r+1}) - P_{\mathfrak{H}}\left(\left[\bigcup_{i=1}^{r} A_i\right] \cap A_{r+1}\right)$$

$$= \sum_{i=1}^{r+1} P_{\mathfrak{H}}(A_i) - \sum_{1 \le i < j \le r} P_{\mathfrak{H}}(A_i \cap A_j) + \sum_{1 \le i < j < k \le r} P_{\mathfrak{H}}(A_i \cap A_j \cap A_k)$$

$$- \ldots + (-1)^{r-1} P_{\mathfrak{H}}(A_1 \cap A_2 \cap \ldots \cap A_r) - \left[\sum_{i=1}^{r} P_{\mathfrak{H}}\,(A_i A_{r+1})\right.$$

$$- \sum_{1 \le i < j \le r} P_{\mathfrak{H}}(A_i A_j A_{r+1})$$

$$\left. + \ldots + (-1)^{r-1} P_{\mathfrak{H}}(A_1 \cap A_2 \cap \ldots \cap A_{r+1})\right]$$

$$= \sum_{i=1}^{r+1} P_{\mathfrak{H}}(A_i) - \sum_{1 \le i < j \le r+1} P_{\mathfrak{H}}(A_i \cap A_j)$$

$$+ \sum_{1 \le i < j < k \le r+1} P_{\mathfrak{H}}(A_i \cap A_j \cap A_k)$$

$$- \ldots + (-1)^{n} P_{\mathfrak{H}}(A_1 \cap A_2 \cap \ldots \cap A_{r+1})$$

Thus, the equation (2.1) is also true for $n = r + 1$ and hence this proves the theorem.

This result is the extension of 'Addition Theorem' in probability.

(iii) As $B = B \cap \Omega = B \cap (A \cup A^C) = (B \cap A) \cup (B \cap A^C) = A \cup (A^C \cap B)$. Since, $A \cap (A^C \cap B) = \varnothing$, thus, $P_{\mathfrak{H}}(B) = P_{\mathfrak{H}}(A) + P_{\mathfrak{H}}(A^C \cap B)$, and also since $P_{\mathfrak{H}}(A^C \cap B) \succeq 0$, so we get $P_{\mathfrak{H}}(A) \preceq P_{\mathfrak{H}}(B)$.

(iv) Since for any two events A_1 and A_2, we have

$$P_{\mathfrak{H}}(A_1 \cup A_2) = P_{\mathfrak{H}}(A_1) + P_{\mathfrak{H}}(A_2) - P_{\mathfrak{H}}(A_1 \cap A_2)$$

$$\preceq P_{\mathfrak{H}}(A_1) + P_{\mathfrak{H}}(A_2) \qquad [\because P_{\mathfrak{H}}(A_1 \cap A_2) \succeq 0]$$

Thus, the result (2.2) is true for $n = 2$.

Let us suppose that the result (2.2) is true for $n = m$.

Now, for $n = m + 1$,

$$P_{\mathfrak{H}}(A_1 \cup A_2 \cup \ldots \cup A_{m+1}) \preceq P_{\mathfrak{H}}(A_1 \cup A_2 \cup \ldots \cup A_m) + P_{\mathfrak{H}}(A_{m+1})$$

$$\preceq P_{\mathfrak{H}}(A_1) + P_{\mathfrak{H}}(A_2) + \ldots + P_{\mathfrak{H}}(A_{m+1})$$

Thus, the result (2.2) is true for all $n \in \mathbb{N}$ and hence the result known as 'Boole's inequality' also hold in $\mathfrak{H}$-probabilistic space.

From the addition theorem, we get

$$P_{\mathfrak{H}}(A_1 \cup A_2 \cup A_3) = P_{\mathfrak{H}}(A_1) + P_{\mathfrak{H}}(A_2) + P_{\mathfrak{H}}(A_3) - P_{\mathfrak{H}}(A_1 \cap A_2)$$

$$- P_{\mathfrak{H}}(A_2 \cap A_3) - P_{\mathfrak{H}}(A_3 \cap A_1) + P_{\mathfrak{H}}(A_1 \cap A_2 \cap A_3)$$

$$P_{\mathfrak{H}}\left(\bigcup_{i=1}^{3} A_i\right) \succeq \sum_{i=1}^{3} P_{\mathfrak{H}}(A_i) \sum_{1 \le i < j \le 3} P_{\mathfrak{H}}(A_i \cap A_j)$$

Thus, the result (2.3) is true for $n = 3$. Now, we will prove the result by the help of mathematical induction.

Let us suppose that the result (2.3) is true for $n = k$, i.e.,

$$P_{\mathfrak{H}}\left(\bigcup_{i=1}^{k} A_i\right) \succeq \sum_{i=1}^{k} P_{\mathfrak{H}}(A_i) - \sum_{1\le i<j\le k} P_{\mathfrak{H}}(A_i \cap A_j)$$

Now, for $n = k + 1$

$$P_{\mathfrak{H}}\left(\bigcup_{i=1}^{k+1} A_i\right) = P_{\mathfrak{H}}\left(\left[\bigcup_{i=1}^{k} A_i\right] \cup A_{k+1}\right)$$

$$= P_{\mathfrak{H}}\left(\bigcup_{i=1}^{k} A_i\right) + P_{\mathfrak{H}}(A_{k+1}) - P_{\mathfrak{H}}\left(\left[\bigcup_{i=1}^{k} A_i\right] \cap A_{k+1}\right)$$

$$= P_{\mathfrak{H}}\left(\bigcup_{i=1}^{k} A_i\right) + P_{\mathfrak{H}}(A_{k+1}) - P_{\mathfrak{H}}\left(\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right)$$

$$\succeq \left[\sum_{i=1}^{k} P_{\mathfrak{H}}(A_i) - \sum_{1\le i<j\le k} P_{\mathfrak{H}}(A_i \cap A_j)\right] + P_{\mathfrak{H}}(A_{k+1}) - P_{\mathfrak{H}}\left(\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right)$$

From the result-(iv), we get

$$- P_{\mathfrak{H}}\left(\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right) \succeq -\sum_{i=1}^{k} P_{\mathfrak{H}}(A_i \cap A_{k+1})$$

and therefore,

$$P_{\mathfrak{H}}\left(\bigcup_{i=1}^{k+1} A_i\right) \succeq \sum_{i=1}^{k+1} P_{\mathfrak{H}}(A_i) - \sum_{1\le i<j\le k} P_{\mathfrak{H}}(A_i \cap A_j)$$

$$-\sum_{i=1}^{k} P_{\mathfrak{H}}(A_i \cap A_{k+1})$$

$$P_{\mathfrak{H}}\left(\bigcup_{i=1}^{k+1} A_i\right) \succeq \sum_{i=1}^{k+1} P_{\mathfrak{H}}(A_i) - \sum_{1 \le i < j \le k+1} P_{\mathfrak{H}}(A_i \cap A_j)$$

Thus, the theorem is also true $n = k + 1$ and hence this proves the result.

(vi) Since the given sequence of events is expanding, so we get

$$\bigcup_{i=1}^{n} A_i = A_n \qquad \text{and} \qquad \lim_{n \to \infty} A_n = \bigcup_{i=1}^{\infty} A_n \tag{2.4}$$

Now if we take some events as

$$B_1 = A_1, B_2 = A_2 \cap A_1^C, \ldots, B_n = A_n \cap A_{n-1}^C ,$$

then $B_i \cap B_j = \phi$, for all $i \neq j$ and $i, \; j = 1, 2, \ldots, n$ and hence

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \quad \text{and} \quad P_{\mathfrak{H}}\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} P_{\mathfrak{H}}(B_n) \tag{2.5}$$

Now, from (2.4) and (2.5), we get

$$P_{\mathfrak{H}}\left(\lim_{n\to\infty} A_n\right) = P_{\mathfrak{H}}\left(\bigcup_{n=1}^{\infty} A_n\right)$$

$$= \sum_{n=1}^{\infty} P_{\mathfrak{H}}(B_n)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} P_{\mathfrak{H}}(B_i)$$

$$= \lim_{n\to\infty} P_{\mathfrak{H}}\left(\bigcup_{i=1}^{n} B_i\right)$$

$$= \lim_{n\to\infty} P_{\mathfrak{H}}\left(\bigcup_{i=1}^{n} A_i\right) = \lim_{n\to\infty} P_{\mathfrak{H}}(A)$$

This completes the proof.

(vii) Since, $A^C \cap B$ and $A \cap B$ are disjoint events and $(A^C \cap B) \cup (A \cap B) = B$, then

$$P_{\mathfrak{H}}(B) = P_{\mathfrak{H}}(A^C \cap B) + P_{\mathfrak{H}}(A \cap B)$$

or,
$$P_{\mathfrak{H}}(A^C \cap B) = P_{\mathfrak{H}}(B) - P_{\mathfrak{H}}(A \cap B).$$

□

Corollary 2.1.1: *Since, for given* $A \in \Sigma$, $A \subseteq \Omega$, *therefore* $P_{\mathfrak{H}}(A)$ *is always comparable with* $P_{\mathfrak{H}}(\Omega)$ *and also,* $P_{\mathfrak{H}}(A) \preceq P_{\mathfrak{H}}(\Omega) = \mathfrak{p}$. *Hence, for any* $A \in \Sigma$, $0 \preceq P_{\mathfrak{H}}(A) \preceq \mathfrak{p}$.

Corollary 2.1.2: *If* $P\mathfrak{H}(\Omega) = e_+$ *then for any random event* A *there holds that* $P_{\mathfrak{H}}(A)$ *is of the form* λe_+ *with* $\lambda \in [0,1]$ *. If* $P\mathfrak{H}(\Omega) = e_-$ *then for any random events* A *there holds that* $P_{\mathfrak{H}}(A)$ *is of the form* μe_- *with* $\mu \in [0,1]$.

Corollary 2.1.3: *For any* n *events* $A_1, A_2, \ldots, A_n$

$$P_{\mathfrak{H}}(A_1 \cap A_2 \cap \ldots \cap A_n) \succeq \mathfrak{p} - \sum_{i=1}^{n} P_{\mathfrak{H}}(A_i^C) \tag{2.6}$$

and
$$P_{\mathfrak{H}}(A_1 \cap A_2 \cap \ldots \cap A_n) \succeq \sum_{i=1}^{n} P_{\mathfrak{H}}(A_i) - (n-1)\mathfrak{p} \tag{2.7}$$

Proof: Applying the 'Boole's inequality' on the events $A_1^C, A_2^C, \dots, A_n^C$, then we get

$$P_{\mathfrak{H}}(A_1^C \cup A_2^C \cup \dots \cup A_n^C) \preceq \sum_{i=1}^{n} P_{\mathfrak{H}}(A_i^C)$$

or,

$$P_{\mathfrak{H}}\left[(A_1 \cap A_2 \cap \dots \cap A_n)^C\right] \preceq \sum_{i=1}^{n} P_{\mathfrak{H}}(A_i^C)$$

or,

$$P_{\mathfrak{H}}(A_1 \cap A_2 \cap \dots \cap A_n) \succeq \mathfrak{p} - \sum_{i=1}^{n} P_{\mathfrak{H}}(A_i^C) \tag{2.8}$$

Again,

$$\sum_{i=1}^{n} P_{\mathfrak{H}}(A_i^C) = \mathfrak{p} - P_{\mathfrak{H}}(A_1)\mathfrak{p} - P_{\mathfrak{H}}(A_2) + \dots \mathfrak{p} - P_{\mathfrak{H}}(A_n)$$

$$= n\mathfrak{p} - \sum_{i=1}^{n} P_{\mathfrak{H}}(A_i^C) \tag{2.9}$$

Now, combining (2.8) and (2.9), we get

$$P_{\mathfrak{H}}(A_1 \cap A_2 \cap \dots \cap A_n) \succeq \sum_{i=1}^{n} P_{\mathfrak{H}}(A_i) - (n-1)\mathfrak{p}.$$

□

The results (2.6) and (2.7) is the 'Bonferroni's inequalities' in $\mathfrak{H}$-valued probabilistic space.

Corollary 2.1.4: *If* $A_1 \supset A_2 \supset \dots \supset A_n \supset \dots$

and

$$A := A_1 \cap A_2 \cap \dots \cap A_n \cap \dots \cap, \textit{ then}$$

$$\lim_{n\to\infty} P_{\mathfrak{H}}(An) = P_{\mathfrak{H}}(A) = P_{\mathfrak{H}}\left(\bigcap_{n=1}^{\infty} A_n\right).$$

3. Future Prospect

In the line of the works as carried out in the paper one may think of to establish the hyperbolic hybrid valued conditional probability and may look into the matter of independency of events and also may try to revisit some well known results like multiplication theorem, law of total probability and Bayes' theorem. This may be regarded as an active area of research to the future workers in this branch.

Acknowledgment

The third author sincerely acknowledges the financial support rendered by DST-FIST 2024-2025 running at the Department of Mathematics, University of Kalyani, P.O.: Kalyani, Dist: Nadia, Pin: 741235, West Bengal, India.

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Journal of Indian Acad. Math.
Vol. 46, No. 2 (2024) pp. 101-116

ISSN: 0970-5120

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A STUDY ON SOME ALGEBRAIC PROPERTIES OF HYBRID NUMBER

Abstract: In this paper, we define a new notion of sets, termed as symmetric hybrid number and skew symmetric hybrid number and study some of their algebraic properties. The key result of this paper is to establish that the set of non-lightlike hybrid number forms a non-abelian group under multiplication and also to find a normal subgroup of it. Moreover, the existence of rings and also their ideals is the prime concern under some additional conditions.

Keywords and Phrases: Hybrid Number, Character, Type, Symmetric Hybrid Number.

Mathematics Subject Classification (2020) No.: 06F25, 05C25.

1. Introduction, Definitions and Notations

In the last century, a lot of researchers work with some two-dimensional systems like complex, dual and hyperbolic number systems which have the most significant roles in algebraic, geometric, physics, engineering, etc. The geometry of the Euclidean plane, the Minkowski plane and the Gallian plane can be described with the help of complex numbers

$$\mathbb{C} = \{a + \mathbf{i}b : a, b \in \mathbb{R}, \mathbf{i}^2 = -1\},$$

dual numbers

$$\mathbb{D} = \{a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0\},$$

and hyperbolic numbers

$$\mathbb{H} = \{a + \mathbf{h}b : a, b \in \mathbb{R}, \mathbf{h}^2 = 1\}.$$

We know that the complex numbers, dual numbers and hyperbolic numbers can be described as the quotient of the polynomial ring $\mathbb{R}[x]$ by the ideal generated by the polynomials $x^2 + 1$, x^2 and $x^2 - 1$ respectively. i.e.,

$$\mathbb{C} = \mathbb{R}[x]/ < x^2 + 1 >,$$

$$\mathbb{D} = \mathbb{R}[x]/ < x^2 > \text{ and } \mathbb{H} = \mathbb{R}[x]/ < x^2 - 1 >.$$

In [1], [2], [3] S. Olariu defined a different generalization of n-dimensional complex numbers terming them twocomplex numbers, threecomplex numbers, etc. Actually, Olariu used the name twocomplex numbers instead of hyperbolic numbers. In these series of papers the geometrical and the algebraic properties of these numbers are thoroughly studied. The set of threecomplex numbers is defined as

$$\mathbb{C}_3 = \{z = a + \mathbf{h}b + \mathbf{k}c : a, b, c \in \mathbb{R} \text{ and } \mathbf{h}^2 = \mathbf{k}, \mathbf{k}^2 = \mathbf{h}, \mathbf{hk} = 1\}.$$

Anthony Harkin and Joseph Harkin [4] generalized the two dimensional complex numbers as

$$\mathbb{C}p = \{z = a + \mathbf{i}b : a, b \in \mathbb{R} \text{ and } \mathbf{i}^2 = p\}.$$

Here they gave some trigonometric relations for this generalization. In [5] Catoni *et al*. defined two dimensional hypercomplex numbers as

$$\mathbb{C}_{\alpha,\beta} = \{z = a + \mathbf{i}b : \mathbf{i}^2 = \alpha + \mathbf{i}\beta, a, b, \alpha, \beta \in \mathbb{R}, \mathbf{i} \notin \mathbb{R}\}$$

and extended the relationship amongst these numbers and Euclidean and semi-Euclidean geometry. This generalization is also expressible as a quotient ring $\mathbb{R}[x]/ < x^2 - \beta x - \alpha >$.

A theory of commutative two dimensional conformal hyperbolic numbers as a generalization of the theory of hyperbolic numbers is presented by Zaripov [6].

Mustafa *Ö*zdemir [7] defined a new generalization of set containing complex, hyperbolic and dual numbers as different from above generalizations. This new number system appears to be four-dimensional and it can be viewed as a two dimensional set of numbers, since it can be represented in a generalized two dimensional plane, called hybridian plane in $\mathbb{R}^4$.

The following definitions are due to Özdemir [7].

Definition 1.1 [7]: *The set of hybrid numbers, denoted by* $\mathbb{K}$, *is defined as any combination of complex, dual and hyperbolic numbers with the relation* $\mathbf{ih} = -\mathbf{hi} = \mathbf{i} + \varepsilon$ *i.e.,*

$$\mathbb{K} = \{Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} : a, b, c, d \in \mathbb{R}, \mathbf{i}^2 = -1, \varepsilon^2 = 0,$$
$$\mathbf{h}^2 = 1, \mathbf{ih} = -\mathbf{hi} = \mathbf{i} + \varepsilon\}.$$

The geometry corresponding to the hybrid numbers is called **Hybridian plane geometry** and it is a two-dimensional subspace of $\mathbb{R}^4$. The real part $'a'$ of the hybrid number $Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ is called **scalar part** and is denoted by $S(Z)$ whereas the remaining part $b\mathbf{i} + c\varepsilon + d\mathbf{h}$ is called **vector part** and is denoted by $\mathbf{V}(\boldsymbol{Z})$.

For any two hybrid numbers $Z_1 = a_0 + a_1\mathbf{i} + a_2\varepsilon + a_3\mathbf{h}$, and $Z_2 = b_0 + b_1\mathbf{i} + b_2\varepsilon + b_3\mathbf{h}$, a_i, $b_i \in \mathbb{R}$, $i = 0, 1, 2, 3$ the equality, addition, subtraction, multiplication by a scalar $s \in \mathbb{R}$ and multiplication of two hybrid numbers are defined as follows

$$Z_1 = Z_2 \Leftrightarrow a_0 = b_0, \ \ a_1 = b_1, \ \ a_2 = b_2, \ \ a_3 = b_3,$$

$$Z_1 \pm Z_2 = (a_0 \pm b_0) + (a_1 \pm b_1)\mathbf{i} + (a_2 \pm b_2)\ \varepsilon + (a_3 \pm b_3)\mathbf{h},$$

$$sZ_1 = sa_0 + sa_1\mathbf{i} + sa_2\varepsilon + sa_3\mathbf{h}$$

and

$$Z_1.Z_2 = (a_0 + a_1\mathbf{i} + a_2\varepsilon + a_3\mathbf{h}).(b_0 + b_1\mathbf{i} + b_2\varepsilon + b_3\mathbf{h})$$

$$= (a_0b_0 - a_1b_1 + a_2b_1 - a_1b_2 + a_3b_3) + (a_0b_2 + a_1b_0 + a_1b_3$$

$$- a_3b_1)\mathbf{i} + (a_0b_2 + a_2b_0 - a_2b_3 + a_3b_2 + a_1b_3 - a_3b_1)\varepsilon$$

$$+ (a_0b_3 + a_3b_0 + a_1b_2 + a_2b_1)\mathbf{h}.$$

The $'+'$ operation is both commutative and associative. The null element is **0** and the inverse element of Z is $-Z$. As a consequence of these properties, $(\mathbb{K}, +)$ forms an abelian group.

For the above multiplication of hybrid numbers the following table of hybrid units can be used.

.	1	**i**	ε	**h**
1	1	i	ε	**h**
i	i	1	1 **h**	ε + **i**
ε	ε	1 + **h**	0	ε
h	**h**	ε **i**	ε	1

Multiplication table of Hybrid units

From the above table it is clear that the multiplication operation in the hybrid number system is not commutative. But the multiplication is associative.

Definition 1.2 [7]: *The* ***conjugate*** *of a hybrid number* $Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ *is defined by*

$$\overline{Z} = S(Z) - V(Z) = a - b\mathbf{i} - c\varepsilon - d\mathbf{h}$$

This conjugation of a hybrid number is additive, involutive and multiplicative operation on $\mathbb{K}$. i.e., for any two hybrid numbers Z_1 and Z_2

a. $\overline{Z_1 + Z_2} = \overline{Z_1} + \overline{Z_2}$

b. $\overline{(\overline{Z_1})} = Z_1$

c. $\overline{Z_1.Z_2} = \overline{Z_1}.\overline{Z_2}$

Definition 1.3 [7]: *The real number* $\mathcal{C}(Z) = Z\overline{Z} = a^2 + (b-c)^2 - c^2 - d^2$ *is called the* ***character*** *of the hybrid number* Z.

Since, $\mathcal{C}(Z) \in \mathbb{R}$, so depending on the value of $\mathcal{C}(Z)$ a hybrid number can be categorized into three parts, **spacelike**, **timelike**, or **lightlike** according as the character is negative, positive or zero.

Also, the real number, $\sqrt{|\mathcal{C}(Z)|}$ is called the **norm** of the hybrid number Z and is denoted by $||Z||$.

Definition 1.4: [7] *The inverse of the hybrid number* $Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$,

$$Z^{-1} = \frac{\overline{Z}}{\mathcal{C}(Z)}.$$

Therefore, we can conclude that a lightlike hybrid number never possesses an inverse.

Definition 1.5 [7]: *For the hybrid number* $Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$, *the vector* $\varepsilon_Z = ((b-c), c, d)$ *is called the* ***hybrid vector*** *of* Z.

Definition 1.6 [7]: *The real number* $\mathcal{C}_\varepsilon(Z) = -(b-c)^2 + c^2 + d^2$ *is called the* ***type*** *of the hybrid number* Z.

Depending on the real value of the type of a hybrid nunber, it is classified as **complike** (**elliptic**), **hyperlike** (**hyperbolic**) or **duallike** (**parabolic**) for $\mathcal{C}_\varepsilon(Z) < 0$, $\mathcal{C}_\varepsilon(Z) > 0$ and $\mathcal{C}_\varepsilon(Z) = 0$ respectively. Also, the real number $\sqrt{|\mathcal{C}_\varepsilon(Z)|}$ is called the norm of the hybrid vector of Z and is denoted by $\mathcal{N}(Z)$.

Continuing the above discussion we may define following sets.

Definition 1.7: *A hybrid number is called symmetric if it's imaginary part and dual part both are same.*

i.e. the hybrid number, $Z_S = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ *is called symmetric hybrid number if* $b = c$ *.*

The following example is a symmetric hybrid number.

Example 1.1: *The hybrid number* $Z_S = 2 + 3\mathbf{i} + 3\varepsilon + \mathbf{h}$ *is a symmetric.*

Definition 1.8: *The set of symmetric hybrid number with free hyperbolic unit is called null-hyperbolic and is denoted by* $Z_{S_{\bar{H}}}$ *.*

Example 1.2: $-1 + i + \varepsilon$ *is an example of null-hyperbolic symmetric hybrid number.*

Definition 1.9: *A hybrid number is called skew-symmetric hybrid number if it's scalar, dual and hyperbolic coefficients are vanish.*

Example 1.3: $Z = 2\mathbf{i}$ *is an example of a skew-symmetric hybrid number.*

2. Lemmas

In this section we present the following lemma which will be needed in the sequel.

Lemma 2.1: *For any two hybrid numbers* Z_1 *,* Z_2 *the following equality holds*

$$\mathcal{C}(Z_1.Z_2) = \mathcal{C}(Z_1).\mathcal{C}(Z_2)$$

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1: *Symmetric hybrid numbers never be elliptic.*

Proof: The type of any symmetric hybrid number $Z_S = a + c\mathbf{i} + c\varepsilon + d\mathbf{h}$ is

$$\mathcal{C}_\varepsilon(Z_S) = -(c-c)^2 + c^2 + d^2 = c^2 + d^2 \geq 0 .$$

So, a symmetric hybrid number is either hyperbolic or parabolic, but it never be elliptic. □

Remark 3.1.1: *The character for any symmetric hybrid number* $Z_S = a + c\mathbf{i} + c\varepsilon + d\mathbf{h}$ *is* $\mathcal{C}(Z_S) = a^2 - c^2 - d^2$. *Thus,* Z_S *may be spacelike, timelike or lightlike for* $a^2 < c^2 + d^2$, $a^2 > c^2 + d^2$ or $a^2 = c^2 + d^2$ *respectively.*

Theorem 3.2: *A skew-symmetric hybrid number never be spacelike as well as hyperbolic.*

Proof: Character of a skew-symmetric hybrid number, $Z = b\mathbf{i}$ is $\mathcal{C}(Z) = b^2 \geq 0$ and therefore it is either timelike or lightlike, it never be spacelike.

Now, the type of a skew-symmetric hybrid number $Z = b\mathbf{i}$ is $\mathcal{C}_\varepsilon(z) = -b^2 \leq 0$ and hence it is either elliptic or parabolic but never be hyperbolic. □

Theorem 3.3: *The set* Z_S *is a subgroup of* $\mathbb{K}$ *under* $'+'$.

Proof: The proof of the theorem is trivial. □

Remark 3.3.1: *The group* $(Z_S, +)$ *is abelian as hybrid addition is commutative.*

Theorem 3.4: *The set of null-hyperbolic symmetric hybrid numbers is a normal subgroup of symmetric hybrid numbers.*

Proof: Clearly, $Z_{S\bar{H}}$ is a non empty subset of Z_S as $\mathbf{0} \in Z_{S\bar{H}}$.

Let,

$$Z_1 = a_1 + c_1\mathbf{i} + c_1\varepsilon,\ Z_2 = a_2 + c_2\mathbf{i} + c_2\varepsilon \in Z_{S\bar{H}} .$$

Now, $Z_1 + Z_2 = (a_1 + a_2) + (c_1 + c_2)\mathbf{i} + (c_1 + c_2)\varepsilon \in Z_{S_{\overline{H}}}$ and $Z_1^{-1} = -a_1 - c_1\mathbf{i} - c_1\varepsilon \in Z_{S_{\overline{H}}}$.

Therefore, $(Z_{S_{\overline{H}}}, +)$ is a subgroup of $(Z_S, +)$ and as $(Z_S, +)$ is abelian implies $(Z_{S_{\overline{H}}}, +)$ is normal in $(Z_S, +)$. □

Theorem 3.4 leads us to the following remarks.

Remark 3.4.1: *The set of null-hyperbolic symmetric hybrid numbers is also a normal subgroup of hybrid numbers.*

Remark 3.4.2: *Let* $G = (\mathbb{K}, +)$ *and* $H = (Z_{S_{\overline{H}}}, +)$. *Then* G/H *forms the quotient group.*

Remark 3.4.3: *Let,* $L = (Z_S, +)$ *and* $H = (Z_{S_{\overline{H}}}, +)$. *Then* L/H *forms the quotient group.*

Remark 3.4.4: *Since,* $H = (Z_{S_{\overline{H}}}, +)$ *is a normal subgroup of* $G = (\mathbb{K}, +)$ *and* $L = (Z_S, +)$ *is a subgroup* $G = (\mathbb{K}, +)$ *with* $H \subset L \subset G$ *then* L/H *is a subgroup of* G/H.

Theorem 3.5: *The set of all symmetric hybrid numbers* Z_S *does not form a group under the operation 'hybrid Multiplication'.*

Proof: Let, $Z_1 = a_1 + c_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}$ and $Z_2 = a_2 + c_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h} \in Z_s$. Then

$$Z_1.Z_2 = a_1a_2 + c_1a_2\mathbf{i} + c_1a_2\varepsilon + d_1a_2\mathbf{h} + a_1c_2\mathbf{i} - c_1c_2 + c_1c_2(\mathbf{h} + 1)$$

$$+ d_1c_2(-\mathbf{i} - \varepsilon) + a_1c_2\varepsilon + c_1c_2(1 - \mathbf{h}) + d_1c_2\varepsilon + a_1d_2\mathbf{h}$$

$$+ c_1d_2(\mathbf{i} + \varepsilon) - c_1d_2\varepsilon + d_1d_2$$

$$= (a_1a_2 + c_1c_2 + d_1d_2) + (c_1a_2 + a_1c_2 - d_1c_2 + c_1d_2)\mathbf{i}$$
$$+ (c_1a_2 + a_1c_2)\varepsilon + (d_1a_2 + a_1d_2)\mathbf{h}\,.$$

Clearly, $Z_1.Z_2 \notin Z_S$.

Hence, $(Z_S, .)$ is not a group. □

The following example ensures the above fact.

Example 3.1: *Let,* $Z_1 = 1 + \mathbf{i} + \varepsilon + \mathbf{h}$, $Z_2 = \mathbf{i} + \varepsilon + 2\mathbf{h} \in Z_S$ *Then,*

$$Z_1.Z_2 = \mathbf{i} - 1 + \mathbf{h} + 1 - \varepsilon - \mathbf{i} + \varepsilon + 1 - \mathbf{h} + \varepsilon + 2\mathbf{h}$$
$$+ 2(\mathbf{i} + \varepsilon) - 2\varepsilon + 2$$
$$= 3 + 2\mathbf{i} + \varepsilon + 2\mathbf{h}\,,$$

not a symmetric hybrid number.

The following remark is immediate of the above.

Remark 3.5.1: *The set of non-lightlike symmetric hybrid number whose dual and hyperbolic coefficients are in a constant ratio forms an abelian group under hybrid multiplication.*

Theorem 3.6: *The set of all non-lightlike hybrid number forms a group under the hybrid multiplication.*

Proof: Let $\mathbb{K}_{NL} = \{Z \in \mathbb{K} : \mathcal{C}(Z) \neq 0\}$ and $Z_1, Z_2 \in \mathbb{K}_{NL}$. Then, $\mathcal{C}(Z_1) \neq 0$, $\mathcal{C}(Z_2) \neq 0$.

Using Lemma 2.1, $\mathcal{C}(Z_1.Z_2) \neq 0$ and hence, $Z_1.Z_2 \in \mathbb{K}_{NL}$.

Here $Z_e = 1 + 0.\mathbf{i} + 0.\varepsilon + 0.\mathbf{h} \in \mathbb{K}_{NL}$ acts as the identity in $\mathbb{K}_{NL}$. Let Z^{-1} be the inverse of $Z \in \mathbb{K}_{NL}$.

Therefore, $\mathcal{C}(Z.Z^{-1}) = \mathcal{C}(Z_e) = 1 \Rightarrow \mathcal{C}(Z).\mathcal{C}(Z^{-1}) = 1$ and $\mathcal{C}(Z^{-1}) = \dfrac{1}{\mathcal{C}(Z)} \neq 0$ [since $\mathcal{C}(Z) \neq 0$].

Thus, $Z^{-1} \in \mathbb{K}_{NL}$.

As the associative property is the hereditary property implies the set of non-lightlike hybrid numbers $\mathbb{K}_{NL}$ forms a group under multiplication. □

The following corollary is immediate of the above.

Corollary 3.6.1: *The group $\mathbb{K}_{NL}$ is non-commutative.*

Proof: We know that $(\mathbb{K}_{NL}, .)$ is a group.

Now, let $Z_1 = a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}$, $Z_2 = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} \in \mathbb{K}_{NL}$.

Then

$$Z_1.Z_2 = (a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}).(a_2 + b_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h})$$

$$= (a_2a_1 - b_2b_1 + c_2b_1 + b_2c_1 + d_2d_1) + (b_2a_1 + a_2b_1 + d_2b_1 - b_2d_1)\mathbf{i}$$

$$+ (c_2a_1 + d_2b_1 + a_2c_1 - d_2c_1 - b_2d_1 + c_2d_1)\varepsilon$$

$$+ (d_2a_1 - c_2b_1 + b_2c_1 + a_2d_1)\mathbf{h}$$

whereas

$$Z_2.Z_1 = (a_2a_1 - b_2b_1 + c_2b_1 + b_2c_1 + d_2d_1)$$

$$+ (b_2a_1 + a_2b_1 - d_2b_1 + b_2d_1)\mathbf{i}$$

$$+ (c_2a_1 - d_2b_1 + a_2c_1 + d_2c_1 + b_2d_1 - c_2d_1)\varepsilon$$

$$+ (d_2a_1 + c_2b_1 - b_2c_1 + a_2d_1)\mathbf{h}$$

Thus, $Z_1.Z_2 \neq Z_2.Z_1$ and hence, $(\mathbb{K}_{NL}, .)$ is a non-abelian group. □

Theorem 3.7: *The set of all hybrid numbers of unit character is normal in* $\mathbb{K}_{NL}$.

Proof: Let $\mathbb{K}_{\mathcal{C}1} = \{Z \in \mathbb{K} : \mathcal{C}(Z) = 1\}$,

which is non-empty as $\mathbf{1} = 1 + 0.\mathbf{i} + 0.\varepsilon + 0.\mathbf{h} \in \mathbb{K}_{\mathcal{C}1}$. Let $Z_1, Z_2 \in \mathbb{K}_{\mathcal{C}1}$, then $\mathcal{C}(Z_1.Z_2) = 1$.

Thus, $Z_1.Z_2 \in \mathbb{K}_{\mathcal{C}1}$.

Let Z_1^{-1} be the inverse of $Z_1 \in \mathbb{K}_{\mathcal{C}1}$.

Now, $\mathcal{C}(Z_1.Z_1^{-1}) = \mathcal{C}(Z_e) = 1$

i.e., $\mathcal{C}(Z_1^{-1}) = \dfrac{1}{\mathcal{C}(Z_1)} = 1$ [as $\mathcal{C}(Z_1) = 1$]

Thus, $Z_1^{-1} \in \mathbb{K}_{\mathcal{C}1}$.

Hence, $K_{\mathcal{C}1}$ is a subgroup of $\mathbb{K}_{NL}$. Let $W \in \mathbb{K}_{NL}$ and $Z \in \mathbb{K}_{\mathcal{C}1}$

Now, $\mathcal{C}(WZW^{-1}) = \mathcal{C}(W)\mathcal{C}(Z)\mathcal{C}(W^{-1}) = \mathcal{C}(W)\mathcal{C}(W^{-1}) = \mathcal{C}(\mathbf{1}) = 1$, non zero.

Therefore, $WZW^{-1} \in \mathbb{K}_{\mathcal{C}1}$ and hence $\mathbb{K}_{\mathcal{C}1}$ is a normal subgroup of $\mathbb{K}_{NL}$

This proves the theorem. □

Remark 3.7.1: *The set of lightlike hybrid numbers does not form a group under hybrid multiplication as the inverse of any element does not exist.*

Theorem 3.8: *The set* $\mathbb{K}_{\mathbb{Q}} = \{Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} : a, b, c, d \in \mathbb{Q}\}$ *forms a subring of* $\mathbb{K}$.

Proof: Clearly $\mathbf{0} \in \mathbb{K}_{\mathbb{Q}}$ and let $Z_1 = a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}$ and $Z_2 = a_2 + b_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h} \in \mathbb{K}_{\mathbb{Q}}$

Now, $Z_1 - Z_2 = (a_1 - a_2) + (b_1 - b_2)\mathbf{i} + (c_1 - c_2)\varepsilon + (d_1 - d_2)\mathbf{h} \in \mathbb{K}_{\mathbb{Q}}$ and also

$$Z_1.Z_2 = (a_1a_2 - b_1b_2 + c_1b_2 + b_1c_2 + d_1d_2) + (b_1a_2 + a_1b_2 - d_1b_2 + b_1d_2)\mathbf{i}$$
$$+ (c_1a_2 - d_1b_2 + a_1c_2 + d_1c_2 + b_1d_2 - c_1d_2)\varepsilon$$
$$+ (d_1a_2 + c_1b_2 - b_1c_2 + a_1d_2)\mathbf{h} \in \mathbb{K}_{\mathbb{Q}}$$

Therefore, $\mathbb{K}_{\mathbb{Q}}$ is a subring of the ring of $\mathbb{K}$.

This completes the proof. □

Remark 3.8.1: *The set* $\mathbb{K}_{\mathbb{Z}} = \{z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} : a, b, c, d \in \mathbb{Z}\}$ *is also a subring of* $\mathbb{K}$.

Proof: The proof is omitted as it is similar to previous. □

Theorem 3.9: *The set* $\mathbb{K}_{RI} = \{Z = \frac{a}{2} + \frac{a+b}{2}\mathbf{i} + \frac{a}{2}\varepsilon + \frac{b}{2}\mathbf{h} : a, b \in \mathbb{Z}\}$ *is a right ideal of* $\mathbb{K}$.

Proof: Clearly $\mathbb{K}_{RI}$ is a subring of $\mathbb{K}$.

Let $Z = \frac{a}{2} + \frac{a+b}{2}\mathbf{i} + \frac{a}{2}\varepsilon + \frac{b}{2}\mathbf{h} \in \mathbb{K}_{RI}$ and $W = p + q\mathbf{i} + r\varepsilon + s\mathbf{h} \in K$.

Then, $$Z.W = \frac{ap}{2} + \frac{ap+bp}{2}\mathbf{i} + \frac{ap}{2}\varepsilon + \frac{bp}{2}\mathbf{h} + \frac{aq}{2}\mathbf{i} - \frac{aq+bq}{2}$$

$$+ \frac{aq}{2}(\mathbf{h}+1) + \frac{bq}{2}(-\varepsilon-\mathbf{i}) + \frac{ar}{2}\varepsilon + \frac{ar+br}{2}(1-\mathbf{h})$$

$$+ \frac{br}{2}\varepsilon + \frac{as}{2}\mathbf{h} + \frac{as+bs}{2}(\varepsilon+\mathbf{i}) + \frac{as}{2}(-\varepsilon) + \frac{bs}{2}$$

$$= (\frac{ap-bq+ar+bs+br}{2}) + (\frac{ap+aq+as+bs+bp-bq}{2})\mathbf{i}$$

$$= (\frac{ap-bq+ar+br+bs}{2})\varepsilon + (\frac{bp+aq-ar-br+as}{2})\mathbf{h}$$

Therefore, $Z.W \in \mathbb{K}_{RI}$ and hence, $\mathbb{K}_{RI}$ is a right ideal of $\mathbb{K}$. □

Corollary 3.9.1: *The subring* $\mathbb{K}_{RI}$ *is not an left ideal of* $\mathbb{K}$.

Proof: Let $Z = \frac{a}{2} + \frac{a+b}{2}\mathbf{i} + \frac{a}{2}\varepsilon + \frac{b}{2}\mathbf{h} \in \mathbb{K}_{RI}$ and $W = p + q\mathbf{i} + r\varepsilon + s\mathbf{h} \in \mathbb{K}$

Then

$$W.Z = \frac{pa}{2} + \frac{aq}{2}\mathbf{i} + \frac{ar}{2}\varepsilon + \frac{as}{2}\mathbf{h} + \frac{ap+bq}{2}\mathbf{i} - \frac{aq+bq}{2}$$

$$+ \frac{ar+br}{2}(\mathbf{h}+1) + \frac{as+bs}{2}(-\varepsilon-\mathbf{i}) + \frac{ap}{2}\varepsilon + \frac{aq}{2}(1-\mathbf{h})$$

$$+ \frac{as}{2}\varepsilon + \frac{pb}{2}\mathbf{h} + \frac{qb}{2}(\varepsilon+\mathbf{i}) + \frac{br}{2}(-\varepsilon) + \frac{bs}{2}$$

$$= (\frac{ap-bq+ar+br+bs}{2}) + (\frac{aq+ap+bp-as-bs+bq}{2})\mathbf{i}$$

$$= (\frac{ar-bs+ap+bq-br}{2})\varepsilon + (\frac{as+ar+br-aq+bp}{2})\mathbf{h}$$

$$= \frac{u}{2} + \frac{v}{2}\mathbf{i} + \frac{x}{2}\varepsilon + \frac{y}{2}\mathbf{h} \qquad (say)$$

where $u = ap - bq + ar + br + bs$; $v = aq + ap + bp - as - bs + bq$;
$x = ar - bs + ap + bq - br$; and $y = as + ar + br - aq + bp$

Clearly, $u \neq x$ and so $W.Z \notin \mathbb{K}_{RI}$.

Therefore, $\mathbb{K}_{RI}$ is not left ideal. □

The following example validate the above result.

Example 3.2: *Consider* $Z = \mathbf{i} + \mathbf{h} \in \mathbb{K}$, $W = (-\frac{1}{2})\mathbf{i} + \frac{1}{2}\mathbf{h} \in \mathbb{K}_{RI}$.

But, $W.Z = 1 - \mathbf{i} - \varepsilon \notin \mathbb{K}_{RI}$.

Theorem 3.10: *The set* $\mathbb{K}_{LI} = \{Z = \frac{a}{2} + \frac{a-b}{2}\mathbf{i} + \frac{a}{2}\varepsilon + \frac{b}{2}\mathbf{h} : a, b \in \mathbb{K}\}$ *is a left ideal of* $\mathbb{K}$.

Proof: Let $Z = \frac{a}{2} + \frac{a-b}{2}\mathbf{i} + \frac{a}{2}\varepsilon + \frac{b}{2}\mathbf{h} \in \mathbb{K}_{LI}$ and $W = p + q\mathbf{i} + r\varepsilon + s\mathbf{h} \in \mathbb{K}$. Then

$$W.Z = \frac{ap + bq + ar - br + aq + bs}{2} + \frac{aq + ap - bp - as + bs + bq}{2}\mathbf{i}$$
$$+ \frac{ar - as + bs + ap + as + bq - br}{2}\varepsilon$$
$$+ \frac{as + ar - br - aq + bp}{2}\mathbf{h}$$

Therefore, $W.Z \in \mathbb{K}_{LI}$ and hence, $\mathbb{K}_{LI}$ is a left ideal of $\mathbb{K}$.

This proves the theorem. □

Corollary 3.10.1: *The subring* $\mathbb{K}_{LI}$ *is not a right ideal of* $\mathbb{K}$.

Proof:

$$W.Z = \frac{ap + bq + ar - br + bs}{2} + \frac{ap - bp + aq - bq + as - bs}{2}\mathbf{i}$$

$$+ \frac{ap - bq + ar + br - bs}{2}\varepsilon + \frac{bp + aq - ar + br - as}{2}\mathbf{h}$$

$$= \frac{u_1}{2} + \frac{v_1}{2}\mathbf{i} + \frac{x_1}{2}\varepsilon + \frac{y_1}{2}\mathbf{h}$$

Clearly, $u_1 \neq x_1$ and so $ZW \notin \mathbb{K}_{LI}$.

So, $\mathbb{K}_{LI}$ is not right ideal.

Example 3.3: *Consider* $Z = \frac{1}{2} - \mathbf{i} + \frac{1}{2}\varepsilon + \frac{3}{2}\mathbf{h}$; $W = \mathbf{i} + \mathbf{h}$, *then* $ZW = 3 - 2\mathbf{i} - 3\varepsilon + \mathbf{h} \notin \mathbb{K}_{LI}$.

Clearly, $u1 = x1$ and so $ZW \in/ \mathrm{K}LI$.

Conclusion

In this work we define a pair of sets called symmetric & skew-symmetric hybrid number and we show that the set of null-hyperbolic hybrid number is a normal subgroup of the group of symmetric hybrid numbers under the hybrid addition whereas the symmetric hybrid number does not form a subgroup under hybrid multiplication. After that, we prove that the set of all non-lightlike hybrid numbers forms an abelian group under hybrid multiplication, not only that the set of hybrid numbers of unit character is normal in the set of all hybrid number with non-zero character. Then, we find some subgroups of the set of hybrid numbers. In addition, we concentrated on finding right ideal, left ideal of this system.

4. Future Prospects

In the line of work as carried out in the paper one may think of to explore some properties by taking into account some different kinds of ideals and also may try to investigate the results in higher dimensional system.

Acknowledgement

The third author sincerely acknowledges the financial support rendered by DST-FIST 2024-2025 running at the Department of Mathematics, University of Kalyani, P.O.: Kalyani, Dist: Nadia, Pin: 741235, West Bengal, India.

Conflict of Interest

All the authors of this paper declare that they have no conflict of interest.

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Journal of Indian Acad. Math. ISSN: 0970-5120
Vol. 46, No. 2 (2024) pp. 117-131

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ALGORITHMIC APPROACH FOR SPLIT DOMINATION NUMBER OF GRAPH

Abstract: This article presents a novel algorithm for determining the split domination number of various graph classes such as Cartesian product of Complete graph and Path graph, its Line graph, Tadpole graph, Line graph of Tadpole graph, web graph and its dual graph etc. To enable the functioning of algorithm, the vertices of these graphs are divided into sets resembling some specific graph types such as K_1, K_2, $K_{1,2}$, $K_{1,3}$ etc.. By examining these divisions, the algorithm ascertains the bare minimum of vertices required to dominate every other vertex, thereby disconnecting the graph upon removal. This process, known as split domination, is crucial in understanding the connectivity and the structure of graph. The algorithm is versatile and can be applied to any graph that allows for partitioning vertices in the similar fashion.

Keywords: Partitions, Domination Number, Split Domination Number, Split Adjacency Matrix.

Mathematics Subject Classification (2020) No.: 05C69, 05C85, 05C90, 68R10.

1. Introduction

A key idea in graph theory [4] that aids in comprehending the composition and characteristics of graphs is the notion of domination number $\gamma(G)$ of G [1]. The domination number of a graph is the smallest number of vertices required to dominate all other vertices in the graph. It is said that a vertex dominates all of the vertices around it.

In 1997 [2], V R Kulli *et al*. introduced the concepts Split domination number and afterwards studied for research. The notions of the dominance number and vertices connectivity are combined to define the split dominating set and split dominating number. The dominating set D of a graph is known as a split dominating set if on removal of D graph becomes disconnected, and the split domination number $\gamma_s(G)$ of G is the minimum cardinality of the split dominating set.

The computation of the Split Domination Number of $K_2 \times P_n$, Line graph of $K_2 \times P_n$, Tadpole graph $T(m, n)$ and Line graph of Tadpole graph is analysed in [5, 6]. We looked at the split domination number of some graphs' dual graphs in [10].

In [7] Authors presents present two algorithms for finding a dominating set of the $m \times n$ complete toroidal grid graph $C_m \times C_n$. Algorithmic aspects of the k-domination problem in graphs is provided in [8].

To exactly find the split domination number for any graph is known to be NP-hard problem. Some approximation algorithms and heuristic methods are available, which can be used to find or estimate the split domination number of specific graph.

In this article, we present a new algorithmic approach to obtain the split domination number of some specific graphs, such as $K_2 \times P_n$, line graph of $K_2 \times P_n$, Tadpole graph $T(m, n)$, line graph of tadpole graph $L(T(m, n))$, Web graph, and Dual of Web graph by partitioning the vertex set in terms of $K_1, K_2, K_{1,n}$, with n in ascending order.

The proposed algorithm can be applied to any graph that allows for defined graph partitioning.

Here we are using the concept of graph partitioning, which is a fundamental problem in computer science and mathematics that involves dividing a graph into subsets of vertices, with the goal of optimizing certain objectives or constraint. Graph partitioning is performed by considering all the vertices or it is possible to perform graph partitioning without considering all vertices in certain contexts or in specific algorithms.

We are following Constraint-Based Partitioning approach in our algorithm by considering partitions in form of $K_2, K_1, K_{1,n}$ specifically $K_{1,2}, K_{1,3}$.

The paper is organized as: section 2 contains some elementary theory and statements defined by us, section 3 contains an algorithm for determining split dominance number, and section 4 justifies our findings with some illustrative examples. Section 5 contains our work's concluding remarks.

2. Preliminaries

Definition 2.1 Line Graph: Graph $G's$ Line graph $L(G)$ [3] is a graph with its vertices as points, which are mapped by the edges of G. If two edges of G have a common vertex, then the two vertices of $L(G)$ that are mapped to those two edges of G are connected.

Definition 2.2 Cartesian product of graph $K_2 \times P_n$: Complete graph K_2 and path graph P_n are graphs with vertices set $V(K_2) = \{u_1, u_2\}$ and $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ respectively.

The Cartesian product of K_2 and P_n denoted by $K_2 \times P_n$ is the graph with vertex set $V(K_2 \times P_n) = V(K_2) \times V(P_n) = \{(u_i, v_j) \mid u_i \epsilon V(K_2), v_j \epsilon V(P_n)\}$, edge set $E = \{e\}$ such that ' e ' is an edge of $K_2 \times P_n$ if and only if $e = \langle (u_i, v_j),(u_k, v_l) \rangle$

Where either

1. $i = k$ and $v_j v_l \epsilon E(P_n)$
2. $j = l$ and $v_j v_l \epsilon E(K_2)$

Definition 2.3 Graph Partition: To generalize the algorithm, we define a graph partition by considering all the vertices of the graph, as follows: Let $P^1, P^2, \ldots, P^n$ be any of the induced sub graphs of G, like $K_1, K_2, K_{1,n}, n \geq 2$, in ascending order $A = \{P^1, P^2, \ldots, P^n\}$ is the partition of Graph G, if A is minimum satisfying following conditions

1. $V(P^1) \cap V(P^2) \cap ... \cap V(P^n) = \phi$

2. $E(P^i) \cap E(P^j) = \phi$ for any $i \neq j$.

3. $\forall v_i \in G, \exists j : v_i \in P^j$ but not all $e_j \in P^i$.

Illustration

To explain the partition consider graph of $K_2 \times P_n$: The observed partition follows a pattern as shown in Figure 1.

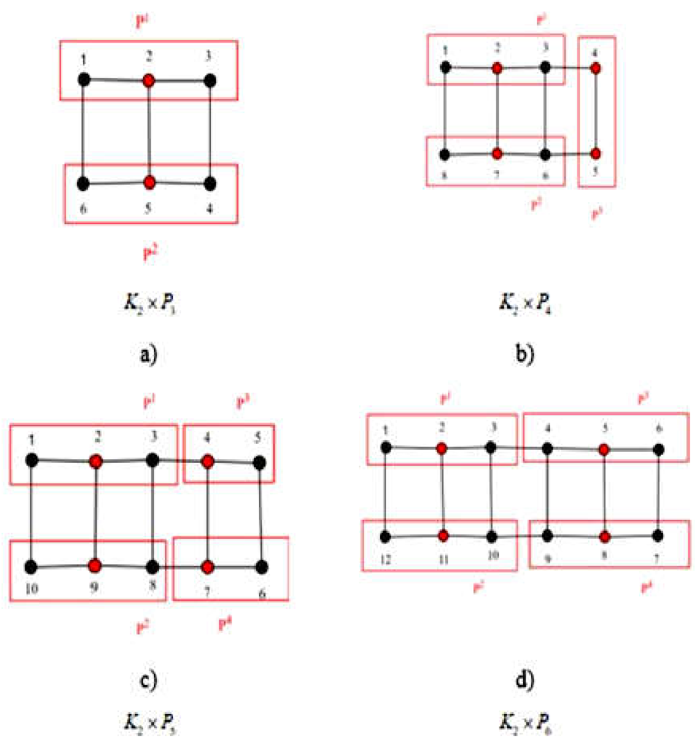

Figure 1 Partition of $K_2 \times P_n$

(a) $K_2 \times P_3 : \{K_{1,2}, K_{1,2}\}$ as $\{P^1, P^2\}$

(b) $K_2 \times P_4 : \{K_{1,2}, K_{1,2}, K_2\}$ as $\{P^1, P^2, P^3\}$

(c) $K_2 \times P_5 : \{K_{1,2}, K_{1,2}, K_2, K_2\}$ as $\{P^1, P^2, P^3, P^4\}$

(d) $K_2 \times P_6 : \{K_{1,2}, K_{1,2}, K_{1,2}, K_{1,2}\}$ as $\{P^1, P^2, P^3, P^4\}$

Similarly for $K_2 \times P_7 : \{K_{1,2}, K_{1,2}, K_{1,2}, K_{1,2}, K_2\}$ as $\{P^1, P^2, P^3, P^4, P^5\}$ and so on.

Such partition can be observed for many other classes of graph.

Table 1 illustrate the observed partitions for some of the observed graph classes, used by us for practical purpose. Where $T(m, n)$ is the tadpole graph and $L(T(m, n))$ is $T(m, n)'s$ line graph. $K_2 \times P_n$ is graph of Cartesian product of K_2 and P_n, $L(K_2 \times P_n)$ is Line graph of Cartesian product graph of complete graph K_2 and path P_n.

Such partition patterns can also be obtained for other graphs like cycle, path, wheel and web etc. along with their line and dual graphs.

Definition 2.4 Partition Matrix: For a graph G where partition of graph is done as per definition 2.3, we define the partition matrix $P = [P_{ij}]$ in which the rows corresponds to the partition sets and the columns to the vertices of the graph defined as follows:

$$P_{ij} = \begin{cases} d_j \text{ if } i^{th} \text{ partition contains vertex which is of degree } d_j \\ 0, \text{ Otherwise} \end{cases}$$

It is required to specify that d_j is degree of vertex v_j in the induced partition, not in the graph.

Also for specific partition $\boldsymbol{K}_1$, the degree of that vertex is considered as one for algorithmic purpose.

Table 1: Partition set of graph

Some graph and their obtained Partitioned set as per definition 2.3

Graph	**Partitioned set**
$K_2 X P_3$	$K_{1,2}, K_{1,2}$ as P^1, P^2
$K_2 X P_4$	$K_{1,2}, K_{1,2}, K_2$ as P^1, P^2, P^3
$L(K_2 \times P_3)$	$K_1, K_{1,2}, K_{1,2}$ as P^1, P^2, P^3
$L(K_2 \times P_4)$	$K_1, K_{1,2}, K_{1,2}, K_{1,2}$ as P^1, P^2, P^3, P^4
$L(K_2 \times P_n)$	$K_1, K_{1,2}, K_{1,2}, K_{1,2} \ldots K_{1,2}$ as $P^1, P^2, P^3, P^4, \ldots, P^n$
$T(4,4)$	$K_1, K_{1,3}, K_{1,2}$ as P^1, P^2, P^3
$T(4,5)$	$K_1, K_{1,3}, K_{1,2}, K_1$ as P^1, P^2, P^3, P^4
$T(4,6)$	$K_1, K_{1,3}, K_{1,2}, K_{1,2}$ as P^1, P^2, P^3, P^4
$T(4,7)$	$K_1, K_{1,3}, K_{1,2}, K_{1,2}, K_1$ as P^1, P^2, P^3, P^4, P^5
$T(4,n)$	$\begin{cases} K_1, K_{1,3}, K_{1,2}, \ldots, K_1 \text{ as } P^1, P^2, \ldots, P^{n-2} & \text{for n odd} \\ K_1, K_{1,3}, K_{1,2}, K_{1,2}, \ldots, K_{1,2} \text{ as } P^1, P^2, \ldots, P^{n-2} & \text{for n even} \end{cases}$
$L(T(3,1))$	K_2, K_2 as P^1, P^2
$L(T(3,2))$	$K_2, K_{1,2}$ as P^1, P^2
$L(T(3,3))$	$K_2, K_{1,2}, K_1$ as P^1, P^2, P^3
$L(T(3,4))$	$K_2, K_{1,2}, K_2$ as P^1, P^2, P^3
$L(T(3,5))$	$K_2, K_{1,2}, K_{1,2}$ as P^1, P^2, P^3
$L(T(3,6))$	$K_2, K_{1,2}, K_{1,2}, K_1$ as P^1, P^2, P^3, P^4
$L(T(3,7))$	$K_2, K_{1,2}, K_{1,2}, K_2$ as P^1, P^2, P^3, P^4
$L(T(3,8))$	$K_2, K_{1,2}, K_{1,2}, K_{1,2}$ as P^1, P^2, P^3, P^4
$L(T(3,9))$	$K_2, K_{1,2}, K_{1,2}, K_{1,2}, K_1$ as P^1, P^2, P^3, P^4, P^5
$L(T(3,10))$	$K_2, K_{1,2}, K_{1,2}, K_{1,2}, K_2$ as P^1, P^2, P^3, P^4, P^5

From figure 1 (c)

The partition set of the graph $K_2 \times P_5 \ : A = \{P^1, P^2, P^3, P^4\}$

Partition Matrix P obtained as follows

$$\begin{array}{c} \\ P^1 \\ P^2 \\ P^3 \\ P^4 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

(1)

2.5 Deduction from Partition matrix: ***2.5.1 Dominating Set D*****:** Select the vertices with maximum degree from each row of Partition matrix this gives the set D may be defines mathematically as

$$D = \{v_i \in \langle V(Pi) \rangle, P_i \in A \, / \, v_i, \ is\ of \max \deg of \ \langle V(P_i) \rangle\}$$

For Specified matrix (1) of 2.4 section

Using maximum degree in each row of P, we get

Dominating set $D = \{2, 9, 4, 7\}$ (2)

2.5.2 Split Adjacency Matrix: Find out the Adjacency matrix $A = [a_{ij}]$ of dimension $n \times n$ between vertices of graph G, defined with

$$a_{ij} \begin{cases} 1, & \text{if } j^{\text{th}} \text{ vertex is adjacent to } i^{\text{th}} \text{ vertex} \\ 0, & \text{Otherwise} \end{cases}.$$

Here graph G under consideration have no parallel edges.

Now update $a_{ij}{}^{th}$ element of matrix A by replacing one by zero if vertex v_i is adjacent to v_j and $v_j \in D$ the reduced matrix is split dominating matrix given by $S = [S_{ij}]$ of dimension $n \times n$ between vertices of graph G, where

$$S_{ij} = \begin{cases} 1, \text{ if } v_i \text{ is adjacent to } v_j \text{ and } vj \notin D \\ 0, \quad \text{Otherwise} \end{cases}.$$

Using matrix (1) and dominating set (2), the Split Adjacency Matrix S corresponding to figure 1(c), given by

	1	2	3	4	5	6	7	8	9	10
1	0	0	0	0	0	0	0	0	0	1
2	1	0	1	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	1	0	0
4	0	0	1	0	1	0	0	0	0	0
5	0	0	0	0	0	1	0	0	0	0
6	0	0	0	0	1	0	0	0	0	0
7	0	0	0	0	0	1	0	1	0	0
8	0	0	1	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	1	0	1
10	1	0	0	0	0	0	0	0	0	0

(3)

3. Algorithm to find the Split domination number

Step 1: Divide the specified graph by partitioning the vertex set in terms of $K_1, K_2, K_{1,2}, K_{1,3}$ and so on, to obtain the partition set A as mentioned definition 2.3.

Step 2: Generate the partition matrix P as per definition 2.4.

Step 3: Find out the dominating Set D as defined in deduction 2.5.1.

Step 4: Create Split adjacency matrix S as discussed in deduction 2.5.2.

Step 5: Update the Matrix S by deleting rows and columns corresponding to vertices in set D.

Step 6: After deleting the corresponding row and columns, we get a adjacency matrix between vertices of $G - D$, containing block of matrices or containing row with all zero that indicates the isolated components.

In either case the resultant matrix represents the disconnected graph [9]. The total of number of blocks and the zeros rows say N represents that on removal of D, graph get splits into N components.

Step 7: Display the split domination number of $G : \gamma_s(G) = |D|$, splited components: N.

Step 8: END

Illustrative Example

3.1 Split domination number of $L(K_2 \times P_5)$

Find out the partition set $A = \{P^1, P^2, P^3, P^4, P^5\}$ as specified in section 2.

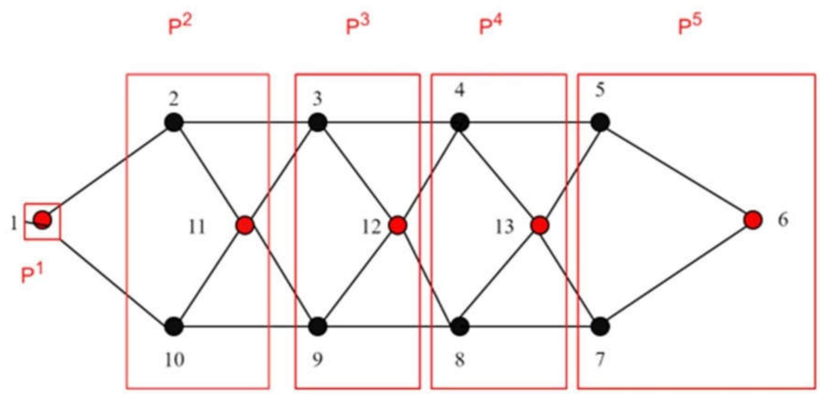

Figure 2: Line graph of $K_2 \times P_5$

Creat the Partition Matrix P

	1	2	3	4	5	6	7	8	9	10	11	12	13
P^1	1	0	0	0	0	0	0	0	0	0	0	0	0
P^2	0	1	0	0	0	0	0	0	0	1	2	0	0
P^3	0	0	1	0	0	0	0	0	1	0	0	2	0
P^4	0	0	0	1	0	0	0	1	0	0	0	0	2
P^5	0	0	0	0	1	2	1	0	0	0	0	0	0

Using maximum degree in each row of P, we get

Dominating set $D = \{1, 11, 12, 13, 6\}$

Obtain the Split adjacency matrix S as

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
1	0	1	0	0	0	0	0	0	0	1	0	0	0
2	0	0	1	0	0	0	0	0	0	0	0	0	0
3	0	1	0	1	0	0	0	0	0	0	0	0	0
4	0	0	1	0	1	0	0	0	0	0	0	0	0
5	0	0	0	1	0	0	0	0	0	0	0	0	0
6	0	0	0	0	1	0	1	0	0	0	0	0	0
7	0	0	0	0	0	0	0	1	0	0	0	0	0
8	0	0	0	0	0	0	1	0	1	0	0	0	0
9	0	0	0	0	0	0	0	1	0	1	0	0	0
10	0	0	0	0	0	0	0	0	1	0	0	0	0
11	0	1	1	0	0	0	0	0	1	1	0	0	0
12	0	0	1	1	0	0	0	1	1	0	0	0	0
13	0	0	0	1	1	0	1	1	0	0	0	0	0

Now delete the rows and columns of S, that corresponds to vertex of set D.

The Obtained adjacency matrix of $L(K_2 \times P_5) - D$ contains 2 block matrices, it indicates the splitting of graph in 2 components.

	2	3	4	5	7	8	9	10
2	0	1	0	0	0	0	0	0
3	1	0	1	0	0	0	0	0
4	0	1	0	1	0	0	0	0
5	0	0	1	0	0	0	0	0
7	0	0	0	0	0	1	0	0
8	0	0	0	0	1	0	1	0
9	0	0	0	0	0	1	0	1
10	0	0	0	0	0	0	1	0

As per defined partition this set D is set with minimum number of vertices that dominates all other vertices of $L(K_2 \times P_5)$, and on removal of D the $L(K_2 \times P_5)$ get splits into two components.

Thus, $\gamma_s(L(K_2 \times P_5)) = |D| = 5$.

3.2 To find the split domination number of line graph of $T(4, 4) : L(T(4, 4))$

As discussed in the previous example, we get following

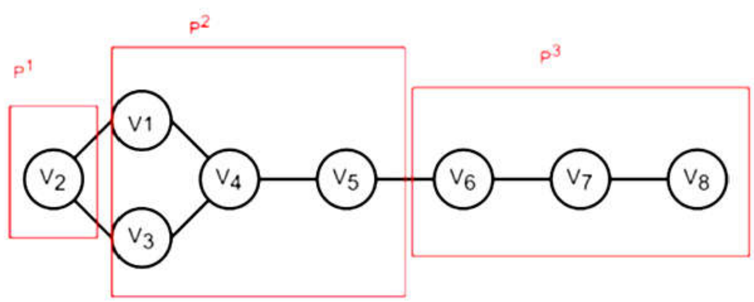

Figure 3: Line Graph of T(4,4)

$A = \{P^1, P^2, P^3\}$

Partition Matrix P:

$$
\begin{array}{c|cccccccc}
 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \\
P^1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
P^2 & 1 & 0 & 1 & 3 & 1 & 0 & 0 & 0 \\
P^3 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1
\end{array}
$$

Using maximum degree in each row of P:

Dominating set $D = \{v_2, v_4, v_7\}$

The Split adjacency matrix is S:

$$
\begin{array}{c|cccccccc}
 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \\
v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_4 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
v_5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
v_6 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
v_7 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
v_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Now delete the rows and columns of S, that corresponds to vertex of set D, we get Adjacency matrix of set of vertices of $L(T(4, 4) - D$ as

$$\begin{array}{c} \\ v_1 \\ v_3 \\ v_5 \\ v_6 \\ v_8 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_3 & v_5 & v_6 & v_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

In above reduced matrix three zero rows, represents 3 isolated vertex v_1, v_3 and v_8. One block of matrix represents connected component of vertex v_5 and v_6. Thus, after removing the set D the graph $L(T(4, 4))$ splits into 4 components that contains 3 isolated vertices.

As per defined partition this obtained set D is minimum dominating that split's the Graph $L(T(4, 4))$ in 4 components.

Thus, $\gamma_s(L(T(4, 4)) = |D| = 3$

Obtained results for $\gamma_s(L(T(m, n)))$, corresponding to various values of m and n are listed below in Table 2, $m \geq 3$.

Table 2: Split domination number of Line graph of Tadpole graph

m/n	1	2	3	4	5	6	7	8	9	10	11	12	13
3	2	2	2	2	3	3	3	4	4	4	5	5	5
4	2	2	3	3	4	4	4	5	5	5	6	6	6
5	2	2	3	3	3	4	4	4	5	5	5	6	6
6	2	3	3	3	4	4	4	5	5	5	6	6	6
7	3	3	4	4	4	5	5	5	6	6	6	7	7
8	3	3	4	4	4	5	5	5	6	6	6	7	7

That matches the exact values as per our generalized formulae derived in [6] and mentioned bellow in section 4.

4. Exact values

Theorem 1: *Let $T(m,n)$ is Tadpole graph $m \geq 3$ then $\gamma_s(L(T(m,n)))$ is given by*

(a) $\gamma_s(L(T(3,1))) = 2$

(b) $\gamma_s(L(T(3,n))) = 1 + \left\lceil \frac{n-1}{3} \right\rceil, n > 1$

(c) $\gamma_s(L(T(m,1))) = 2\left\lceil \frac{m-1}{3} \right\rceil, m > 3$

(d) $$\gamma_s(L(T(3,1))) = \begin{cases} 1 + \left\lceil \frac{m-2}{3} \right\rceil + \left\lceil \frac{n-2}{3} \right\rceil, m \neq 3k \\ 1 + \left\lceil \frac{m-3}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil, m = 3k \end{cases} \text{ for } m \geq 4 \text{ and } k \geq 2$$

Theorem 2: $\gamma_s(L(K_2 \times P_n)) = n$

The proof of above theorems is mentioned in our previous work [6].

5. Conclusion

Finding the split domination number of any graph is a NP hard problem, there is no specific algorithm that can be used to find the split domination number of any graph. The proposed partition-based algorithmic approach is applicable not only to the specified graph but also to any graph where such defined partitions exist. Secondly, in many programmable languages, the matrix used for algorithm implementation can be declared as sparse, which uses less memory and is easier to construct with a lower time complexity. One additional feature of the suggested technique is that it provides the number of components in which the provided graph splits after deleting the split dominating set.

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(*Received*, July 18, 2024)
(*Revised*, August 14, 2024)

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Journal of Indian Acad. Math.
Vol. 46, No. 2 (2024) pp. 133-152
ISSN: 0970-5120

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MAGNETO-CONVECTION AND SORET EFFECT IN NANOFLUID LAYERS: A DOUBLE DIFFUSIVE PERSPECTIVE

Abstract: This mathematical study aims at investigating the combined effect of a uniform vertical magnetic field and Soret effect on the onset of double diffusive convection in a nanofluid layer. The linear stability analysis is based on normal mode technique. Galerkin method has been applied to find the critical Rayleigh number and the corresponding wave number in terms of various parameters numerically. The effects of Soret parameter, magnetic field, Lewis number, Modified diffusivity ratio, Concentration Rayleigh-Darcy number, Solutal Rayleigh number on the stability of the system have been investigated. In this double diffusive convection under magnetic field, Darcy number also comes into play and has been observed to provide the stabilizing effect on stationary as well as oscillatory modes. The effects of other different parameters on the stability of the system have been investigated analytically as well as graphically. The comparison of results obtained has been done with the existing relevant studies.

Keywords: Nanofluid, Magnetic Field, Double Diffusion, Soret Parameter, Rayleigh Number.

Mathematics Subject Classification (2010) No.: 76E06, 76E25, 76R50, 76W05.

1. Introduction

Heat transfer mechanism has been improved by replacing micro sized particles with nano sized particles in conventional fluids. In 1992, Choi [7] observed that heat transfer is very excellent only when pumping power is increased and an expensive cryogenic system is maintained. The term nanofluid was first coined by Choi, he described the future and hope of the application of nanotechnology. The nanoparticles differ from conventional particles (milli-meter or micro scale) in sense that they stay in suspension in the fluid without sedimentation. Kumar *et al*. [15] established the utility of a particular nanofluid for its heat transfer application. Wong and Leon [28] focussed on giving the broad range of present and future applications of nanofluids.

The problem of thermal convection for a Newtonian fluid layer was discussed by Chandrasekhar [6] taking varying assumptions of hydro-dynamics and hydro-magnetism. In a horizontal porous layer of nanofluid, convection was studied by Nield and Kuznetsov [21], incorporating Brownian Motion and Thermophoresis. He observed that the critical Rayleigh number decreases or increases by a significant amount when the basic nanoparticle concentration is increased at the top or bottom.

Study of convective instability of the nanofluids has many uses in astrophysics and geophysics etc. Buongiorno [5] proposed the model for nanofluid convection. Later, Nield and Kuznetsov [22] revisited this problem by taking different types of non-dimensional variables. It was observed that nanofluids are more unstable than the pure fluids. Maxwell [18] gave mathematical model for non-Newtonian fluids exhibiting the elastic and viscous behaviour simultaneously. Khuzhayorov *et al*. [14] introduced law for linear flow of variety of viscoelastic fluids flow. Problem for Maxwell nanofluid taking into account thermophoresis and Brownian diffusion was studied by Jaimala *et al.* [13].

The study of magnetic field effects on the onset of convection has important applications in physics and engineering. In metal casting and in cooling systems of electronic devices, magnetic field effects are of great importance. The nanofluid can be taken as a working medium in order to get effective heat performance of such devices. Rayleigh Benard Magneto-convection arises due to combined effect of buoyancy force and magnetic field induced Lorentz force. A non-dimensional parameter called Chandrasekhar number gets introduced due to Lorentz force. Heris *et al*. [10] observed the increase in thermal efficiency of a two-phase closed thermosyphon while experimental study in presence of magnetic field. The combined effect of a vertical magnetic field and the boundaries on the onset of convection in an electrically nanofluid layer heated from below was investigated by Yadav *et al*. [29]. Effect of magnetic field considering internal heating after filling the space between plates with nanofluid was also studied by Yadav *et al*. [30].

Double diffusion has become very important now days due to its vast applications. The temperature gradient affects the buoyancy force in double diffusive convection, but also the concentration gradient of the fluid has a significant effect on the buoyancy force in double diffusion. Detailed discussion on double diffusive convection has been done in the books by Nield and Bejan [20], Ingham and Pop [12] and Vafai [26]. Research work on fluid instability in a horizontal layer was done by Horton and Rogers [11] and Lapwood [16]. Further Haajizadeh *et al*. [9], Gaikwad *et al.* [8], Malashetty and Swamy [17] also analysed stability of double diffusion in different cases.

In fluid flow problems, the phenomenon of generation of the concentration flux by temperature gradient is termed as Soret effect. Study on the Soret induced convective instability of a regular Newtonian fluid saturated in a porous medium has been done by many researchers. Wang and Tan [27] analysed the convctive instability in Benard cells in a non-Newtonian fluid incorporating Soret factor. The impact of Soret parameter induced by the temperature gradient was studied by Singh *et al*. [25]. Bahlowl *et al*. [4] and Mansour *et al*. [19] also have much research work on Soret effect in different forms of fluid layers. Postelnicu [23] and Rajput and Shareef [24] also studied the combined effect of magnetic field and Soret parameter. Agarwal *et al*. [1] studied the phenomenon in a horizontal porous layer taking the base fluid of the nanofluid as binary fluid. These nanofluids proved importance in electroplating and as a transfer medium in medical treatment by Buongiorno [5].

The literature survey indicates that no study has investigated the effect of magnetic field on double diffusive convection in a nanofluid layer with Soret factor. The present study examines the effect of vertical magnetic field on Soret induced double diffusive convection in a nanofluid layer.

2. Mathematical Formulation

We consider a layer of nanofluid confined between two infinite horizontal surfaces separated by a distance a, with z-axis vertically upward. Lower surface is maintained at higher temperature T_l^* and upper surface is maintained at temperature T_r^*. A uniform vertical magnetic field $M^* = (0, 0, M_0^*)$ is applied (See Fig. 1).

The governing equations for conservation of mass, momentum, energy and concentration of salt and nanoparticles are as follows:

$$\nabla^*.\boldsymbol{q}_d^* = 0 \tag{2.1}$$

$$\frac{\mu}{K}\boldsymbol{q}_d^* = (1+\lambda^*\frac{\partial}{\partial t^*})[-\nabla^* p^* + (\psi^*\rho_p + (1-\psi^*)\{\rho(1-\beta_t(T^*-T_r^*) -\beta_c(S^*-S_r^*))\})\boldsymbol{g} + \frac{\mu_c}{4\pi}(\nabla^*\times\boldsymbol{M}^*)\times\boldsymbol{M}^* \tag{2.2}$$

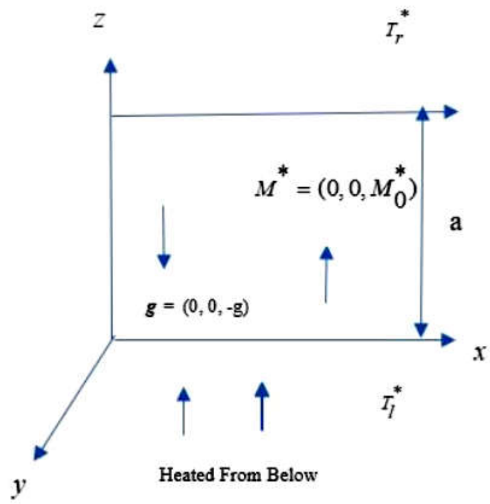


Fig. 1: Physical configuration of the problem

$$(\rho c)_M \frac{\partial T^*}{\partial t^*} + (\rho c)_F \boldsymbol{q}_d^*.\nabla^* T^* = k_m {\nabla^*}^2 T^* + \in (\rho c)_P \left[B_d \nabla^* \psi^*.\nabla^* T^* + (\tfrac{B_t}{T_c^*}) \nabla^* T^*.\nabla^* T^* \right] \quad (2.3)$$

$$\frac{\partial S^*}{\partial t^*} + \frac{1}{\in} \boldsymbol{q}_d^*.\nabla^* S^* = S_d {\nabla^*}^2 S^* + S_{ct} {\nabla^*}^2 T^* \quad (2.4)$$

$$\frac{\partial \psi^*}{\partial t^*} + \frac{1}{\in} \boldsymbol{q}_d^*.\nabla^* \psi^* = B_d {\nabla^*}^2 \psi^* + \frac{B_t}{T_c^*} \nabla^{*2} T^* \quad (2.5)$$

Maxwell equations in modified form are

$$\left(\frac{\partial}{\partial t^*} + \frac{1}{\in} (\boldsymbol{q}_d^*.\nabla^*) \right) \boldsymbol{M}^* = (\boldsymbol{M}^*.\nabla^*) \frac{1}{\in} \boldsymbol{q}_d^* + \eta {\nabla^*}^2 \boldsymbol{M}^* \quad (2.6)$$

where $\nabla^*.\boldsymbol{M}^* = 0,\ \eta = \dfrac{1}{4\pi\mu_e \sigma'}$ and $q_d^* = (u_{1d}^*, u_{2d}^*, u_{3d}^*)$.

For constant temperature and salt concentrations at the boundaries and zero nanoparticle flux, the boundary conditions are taken as

$$\boldsymbol{q}_d^* = 0,\ T = T_l^*,\ S^* = S_l^*,\ B_d \frac{\partial \psi^*}{\partial z^*} + \frac{B_t}{T_c^*}\frac{\partial T^*}{\partial z^*} = 0 \quad \text{at} \quad z^* = 0 \tag{2.7}$$

$$\boldsymbol{q}_d^* = 0,\ T = T_r^*,\ S^* = S_r^*,\ B_d \frac{\partial \phi^*}{\partial z^*} + \frac{B_t}{T_c^*}\frac{\partial T^*}{\partial z^*} = 0 \quad \text{at} \quad z^* = a \tag{2.8}$$

Non-dimensional scheme is given below as

$$(X, Y, Z) = \frac{(x^*, y^*, z^*)}{a}, \qquad t = \frac{t^*\alpha_m}{\sigma a^2}, \qquad (u_{1d}, u_{2d}, u_{3d}) = \frac{(u_{1d}^*, u_{2d}^*, u_{3d}^*)a}{\alpha_m},$$

$$p = \frac{p^*K}{\mu\alpha_m}, \qquad \psi = \frac{\psi^* - \psi_0^*}{\psi_0^*}\ T = \frac{T^* - T_r^*}{T_l^* - T_r^*}, \qquad \lambda = \frac{\lambda^*\alpha_m}{a^2} \quad \text{and}$$

$$(M_X, M_Y, M_Z) = \frac{(M_X^*, M_Y^*, M_Z^*)}{M_0^*}, \qquad (X, Y, Z) = \frac{(x^*, y^*, z^*)}{a}, \qquad t = \frac{t^*\alpha_m}{\sigma a^2},$$

$$(u_{1d}, u_{2d}, u_{3d}) = \frac{(u_{1d}^*, u_{2d}^*, u_{3d}^*)a}{\alpha_m}, \quad p = \frac{p^*K}{\mu\alpha_m}, \quad \psi = \frac{\psi^* - \psi_0^*}{\psi_0^*}, \quad S = \frac{S^* - S_r^*}{S_l^* - S_r^*}$$

$$T = \frac{T^* - T_r^*}{T_l^* - T_r^*},\ \lambda = \frac{\lambda^*\alpha_m}{a^2} \ \text{ and } \ (M_X, M_Y, M_Z) = \frac{(M_X^*, M_Y^*, M_Z^*)}{M_0^*}.$$

On replacing $\boldsymbol{q}_d$ by $\boldsymbol{q}$

$$\nabla.\boldsymbol{q} = 0 \tag{2.9}$$

$$\boldsymbol{q} = (1 + \frac{\lambda}{\sigma}\frac{\partial}{\partial t})[\{-\nabla p - R_m\hat{\boldsymbol{e}}_z - R_n\phi\hat{\boldsymbol{e}}_z + R_aT\hat{\boldsymbol{e}}_z + \frac{R_s}{Ln}S\hat{\boldsymbol{e}}_z\} + \frac{P_1}{P_{1m}}QD_a(\nabla \times M) \times M] \tag{2.10}$$

$$\frac{\partial T}{\partial t} + (\boldsymbol{q}.\nabla)T = \nabla^2 T + \frac{N_b}{L_e}\nabla\psi.\nabla T + \frac{N_aN_b}{L_e}\nabla T.\nabla T \tag{2.11}$$

$$\frac{1}{\sigma}\frac{\partial S}{\partial t}+\frac{1}{\in}\boldsymbol{q}.\nabla S=\frac{1}{Ln}\nabla^2 S+N_{ct}\nabla^2 T \tag{2.12}$$

$$\frac{1}{\sigma}\frac{\partial \psi}{\partial t}+\frac{1}{\in}(\boldsymbol{q}.\nabla)\psi=\frac{1}{Le}\nabla^2\psi+\frac{N_a}{Le}\nabla^2 T \tag{2.13}$$

$$\frac{1}{\sigma}\frac{\partial \boldsymbol{M}}{\partial t}+\frac{1}{\in}(\boldsymbol{q}.\nabla)\boldsymbol{M}=\frac{1}{\in}(\boldsymbol{M}.\nabla)\boldsymbol{q}+\frac{P_1}{P_{1m}}\nabla^2\boldsymbol{M} \tag{2.14}$$

Here $R_a(=\dfrac{\rho g\beta Kd(T_l^*-T_r^*)}{\mu\alpha_m})$, $R_n(=\dfrac{(\rho_P-\rho)\psi_0^* gKd}{\mu\alpha_m})$, $R_m(=\dfrac{\rho_P\psi_0^*+\rho(1-\psi_0^*)gKd}{\mu\alpha_m})$, $R_s(=\dfrac{\rho\beta_c gdK(S_l^*-S_r^*)}{\mu S_d})$ are thermal, concentration, basic density and solutal Rayleigh Darcy number respectively, $P_1(=\dfrac{\mu}{\rho\alpha_m})$ and $P_{1m}(=\dfrac{\mu}{\rho\eta})$ are Prandtl numbers, $Q(=\dfrac{\mu_e M_0^{*2}d^2}{4\pi\mu\eta})$ is Magnetic Chandrasekhar number, $D_a(=\dfrac{K}{a^2})$ is Darcy number, $N_{ct}(=\dfrac{S_{ct}(T_l^*-T_r^*)}{\alpha_m(S_l^*-S_r^*)})$ is Soret parameter, $N_a(=\dfrac{B_t(T_l^*-T_r^*)}{B_d T_r^* Q_0^*})$ and $N_b(=\dfrac{(\rho c)_p\in Q_0^*)}{(\rho c)_F})$ are modified diffusivity ratio and modified particle density increment respectively, $Le(=\dfrac{\alpha_m}{B_d})$ and $Ln=\dfrac{\alpha_m}{S_d}$ are Lewis numbers for nanofluid and salt respectively.

The basic time independent flow state is given by

$$\boldsymbol{q}=0,\ p=p_{bs}(Z),\psi=\psi_{bs}(Z),T=T_{bs}(Z),S=S_{bs}(Z),\boldsymbol{M}\hat{\boldsymbol{e}}_z \tag{2.15}$$

where the suffix ‘bs’ refers to the basic flow. The basic volume fraction and temperature equations provide

$$T_{bs}=1-Z,\ \psi_{bs}=\psi_0+N_a Z,\ S_{bs}=1-Z.$$

On the basic state, we superimpose perturbations in the form

Let $\boldsymbol{q} = \boldsymbol{q}'$, $p = p_{bs} + p'$, $T = T_{bs} + T'$, $S = S_{bs} + S'$, $\phi = \phi_{bs} + \phi'$, $\boldsymbol{M} = \hat{\boldsymbol{e}}_z + \boldsymbol{M}'$,

where the primes denote infinitesimal small quantities. Ignoring the products of primed quantities and their derivatives, following linearised form of equations is obtained:

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \frac{P_1}{P_{1m}}\nabla^2\right)\left[\nabla^2 u_3' - \left(1 + \frac{\lambda}{\sigma}\frac{\partial}{\partial t}\right)\left(R_a\nabla_H^2 T' - R_n\nabla_H^2\phi' + \frac{R_s}{L_n}\nabla_H^2 S'\right)\right]$$
$$= (1 + \frac{\lambda}{\sigma}\frac{\partial}{\partial t})Q\,\frac{P_r}{P_{rM}}\,\frac{D_a}{\in}\,\nabla^2\,\frac{\partial^2 u_3'}{\partial Z^2} \qquad (2.16)$$

$$\frac{\partial T'}{\partial t} - \mu_3' = \nabla^2 T' - \frac{N_a N_b}{Le}\frac{\partial T'}{\partial Z} - \frac{N_b}{Le}\frac{\partial \phi'}{\partial Z} \qquad (2.17)$$

$$\frac{1}{\sigma}\frac{\partial S'}{\partial t} - \frac{u_3'}{\in} = \frac{1}{Ln}\nabla^2 S' + N_{ct}\nabla^2 T' \qquad (2.18)$$

$$\frac{1}{\sigma}\frac{\partial \psi'}{\partial t} + \frac{1}{\in}(\boldsymbol{V}'.\nabla)\psi' + \frac{1}{\in}N_a u_3' = \frac{1}{Le}\nabla^2\psi' + \frac{N_a}{Le}\nabla^2 T' \qquad (2.19)$$

with boundary conditions

$$u_3' = 0,\ T' = 0,\ S' = 0,\ \frac{\partial \psi'}{\partial Z} + N_a\frac{\partial T'}{\partial Z} = 0 \quad \text{and } Z = 1 \qquad (2.20)$$

3. Linear Study

Following the linear stability theory by Chandrasekhar [6], the perturbations are taken of the form

$$(\psi', T', u_3', S') = [\Phi(Z), \Theta(\mathrm{Z}), \Omega(\mathrm{Z}), \Psi(\mathrm{Z})]e^{st+iLX+iMY} \qquad (3.1)$$

where *L* and *M* are dimensionless wave numbers in X and *Y* directions respectively.

On substituting the above values and employing Galerkin method to solve equations (2.16) to (2.19) together with the boundary condition (2.20) and taking first estimation as N = 1, we have

$$\Omega = A_1 \sin \pi Z,\ \Theta = B_1 \sin \pi Z, \Phi = -N_a C_1 \sin \pi Z,\ \Psi = D_1 \sin \pi Z\ .$$

Taking the determinant of the resulting matrix equation as zero, the following Rayleigh number is obtained

$$R_a = \frac{\sigma}{\in \alpha^2}\left[\frac{\begin{array}{l} R_s\alpha^2(\lambda s+\sigma)(\sigma A\chi^2+s)(\sigma\chi^2+sLe)(\chi^2+s)\{\chi^2(\in N_{ct}-1)-s\} \\ -R_n N_a\alpha^2(\lambda s+\sigma)(\chi^2\sigma+sLn)(A\chi^2\sigma+s)\{\chi^2(\in+Le)+sLe\} \\ +\in(\chi^2\sigma+sLn)(\chi^2\sigma+sLe)(\chi^2+s)(A\sigma\chi^4+B\pi^2\chi^2(\sigma+\lambda s)+s\chi^2\} \end{array}}{(\sigma\chi^2+sLn)(\sigma\chi^2+sLe)(\lambda s+\sigma)(A\chi^2\sigma+s)}\right] \tag{3.2}$$

where, $A = \dfrac{P_1}{P_{1m}}$ and $B = Q\dfrac{P_1}{P_{1m}}\dfrac{D_a}{\in}$.

4. Graphical results and discussions

4.1 Stationary Convection: Taking s = 0 in equation (3.2),

$$R_a^{st} = \frac{\chi^4}{\alpha^2} - \left(1+\frac{Le}{\in}\right)R_n N_a + \frac{QD_a\pi^2\chi^2}{\in \alpha^2} - \frac{R_s}{\in}(1- \in N_{ct}) \tag{4.1}$$

From equation (4.1), the minimum Rayleigh number is given as

$$R_{ac}^{st} = \pi^2\left[1+\left(1+\frac{QD_a}{\in}\right)^{1/2}\right]^2 - R_n N_a\left(1+\frac{Le}{\in}\right) - \frac{R_s}{\in}(1- \in N_a) \tag{4.2}$$

at critical wave number $\alpha_c = \pi\left(1+\dfrac{QD_a}{\in}\right)^{1/4}$. In absence of magnetic field, eq. (4.2) converts to

$$R_{ac}^{st} = 4\pi^2 - R_n N_a\left(1+\frac{Le}{\in}\right) - \frac{R_S}{\in}(1- \in N_{ct})$$

which is same as obtained by Singh *et al*. [25]. The stationary convection curves for Rayleigh number R_a versus the wave number α are shown in Fig. 2(a)-(f) by assigning fixed values.

$$N_a = 4,\ D_a = 0.2,\ Le = 10,\ R_n = 4,\ \in = 0.4,\ Q = 800,\ R_s = 5,\ N_{ct} = 0.1$$

with variations in one of these parameters.

Fig 2(a) shows the effect of Darcy number. The increase in Darcy Number increases the Rayleigh Number resulting in delay in convection. Fig 2(b) displays the effect of porosity parameter $\in$. Porosity has stabilizing as well as destabilizing effect in presence of Q. Initially there is decrease in Rayleigh no. with increase in porosity and after a certain wave no. behaviour gets reversed. Fig 2(c) illustrates the behaviour of Rayleigh Number for different values of Lewis number. There is decrease in Rayleigh number with Le.

The effect of R_n on Rayleigh Number is shown in Fig 2(d). Different Curves show that Rayleigh Number is decreased with increase in R_n. The graphs for Rayleigh Number R_a against the wave number α for various values of N_a and fixed values of other parameters are in Fig 2(e). It is evident that N_a advances the onset of stationary convection. Fig 2(f) shows the variation of Rayleigh Number for different values of *Q*. It is clear from the figure that there is a significant increase in the value of critical Rayleigh Number with increase in *Q*. Thus, the magnetic field stabilises the nanofluid layer and the increase in magnetic field increases the stabilising effect. Fig. 2(g) shows the effect of Soret parameter. Critical Rayleigh no. increases with increase in Soret parameter and hence responsible for promoting the stability of the flow. It is clear from Fig. 2(h) on increasing the solutal Rayleigh number critical Rayleigh no. is decreased, thus resulting in an early convection.

4.2. Oscillatory Convection: Taking $s = i\omega$ in equation (35), we get the following Rayleigh Number:

$$R_a^{osc} = \frac{\sigma}{\alpha^2 \in}\left[\frac{(X_1 + X_2 + X_3)U - (Y_1 + Y_2 + Y_3)V}{U^2 + \omega^2 V^2}\right] \tag{4.3}$$

where $A = \dfrac{P_r}{P_{rM}}$, $\quad B = Q\dfrac{P_r}{P_{rM}}\dfrac{D_a}{\in}$

$$X_1 = \begin{bmatrix} R_s\alpha^2[\sigma\chi^2(\in N_{ct} - 1)\{A\chi^2(\sigma^2\chi^2 - \lambda\omega^2 Le) - \omega^2(Le + \lambda\chi^2)\} \\ -\omega^2\{\lambda\omega^2 Le - \sigma^2 A\chi^2(Le + \lambda\chi^2) - \sigma^2\chi^2\}] \end{bmatrix},$$

$$X_2 = \begin{bmatrix} -R_n N_a\alpha^2[(\in + Le)\{A\chi^4\sigma(\sigma^2\chi^2 - \lambda\omega^2 Ln) - \omega^2\sigma\chi^2(\lambda\chi^2 + Ln)\} \\ -\omega^2\chi^2\sigma^2 Le(1 + \lambda A\chi^2) + \omega^2 LeLn(\lambda\omega^2 - \sigma^2 A\chi^2)] \end{bmatrix},$$

$$X_3 = \begin{bmatrix} \in[\sigma\chi^4(A\chi^2 + B\pi^2)\{\sigma^2\chi^4 - \omega^2 LnLe - \sigma\omega^2(Ln + Le)\} \\ -\omega^2\chi^2(1 + \lambda\pi^2 B)\{\sigma^2\chi^4 - \omega^2 LnLe + \sigma\chi^4(Ln + Le)\}] \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} R_s\alpha^2[(\in N_{ct} - 1)\{A\chi^4\sigma^2\omega^2(\lambda\chi^2 + Le) + \omega^2\chi^2(\sigma^2\chi^2 - \lambda\omega^2 Le)\} \\ +\omega^2\sigma(\lambda\omega^2 A\chi^2 Le + \omega^2 Le - \sigma^2 A\chi^4 + \lambda\omega^2\chi^2)] \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} -R_n N_a\alpha^2[\omega^2\chi^2(\in + Le)(A\chi^4\lambda\sigma^2 + A\chi^2\sigma^2 Ln + \sigma^2\chi^2 - \lambda\omega^2 Ln) \\ -\omega^2 Le\{\lambda\chi^2\omega^2\sigma(1 + ALn) + \sigma\omega^2 Ln - A\sigma^3\chi^4\}] \end{bmatrix},$$

$$Y_3 = \begin{bmatrix} \in[\sigma\chi^2\omega^2(A\chi^2 + B\pi^2)\{\sigma^2\chi^4 - \omega^2 LnLe + \sigma\chi^4(Ln + Le)\} \\ -\omega^2\chi^4(1 + \lambda\pi^2 B)\{\sigma\omega^2(Ln + Le) - \chi^4\sigma^2 + \omega^2 LnLe\}] \end{bmatrix},$$

$$U = \begin{bmatrix} \omega^2\{\lambda\omega^2 LnLe - \chi^2\sigma^2(\chi^2\lambda + Ln + Le)\} \\ +A\chi^2\sigma^2\{\sigma^2\chi^4 - \omega^2 LnLe - \lambda\omega^2\chi^2(Ln + Le)\} \end{bmatrix},$$

$$V = \begin{bmatrix} \sigma\omega^2\{\lambda\chi^2(Ln + Le) + LeLn\} - \chi^4\sigma^3 \\ -A\chi^2\sigma\{\sigma^2\chi^2(Ln + Le) + \lambda(\sigma^2\chi^4 - \omega^2 LnLe)\} \end{bmatrix}.$$

The frequency of oscillation is given by

$$X_1'L_1 - X'L')\omega^6 + (X_1'M_1 + Z_1'L_1 - Y'L' - X'N')\omega^4$$
$$+ (Y_1'L_1 + Z_1'M_1 - X'M' - Y'N')\omega^2 + (Y_1'M_1 - Y'M') = 0. \quad (4.4)$$

Here

$$X_1' = \left[-R_n N_a \alpha^2 \lambda Le Ln + \chi^2 \in Le Ln(1+\lambda\pi^2 B) - \lambda R_s \alpha^2 Le\right]$$

$$Y_1' = [\sigma^3 A\chi^6 R_s \alpha^2 (\in N_{ct} - 1) + \in \sigma^3 \chi^8 (A\chi^2 + B\pi^2) - R_n N_a \alpha^2 A\chi^6 \sigma^2 (\in + Le)],$$

$$Z_1' = \begin{bmatrix} R_n N_a \alpha^2 [\sigma\chi^2\{\lambda\chi^2(\in + Le) + \sigma Le + \in Ln + Le Ln\} \\ + A\chi^4 \lambda\sigma Ln(\in + Le) + \lambda A\chi^4 \sigma^2 Le] \\ + \sigma^2 A\chi^2 Le Ln] + R_s \alpha^2 [\chi^2 \sigma Le - \in N_{ct} Le + \sigma - \lambda\chi^2 \in N_{ct} + \lambda\chi^2) \\ + \sigma^2 A\chi^2 Le - \lambda A\chi^4 \sigma Le(\in N_{ct} - 1) + \lambda\sigma^2 A\chi^4 \\ - \in \chi^4 \sigma[\chi^2(1+\lambda\pi^2 B)(Le + \sigma + Ln) + (A\chi^2 + B\pi^2)\{\sigma(Ln + Le) + Ln Le\} \end{bmatrix},$$

$$L_1 = \sigma\lambda\chi^2(Ln + Le) + \sigma Le Ln + A\sigma\chi^2 \lambda Ln Le,$$

$$M_1 = -A\sigma\chi^2\{\sigma^2\chi^2(Ln + Le) + \lambda\sigma^2\chi^4\} - \chi^4\sigma^3,$$

$$X' = \begin{bmatrix} -R_n N_a \alpha^2 [\lambda\chi^2\{\sigma Le + Ln)(\in + Le)\} + \sigma Le Ln(1 + \lambda A\chi^2)] \\ + \in [\chi^4(1+\lambda\pi^2 B)\{\sigma(Ln + Le) + Le Ln\} + \sigma\chi^2 Ln Le(A\chi^2 + B\pi^2)] \\ - R_s \alpha^2 [\chi^2 \lambda\sigma + Le\{\sigma + \chi^2\lambda(1 - \in N_{ct})\} + \lambda\sigma A\chi^2 Le] \end{bmatrix},$$

$$Y' = \begin{bmatrix} R_n N_a \alpha^2 [\chi^4 \sigma^2 (\in + Le) + A\sigma^2 \chi^4 \{(\in + Le)(Ln + \lambda\chi^2) + \sigma Le\}] \\ - \in \sigma^2 \chi^6 \{\chi^2(1+\lambda\pi^2 B) + (A\chi^2 + B\pi^2)(Ln + Le + \sigma)\} \\ + R_s \alpha^2 \chi^4 \sigma^2 [(1 - \in N_{ct})\{1 + A(Le + \lambda\chi^2) + \sigma A\}] \end{bmatrix},$$

$$L' = \lambda Ln Le,$$

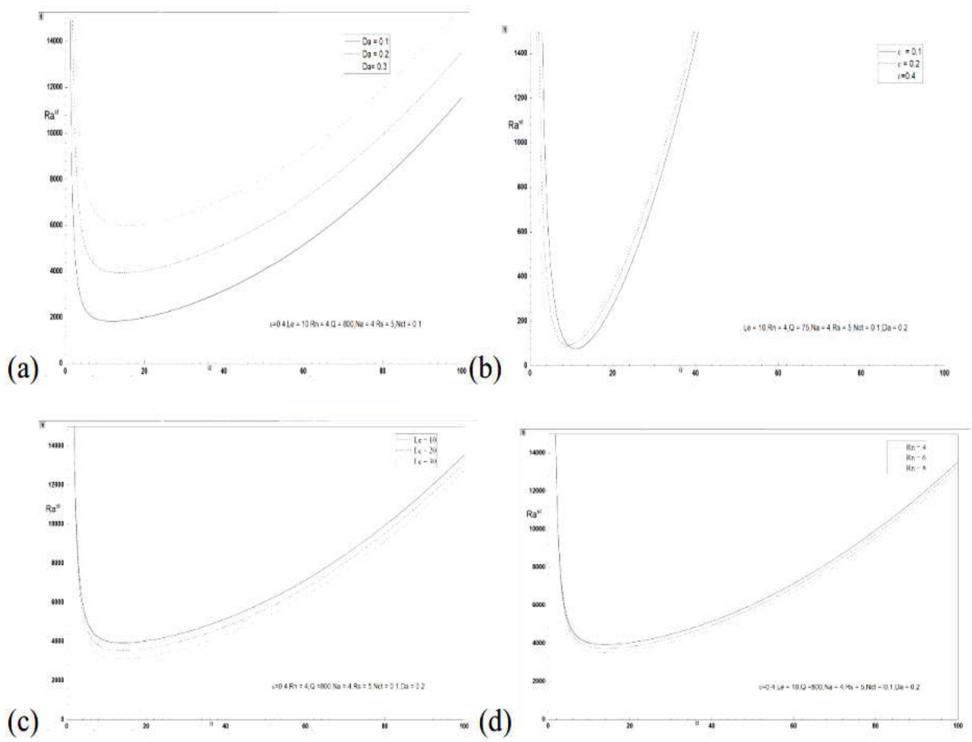
$$M' = A\chi^6\sigma^4,$$

and $N' = -\chi^2\sigma^2\{\chi^2\lambda + Ln + Le + A Ln Le + A\chi^2\lambda(Ln + Le)\}$.

For oscillatory convection, R_a vs α curves are shown in Figure 3 (a)-(k) for fixed value of $R_n, Le, Q, D_a, \varepsilon, N_a, \sigma, \lambda, Ln, R_s$ and N_{ct} with variations in one of these parameters. No critical Rayleigh number is obtained for oscillatory convection.

Fig 3(a), (d), (e), (f), (j) and (k) depict the effects of D_a, N_a, R_n, Q, R_s and λ. In each of these graphs, Rayleigh no. increases with increase in each of the respective parameters.

Fig 3(b), (g), (h) and (i) show the effects of ε, Ln, σ and N_{ct}. In each of these graphs, Rayleigh no. decreases with increase in each of these respective parameters. Fig. 3(c) shows the dual effect of Lewis number on oscillatory Rayleigh number. Initially Rayleigh number decreases with increase in Lewis no. but after a certain wave no. the effect gets reversed.



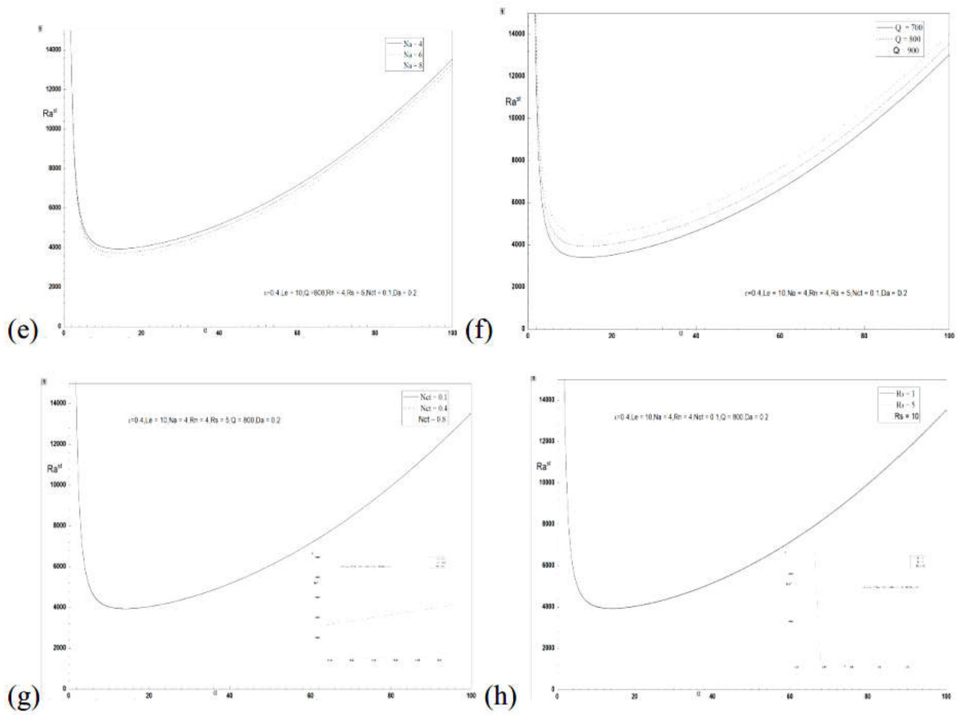
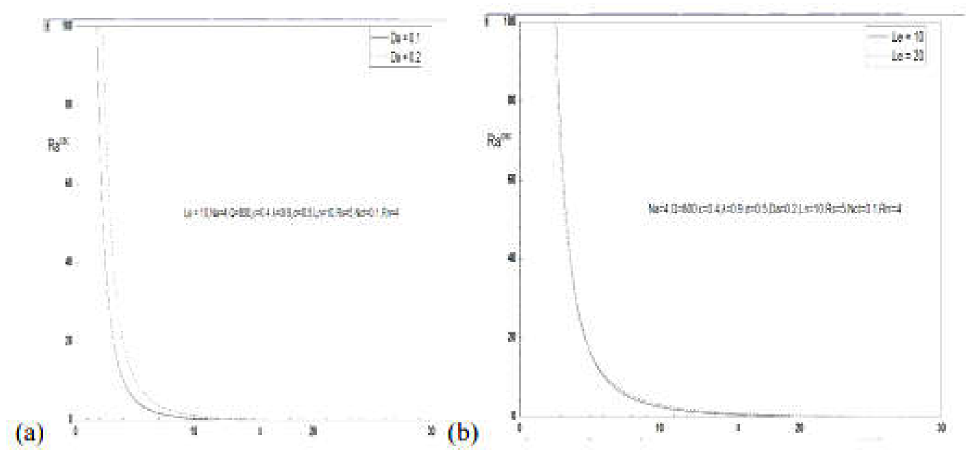
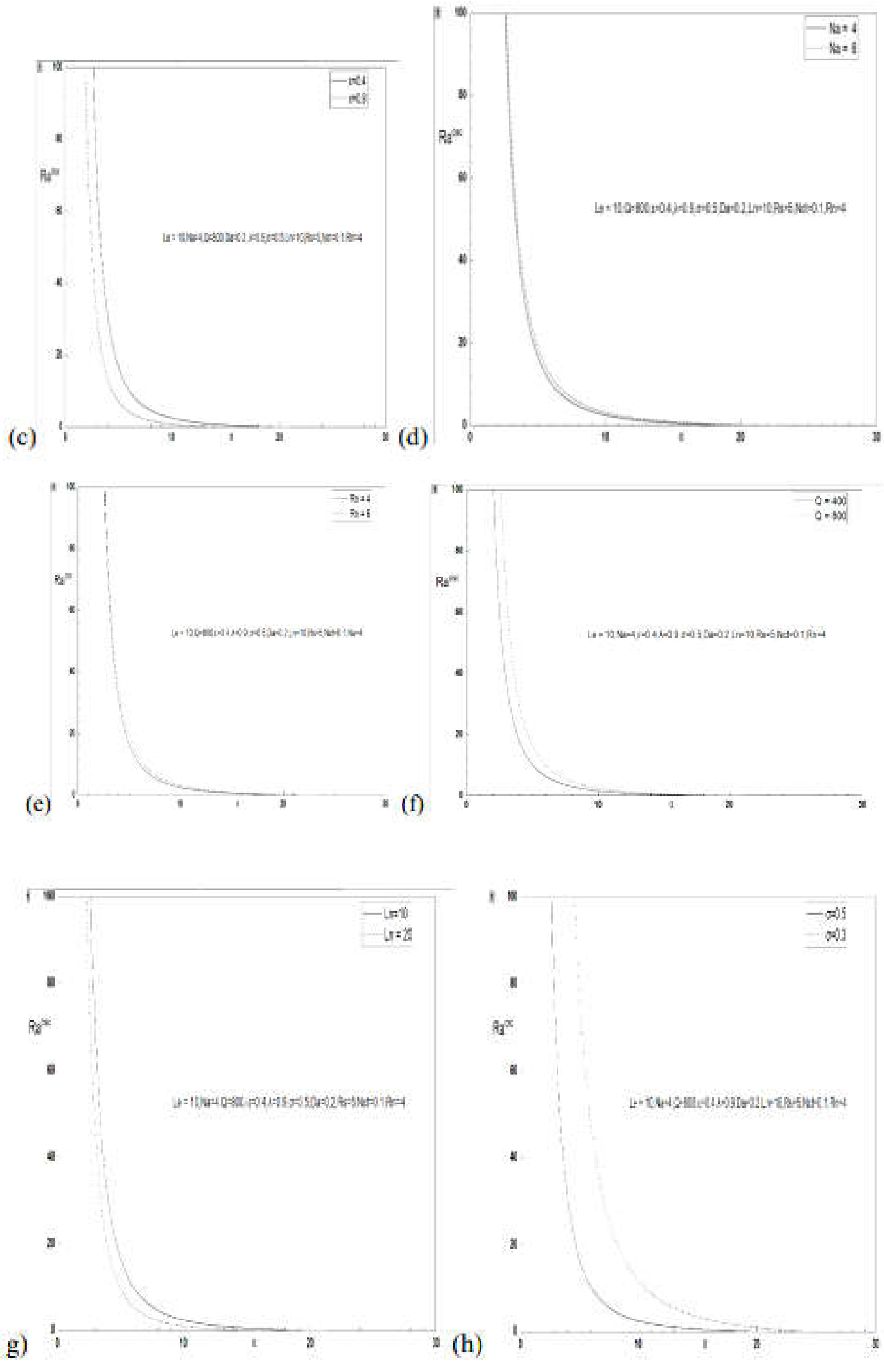


Fig. 2: Stationary convection for different values of
(a) D_a (b) ε (c) Le (d) R_n (e) N_a (f) Q (g) N_{ct} (h) R_s




Na = 4
Na = 6
Rs = 4
Rs = 6
Q = 400
Q = 800
Ln=10
Ln = 20

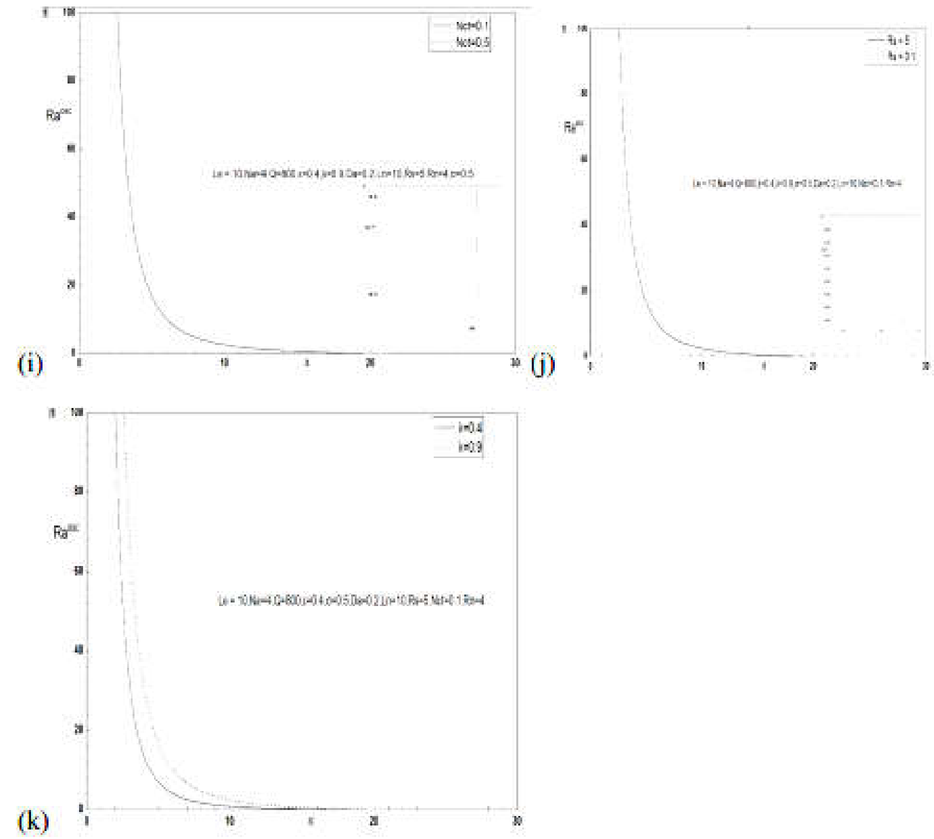


Fig. 3: Oscillatory convection for different values of
(a) D_a (b) ε (c) Le (d) N_a (e) R_n (f) Q (g) Ln (h) σ (i) N_{ct} (j) R_s (k) λ

5. Conclusion

In this paper, we have determined how the presence of the magnetic field affects double diffusive convection in Soret induced Darcy Maxwell nanofluid layer. The layer was soluted and heated from below and uniform magnetic field was applied in vertical direction. The comparison of results obtained has been done with the existing relevant studies. The main conclusions of the present analysis are as follows:

R_a^{st} has been observed to be function of parameters $Da,\ \varepsilon,\ Le, R_n,\ N_a,$ Q, N_{ct}, R_s whereas R_a^{osc} is function of Ln, σ and λ in addition to above parameters.

The effect of Lewis number Le is to decrease R_a^{st} but has twin effect on oscillatory convection.

An increase in porosity decreases R_a^{osc} but dual effect on stationary Rayleigh number.

A positive Soret coefficient N_{ct} has stabilizing effect on convection as obtained by Gaikwad *et al*. [8] for a regular fluid as well as obtained by Singh *et al*. [25] for a nanofluid but here in presence of magnetic field the effect is found to destabilize the oscillatory Rayleigh number as increase in N_{ct} decreases R_a^{osc}.

The influence of magnetic field is to stabilise the Soret induced double diffusive convection as was found by Yadav [30] in nanofluid convection induced by internal heating.

In this convection under magnetic field, Darcy number also comes into play and and has been observed to provide stabilizing effect on stationary and oscillatory modes.

An increase in Solutal Rayleigh Darcy number R_s was observed to cause increase in R_a^{st} and R_a^{osc} by Singh *et al*. [25] but the presence of magnetic field here causes the effect of R_s to be stabilize the oscillatory convection.

The increase in parameters Ln, σ increases the oscillatory Rayleigh number. This behaviour is opposite to be observed by Singh *et al*. [25] in absence of magnetic field.

Parameters R_n and N_a destabilizes the stationary mode but are found to stabilize the oscillatory convection.

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Nomenclature			
B_d	Brownian diffusion coefficient	$(\rho c)_M$	Medium's effective heat capacity
B_t	Thermophoretic diffusion coefficient	$(\rho c)_F$	Fluid's effective heat capacity
C	Specific heat of nanofluid	$(\rho c)_P$	Effective heat capacity of nanoparticles
G	Gravitational acceleration	α	Wave number
k_m	Effective thermal conductivity	α_m	Thermal diffusivity of the porous medium
K	Permeability	β_c	Solutal Volumetric Coefficient
S	Solute concentration	β_t	Thermal Volumetric Coefficient
$S*$	Dimensionless solute concentration	λ	Relaxation time
S_d	Diffusion coefficient	μ_e	Magnetic permeability
S_{ct}	Soret coefficient of salt	σ	Heat capacity ratio
t	Dimensionless time	σ'	Electrical conductivity of nanofluid
t^*	Time	ω	Frequency of oscillation
T^*	Nanofluid temperature	ψ^*	Volume fraction of Nanoparticle
(x^*, y^*, z^*)	Space coordinates	ψ_0^*	Reference value of nanoparticle volume fraction
(X, Y, Z)	Dimensionless space coordinates	$\in$	Porosity

Journal of Indian Acad. Math. ISSN: 0970-5120
Vol. 46, No. 2 (2024) pp. 153-157

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SOLVING THE KEPLER EQUATION USING ITERATIONS

Abstract: The Kepler equation, which describes how planets and other celestial bodies move in gravitational fields, is a key concept in celestial mechanics. For many uses in space research and engineering, an accurate and speedy solution of the Kepler equation is necessary. In this study, we provide an iterative approach to resolving the Kepler equation. The technique is based on a numerical methodology that iteratively converges to a solution and yields precise and trustworthy findings. The outcomes show how effective our iteration method is in resolving the Kepler equation.

Keywords: Kepler Equation, Celestial Mechanics, Iteration Method.

Mathematical Subject Classification (2010) No.: 83C40, 83C45.

1. Introduction

The Kepler equation, which derives from Newton's law of gravitation, describes the connection between a celestial body's orbital components and the passage of time during its orbit. The equation is stated as

$$E = M + e\sin E \tag{1}$$

Here, the mean anomaly is denoted by M, eccentric anomaly by E, and orbit's eccentricity by e.

The Kepler equation links the nonlinear relative angular position of two entities in Keplerian motion around their centers of mass to the linearly rising time.

$$M(t) = E(t) - e \sin E(t) \tag{2}$$

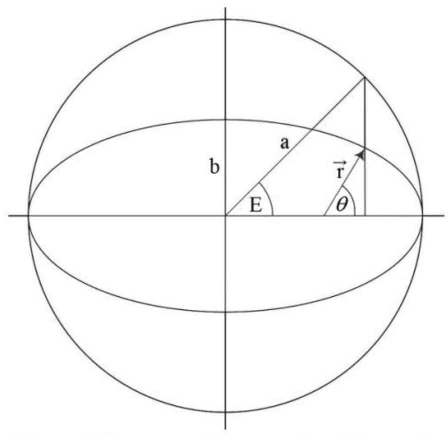

Figure 1. The geometric connection between the eccentric anomaly E and the actual anomaly θ

where the mean anomaly is $M(t) = n(t - \iota)$ and $n = \sqrt{\dfrac{G(m_1 + m_2)}{a^3}}$ is two-body mean motion with α being the semimajor axis of the orbital ellipse and /time of pericenter passage. Fig. 1 depicts the geometric correlation between the actual and eccentric anomalies E.

Here, we shall go through how a numerical approach to solving the Kepler equation was created. The Kepler equation is transcendental, iterative solutions are required. Therefore, each numerical process has two tasks. The 1st step is the iterative loop, which repeats a refining process until a desired convergence is obtained. In general, fewer repeated passes are required the higher the algorithm's rank.

The iterative loop's starting value must be chosen as the second task. The loop will converge more quickly the better the starting approximation. It is not necessary for the starting value method to be identical to or even closely related to the iteration technique.

2. An Initial Value Approach

The more precise initial value is provided to the iterative loop seems reasonable given that we must solve the Kepler equation repeatedly, at least till the expression complexity gets disagreeable. If, we only have *E* = *M* in the limit of zero eccentricity. Thus, eq. (1) proposes a straightforward iterative method for enhancing the initial approximation $E = M$. It forms as:

$$E_k = M + e\sin E_{k-1} \tag{3}$$

Here $E_0 = M$. The recursive phrase eq. (3) can then be repeated to any number of orders in e. The third-order approximation, for instance, is given by

$$E = M + e\sin M + e^2 \sin M \cos M + \frac{1}{2} e^3 \sin M (3\cos^2 M - 1) \tag{4}$$

3. Using Iterations

Either numerically or analytically, eq. (2) must be solved iteratively because it is a transcendental equation. Therefore, we must look for an expression that, given an error-containing value of E, provides an approximate value that minimizes the error. It must converge as well. To do this, write equation (2) in the following expression

$$f(x) = x - e\sin x - M \tag{5}$$

Here, solution to $f(x) = 0$ is $x = E$. Let $\varphi = x - E$ be approximation error of E provided by x. $x = E$ is the center of the Taylor expansion.

$$f(E) = f(x-e) = x - e\sin x - M - (1 - e\cos x)\varphi + \frac{1}{2}\varphi^2 e\sin x - \frac{1}{6} e\cos x + \ldots \tag{6}$$

assuming Π is very small. We may find

$$\varphi = \frac{x - e\sin x - M}{1 - e\cos x} \tag{7}$$

by solving this first-order term in equation (6) for Π. This serves as the basis for a first-order iterative strategy. Let's assume that our original hunch is that $x = x_0$. In such case, for E, $x_1 = x_0 + \varphi$ ought to be a more accurate estimation than x_0, and so on. As a result, we propose the 1st order iterative method

$$\varphi_{n+1} = \frac{x_n - e\sin x_n - M}{1 - e\cos x_n} \tag{8}$$

where a more detailed discussion of the initial value for x_0 will follow. Equation (8) provides a first-order iterative single-step calculation method i.e. $E_{n+1} = E_n - \varphi_n$.

At second order Equation (6) in form of Horner is

$$f(x-\varphi) = x - e\sin x - M - \left(1 - e\cos x - \frac{1}{2}e\sin x\varphi\right)\varphi \tag{9}$$

We can rearrange the expression $f(x-\varphi)=0$, then equation (9) written as

$$\varphi = \frac{x - e\sin x - M}{1 - e\cos x - \frac{1}{2}e\sin x} \tag{10}$$

So, let's write to establish a 2nd-order iterative system in the analogy to (8)

$$\varphi_{n+1} = \frac{x_n - e\sin x_n - M}{1 - e\cos x_n - \frac{1}{2}\varphi_n e\sin x_n} \tag{11}$$

This can be turned into a two-step iterative process by initially determining φ_n as expressed in equation (8) and then φ_{n+1} as presented in equation (11). By replacing φ_n in equation (11), simply substituting equation (7) for φ_n, we can also omit the intermediary step. The single-step iteration is then

$$\varphi_{n+1} = \frac{x_n - e\sin x_n - M}{1 - e\cos x_n - \frac{1}{2}e\sin x_n \dfrac{x_n - e\sin x_n - M}{1 - e\cos x_n}} \tag{12}$$

The third-order approximation of $f(E) = f(x-\varphi)$ in Horner form is

$$f(x-\varphi) = x - e\sin x - M - \left(1 - e\cos x - \left(\frac{1}{2}e\sin x - \frac{1}{6}e\cos x\varphi\right)\varphi\right)\varphi \tag{13}$$

After solving the "top-level" on the right and setting this to zero, and deduced to

$$\varphi_{n+1} = \frac{x_n - e\sin x_n - M}{1 - e\cos x_n - \frac{1}{2}\left(e\sin x_n - \frac{1}{3}e\cos x_n \varphi_n\right)\varphi_n} \tag{14}$$

For a two-step technique, we can utilize equation (12); for a three-step method, equation (8); and the resulting equation (14) for φ_n. As an alternative, we may simply put equation (12) for φ_n a one-step third-order technique in equation (14).

4. Conclusion

In this paper, we proposed an iterative approach for solving the Kepler equation that yields accurate and efficient answers for a wide variety of orbital conditions.

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(*Received*, August 3, 2024)

Journal of Indian Acad. Math. ISSN: 0970-5120
Vol. 46, No. 2 (2024) pp. 159-169

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VOLATILITY DYNAMICS ANALYSIS OF NIFTY 500 INDEX USING ARCH AND GARCH MODELS: A COMPREHENSIVE STUDY

Abstract: This paper investigates the volatility dynamics of the NIFTY 500 Index using historical data from 2005 to 2024. Weemploy Autoregressive Conditional Volatility (ACV) and Generalized Autoregressive Conditional Volatility (GACV) models to analyze and model the volatility of dailylog returns. Our findings indicate significant ARCH effects, and the fitted ARCH (2) model demonstrates a robust framework for capturing the conditional volatility in the financial time series data.

Keywords: ARCH, GARCH, Volatility Modeling, NIFTY500, Financial Time Series, Econometrics.

Mathematics Subject Classification (2010) No.: 91B24, 91B26, 91B30.

1. Introduction

Volatility modeling is crucial in financial econometrics, offering insights into risk management and derivative pricing. The NIFTY 500 Index, representing a broad spectrum of the Indian stock market, provides an ideal subject for studying volatility dynamics. Financial markets are characterized by fluctuating prices that exhibit periods of high volatility followed by periods of relative calm, a phenomenon known as volatility clustering. Understanding these patterns is essential for investors, policy makers, and researchers.

In this paper, we aim to analyze the volatility patterns in the NIFTY 500 Index using ARCH and GARCH models. We employ historical data spanning from 2005 to 2024 to comprehensively analyze the volatility dynamics of daily log returns.

By doing so, we contribute to the existing literature by providing a detailed study of nearly two decades of data from the NIFTY 500 Index.

2. Data

We use historical data on the NIFTY 500 Index, obtained from Yahoo Finance, spanning from September 26, 2005, to June 10, 2024. The data includes daily adjusted closing prices, from which we compute the daily log returns. The choice of this time frame ensures a comprehensive analysis that encompasses various market cycles, including bull and bear markets, economic expansions, and recessions.

A. Adjusted Close Prices

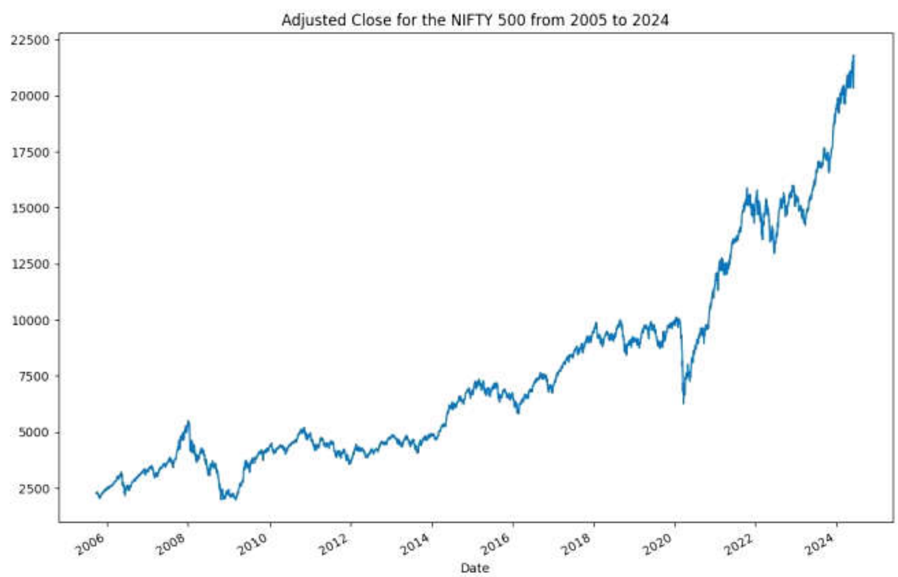

Fig. 1. Adjusted Close Prices of NIFTY 500 Index

B. Daily Log Returns: Daily log returns are computed as follows:

$$\text{Log Return}_t = \log \frac{Price_t}{Price_{t-1}}$$

This transformation helps to stabilize the variance and achieve stationarity, which are prerequisites for applying ARCH and GARCH models.

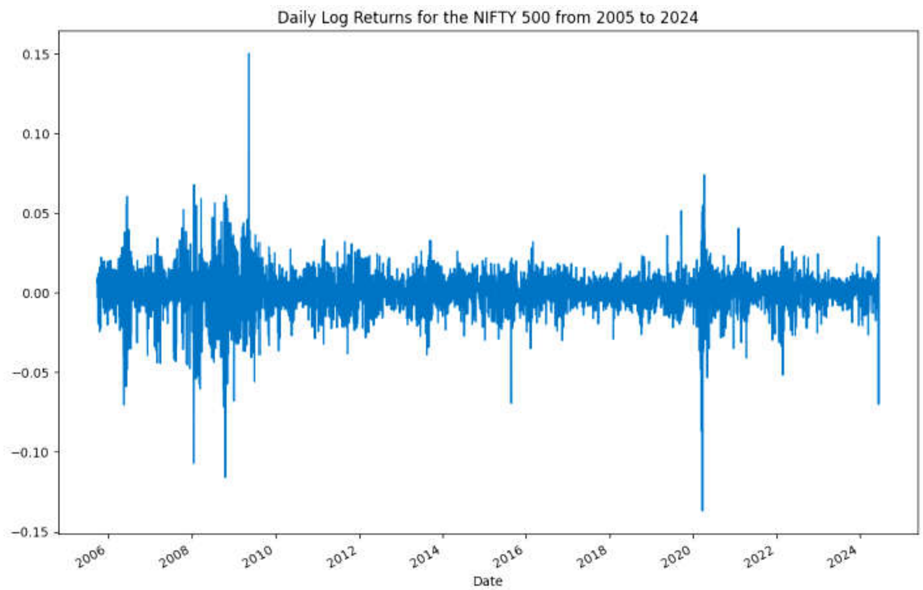

Fig. 2: Daily Log Returns of NIFTY 500 Index

3. Data Preprocessing

Data preprocessing plays a pivotal role in the analysis of time series data. The dataset was cleaned by removing any missing values, which could otherwise distort the results. The adjusted closing prices were then used to calculate the daily log returns, a common practice in financial econometrics to stabilize variance and achieve stationarity.

4. Methodology

A. Stationarity Test: To ascertain the suitability of the time series data for advanced analysis, we conducted the Augmented Dickey-Fuller (ADF) test. This statistical test is instrumental in assessing the stationarity of a series by detecting the presence of a unit root, which suggests non-stationarity. Stationarity is pivotal in time series analysis as it ensures reliable and meaningful statistical inferences, mitigating the risk of spurious findings.

B. Log Returns Calculation: The daily log returns were calculated from the adjusted closing prices. This transformation helps in stabilizing the variance and making the series stationary.

C. Modeling Volatility Dynamics: Autoregressive Conditional Heteroskedasticity (ARCH): The ARCH model, pioneered by Engle in 1982, is a statistical framework within time series analysis. It focuses on modeling the volatility of the current error term based on past error terms. This model is particularly adept at capturing volatility clustering, a prevalent characteristic in financial data where periods of high volatility tend to be followed by similar periods, and periods of low volatility likewise persist.

1) ***ARCH Model Specification*****:** The fundamental equations of the ARCH model are as follows:

$$y_t = \mu + n_t$$

$$= \gamma_t v_t$$

$$\gamma_t^2 = \beta_0 + \beta_1 n_{t-1}^2 + \beta_2 n_{t-2}^2 + \dots + \beta_q n_{t-q}^2$$

where:

- y_t represents the return at time t.
- μ denotes the mean return.
- η_t is the error term at time t, representing the deviation of the actual return from the mean.
- γ^2 signifies the conditional variance at time t, indicating the expected variability of $\eta_{t.}$
- v_t is a white noise process with zero mean and unit variance, ensuring that η t has the desired properties.
- β_0 is a constant term representing the baseline level of volatility.
- βi for i = 1, 2, . . . , q are coefficients that measure the influence of past squared errors on current volatility.

2) ***Estimation of ARCH Framework*****:** The parameters of the ARCH model can be estimated using the method of maximum likelihood. The log-likelihood function for the ARCH model is formulated as follows:

$$L(\psi) = \quad 2^{m \ln(2\pi) - \sum_{2t=1}^{m} \ln(y_t^2) + t} \frac{2}{y_t^2}$$

where:

- m is the total number of observations.
- ψ represents the vector of parameters (μ, β_0 , β_1 , . . . , β_q).
- γ_t^2 is the conditional variance at time t as defined by theARCH model.
- η_t is the residual term at time t.

3) ***Model Diagnostics*****:** To assess the adequacy of the fitted ARCH model, it is important to perform diagnostic checks. One common diagnostic check is to investigate the existence of remaining ARCH impacts within the residuals. This can be done using the Lagrange Multiplier (LM) test, also known as the ARCH-LM test. The test involves regressing the squared residuals on lagged squared residuals and checking for significant coefficients.

The null hypothesis of the ARCH-LM test is that there are no remaining ARCH effects, i.e., all coefficients of the lagged squared residuals are zero. A significant test statistic indicates the presence of remaining ARCH effects, suggesting that a higher-order ARCH model or a GARCH model may be more appropriate.

Statistic	Value
LM Statistic	30.33
p-value(L.M)	0.00076

Table 1 - ARCH-LM Test Result

4) *Model Selection*: When evaluating the suitable ARCH model, it is essential to take into account the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC).These measures assist in comparing various models by penalizing complexity while favoring models that provide a better balance of explanatory power and parsimony. Optimal model selection is guided by minimizing both AIC and BIC values.

$$AIC_{new} = 2m \quad 2\log(L)$$

$$BIC_{new} = m\log(n) \quad 2\log(L)$$

Where m denotes the number of parameters in the model, L represents the maximum likelihood estimate of the likelihood function, and n indicates the sample size.

By carefully comparing AIC and BIC values, we can identify the most suitable model for the data.

D. ARCH Effect Test: We examined the existence of ARCH effects in the log returns series using the ARCH test. This test evaluates whether there is substantial autocorrelation in the squared residuals of the log returns, which suggests the occurrence of volatility clustering.

E. Model Fitting: Three models were fitted to the log returns series:

1) ARCH (1) Model: This model uses one lag of the squared residuals to model the conditional variance.

2) ARCH (2) Model: This model uses two lags of the squared residuals.

3) GARCH (1,1) Model: This model incorporates an ARCH term based on the lagged squared residuals and a GARCH term utilizing the lagged conditional variance to model the variance dynamics.

5. Results

A. Stationarity Test: The ADF test yielded a p-value indicating whether the adjusted closing prices were stationary. In our analysis, the ADF test results confirmed that the log returns of the NIFTY 500 Index are stationary, making them suitable for further analysis using ARCH and GARCH models.

B. Log Returns: The mean of the daily log returns was calculated, and the log returns series was plotted to visualize the data.

C. ARCH Effect Test: The ARCH test results indicated significant ARCH effects in the log returns series, justifying the use of ARCH and GARCH models for further analysis. Significant ARCH effects imply that past squared residuals have predictive power for future volatility.

D. Model Results: 1) *ARCH* (1) *Model*: Mean of Daily Log Returns = 0.00049

Statistic	Value
LM Statistic	30.33
p-value(L.M)	0.00076
F Statistic	3.05
p-value (F)	0.00074

Table 2: ARCH (1) Test Results

Parameter	Coefficient	p-value
μ	0.00072696	0.0001046
ω	0.00010542	9.154e-84
α_1	0.4001	2.176e-12

Table 3: ARCH (1) Model Coefficients

2) *ARCH* (2) *Model*: Mean of Daily Log Returns = 0.00049

Statistic	Value
LM Statistic	30.59
p-value(L.M)	0.00068
F Statistic	3.07
p-value (F)	0.00067

Table 4: ARCH (2) Test Results

Parameter	Coefficient	p-value
μ	0.00094068	6.958e-08
ω	0.000070296	6.430e-84
α_1	0.3000	3.960e-07
α_2	0.3000	7.020e-14

Table 5: ARCH (2) Model Coefficients

3) *GARCH*(1,1) *Model*: Mean of Daily Log Returns = 0.00049

Statistic	Value
LM Statistic	30.38
p-value(L.M)	0.00074
F Statistic	3.05
p-value (F)	0.00072

Table 6: GARCH (1,1) Test Result

Parameter	Coefficient	p-value
μ	0.00084699	7.572e-10
ω	0.0000035148	0.000
α_1	0.1000	0.000
β_1	0.8800	0.000

Table 7: GARCH (1,1) Modle Coefficients

E. Model Comparison: To determine the best model, we compare the AIC and BIC values for each model. Here are the AIC and BIC values for the NIFTY 500 data:

Model	AIC	BIC
ARCH(1)	-151800	-151780
ARCH(2)	-151811	-151779
GARCH(1,1)	-151805	-151788

Table 8: AIC and BIC Values for Models

F. Forecasting with ARCH (2) Model: Using the fitted ARCH (2) model, we forecast the conditional variance for the next 30 days. The forecasted conditional variances provide insights into the expected volatility of the NIFTY 500 Index.

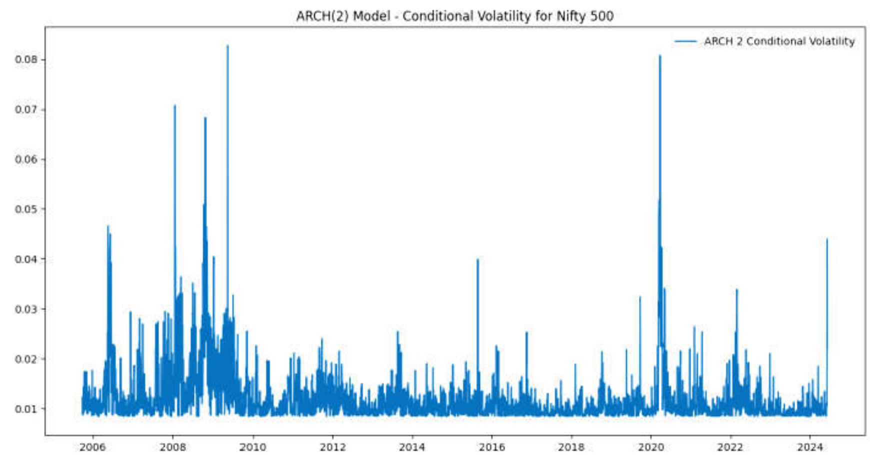

Fig. 3: 30-Day Volatility Forecast Using ARCH (2) Model

4. Conclusion

By observing the AIC and BIC values, the ARCH (2) model has the lowest values, indicating the best fit to the NIFTY 500 data. The ARCH (2) model has an AIC of -151811and a BIC of -151779, making it the preferred model for forecasting volatility in the NIFTY 500 Index.

5. Conclusion

This paper provides a comprehensive analysis of the volatility dynamics in the NIFTY 500 Index using ARCH and GARCH models. Through rigorous testing and model fitting, we found significant ARCH effects in the daily log returns series, confirming the suitability of these models for analyzing volatility.

The ARCH (2) model demonstrated superior performance over other models, as evidenced by its favorable Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) scores. Its capability to effectively capture the conditional variance underscores its robustness in modeling the volatility of the NIFTY 500 Index.

The insights gained from this analysis are valuable for risk management and financial decision-making. Investors and policymakers can use these models to better understand market volatility, anticipate potential risks, and make informed decisions. “This research adds to the extensive literature on financial econometrics by conducting a comprehensive analysis of volatility in the Indian stock market spanning nearly two decades.”

Future research could extend this work by exploring higher-order GARCH models or incorporating exogenous variables to further enhance volatility forecasting. Additionally, applying these models to other financial indices or markets could provide comparative insights and broaden the applicability of the findings.

Acknowledgement

The authors would like to thank Yahoo Finance for providing the historical data used in this analysis.

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(*Received*, August 22, 2024)
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Journal of Indian Acad. Math.
Vol. 46, No. 2 (2024) pp. 171-194

ISSN: 0970-5120

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NEW ALGEBRAIC STRUCTURE OF MULTI SPHERICAL FUZZY SETS

Abstract: The main aim of the present paper is to introduce the concept of multi spherical fuzzy (briefly, MSF) Lie sub algebras and multi spherical fuzzy (briefly, MSF) Lie ideals of Lie algebra. Some of their fundamental properties and operations like intersection and Cartesian product are investigated. Moreover, the relationship between multi spherical fuzzy Lie sub algebras and multispherical fuzzy Lie ideals are also established. Moreover, the images and the inverse images of both multi spherical fuzzy Lie sub algebras and multi spherical fuzzy Lie ideals under Lie homomorphisms are also studied.

Key words and phrases: Lie Algebra, Lie Subalgebra, Multi Spherical Fuzzy Set, Multi Spherical Fuzzy Lie Subalgebras, Multi Spherical Fuzzy Lie Ideals.

Mathematics Subject Classification (2020) No.: 08A30, 03E72, 08A72, 08B20.

1. Introduction

The idea of Lie algebra was first introduced by Sophus Lie (1842-1899) in an effort to categorise certain smooth subgroups of general linear groups [13].Lie algebras, so named because they were invented by Sophus Lie, are a specific instance of general linear algebra. Following the introduction of this theory, Lie groups were used in mathematics and physics to categorise smooth subgroups. In [[3], [16], [17], [21], [22], [23]], Lie subalgebras and their properties were created and explored in more detail. Zadeh [15] proposed the concept of fuzzy set (FS) in situations that are vague, imprecise and uncertain. As a generalisation of fuzzy set, Atanassov [14] created intuitionistic fuzzy set (IFS) in 1986. His theory thereafter became widely acknowledged as an essential resource in the fields of science, technology, engineering, medicine, etc. In 1995, neutrosophic set (in short, NS) was introduced by Smarandache [8] as a generalization of not only intuitionistic fuzzy set but also Pythagorean fuzzy set, Farmatean fuzzy set, spherical fuzzy set and so on. In 2023, Smarandache [7] proposed the model of multi spherical fuzzy set which is the genralization of multi pythagorean fuzzy set as well as spherical fuzzy set model which has wider scope of applications in multi criteria decision making problems where as spherical fuzzy set has great importance in real life application for single criteria decision making problems.

One such application is due to Ajoy, Ganeshsree, Aldring, Pham Huy, Hoang, Cong [2]. In order to create model for vague and imprecise information, the model of Pythagorean fuzzy set (PFS) was introduced by Yager [19, 20] which is some how different from IFS model since it involves the condition $0 \leq \mathrm{T}^2 + \mathrm{F}^2 \leq 1$ where T and F stand for membership and non-membership function respectably. In decision making problems, PFS model has significant application proposed by Garg [11, 12]. The application range of solving real life problems such as decision making problems in spherical fuzzy set model is popularly increasing than that of the PFS model because number of triplets satisfying the condition $0 \leq \tau^2 + \lambda^2 + \eta^2 \leq 1$ is higher than that of the pair satisfying the condition $0 \leq \tau^2 + \eta^2 \leq 1$. Recently, Debnath and Smarandache [4] introduced the notion of Multi Intuitionistic Fuzzy Lie Algebras and Debnath [5]initiated Multi Fermatean Fuzzy Lie Ideals of Lie algebras. In the modern age of artificial intelligence (AI) , science and technology significantly deal with intricate phenomena and processes for which there is inadequate information. In such kind of situations, mathematical models are created for dealing with different types of systems that have uncertain and imprecise components. One of such models is multi spherical fuzzy set which was built on the extensions of standard set theory. In this paper we cover the core features of a multi spherical fuzzy Lie subalgebra and Lie ideals of Lie algebra. The numerous domains along with signal processing, artificial intelligence, multiagent systems, computer networks, robotics, genetic algorithms, expert systems, neural networks, decision making, medical diagnosis and automata theory shall be benefited with the acquired outcomes.

We organized the work as follows: we presented the introduction and literature review in the first section. Section 2 focuses into common definitions and preliminaries. Section 3 describes the concept of multi spherical fuzzy Lie subalgebras and multi spherical fuzzy Lie ideals of Lie algebra. Some of their fundamental properties and operations like intersection and generalized cartesian product are investigated. Moreover, the relationship between multi spherical fuzzy Lie subalgebras and multi spherical fuzzy Lie ideals are also established. In section 4, we investigate the images and the inverse images of multi spherical fuzzy Lie subalgebras and multi spherical fuzzy Lie ideals under Lie homomorphisms. In section 5, we give the conclusion of the newly defined concept of multi spherical fuzzy Lie subalgebras and Lie ideals.

2. Preliminaries

This section consists of some common notations and definitions which have been involved in the course of the paper.

A Lie algebra is a vector space $\mathcal{L}$ over the field $\mathcal{F}(\mathcal{L} = \mathrm{R}$ or $\mathrm{C})$ on which $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ defined by $(\gamma, \mu) = [\gamma, \mu]$ for all $\gamma, \mu \in \mathcal{L}$, where, $[\gamma, \mu]$ is called Lie Bracket satisfying the following conditions:

(1) $[\gamma, \mu]$ is bilinear

(2) $[\gamma, \gamma] = 0$, for all $\gamma \in \mathcal{L}$

(3) $[[\gamma, \mu], \lambda] + [[\mu, \lambda], \gamma] + [[\lambda, \gamma], \mu] = 0$, for all $\gamma, \mu, \lambda \in \mathcal{L}$.
(This is called Jacobi identity).

Throughout the paper $\mathcal{L}$ will denote Lie algebra and also we note that the operation Lie bracket $[.,.]$ is neither associative nor commutative i.e., $[[\gamma, \mu], \lambda] \neq [[\mu, \lambda], \gamma]$ and $[\gamma, \mu] \neq [\mu, \gamma]$. But the operation Lie bracket $[.,.]$ is anti-commutative i.e., $[\gamma, \mu] = -[\mu, \gamma]$.

A subspace H of $\mathcal{L}$ is called a Lie subalgebra if it is closed under $[.,.]$.

A subspace I of $\mathcal{L}$ is called a Lie ideal if $[\mathrm{I}, \mathrm{L}] \subset \mathrm{I}$. It is always true that every Lie ideal is Lie subalgebra.

Definition 2.1: [15] Let U be a universe of discourse. Then the fuzzy set on U is described as $\mathrm{F} = \{(x, \mu(x)) : x \in \mathrm{X}$, where, $\mu(x) \in [0, 1]$, denotes the degree of membership of $x \in \mathrm{X}\}$.

Definition 2.2: [14] Let U be a universe of discourse. Then the intuitonistic fuzzy set on U is described as $\mathrm{F} = \{(x, \mu(x), \nu(x)) : x \in \mathrm{X}$, where, $\mu(x)$, $\nu(x) \in [0, 1]$, indicating the degrees of membership and non-membership respectively such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1\}$.

Definition 2.3: [8] Let X be a universe of discourse. Then the neutrosophic set is defined by $\mathrm{N} = \{(x, \tau(x), \lambda(x), \eta(x)), x \in \mathrm{X}$, where, $\tau, \lambda, \eta \in [0, 1]$, indicating the degrees of truth, indeterminacy and falsehood respectively that satisfy $0 \leq \inf(\tau) + \inf(\lambda) + \inf(\eta) \leq \sup(\tau) + \sup(\lambda) + \sup(\eta) \leq 3\}$.

Definition 2.4: [14] Let U be a universe of discourse. Then the Pythagorean fuzzy set on U is described as, $\mathrm{P} = \{(x, \mu(x), \nu(x)) : x \in \mathrm{X}$ where, $\mu(x), \nu(x) \in [0, 1]$, indicating the degrees of membership and non-membership respectively satisfying $0 \leq \mu_P^2(x) + \nu_P^2(x) \leq 1\}$.

Definition 2.5: [10] Let X be a universe of discourse. Then the spherical fuzzy set is defined by $\mathrm{S} = \{(x, \tau(x), \lambda(x), \eta(x)), x \in \mathrm{X}$, where, $\tau, \lambda, \eta \in [0, 1]$, indicating the degrees of truth, indeterminacy and falsehood respectively that satisfy $0 \leq \mu_S^2(x) + \lambda_S^2(x) + \nu_S^2(x) \leq 1\}$.

Definition 2.6: [21] A neutrosophic set $\mathrm{W} = (\tau, \lambda, \eta)$ on $\mathcal{L}$ is said to be neutrosophic subalgebra if, $\forall \gamma, \mu \in \mathcal{L}; \forall \mathrm{c} \in \mathcal{F}$, the following assumptions hold good:

(1) $\tau_W(\gamma + \mu) \geq \mathrm{Min}\ \{\tau_W(\gamma),\ \ \tau_W(\mu)\}$, $\lambda_W(\gamma + \mu) \geq \mathrm{Min}\ \{\lambda_W(\gamma)$, $\lambda_W(\mu)\}$, $\eta_W(\gamma + \mu) \leq \mathrm{Max}\{\eta_W(\gamma),\ \eta_W(\mu)\}$.

(2) $\tau_W(c\gamma) \geq \tau_W(\gamma)$, $\lambda_W(c\gamma) \geq \lambda_W(\gamma)$, $\eta_W(c\gamma) \leq \eta_W(\gamma)$.

(3) $\tau_W[\gamma, \mu] \geq \mathrm{Min}\ \{\tau_W(\gamma), \tau_W(\mu)\}$, $\lambda_W[\gamma, \mu] \geq \mathrm{Min}\{\lambda_W(\gamma), \lambda W(\mu)\}$, $\eta_W[\gamma, \mu] \leq \mathrm{Max}\{\eta_W(\gamma), \eta_W(\mu)\}$

Definition 2.7: [21] A neutrosophic set $\mathrm{W} = (\tau, \lambda, \eta)$ on $\mathcal{L}$ is said to be neutrosophic Lie ideal if, $\forall \gamma, \mu \in \mathcal{L}; \forall \mathrm{c} \in \mathcal{F}$, the conditions (1) and (2) of definition (2.4) and the conditions: $\tau_W([\gamma, \mu]) \geq \tau_W(\gamma)$, $\lambda_W([\gamma, \mu]) \geq \lambda_W(\gamma)$, $\eta_W([\gamma, \mu]) \leq \eta_W(\gamma)$, hold good.

Definition 2.8: [7] Let X be a universe of discourse. Then, a multi neutrosophic set on X is defined by

$$M = \{(x, x(\tau_1, \tau_2, ..., \tau_p \lambda_1, \lambda_2, ..., \lambda q; \eta_1, \eta_2, ..., \eta_s)) : x \in X,$$

where p,q,s are integers ≥ 0, $p + q + s = n \geq 2$, and at least one of p,q,s is ≥ 2, in order to ensure the existence of multiplicity of at least one neutrosophic component: truth, indeterminacy, or falsehood; all subsets $\tau_1, \tau_2, ..., \tau_p$; $\lambda_1, \lambda_2, ..., \lambda_q$; $\eta_1, \eta_2, ..., \eta_s \subseteq [0,1]$; $0 \leq \sum_{i=1}^{p} \inf(\tau_i) + \sum_{j=1}^{q} \inf(\lambda_j)$

$$+\sum_{k=1}^{s} \inf(\eta_k) \leq \sum_{i=1}^{p} \sup(\tau_i) + \sum_{j=1}^{q} \sup(\lambda_j) + \sum_{k=1}^{s} \sup(\eta_k) \leq n\}.$$

Definition 2.9: [7] Let X be a universe of discourse. Then, a multi spherical fuzzy set on X is defined by

$$M = \{(x, x(\tau_1, \tau_2, ..., \tau_p; \lambda_1, \lambda_2, ..., \lambda_q; \eta_1, \eta_2, ..., \eta_s)) : x \in X,$$

where p, q, s are integers ≥ 1, $p + q + s \geq 4$, in order to ensure the existence of multiplicity of at least one neutrosophic component: truth, indeterminacy, or falsehood; all subsets $\tau_1, \tau_2, ..., \tau_p$; $\lambda_1, \lambda_2, ..., \lambda_q$; $\eta_1, \eta_2, ..., \eta_s \subseteq [0,1]$; $0 \leq \tau_i^2 + \lambda_j^2 + \eta_k^2 \leq 1$, $R_{ijk} = (1 - \tau_i^2 - \lambda_j^2 - \eta_k^2)^{\frac{1}{2}}$, for all $i = 1, 2, 3, ..., p$; $j = 1, 2, ..., q$ and $k = 1, 2, 3, ..., s\}$.

3. Multi Spherical Fuzzy Lie Algebraic Structure and its Properties

In this section, the notion of multi spherical fuzzy Lie subalgebra is initiated as a generalized version of spherical fuzzy Lie subalgebra. Some characterizations, counter examples, and basic properties are also investigated.

Definition 3.1: Let $N_l = \{1, 2, ..., 1\}$, $N_m = \{1, 2, ..., m\}$, $N_n = \{1, 2, ..., n\}$ and $\mathcal{L}$ be a Lie Algebra of vectors over the field $\mathcal{F}$. A multi spherical fuzzy set $W = (\tau_W^1, \tau_W^2, ..., \tau_W^l; \lambda_W^1, \lambda_W^2 ...., \lambda_W^m; \eta_W^1, \eta_W^2, ..., \eta_W^n)$ on $\mathcal{L}$ is said to be multi spherical fuzzy Lie subalgebra over $\mathcal{L}$ if, $\forall \gamma, \mu \in \mathcal{L}$; $\forall i \in N_l$, $\forall j \in N_m$, $\forall k \in N_n$, $\forall c \in \mathcal{F}$, the following assumptions hold good:

(1) $\tau_W^i(\gamma+\mu) \geq \text{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\}$, $\lambda_W^j(\gamma+\mu) \geq \text{Min}\{\lambda_W^j(\gamma), \lambda_W^j(\mu)\}$,
$\eta_W^k(\gamma+\mu) \leq \text{Max}\{\eta_W^k(\gamma), \eta_W^k(\mu)\}$

(2) $\tau_W^i(c\gamma) \geq \tau_W^i(\gamma)$, $\lambda_W^j(c\gamma) \geq \lambda_W^j(\gamma)$, $\eta_W^k(c\gamma) \leq \eta_W^k(\gamma)$

(3) $\tau_W^i[\gamma, \mu] \geq \text{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\}$, $\lambda_W^j[\gamma, \mu] \geq \text{Min}\{\lambda_W^j(\gamma), \lambda_W^j(\mu)\}$,
$\eta_W^k[\gamma, \mu] \leq \text{Max}\{\eta_W^k(\gamma), \eta_W^k(\mu)\}$

Definition 3.2: Let $\mathrm{N}_l = \{1, 2, ..., 1\}$, $\mathrm{N}_m = \{1, 2, ..., m\}$, $\mathrm{N}_n = \{1, 2, ..., n\}$ and $\mathcal{L}$ be a Lie Algebra of vectors over the field $\mathcal{F}$. A multi spherical fuzzy set $\mathrm{W} = (\tau_W^1, \tau_W^2, ..., \tau_W^l; \lambda_W^1, \lambda_W^2 ...., \lambda_W^m; \eta_W^1, \eta_W^2, ..., \eta_W^n)$ on $\mathcal{L}$ is said to be multi spherical fuzzy Lie ideal if, $\forall \gamma, \mu \in \mathcal{L}; \forall \mathrm{i} \in \mathrm{N}_l$, $\forall \mathrm{j} \in \mathrm{N}_m$, $\forall \mathrm{k} \in \mathrm{N}_n$, $\forall \mathrm{c} \in \mathcal{F}$, the conditions (1)-(2) of definition 3.1 along with the following conditions are satisfied:

(4) $\tau_W^i[\gamma, \mu] \geq \tau_W^i(\gamma), \lambda_W^j[\gamma, \mu] \geq \lambda_W^j(\gamma), \eta_W^k[\gamma, \mu] \leq \eta_W^k(\gamma)$

It follows from condition (2) that

(5) $\tau_W^i(0) \geq \tau_W^i(\gamma), \lambda_W^j(0) \geq \lambda_W^j(\gamma), \eta_W^k(0) \leq \eta_W^k(\gamma)$

(6) $\tau_W^i(-\gamma) \geq \tau_W^i(\gamma), \lambda_W^j(-\gamma) \geq \lambda_W^j(\gamma), \eta_W^k(-\gamma) \leq \eta_W^k(\gamma)$

Theorem 3.3: *Let* $\mathrm{M} = (\mathcal{L}, \tau_M^1, \tau_M^2, ..., \tau_M^l; \lambda_M^1, \lambda_M^2 ...., \lambda_M^m; \eta_M^1, \eta_M^2, ..., \eta_M^n)$ *be a multi spherical fuzzy Lie ideal over* $\mathcal{L}$ *and let* $N_l = \{1, 2, ..., 1\}$, $\mathrm{N}_m = \{1, 2, ..., m\}$, $\mathrm{N}_n = \{1, 2, ..., n\}$, *then* $\tau_M^i(0) = \text{Sup}\{\tau_M^i(\gamma) : \gamma \in \mathcal{L}\}$, $\lambda_M^j(0) = \text{Sup}\{\lambda_M^j(\gamma) : \gamma \in \mathcal{L}\}$ *and* $\eta_M^k(0) = \text{Inf}\{\eta_M^k(\gamma) : \gamma \in \mathcal{L}\}$, $\forall \mathrm{i} \in \mathrm{N}_l$, $\forall \mathrm{j} \in \mathrm{N}_m$, $\forall \mathrm{k} \in \mathrm{N}_n$.

Proof: From condition (5) of definition 3.2, we have,

$$\tau_M^i(0) \geq \tau_M^i(\gamma) \tag{1}$$

$$\lambda_M^j(0) \geq \lambda_M^j(\gamma) \tag{2}$$

$$\eta_M^k(0) \leq \eta_M^k(\gamma) \tag{3}$$

As γ runs over $\mathcal{L}$, the results follow just taking supremum on both sides of above inequalities (1) and (2) and infimum on (3). □

Theorem 3.4: *Let* $\mathrm{M} = (\mathcal{L}, \{\tau_M^i\}_{i=1}^l, \{\lambda_M^j\}_{j=1}^m, \{\eta_M^k\}_{k=1}^n)$ *be a multi spherical fuzzy Lie ideal over* $\mathcal{L}$*. Then for each,* $\psi, \xi, \sigma \in [0,1]$ *satisfying* $\tau_M^i(0) \geq \psi, \lambda_M^j(0) \geq \xi, \eta_M^k(0) \leq \sigma$ *and* $0 \leq \psi + \xi + \sigma \leq 1$*, the* (ψ, ξ, σ)*-level subset* $\mathcal{L}_M^{(\psi;\xi;\sigma)}$ *is a multi spherical fuzzy Lie ideal of* $\mathcal{L}$*.*

Proof: Straight forward from definition. □

Theorem 3.5: *If* δ *is a fixed element of* $\mathcal{L}$ *and* $\mathrm{W} = (\mathcal{L}, \{\tau_W^i\}_{i=1}^l, \{\lambda_W^j\}_{j=1}^m, \{\eta_W^k\}_{k=1}^n)$ *is a multi spherical fuzzy Lie ideal of* $\mathcal{L}$*. Then the set defined by* $\mathrm{W}^\delta = \{\gamma \in \mathcal{L} : \tau_W^i(\gamma) \geq \tau_W^i(\delta), \lambda_W^j(\gamma) \geq \lambda_W^j(\delta), \eta_W^k(\gamma) \leq \eta_W^k(\delta)\}$ *is a multi spherical fuzzy Lie ideal of* $\mathcal{L}$*.*

Proof: Suppose that $\gamma, \mu \in \mathrm{W}^\delta$, $\mathrm{i} \in \mathrm{N}_l$, $\mathrm{j} \in \mathrm{N}_m$, $\mathrm{k} \in \mathrm{N}_n$.

Then $\forall \gamma, \mu \in \mathrm{W}^\delta$, $\forall \mathrm{i} \in \mathrm{N}_l$, $\forall \mathrm{j} \in \mathrm{N}_m$, $\forall \mathrm{k} \in \mathrm{N}_n$

$$\tau_W^i(\gamma + \mu) \geq \mathrm{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\} \geq \tau_W^i(\delta);$$

$$\lambda_W^j(\gamma + \mu) \geq \mathrm{Min}\{\lambda_W^j(\gamma), \lambda_W^j(\mu)\} \geq \lambda_W^i(\delta);$$

$$\eta_W^k(\gamma + \mu) \leq \mathrm{Max}\{\eta_W^k(\gamma), \eta_W^k(\mu)\} \leq \eta_W^i(\delta)$$

This implies that $\gamma + \mu \in \mathrm{W}^\delta$.

Now, $\forall \gamma, \mu \in \mathrm{W}^{\delta}$, $\forall \mathrm{i} \in \mathrm{N}_l$, $\forall \mathrm{j} \in \mathrm{N}_m$, $\forall \mathrm{k} \in \mathrm{N}_n$, $\forall \mathrm{c} \in \mathcal{F}$,

$$\tau_W^i(c\gamma) \geq \tau_W^i(\gamma) \geq \tau_W^i(\delta); \lambda_W^j(c\gamma) \geq \lambda_W^j(\gamma) \geq \lambda_W^i(\delta);$$

$$\eta_W^k(c\gamma) \leq \eta_W^k(\gamma) \leq \eta_W^i(\delta) \Rightarrow c\gamma \in \mathrm{W}^{\delta}$$

Also for every $\gamma \in \mathrm{W}^{\delta}$ and for every $\mu \in \mathrm{W}^{\delta}$, $\forall \mathrm{i} \in \mathrm{N}_l$, $\forall \mathrm{j} \in \mathrm{N}_m$, $\forall \mathrm{k} \in \mathrm{N}_n$

$$\tau_W^i[\gamma, \mu] \geq \mathrm{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\} \geq \tau_W^i(\delta);$$

$$\lambda_W^j[\gamma, \mu] \geq \mathrm{Min}\{\lambda_W^j(\gamma), \lambda_W^j(\mu)\} \geq \lambda_W^i(\delta);$$

$\eta_W^k[\gamma, \mu] \leq \mathrm{Max}\{\eta_W^k(\gamma), \eta_W^k(\mu)\} \leq \eta_W^i(\delta)$, which shows that $[\gamma, \eta] \in \mathrm{W}^{\delta}$

Hence, W^{δ} is a multi spherical fuzzy Lie ideal of $\mathcal{L}$. □

Theorem 3.6: *If* $\mathrm{W} = (\mathcal{L}, \{\tau_W^i\}_{i=1}^l, \{\lambda_W^j\}_{j=1}^m, \{\eta_W^k\}_{k=1}^n)$ *is a multi spherical fuzzy Lie ideal of* $\mathcal{L}$. *Then the set defined by* $\mathrm{W}^0 = \{\gamma \in L : \tau_W^i(\gamma) \geq \tau_W^i(0),\ \lambda_W^j(\gamma) \geq \lambda_W^j(0),\ \eta_W^k(\gamma) \leq \eta_W^k(0),\ \forall \mathrm{i} \in \mathrm{N}_l,\ \forall \mathrm{j} \in \mathrm{N}_m,\ \forall \mathrm{k} \in \mathrm{N}_n\}$ *is a multi spherical fuzzy Lie ideal of* $\mathcal{L}$.

Proof: Straight forward. □

Theorem 3.7: *Let* $\mathrm{W} = (\{\tau_W^i\}_{i=1}^l, \{\lambda_W^j\}_{j=1}^m, \{\eta_W^k\}_{k=1}^n)$ *be a multi spherical fuzzy Lie subalgebra of a Lie algebra* $\mathcal{L}$ *and* $\mathcal{R} \subseteq \mathcal{L} \times \mathcal{L}$ *be a binary relation on* $\mathcal{L}$ *defined by* $\mathcal{R} = \{(\gamma, \mu) \in \mathcal{L} \times \mathcal{L} \mid \tau_W^i(\gamma - \mu) = \tau_W^i(0),\ \lambda_W^j(\gamma - \mu) = \lambda_W^j(0),\ \eta_W^k(\gamma - \mu) = \eta_W^k(0),\ \gamma, \mu \in \mathcal{L},\ \mathrm{i} \in \mathrm{N}_l,\ \mathrm{j} \in \mathrm{N}_m,\ \mathrm{k} \in \mathrm{N}_n\}$, *then* $\mathcal{R}$ *is a congruence relation on* $\mathcal{L}$.

Proof: First of all we need to prove that the relation $\mathcal{R}$ is equivalence relation on $\mathcal{L}$.

(i) **Reflexivity:** Since $\forall \gamma \in \mathcal{L}$, $\tau_W^i(\gamma - \gamma) = \tau_W^i(0)$, $\lambda_W^j(\gamma - \gamma) = \lambda_W^j(0)$, $\eta_W^k(\gamma - \gamma) = \eta_W^k(0)$, thus, $(\gamma, \gamma) \in \mathcal{R}$, $\forall \gamma \in \mathcal{L}$, $\forall \mathrm{i} \in \mathrm{N}_l$, $\forall \mathrm{j} \in \mathrm{N}_m$, $\forall \mathrm{k} \in \mathrm{N}_n$, and consequently $\mathcal{R}$ is reflexive relation on $\mathcal{L}$.

(ii) **Symmetric:** Let, $(\gamma, \mu) \in \mathcal{R}$. Then $\tau_W^i(\gamma - \mu) = \tau_W^i(0)$

$$\Rightarrow \tau_W^i(-(\mu - \gamma)) \geq \tau_W^i((\mu - \gamma)) = \tau_W^i(0),$$

$$\lambda_W^j(\gamma - \mu) = \lambda_W^j(0) \Rightarrow \lambda_W^j(-(\mu - \gamma)) \geq \lambda_W^j((\mu - \gamma)) = \lambda_W^j(0),$$

$$\eta_W^k(\gamma - \mu) = \eta_W^k(0), \Rightarrow \eta_W^k(-(\mu - \gamma)) \leq \eta_W^j((\mu - \gamma)) = \eta_W^j(0),$$

Thus, $(\mu, \gamma) \in \mathcal{R}$, $\forall \gamma, \mu \in \mathcal{L}$, $\forall \mathrm{i} \in \mathrm{N}_l$, $\forall \mathrm{j} \in \mathrm{N}_m$, $\forall \mathrm{k} \in \mathrm{N}_n$, so that $\mathcal{R}$ is symmetric relation on L.

(iii) **Transitive:** Let, $(\gamma, \mu), (\mu, \sigma) \in \mathcal{R}$. Then $\tau_W^i(\gamma - \mu) = \tau_W^i(0)$,

$$\tau_W^i(\mu - \sigma) = \tau_W^i(0)$$

$$\lambda_W^j(\gamma - \mu) = \lambda_W^j(0),\ \lambda_W^j(\mu - \sigma) = \lambda_W^j(0)$$

$$\eta_W^k(\gamma - \mu) = \eta_W^k(0),\ \eta_W^k(\mu - \sigma) = \eta_W^k(0)$$

From which we have,

$$\tau_W^i(\gamma - \sigma) = \tau_W^i\{(\gamma - \mu) + (\mu - \sigma)\} \geq \mathrm{Min}\{\tau_W^i(\gamma - \mu), \tau_W^i(\mu - \sigma)\} = \tau_W^i(0)\ ,$$

$$\lambda_W^j(\gamma - \sigma) = \lambda_W^j\{(\gamma - \mu) + (\mu - \sigma)\} \geq \mathrm{Min}\{\lambda_W^k(\gamma - \mu), \lambda_W^j(\mu - \sigma)\} = \lambda_W^j(0),$$

$$\eta_W^k(\gamma - \sigma) = \eta_W^k\{(\gamma - \mu) + (\mu - \sigma)\} \leq \mathrm{Max}\{\eta_W^k(\gamma - \mu), \eta_W^k(\mu - \sigma)\} = \eta_W^k(0).$$

Hence, $(\gamma, \sigma) \in \mathcal{L}$, $\forall \mathrm{i} \in \mathrm{N}_l$, $\forall \mathrm{j} \in \mathrm{N}_m$, $\forall \mathrm{k} \in \mathrm{N}_n$ and consequently, $\mathcal{R}$ is transitive relation on $\mathcal{L}$.

Hence, $\mathcal{R}$ is an equivalence relation on $\mathcal{L}$.

We now verify that $\mathcal{R}$ is an congruence relation on $\mathcal{L}$ and for that let us take $(\gamma, \mu), (\mu, \sigma) \in \mathcal{R}$. Then

$$\tau_W^i(\gamma - \mu) = \tau_W^i(0), \tau_W^i(\mu - \sigma) = \tau_W^i(0)$$

$$\lambda_W^j(\gamma - \mu) = \lambda_W^j(0), \lambda_W^j(\mu - \sigma) = \lambda_W^j(0)$$

$$\eta_W^k(\gamma - \mu) = \eta_W^k(0), \eta_W^k(\mu - \sigma) = \eta_W^k(0)$$

Now if $\gamma_1, \gamma_2, \mu_1, \mu_2 \in \mathcal{R}$, then we must have,

$$\tau_W^i\{(\gamma_1 + \gamma_2) - (\mu_1 + \mu_2)\} = \tau_W^i\{(\gamma_1 - \mu_1) + (\gamma_2 - \mu_2)\},$$
$$\geq \mathrm{Min}\{\tau_W^i(\gamma_1 - \mu_1), \tau_W^i(\gamma_2 - \mu_2)\} = \tau_W^i(0)$$

$$\lambda_W^j\{(\gamma_1 + \gamma_2) - (\mu_1 + \mu_2)\} = \lambda_W^j\{(\gamma_1 - \mu_1) + (\gamma_2 - \mu_2)\},$$
$$\geq \mathrm{Min}\{\lambda_W^j(\gamma_1 - \mu_1), \lambda_W^j(\gamma_2 - \mu_2)\} = \lambda_W^j(0)$$

$$\eta_W^k\{(\gamma_1 + \gamma_2) - (\mu_1 + \mu_2)\} = \eta_W^k\{(\gamma_1 - \mu_1) + (\gamma_2 - \mu_2)\},$$
$$\leq \mathrm{Max}\{\eta_W^k(\gamma_1 - \mu_1), \eta_W^k(\gamma_2 - \mu_2)\} = \eta_W^k(0)$$

$$\tau_W^i(c\gamma_1 - c\mu_1) = \tau_W^i\{c(\gamma_1 - \mu_1)\} \geq \tau_W^i(\gamma_1 - \mu_1) = \tau_W^i(0),$$

$$\lambda_W^j(c\gamma_1 - c\mu_1) = \lambda_W^j\{c(\gamma_1 - \mu_1)\} \geq \lambda_W^j(\gamma_1 - \mu_1) = \lambda_W^i(0),$$

$$\eta_W^k(c\gamma_1 - c\mu_1) = \eta_W^k\{c(\gamma_1 - \mu_1)\} \geq \eta_W^k(\gamma_1 - \mu_1) = \eta_W^k(0),$$

$$\tau_W^i\{[\gamma_1, \gamma_2] - [\mu_1, \mu_2]\} = \tau_W^i[(\gamma_1 - \mu_1), (\gamma_2 - \mu_2)]$$
$$\geq \mathrm{Min}\{\tau_W^i(\gamma_1 - \mu_1), \tau_W^i(\gamma_2 - \mu_2)\} = \tau_W^i(0),$$

$$\lambda_W^j\{[\gamma_1, \gamma_2] - [\mu_1, \mu_2]\} = \lambda_W^j[(\gamma_1 - \mu_1), (\gamma_2 - \mu_2)]$$
$$\geq \mathrm{Min}\{\lambda_W^j(\gamma_1 - \mu_1), \lambda_W^j(\gamma_2 - \mu_2)\} = \lambda_W^j(0),$$
$$\eta_W^k\{[\gamma_1 + \gamma_2] - [\mu_1 + \mu_2]\} = \eta_W^k[(\gamma_1 - \mu_1), (\gamma_2 - \mu_2)]$$
$$\leq \mathrm{Max}\{\eta_W^k(\gamma_1 - \mu_1), \eta_W^k(\gamma_2 - \mu_2)\} = \eta_W^k(0),$$

Thus, $(\gamma_1 + \gamma_2)\ \mathcal{R}\ (\mu_1 + \mu_2), c\gamma_1\ \mathcal{R}\ c\mu_1$ and $[\gamma_1, \gamma_2]\ R\ [\mu_1, \mu_2]$.

Hence, $\mathcal{R}$ is a congruence relation on $\mathcal{L}$. □

Theorem 3.8: *Every multi spherical fuzzy Lie ideal is multi spherical fuzzy Lie sub-algebra.*

Proof: Straight forward from definition. □

The converse of the above theorem (3.8) is not true which can be seen from the following example:

Example 3.9: Suppose that $F = \mathbb{R}$, the set of real numbers and $\mathcal{L} = \{(\alpha, \beta, \gamma) : \alpha, \beta, \gamma \in \mathbb{R}\}$ be the Lie algebra. Let us define the mapping $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ by $[u, v] = u \times v + v$, where $\times$ denotes the vector product (or cross product). Consider the multi spherical fuzzy set $S = (\tau_S^1, \tau_S^2, \tau_S^3, \lambda_S^1, \lambda_S^2, :$ $\eta_S^1, \eta_S^2, \ldots, \eta_S^4)\ \mathcal{L} \to [0,1] \times [0,1] \times [0,1] \times [0,1] \times [0,1] \times [0,1] \times [0,1] \times [0,1] \times [0,1]$ described by $\tau_S^i(\alpha, \beta, \gamma) = \{\frac{o.8}{i}$, if $\alpha = \beta = \gamma = 0, \frac{o.6}{i}$, if $\alpha \neq 0,\ \beta = \gamma = 0,\ 0$, otherwise$\}$ for $i = 1, 2, 3$.

$\lambda_S^j(\alpha, \beta, \gamma) = \{\frac{o.3}{j}$, if $\alpha = \beta = \gamma = 0$, $\frac{o.2}{j}$, if $\alpha \neq 0,\ \beta = \gamma = 0,\ 0$, otherwise$\}$ for $j = 1, 2$.

$\eta_S^k(\alpha, \beta, \gamma) = \{0$, if $\alpha = \beta = \gamma = 0$, $\frac{o.4}{k}$, if $\alpha \neq 0,\ \beta = \gamma = 0$, $\frac{1}{k}$, otherwise $\}$ for $k = 1, 2, 3, 4$.

Then it is easy to verify that S is a multi spherical fuzzy Lie subalgebra of $\mathcal{L}$ but it is not multi spherical fuzzy Lie ideal of $\mathcal{L}$ because $\lambda_S^j[(2,0,0),(j,-j,j)] = \lambda_S^j(j,-3j,-j) = 0 \ngeq \frac{o.2}{j} = \lambda_S^j(2,0,0)$, for all $j = 1, 2$. □

Theorem 3.10: *Let* $\mathrm{N}_l = \{1,2,\ldots.,1\}$, $\mathrm{N}_m = \{1,2,\ldots.,m\}$, $\mathrm{N}_n = \{1,2,\ldots.,n\}$ *and* $\mathcal{L}$ *be a Lie Algebra of vectors over the field* $\mathcal{F}$. *The necessary and sufficient condition for a multi spherical fuzzy set* $\mathrm{W} = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \lambda_W^1, \lambda_W^2 \ldots., \lambda_W^m; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$ *to be a multi spherical fuzzy Lie subalgebra over* $\mathcal{L}$ *is that,* $\forall \mathrm{r} \in [0,1]$, *non-empty upper* r-*level cut* $\mathrm{U_r}(\tau_W^i) = \{\gamma \in \mathcal{L} : \tau_W^i(\gamma) \geq \mathrm{r}, \quad \forall \mathrm{i} \in \mathrm{N}_l\}$, $\mathrm{U}_r(\lambda_W^j) = \{\gamma \in \mathcal{L} : \lambda_W^j(\gamma) \geq \mathrm{r}, \forall \mathrm{j} \in \mathrm{N}_m\}$, *and non-empty lower r-level cut* $\mathrm{V}_r(\eta_W^k) = \{\gamma \in \mathcal{L} : \eta_W^k(\gamma) \leq \mathrm{r}, \forall \mathrm{k} \in \mathrm{N}_n\}$ *are Lie subalgebra over* $\mathcal{L}$.

Proof: Suppose that $\mathrm{W} = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \lambda_W^1, \lambda_W^2 \ldots., \lambda_W^m; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$ is a multi spherical fuzzy Lie subalgebra over $\mathcal{L}$ and $r \in [0,1]$ is such that $\mathrm{U}_r(\tau_W^i) \neq \phi$. Let $\gamma, \mu \in \mathrm{U}_r(\tau_W^i)$.

Then $\forall \gamma, \mu \in \mathcal{L}; \forall \mathrm{i} \in \mathrm{N}_l, \forall \mathrm{j} \in \mathrm{N}_m, \forall \mathrm{k} \in \mathrm{N}_n, \forall \mathrm{c} \in \mathcal{F}$,

$$\tau_W^i(\gamma+\mu) \geq \mathrm{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\} \geq \mathrm{r}, \lambda_W^j(\gamma+\mu) \geq \mathrm{Min}\{\lambda_W^j(\gamma), \lambda_W^j(\mu)\} \geq \mathrm{r},$$

$$\eta_W^k(\gamma+\mu) \leq \mathrm{Max}\{\eta_W^k(\gamma), \eta_W^k(\mu)\} \leq \mathrm{r}$$

$$\tau_W^i(c\gamma) \geq \tau_W^i(\gamma) \geq \mathrm{r}, \lambda_W^j(c\gamma) \geq \lambda_W^j(\gamma) \geq \mathrm{r}, \eta_W^k(c\gamma) \leq \eta_W^k(\gamma) \leq \mathrm{r},$$

$$\tau_W^i[\gamma,\mu] \geq \mathrm{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\} \geq \mathrm{r}$$

$$\lambda_W^j[\gamma,\mu] \geq \mathrm{Min}\{\lambda_W^j(\gamma), \lambda_W^j(\mu)\} \geq \mathrm{r}$$

$$\eta_W^k[\gamma,\mu] \leq \mathrm{Max}\{\eta_W^k(\gamma), \eta_W^k(\mu)\} \leq \mathrm{r}.$$

Thus,
$\gamma + \mu, c\gamma, [\gamma, \mu] \in \mathrm{U}_r(\tau_W^i), \gamma + \mu, c\gamma, [\gamma, \mu] \in \mathrm{U}_r(\lambda_W^j), \gamma + \mu, c\gamma, [\gamma, \mu] \in \mathrm{V}r(\eta_W^k)$.

Thus, $\mathrm{U}_r(\tau_W^i), \mathrm{U}_r(\lambda_W^j)$ and $\mathrm{V}_r(\eta_W^k)$ constitute Lie sub algebra over $\mathcal{L}$.

Conversely, suppose that $\forall \mathrm{i} \in \mathrm{i} \in \mathrm{N}_l$ and $\forall \mathrm{r} \in [0,1]$, $\mathrm{U}_r(\tau_W^i) \neq \phi$ is a Lie sub algebra over $\mathcal{L}$ and if possible suppose that $\tau_W^i(\gamma + \mu) \operatorname{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\}$ for some $\gamma, \mu \in \mathcal{L}$. If we choose $\mathrm{r}_0 = \frac{1}{2}(\tau_W^i(\gamma + \mu) + \operatorname{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\})$, the by properties of inequality we must have, $\tau_W^i(\gamma + \mu) < \mathrm{r}_0 < \operatorname{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\}$.

This implies that, $\gamma + \mu = \notin \mathrm{U}_r(\tau_W^i), \gamma, \mu \in \mathrm{U}_r(\tau_W^i)$, which is a contradiction. Hence, $\tau_W^i(\gamma + \mu) \geq \operatorname{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\}, \forall\ \gamma, \mu \in \mathcal{L}$.

In a similar manner we can prove that $\tau_W^i(c\gamma) \geq \tau_W^i(\gamma)$ and $\tau_W^i[\gamma, \mu] \geq \operatorname{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\}, \forall \mathrm{c} \in \mathcal{F}, \forall \mathrm{i} \in \mathrm{N}_l$.

The proof is similar for cases $\mathrm{U}_r(\lambda_W^j)$ and $\mathrm{V}_r(\eta_W^k) \precsim$. This completes the proof. □

Theorem 3.11: *Let* $\mathrm{N}_l = \{1, 2, \ldots, \mathrm{l}\}$, $\mathrm{N}_m = \{1, 2, \ldots, \mathrm{m}\}$, $\mathrm{N}_n = \{1, 2, \ldots, \mathrm{n}\}$ *and* $\mathcal{L}$ *be a Lie Algebra of vectors over the field* $\mathcal{F}$. *If* $\mathrm{V} = (\tau_V^1, \tau_V^2, \ldots, \tau_V^l; \lambda_V^1, \lambda_V^2 \ldots, \lambda_V^m; \eta_V^1, \eta_V^2, \ldots, \eta_V^n)$; *and* $\mathrm{W} = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \lambda_W^1, \lambda_W^2 \ldots, \lambda_W^m; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$ *are two multi spherical fuzzy Lie subalgebra over* $\mathcal{L}$, *then their intersection* $\mathrm{V} \cap \mathrm{W} = \mathrm{H} = (\tau_H^1, \tau_H^2, \ldots, \tau_H^l; \lambda_H^1, \lambda_H^2 \ldots, \lambda_H^m; \eta_H^1, \eta_H^2, \ldots, \eta_H^n)$ *is also multi spherical fuzzy Lie subalgebra over* $\mathcal{L}$.

Proof: Suppose that $\gamma, \mu \in \mathcal{L}$ be arbitrary. Then $\forall \mathrm{i} \in \mathrm{N}_l$, $\forall \mathrm{j} \in \mathrm{N}_m$, $\forall \mathrm{k} \in \mathrm{N}_n$, $\forall \mathrm{c} \in \mathcal{F}$, we have,

$\tau_H^i(\gamma + \mu) = \text{Min}\{\tau_V^i(\gamma + \mu), \tau_W^i(\gamma + \mu)\}$

$\geq \text{Min}\{\text{Min}\{\tau_V^i(\gamma), \tau_V^i(\mu)\}, \text{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\}\}$

$= \text{Min}\{\text{Min}\{\tau_V^i(\gamma), \tau_W^i(\gamma)\}, \text{Min}\{\tau_V^i(\mu), \tau_W^i(\mu)\}\} = \text{Min}\{\tau_H^i(\gamma), \tau_H^i(\mu)\}$

$\lambda_H^j(\gamma + \mu) = \text{Min}\{\lambda_V^j(\gamma + \mu), \lambda_W^j(\gamma + \mu)\}$

$\geq \text{Min}\{\text{Min}\{\lambda_V^j(\gamma), \lambda_V^j(\mu)\}, \text{Min}\{\lambda_W^j(\gamma), \lambda_W^j(\mu)\}\}$

$= \text{Min}\{\text{Min}\{\lambda_V^j(\gamma), \lambda_W^j(\gamma)\}, \text{Min}\{\lambda_V^i(\mu), \lambda_W^i(\mu)\}\} = \text{Min}\{\lambda_H^j(\gamma), \lambda_H^j(\mu)\}$

$\eta_H^k(\gamma + \mu) = \text{Max}\{\eta_V^k(\gamma + \mu), \eta_W^k(\gamma + \mu)\} \leq \text{Max}\{\text{Max}\{\eta_V^k(\gamma), \eta_V^k(\mu)\},$

$\text{Max}\{\eta_W^k(\gamma), \eta_W^k(\mu)\}\}$

$= \text{Max}\{\text{Max}\{\eta_V^k(\gamma), \eta_W^k(\gamma)\}, \text{Max}\{\eta_V^k(\mu), \eta_W^k(\mu)\}\} = \text{Max}\{\eta_H^k(\gamma), \eta_H^k(\mu)\}$

$\tau_H^i(c\gamma) = \text{Min}\{\tau_V^i(c\gamma), \tau_W^i(c\gamma)\} \geq \{\text{Min}\{\tau_V^i(\gamma), \tau_W^i(\gamma)\} = \tau_H^i(\gamma)$

$\lambda_H^j(c\gamma) = \text{Min}\{\lambda_V^j(c\gamma), \lambda_W^j(c\gamma)\} \geq \{\text{Min}\{\lambda_V^j(\gamma), \lambda_W^j(\gamma)\} = \lambda_H^j(\gamma)$

$\eta_H^k(c\gamma) = \text{Max}\{\eta_V^k(c\gamma), \eta_W^k(c\gamma)\} \leq \{\text{Max}\{\eta_V^k(\gamma), \eta_W^k(\gamma)\} = \eta_H^k(\gamma)$

$\tau_H^i[\gamma, \mu] = \text{Min}\{\tau_V^i[\gamma, \mu], \tau_W^i[\gamma, \mu]\}$

$\geq \text{Min}\{\text{Min}\{\tau_V^i(\gamma), \tau_V^i(\mu)\}, \text{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\}\}$

$= \text{Min}\{\text{Min}\{\tau_V^i(\gamma), \tau_W^i(\gamma)\}, \text{Min}\{\tau_V^i(\mu), \tau_W^i(\mu)\}\} = \text{Min}\{\tau_H^i(\gamma), \tau_H^i(\mu)\}$

$\lambda_H^j[\gamma, \mu] = \text{Min}\{\lambda_V^j[\gamma, \mu], \lambda_W^j[\gamma, \mu]\}$

$\geq \text{Min}\{\text{Min}\{\lambda_V^j(\gamma), \lambda_V^j(\mu)\}, \text{Min}\{\lambda_W^j(\gamma), \lambda_W^j(\mu)\}\}$

$= \text{Min}\{\text{Min}\{\lambda_V^j(\gamma), \lambda_W^j(\gamma)\}, \text{Min}\{\lambda_V^j(\mu), \lambda_W^j(\mu)\}\} = \text{Min}\{\lambda_H^j(\gamma), \lambda_H^j(\mu)\}$

$\eta_H^k[\gamma, \mu] = \text{Max}\{\eta_V^k[\gamma, \mu], \eta_W^k[\gamma, \mu]\}$

$\leq \text{Max}\{\text{Max}\{\eta_V^k(\gamma), \eta_V^k(\mu)\}, \text{Max}\{\eta_W^k(\gamma), \eta_W^k(\mu)\}\}$

$= \text{Max}\{\text{Max}\{\eta_V^k(\gamma), \eta_W^k(\gamma)\}, \text{Max}\{\eta_V^k(\mu), \eta_W^k(\mu)\}\} = \text{Max}\{\eta_H^k(\gamma), \eta_H^k(\mu)\}$

Hence, $V \cap W = H$ is multi spherical fuzzy Lie subalgebra over $\mathcal{L}$. □

Definition 3.12: *Let* $N_l = \{1, 2, ..., l\}$, $N_m = \{1, 2, ..., m\}$,

$N_n = \{1, 2, ..., n\}$ *and* $\mathcal{L}$ *be a Lie Algebra of vectors over the field* $\mathcal{F}$. *If* $V = (\tau_V^1, \tau_V^2, ..., \tau_V^l; \lambda_V^1, \lambda_V^2 ...., \lambda_V^m; \eta_V^1, \eta_V^2, ..., \eta_V^n)$ *and* $W = (\tau_W^1, \tau_W^2, ..., \tau_W^l; \lambda_W^1, \lambda_W^2 ...., \lambda_W^m; \eta_W^1, \eta_W^2, ..., \eta_W^n)$ *are two multi spherical fuzzy sets on* $\mathcal{L}$, *then the product* $H = V \times W$ *defined on* $\mathcal{L} \times \mathcal{L}$ *will be known as generalized cartasian product if* $(V\times W)(\gamma, \mu) = [(\tau_V^1, \tau_V^2, ..., \tau_V^l; \lambda_V^1, \lambda_V^2 ...., \lambda_V^m; \eta_V^1, \eta_V^2, ..., \eta_V^n) \times (\tau_W^1, \tau_W^2, ..., \tau_W^l; \lambda_W^1, \lambda_W^2 ...., \lambda_W^m; \eta_W^1, \eta_W^2, ..., \eta_W^n)](\gamma, \mu) = (\tau_H^1(\gamma, \mu), \tau_H^2(\gamma, \mu), ..., \tau_H^l(\gamma, \mu); \lambda_H^1(\gamma, \mu), \lambda_H^2(\gamma, \mu), ..., \lambda_H^m(\gamma, \mu); \eta_H^1(\gamma, \mu), \eta_H^2(\gamma, \mu), ..., \eta_H^n(\gamma, \mu)), \forall (\gamma, \mu)\ \mathcal{L} \times \mathcal{L}$ *where,* $\tau_H^i(\gamma, \mu) = (\tau_V^i \times \tau_W^i)(\gamma, \mu) = \text{Min}\{\tau_V^i(\gamma), \tau_W^i(\mu)\}, \forall i \in N_l$.

$$\lambda_H^j(\gamma, \mu) = (\lambda_V^j \times \lambda_W^j)(\gamma, \mu) = \text{Min}\{\lambda_V^j(\gamma), \lambda_W^j(\mu)\}, \forall j \in N_m.$$
$$\eta_H^k(\gamma, \mu) = (\eta_V^k \times \eta_W^k)(\gamma, \mu) = \text{Max}\{\eta_V^k(\gamma), \eta_W^k(\mu)\}, \forall k \in N_n.$$

Evidently the generalized cartasian product $(V \times W)$ is multi spherical fuzzy set on $\mathcal{L} \times \mathcal{L}$ if $0 \le \{\tau_H^i(\gamma, \mu)\}^2 + \lambda_H^i(\gamma, \mu)\}^2 + \{\eta_H^k(\gamma, \mu)\}^2 \le 1$
i.e.,
$0 \le \{\text{Min}\{\tau_V^i(\gamma), \tau_W^i(\mu)\}\}^2 + \text{Min}\{\lambda_V^i(\gamma), \tau_W^i(\mu)\}\}^2 + \{\text{Max}\{\eta_V^k(\gamma), \eta_W^k(\mu)\}\}^2 \le 1,$
where, $i \in N_l$, $j \in N_m$, $k \in N_n$.

Theorem 3.13: *Let* $\mathcal{L}$ *be the Lie Algebra of vectors over the field* $\mathcal{F}$. *If* $V = (\tau_V^1, \tau_V^2, ..., \tau_V^l; \lambda_V^1, \lambda_V^2 ...., \lambda_V^m; \eta_V^1, \eta_V^2, ..., \eta_V^n)$ *and* $W = (\tau_W^1, \tau_W^2, ..., \tau_W^l; \lambda_W^1, \lambda_W^2 ...., \lambda_W^m; \eta_W^1, \eta_W^2, ..., \eta_W^n)$ *are two multi spherical fuzzy Lie subalgebra of* $\mathcal{L}$, *then the generalized cartasian product* $(V\times W)$ *is multi spherical fuzzy Lie subalgebra of* $\mathcal{L} \times \mathcal{L}$.

Proof: Let $N_l = \{1, 2, ..., l\}$, $N_m = \{1, 2, ..., m\}$, $N_n = \{1, 2, ..., n\}$ and $\mathcal{L}$ be a Lie Algebra of vectors over the field $\mathcal{F}$. Then, $\forall i \in N_l$, $\forall j \in N_m$, $\forall k \in N_n$, $\forall \gamma = (\gamma_1, \gamma_2)$, $\mu = (\mu_1, \mu_2) \in \mathcal{L} \times \mathcal{L}$ and $c \in \mathcal{F}$, we have,
$(\tau_V^i \times \tau_W^i)(\gamma + \mu) = (\tau_V^i \times \tau_W^i)((\gamma_1, \gamma_2) + (\mu_1, \mu_2))$

$= (\tau_V^i \times \tau_W^i)\ ((\gamma_1 + \mu_1),(\gamma_2 + \mu_2)) = \text{Min}\{\tau_V^i(\gamma_1 + \mu_1), \tau_W^i(\gamma_2 + \mu_2)\}$

$\geq \text{Min}\{\text{Min}\{\tau_V^i(\gamma_1), \tau_V^i(\mu_1)\}, \text{Min}\{\tau_W^i(\gamma_2), \tau_W^i(\mu_2)\}\}$

$= \text{Min}\{\text{Min}\{\tau_V^i(\gamma_1), \tau_W^i(\gamma_2)\}, \text{Min}\{\tau_V^i(\mu_1), \tau_W^i(\mu_2)\}\}$

$= \text{Min}\{(\tau_V^i \times \tau_W^i)(\gamma_1, \gamma_2), (\tau_V^i \times \tau_W^i)(\mu_1, \mu_2)\} = \text{Min}\{(\tau_V^i \times \tau_W^i)(\gamma), (\tau_V^i \times \tau_W^i)(\mu)\}$

$(\lambda_V^j \times \lambda_W^j) = (\gamma + \mu) = (\lambda_V^j \times \lambda_W^j)((\gamma_1, \gamma_2) + (\mu_1, \mu_2))$

$= (\lambda_V^j \times \lambda_W^j)\ ((\gamma_1 + \mu_1), (\gamma_2 + \mu_2))$

$= \text{Min}\{\lambda_V^j(\gamma_1 + \mu_1), \lambda_W^j(\gamma_2 + \mu_2)\}$

$\geq \text{Min}\{\text{Min}\{\lambda_V^j(\gamma_1), \lambda_V^j(\mu_1)\}, \text{Min}\{\lambda_W^j(\gamma_2), \lambda_W^j(\mu_2)\}\}$

$= \text{Min}\{\text{Min}\{\lambda_V^j(\gamma_1), \lambda_W^j(\gamma_2)\}, \text{Min}\{\lambda_V^j(\mu_1), \lambda_W^j(\mu_2)\}\}$

$= \text{Min}\{(\lambda_V^j \times \lambda_W^j)(\gamma_1, \gamma_2), (\lambda_V^j \times \lambda_W^j)(\mu_1, \mu_2)\}\} = \text{Min}\{(\lambda_V^j \times \lambda_W^j)(\gamma), (\lambda_V^j \times \lambda_W^j)(\mu)\}$

$(\eta_V^k \times \eta_W^k)(\gamma + \mu) = (\eta_V^k \times \eta_W^k)((\gamma_1, \gamma_2) + (\mu_1, \mu_2)) = (\eta_V^k \times \eta_W^k)((\gamma_1 + \mu_1), (\gamma_2 + \mu_2))$

$= \text{Max}\{\eta_V^k(\gamma_1 + \mu_1), \eta_W^k(\gamma_2 + \mu_2)\}$

$\leq \text{Max}\{\text{Max}\{\eta_V^k(\gamma_1), \eta_V^k(\mu_1)\}, \text{Min}\{\eta_W^k(\gamma_2), \eta_W^k(\mu_2)\}\}$

$= \text{Max}\{\text{Max}\{\eta_V^k(\gamma_1), \eta_W^k(\gamma_2)\}, \text{Max}\{\eta_V^k(\mu_1), \eta_W^k(\mu_2)\}\}$

$= \text{Max}\ \{(\eta_V^k \times \eta_W^k)(\gamma_1, \gamma_2), (\eta_V^k \times \eta_W^k)(\mu_1, \mu_2)\}\}$

$= \text{Max}\{(\eta_V^k \times \eta_W^k)(\gamma), (\eta_V^k \times \eta_W^k)(\mu)\}$

$(\tau_V^i \times \tau_W^i)(c\gamma) = (\tau_V^i \times \tau_W^i)\{c(\gamma_1, \gamma_2)\} = (\tau_V^i \times \tau_W^i)(c\gamma_1, c\gamma_2)$

$= \text{Min}\{\tau_V^i(c\gamma_1), \tau_W^i(c\gamma_2)\}$

$\geq \text{Min}\{\tau_V^i(\gamma_1), \tau_W^i(\gamma_2)\} = (\tau_V^i \times \tau_W^i)(\gamma_1, \gamma_2) = (\tau_V^i \times \tau_W^i)(\gamma)$

$(\lambda_V^j \times \lambda_W^j)(c\gamma) = (\lambda_V^j \times \lambda_W^j)\{c(\gamma_1, \gamma_2)\} = (\lambda_V^j \times \lambda_W^j)(c\gamma_1, c\gamma_2)$

$= \text{Min}\{\lambda_V^j(c\gamma_1), \lambda_W^j(c\gamma_2)\}$

$\geq \text{Min}\{\lambda_V^j(\gamma_1), \lambda_W^j(\gamma_2)\} = (\lambda_V^j \times \lambda_W^j)(\gamma_1, \gamma_2) = (\lambda_V^j \times \lambda_W^j)(\gamma)$

$(\eta_V^k \times \eta_W^k)\{(c\gamma)\} = (\eta_V^k \times \eta_W^k)\{c(\gamma_1, \gamma_2)\} = (\eta_V^k \times \eta_W^k)(c\gamma_1, c\gamma_2)$

$= \text{Max}\{\eta_V^k(c\gamma_1), \eta_W^k(c\gamma_2)\} \leq \text{Max}\{\eta_V^k(\gamma_1), \eta_W^k(\gamma_2)\} = (\eta_V^k \times \eta_W^k)(\gamma_1, \gamma_2)$

$= (\eta_V^k \times \eta_W^k)(\gamma)\ (\tau_V^i \times \tau_W^i)[\gamma, \mu] = (\tau_V^i \times \tau_W^i)[(\gamma_1, \gamma_2), (\mu_1, \mu_2)]$

$\geq \mathrm{Min}\{\mathrm{Min}\{\tau_V^i(\gamma_1), \tau_W^i(\gamma_2)\}, \{\mathrm{Min}\{\tau_V^i(\mu_1), \tau_W^i(\mu_2)\}\}$

$= \mathrm{Min}\{(\tau_V^i \times \tau_W^i)(\gamma_1, \gamma_2)\}, \{(\tau_V^i \times \tau_W^i)(\mu_1, \mu_2)\}$

$= \mathrm{Min}\{(\tau_V^i \times \tau_W^i)(\gamma), (\tau_V^i \times \tau_W^i)(\mu)\}$

$(\lambda_V^j \times \lambda_W^j)[\gamma, \mu] = (\lambda_V^j \times \lambda_W^j)[(\gamma_1, \gamma_2), (\mu_1, \mu_2)]$

$\geq \mathrm{Min}\{\mathrm{Min}\{\lambda_V^j\ (\gamma_1), \lambda_W^j\ (\gamma_2)\}, \{\mathrm{Min}\{\lambda_V^j\ (\mu_1), \lambda_W^j\ (\mu_2)\}\}$

$= \mathrm{Min}\{(\lambda_V^j \times \lambda_W^j)(\gamma_1, \gamma_2)\}, \{(\lambda_V^j \times \lambda_W^j)(\mu_1, \mu_2)\} = \mathrm{Min}\{(\lambda_V^j \times \lambda_W^j)(\gamma), (\lambda_V^j \times \lambda_W^j)(\mu)\}$

$(\eta_V^k \times \eta_W^k)[\gamma, \mu] = (\eta_V^k \times \eta_W^k)[(\gamma_1, \gamma_2), (\mu_1, \mu_2)]$

$\leq \mathrm{Max}\{\mathrm{Max}\{\eta_V^k(\gamma_1), \eta_W^k(\gamma_2)\}, \{\mathrm{Max}\{\eta_V^k(\mu_1), \eta_W^k(\mu_2)\}\}$

$= \mathrm{Max}\{(\eta_V^k \times \eta_W^k)(\gamma_1, \gamma_2)\}, \{(\eta_V^k \times \eta_W^k)(\mu_1, \mu_2)\} = \mathrm{Max}\{(\eta_V^k \times \eta_W^k)(\gamma), (\eta_V^k \times \eta_W^k)(\mu)\}$

Hence, $(\mathrm{V} \times \mathrm{W})$ is multi spherical fuzzy Lie subalgebra of $\mathcal{L} \times \mathcal{L}$. □

4. Multi Spherical Fuzzy Lie Algebra Homomorphisms

In this section, the properties of multi spherical fuzzy Lie subalgebras and multi spherical fuzzy Lie ideals under homomorphisms of Lie algebras are investigated. Also the preservation aspects are examined.

Definition 4.1: Let L_1 and L_2 be two Lie algebras over the field $\mathcal{F}$. A linear transformation $\varphi : \mathrm{L}_1 \to \mathrm{L}_2$ is said to be Lie homomorphism if the relationship $\varphi([\gamma, \mu]) = [\varphi(\gamma), \varphi(\mu)]$ is true, $\forall \gamma, \mu \in \mathrm{L}_1$.

Definition 4.2: Let L_1 and L_2 be two Lie algebras over the field $\mathcal{F}$. Then a Lie homomorphism $\varphi : \mathrm{L}_1 \to \mathrm{L}_2$ is said have natural extension $\varphi : \mathrm{I}^{L_1} \to \mathrm{I}^{L_2}$ if $\forall \mathrm{W} = (\tau_W^1 \tau_W^2, \ldots, \tau_W^l; \lambda_W^1, \lambda_W^2 \ldots., \lambda_W^m; \eta_W^1, \eta_W^2, \ldots, \eta_W^n) \in \mathrm{I}^{L_1}$ and $\mu \in \mathrm{L}_2$, the followings hold:

$\varphi(\tau_W^i)(\mu) = \mathrm{Sup}\{\tau_W^i(\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in \mathrm{L}_1\}$, for all $\mathrm{i} = 1, 2, 3, \ldots, \mathrm{l}$.

$\varphi(\lambda_W^j)(\mu) = \mathrm{Sup}\{\tau_W^j(\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in \mathrm{L}_1\}$, for all $\mathrm{j} = 1, 2, 3, \ldots, \mathrm{m}$.

$\varphi(\eta_W^k)(\mu) = \mathrm{Inf}\{\tau_W^k(\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in \mathrm{L}_1\}$, for all $\mathrm{k} = 1, 2, 3, \ldots, \mathrm{n}$.

Theorem 4.3: *Let* $\mathrm{W} = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \lambda_W^1, \lambda_W^2, \ldots., \lambda_W^m; \eta_W^1, \eta_W^2, \ldots, \eta_W^n) \in \mathrm{I}^{L_1}$ *be multi spherical fuzzy Lie subalgebras and* $\varphi : \mathrm{L}_1 \to \mathrm{L}_2$ *be Lie homomorphism between* L_1 *and* L_2.

Then $\varphi(\mathrm{W})$ is multi spherical fuzzy Lie subalgebras of L_2.

Proof: Suppose that $\mu_1, \mu_2 \in \mathrm{L}_2$. Then $\{\gamma : \gamma \in \varphi^{-1}(\mu_1 + \mu_2)\} \supseteq \{\gamma_1 + \gamma_2 : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$. Now, for all $\mathrm{i} = 1, 2, 3, \ldots, \mathrm{l}$, we have,

$\varphi(\tau_W^i)(\mu_1 + \mu_2) = \mathrm{Sup}\{\tau_W^i(\gamma) : \gamma \in \varphi^{-1}(\mu_1 + \mu_2), \gamma \in \mathrm{L}_1\}$

$\geq \{\tau_W^i(\gamma_1 + \gamma_2) : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$

$\geq \mathrm{Sup}\{\mathrm{Min}\{\tau_W^i(\gamma_1), \tau_W^i(\gamma_2)\} : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$

$= \mathrm{Min}\{\mathrm{Sup}\{\tau_W^i(\gamma_1) : \gamma_1 \in \varphi^{-1}(\mu_1)\}, \mathrm{Sup}\{\tau_W^i(\gamma_2) : \gamma_2 \in \varphi^{-1}(\mu_2)\}\}$

$= \mathrm{Min}\{\varphi(\tau_W^i)(\mu_1), \varphi(\tau_W^i)(\mu_2)\}$.

Similarly, for all $\mathrm{j} = 1, 2, 3, \ldots, \mathrm{m}$, we can easily prove that,

$\varphi(\lambda_W^j)(\mu_1 + \mu_2) = \mathrm{Sup}\{\lambda_W^j(\gamma) : \gamma \in \varphi^{-1}(\mu_1 + \mu_2), \gamma \in \mathrm{L}_1\}$

$\geq \mathrm{Min}\{\varphi(\lambda_W^j)(\mu_1), \varphi(\lambda_W^j)(\mu_2)\}$ and for all $\mathrm{k} = 1, 2, 3, \ldots, \mathrm{n}$, we have,

$\varphi(\eta_W^k)(\mu_1 + \mu_2) = \mathrm{Inf}\{\eta_W^k(\gamma) : \gamma \in \varphi^{-1}(\mu_1 + \mu_2), \gamma \in \mathrm{L}_1\}$

$\leq \{\eta_W^k(\gamma_1 + \gamma_2) : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$

$\leq \mathrm{Inf}\{\mathrm{Max}\{\eta_W^k(\gamma_1), \eta_W^k(\gamma_2)\} : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$

$= \mathrm{Max}\{\mathrm{Inf}\{\eta_W^k(\gamma_1) : \gamma_1 \in \varphi^{-1}(\mu_1)\}, \mathrm{Inf}\{\eta_W^k(\gamma_2) : \gamma_2 \in \varphi^{-1}(\mu_2)\}\}$

$= \mathrm{Max}\{\varphi(\eta_W^k)(\mu_1), \varphi(\eta_W^k)(\mu_2)\}$

For $\mu \in \mathrm{L}_2$ and $\mathrm{c} \in \mathcal{F}$, we have, $\{\gamma : \gamma \in \varphi^{-1}(\mathrm{c}\mu_1)\} \supseteq \{\mathrm{c}\gamma : \gamma \in \varphi^{-1}(\mu)\}$.

Now, for all $\mathrm{i} = 1, 2, 3, \ldots, \mathrm{l}$, we have,

$$\varphi(\tau_W^i)(\mathrm{c}\mu) = \mathrm{Sup}\{\tau_W^i(\mathrm{c}\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in \mathrm{L}_1\}$$
$$\geq \mathrm{Sup}\{\tau_W^i(\mathrm{c}\gamma) : \gamma \in \varphi^{-1}(\mathrm{c}\mu), \gamma \in \mathrm{L}_1\}$$
$$\geq \mathrm{Sup}\{\tau_W^i(\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in \mathrm{L}_1\} = \varphi(\tau_W^i)(\mu)$$

Similarly, for all $\mathrm{j} = 1, 2, \ldots, \mathrm{m}$ and $\mathrm{k} = 1, 2, 3, \ldots, \mathrm{m}$, we can we prove that

$$\varphi(\lambda_W^j)(\mathrm{c}\mu) = \mathrm{Sup}\{\lambda_W^j(\mathrm{c}\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in \mathrm{L}_1\} \geq \mathrm{Sup}\{\lambda_W^j(\mathrm{c}\gamma) : \gamma \in \varphi^{-1}(\mathrm{c}\mu), \gamma \in \mathrm{L}_1\}$$
$$\geq \mathrm{Sup}\{\lambda_W^j(\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in \mathrm{L}_1\} = \varphi(\lambda_W^j)(\mu)$$
$$\varphi(\eta_W^k)(\mathrm{c}\mu) = \mathrm{Inf}\{\eta_W^k(\mathrm{c}\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in \mathrm{L}_1\} \leq \mathrm{Inf}\{\eta_W^k(\mathrm{c}\gamma) : \gamma \in \varphi^{-1}(\mathrm{c}\mu), \gamma \in \mathrm{L}_1\}$$
$$\leq \mathrm{Inf}\{\eta_W^k(\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in \mathrm{L}_1\} = \varphi(\eta_W^k)(\mu)$$

For, $\mu_1, \mu_2 \in \mathrm{L}_2$, then

$$\{\gamma : \gamma \in \varphi^{-1}(\mu_1 + \mu_2)\} \supseteq \{\gamma_1 + \gamma_2 : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\} .$$

Now, for all $\mathrm{i} = 1, 2, 3, \ldots, \mathrm{l}$, we have,

$$\varphi(\tau_W^i)([\mu_1, \mu_2]) = \mathrm{Sup}\{\tau_W^i(\gamma) : \gamma \in \varphi^{-1}([\mu_1, \mu_2]), \gamma \in \mathrm{L}_1\}$$
$$\geq \mathrm{Sup}\{\tau_W^i([\gamma_1, \gamma_2]) : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$$
$$\geq \mathrm{Sup}\{\mathrm{Min}\{\tau_W^i(\gamma_1), \tau_W^i(\gamma_2)\} : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$$
$$= \mathrm{Min}\{\mathrm{Sup}\{\tau_W^i(\gamma_1) : \gamma_1 \in \varphi^{-1}(\mu_1)\}, \mathrm{Sup}\{\tau_W^i(\gamma_2) : \gamma_2 \in \varphi^{-1}(\mu_2)\}$$
$$= \mathrm{Min}\{\varphi(\tau_W^i)(\mu_1), \varphi(\tau_W^i)(\mu_2)\}$$

Similarly, for all $\mathrm{j} = 1, 2, \ldots, \mathrm{m}$ and $\mathrm{k} = 1, 2, 3, \ldots, \mathrm{m}$, we can we prove that

$$\varphi(\lambda_W^j)([\mu_1, \mu_2]) = \mathrm{Sup}\{\lambda_W^j(\gamma) : \gamma \in \varphi^{-1}([\mu_1, \mu_2]), \gamma \in \mathrm{L}_1\}$$
$$\geq \mathrm{Sup}\{\lambda_W^j([\gamma_1, \gamma_2]) : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$$
$$\geq \mathrm{Sup}\{\mathrm{Min}\{\lambda_W^j(\gamma_1), \lambda_W^j(\gamma_2)\} : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$$

$= \text{Min}\{\text{Sup}\{\lambda_W^j(\gamma_1) : \gamma_1 \in \varphi^{-1}(\mu_1)\}, \text{Sup}\{\lambda_W^j(\gamma_2) : \gamma_2 \in \varphi^{-1}(\mu_2)\}\}$

$= \text{Min}\{\varphi(\lambda_W^j)(\mu_1), \varphi(\lambda_W^j)(\mu_2)\}$

$\varphi(\eta_W^k)([\mu_1, \mu_2]) = \text{Sup}\{\eta_W^k(\gamma) : \gamma \in \varphi^{-1}([\mu_1, \mu_2]), \gamma \in \text{L}_1\}$

$\geq \text{Sup}\{\eta_W^k([\gamma_1, \gamma_2])\ \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$

$\geq \text{Sup}\{\text{Min}\{\eta_W^k(\gamma_1), \tau_W^k(\gamma_2)\} : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$

$= \text{Min}\{\text{Sup}\{\eta_W^k(\gamma_1) : \gamma_1 \in \varphi^{-1}(\mu_1)\}, \text{Sup}\{\eta_W^i(\gamma_2) : \gamma_2 \in \varphi^{-1}(\mu_2)\}\}$

$= \text{Min}\{\varphi(\eta_W^k)(\mu_1), \varphi(\eta_W^k)(\mu_2)\}$.

Hence, $\varphi(\text{W})$ is multi spherical fuzzy Lie subalgebras of L_2.

□

Theorem 4.4: *Let* $\text{W} = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \lambda_W^1, \lambda_W^2, \ldots., \lambda_W^m; \eta_W^1, \eta_W^2, \ldots, \eta_W^n) \in \text{I}^{L_1}$ *be multispherical fuzzy Lie ideal and* $\varphi : \text{L}_1 \to \text{L}_2$ *be Lie homomorphism between* L_1 *and* L_2. *Then* $\varphi(\text{W})$ *is multi spherical fuzzy Lie ideal of* L_2.

Proof: The proof is similar to the proof of theorem (4.3) □

Theorem 4.5: *Let* $\text{W} = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \lambda_W^1, \lambda_W^2, \ldots., \lambda_W^m; \eta_W^1, \eta_W^2, \ldots, \eta_W^n) \in \text{I}^{L_2}$ *be multi spherical fuzzy Lie subalgebras and* $\varphi : \text{L}_1 \to \text{L}_2$ *be Lie homomorphism between* L_1 *and* L_2.

Then $\varphi^{-1}(W)$ is multi spherical fuzzy Lie subalgebra of L_1.

Proof: Suppose that $\mu_1, \mu_2 \in \text{L}_1$. Now, for all $\text{i} = 1, 2, 3, \ldots, \text{l}$; $\text{j} = 1, 2, 3, \ldots, \text{m}$; $\text{k} = 1, 2, 3, \ldots, \text{n}$; we have,

$\varphi^{-1}(\tau_W^i)(\mu_1 + \mu_2) = \tau_W^i[\varphi(\mu_1 + \mu_2)] = \tau_W^i[\varphi(\mu_1) + \varphi(\mu_2)]$

$\geq \operatorname{Min}\{\tau_W^i(\varphi(\mu_1)), \tau_W^i(\varphi(\mu_2))\}$

$= \operatorname{Min}\{\varphi^{-1}(\tau_W^i)(\mu_1), \varphi^{-1}(\tau_W^i)(\mu_2)\}$

$\varphi^{-1}(\lambda_W^j)(\mu_1 + \mu_2) = \lambda_W^j[\varphi(\mu_1 + \mu_2)] = \lambda_W^j[\varphi(\mu_1) + \varphi(\mu_2)]$

$\geq \operatorname{Min}\{\lambda_W^j(\varphi(\mu_1)), \lambda_W^j(\varphi(\mu_2))\}$

$= \operatorname{Min}\{\varphi^{-1}(\lambda_W^j)(\mu_1), \varphi^{-1}(\lambda_W^j)(\mu_2)\}$

$\varphi^{-1}(\eta_W^k)(\mu_1 + \mu_2) = \eta_W^k[\varphi(\mu_1 + \mu_2)] = \eta_W^k[\varphi(\mu_1) + \varphi(\mu_2)]$

$\leq \operatorname{Max}\{\eta_W^k(\varphi(\mu_1)), \eta_W^k(\varphi(\mu_2))\}$

$= \operatorname{Max}\{\varphi^{-1}(\eta_W^k)(\mu_1), \varphi^{-1}(\eta_W^k)(\mu_2)\}$

For all $\mu \in \mathrm{L}_1$ and $\mathrm{c} \in \mathcal{F}$, we have,

$\varphi^{-1}(\tau_W^i)(\mathrm{c}\mu) = \tau_W^i[\varphi(\mathrm{c}\mu)] = \tau_W^i[\mathrm{c}\varphi(\mu)] \geq \tau_W^i(\varphi(\mu)) = \varphi^{-1}(\tau_W^i)(\mu)$

$\varphi^{-1}(\lambda_W^i)(\mathrm{c}\mu) = \lambda_W^i[\varphi(\mathrm{c}\mu) = \lambda_W^i[\mathrm{c}\varphi(\mu)] \geq \lambda_W^i(\varphi(\mu)) = \varphi^{-1}(\lambda_W^i)(\mu)$

$\varphi^{-1}(\eta_W^i)(\mathrm{c}\mu) = \eta_W^i[\varphi(\mathrm{c}\mu)] = \eta_W^i[\mathrm{c}\varphi(\mu)] \leq \eta_W^i(\varphi(\mu)) = \varphi^{-1}(\eta_W^i)(\mu)$

For all $\mu_1, \mu_2 \in \mathrm{L}_1$.

$\varphi^{-1}(\tau_W^i)[\mu_1, \mu_2] = \tau_W^i(\varphi[\mu_1, \mu_2]) = \tau_W^i[\varphi(\mu_1),\varphi(\mu_2)]$

$\geq \operatorname{Min}\{\tau_W^i(\varphi(\mu_1)), \tau_W^i(\varphi(\mu_2))\}$

$= \operatorname{Min}\{\varphi^{-1}(\tau_W^i)(\mu_1), \varphi^{-1}(\tau_W^i)(\mu_2)\}$

$\varphi^{-1}(\lambda_W^i)[\mu_1, \mu_2] = \lambda_W^i(\varphi[\mu_1, \mu_2]) = \lambda_W^i[\varphi(\mu_1), \varphi(\mu_2)]$

$\geq \operatorname{Min}\{\lambda_W^i(\varphi(\mu_1)), \lambda_W^i(\varphi(\mu_2))\}$

$= \operatorname{Min}\{\varphi^{-1}(\lambda_W^i)(\mu_1), \varphi^{-1}(\lambda_W^i)(\mu_2)\}$

$\varphi^{-1}(\eta_W^i)[\mu_1, \mu_2] = \eta_W^i(\varphi[\mu_1, \mu_2]) = \eta_W^i[\varphi(\mu_1), \varphi(\mu_2)]$

$\leq \operatorname{Max}\{\eta_W^i(\varphi(\mu_1)), \eta_W^i(\varphi(\mu_2))\}$

$= \operatorname{Max}\{\varphi^{-1}(\eta_W^i)(\mu_1), \varphi^{-1}(\eta_W^i)(\mu_2)\}$

Hence, $\varphi^{-1}(\mathrm{W})$ is multi spherical fuzzy Lie subalgebra of L_1. □

Theorem 4.6: *Let* $\mathrm{W}=(\tau_W^1,\tau_W^2,\ldots,\tau_W^l;\lambda_W^1,\lambda_W^2,\ldots,\lambda_W^m;\eta_W^1,\eta_W^2,\ldots,\eta_W^n)\in \mathrm{I}^{L_2}$ *be multi spherical fuzzy Lie ideal and* $\varphi:\mathrm{L}_1\to\mathrm{L}_2$ *be Lie homomorphism between* L_1 *and* L_2*. Then* $\varphi^{-1}(\mathrm{W})$ *is multi spherical fuzzy Lie ideal of* L_1.

Proof: The proof is similar to the proof of theorem (4.5) □

5. Conclusion

Multi spherical fuzzy set theory is the generalization of Multi Pythagorean fuzzy set theory as well as spherical fuzzy set theory. In this paper we introduce the concept of multi spherical fuzzy Lie subalgebras and multi spherical fuzzy Lie ideals of Lie algebra. Some of their fundamental properties and basic operations are investigated. Moreover, the relationship between multi spherical fuzzy Lie subalgebras and multi spherical fuzzy Lie ideals are established. Lastly, the image and inverse image of multi spherical fuzzy Lie subalgebras (multi neutrosophic Lie ideals) under Lie homomorphisms are also studied. In future we shall investigate more properties of multi spherical fuzzy Lie subalgebras and Lie ideals. The outcome of the proposed work is applicable in multicriterion decision making problem, pattern recognition medical diagnosis and classification problems etc.

7. Acknowledgement

The author would like to thank the anonymous reviewer for their careful reading of this research paper and for their valuable suggestions toward the improvement of the article.

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(*Received*, September 16, 2024)
(*Revised*, October 19, 2024)

Journal of Indian Acad. Math. ISSN: 0970-5120
Vol. 46, No. 2 (2024) pp. 195-206

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A FAMILY OF GIBONACCI SUMS: GENERALIZATIONS AND CONSEQUENCES

Abstract: We explore a generalization of an infinite sum involving a large family of gibonacci polynomial squares and its Pell and Jacobsthal consequences.

Keywords: Gibonacci Polynomials, Pell Polynomials, Jacobsthal Polynomial.

Mathematics Subject Classification (2020) No.: Primary 11B37, 11B39, 11C08.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*.

They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 3].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively [3]. Pell numbers P_n and Pell-Lucas numbers Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [3].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$ and $j_n(1) = L_n$ [2, 3, 6].

Gibonacci and Jacobsthal polynomials are linked by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [3, 6].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, and $b_n = p_n$ or q_n; $\Delta = \sqrt{x^2 + 4}$ and $2\alpha = x + \Delta$. Gibonacci and Pell polynomials are linked by the relationship $b_n(x) = g_n(2x)$.

1.1 Fundamental Gibonacci Identities: Gibonacci polynomials satisfy the following properties [3, 5, 7]:

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1} f_k^2, & \text{if } g_n = f_n \\ (-1)^{n+k} \Delta^2 f_k^2, & \text{otherwise;} \end{cases} \tag{1}$$

$$g_{n+k+r}g_{n-k} - g_{n+k}g_{n-k+r} = \begin{cases} (-1)^{n+k+1} f_r f_{2k}, & \text{if } g_n = f_n \\ (-1)^{n+k} \Delta^2 f_r f_{2k}, & \text{otherwise;} \end{cases} \tag{2}$$

$$g_{n+k+r}g_{n-k} + g_{n+k}g_{n-k+r} = \begin{cases} \frac{1}{\Delta^2}[2l_{2n+r} - (-1)^{n+k} l_{2k} l_r], & \text{if } g_n = f_n \\ 2l_{2n+r} + (-1)^{n+k} l_{2k} l_r, & \text{otherwise,} \end{cases} \tag{3}$$

where k and r are positive integers. These properties can be confirmed using the Binet-like formulas.

Identity (3) is a polynomial extension of *d'Ocagne identity* [3, 5].

Consequently, we have

$$g_{n+k+r}^2\ g_{n-k}^2 - g_{n+k}^2 g_{n-k+r}^2 = \begin{cases} \frac{(-1)^{n+k+1}}{\Delta^2}[2l_{2n+r} - (-1)^{n+k} l_{2k} l_r] f_{2k} f_r, & \text{if } g_n = f_n \\ (-1)^{n+k} \Delta^2 [2l_{2n+r} + (-1)^{n+k} l_{2k} l_r] f_{2k} f_r, & \text{otherwise.} \end{cases} \tag{4}$$

Again, in the interest of brevity and clarity, we now let

$$A = 2l_{2(2pn+t-p)k+r} - (-1)^{tk} l_{2pk} l_r; \text{ and } B = 2l_{2(2pn+t-p)k+r} + (-1)^{tk} l_{2pk} l_r .$$

It then follows by identities (1) and (4) that [7]

$$g_{(2pn+t)k} g_{(2pn+t-2p)k} - g_{(2pn+t-p)k}^2 = \begin{cases} (-1)^{tk+1} f_{pk}^2, & \text{if } g_n = f_n \\ (-1)^{tk} \Delta^2 f_{pk}^2, & \text{otherwise;} \end{cases} \tag{5}$$

$$g_{(2pn+t)k+r}^2 g_{(2pn+t-2p)k}^2 - g_{(2pn+t)k}^2 g_{(2pn+t-2p)k+r}^2 = \begin{cases} \frac{(-1)^{tk+1}}{\Delta^2} A f_{2pk} f_r, & \text{if } g_n = f_n \\ (-1)^{tk} \Delta^2 B f_{2pk} f_r, & \text{otherwise,} \end{cases} \tag{6}$$

where k, p, r, and t are positive integers, and $t \leq 2p$.

2. A Telescoping Gibonacci Sum

With recursion, we established the following telescoping gibonacci sum in [7]. For the sake of conciseness and expediency, we omit its proof here.

Lemma 1: *Let k, p, r, t, and λ be positive integers, where $t \leq 2p$. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g_{(2pn+t-2p)k+r}^{\lambda}}{g_{(2pn+t-2p)k}^{\lambda}} - \frac{g_{(2pn+t)k+r}^{\lambda}}{g_{(2pn+t)k}^{\lambda}} \right] = \frac{g_{tk+r}^{\lambda}}{g_{tk}^{\lambda}} - \alpha^{\lambda r} . \tag{7}$$

3. Gibonacci Sums

Coupled with identities (5) and (6), Lemma 1 with $\lambda = 2$ plays a major role in our explorations. To this end, in the interest of brevity, we first let

$$\mu = \begin{cases} 1, & \text{if } g_n = f_n \\ \Delta^2, & \text{otherwise;} \end{cases} \qquad \mu^* = \begin{cases} \frac{1}{\Delta^2} & \text{if } g_n = f_n \\ \Delta^2, & \text{otherwise;} \end{cases}$$

$$\nu = \begin{cases} -1, & \text{if } g_n = f_n \\ 1, & \text{otherwise;} \end{cases} \quad \text{and} \quad \nu = \begin{cases} 1, & \text{if } g_n = f_n \\ -1, & \text{otherwise.} \end{cases}$$

With these tools at our finger tips, we now showcase the main result in the following theorem.

Theorem 1: *Let k, p, r, and t be positive integers, where $t \leq 2p$. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk}\mu^*\nu^*[2l_{2(2pn+t-p)k+r} + (-1)^{tk}\nu l_{2pk}l_r]f_{2pk}f_r}{[g^2_{(2pn+t-p)k} + (-1)^{tk}\mu\nu f^2_{pk}]} = \frac{g^2_{tk+r}}{g^2_{tk}} - \alpha^{2r}. \quad (8)$$

Proof: Suppose $g_n = f_n$. With identities (5) and (6), Lemma 1 then yields

$$\frac{(-1)^{tk+1}Af_{2pk}f_r}{\Delta^2[f^2_{(2pn+t-p)k} - (-1)^{tk}f^2_{pk}]^2} = \frac{f^2_{(2pn+t)k+r}f^2_{(2pn+t-2p)k} - f^2_{(2pn+t)k}f^2_{(2pn+t-2p)k+r}}{f^2_{(2pn+t)k}f^2_{(2pn+t-2p)k+r}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk}Af_{2pk}f_r}{\Delta^2[f^2_{(2pn+t-p)k} - (-1)^{tk}f^2_{pk}]^2} = \sum_{n=1}^{\infty}\left[\frac{f^2_{(2pn+t-2p)k+r}}{f^2_{(2pn+t-2p)k}} - \frac{f^2_{(2pn+t)k+r}}{f^2_{(2pn+t)k}}\right]$$

$$= \frac{f^2_{tk+r}}{f^2_{tk}} - \alpha^{2r}$$

On the flip side, suppose $g_n = l_n$. It then follows by the same two identities and Lemma 1 that

$$\frac{(-1)^{tk+1}\Delta^2 B f_{2pk} f_r}{[l^2_{(2pn+t-p)k} + (-1)^{tk}\Delta^2 f^2_{pk}]^2} = \frac{l^2_{(2pn+t)k} l^2_{(2pn+t-2p)k+r} - l^2_{(2pn+t)k+r} l^2_{(2pn+t-2p)k}}{l^2_{(2pn+t)k} l^2_{(2pn+t-2p)k}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1}\Delta^2 B f_{2pk} f_r}{[l^2_{(2pn+t-p)k} + (-1)^{tk}\Delta^2 f^2_{pk}]^2} = \sum_{n=1}^{\infty} \left[\frac{l^2_{(2pn+t-2p)k+r}}{l^2_{(2pn+t-2p)k}} - \frac{l^2_{(2pn+t)k+r}}{l^2_{(2pn+t)k}} \right]$$

$$= \frac{l^2_{tk+r}}{l^2_{tk}} - \alpha^{2r}.$$

Combining the two cases yields the given result, as desired. □

In particular, the theorem yields

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk}[2L_{2(2pn+t-p)k+r} - (-1)^{tk} L_{2pk} L_r] F_{2pk} F_r}{5[F^2_{(2pn+t-p)k} - (-1)^{tk} F^2_{pk}]^2} = \frac{F^2_{tk+r}}{F^2_{tk}} - \alpha^{2r}; \tag{9}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} 5[2L_{2(2pn+t-p)k+r} + (-1)^{tk} L_{2pk} L_r] F_{2pk} F_r}{[F^2_{(2pn+t-p)k} + (-1)^{tk} F^2_{pk}]^2} = \frac{L^2_{tk+r}}{L^2_{tk}} - \alpha^{2r}. \tag{10}$$

With $p \le 2$, $k \le 3$, and $r = 1 = t$, it then follows that

$$\sum_{n=1}^{\infty} \frac{2L_{4n+1} + 3}{(F^2_{2n} + 1)^2} = \frac{5}{2} + \frac{5\sqrt{5}}{2}; \qquad \sum_{n=1}^{\infty} \frac{2L_{4n+1} - 3}{(L^2_{2n} - 5)^2} = \frac{3}{2} - \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{2L_{8n+1} - 7}{(F^2_{4n} - 1)^2} = \frac{25}{6} - \frac{5\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{2L_{8n+1} + 7}{(L^2_{4n} + 5)^2} = -\frac{1}{54} + \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{2L_{12n+1} + 9}{(F^2_{6n} + 4)^2} = -\frac{15}{64} + \frac{5\sqrt{5}}{32}; \qquad \sum_{n=1}^{\infty} \frac{2L_{12n+1} - 9}{(L^2_{6n} - 20)^2} = \frac{5}{256} - \frac{\sqrt{5}}{160};$$

$$\sum_{n=1}^{\infty} \frac{2L_{8n-1}+7}{(F_{4n-1}^2+1)^2} = \frac{5}{6}+\frac{5\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{2L_{8n-1}-7}{(L_{4n-1}^2-5)^2} = \frac{1}{2}-\frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{2L_{16n-3}-47}{(F_{8n-2}^2-9)^2} = \frac{25}{42}-\frac{5\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{2L_{16n-3}+47}{(L_{8n-2}^2+45)^2} = -\frac{1}{378}+\frac{\sqrt{5}}{210};$$

$$\sum_{n=1}^{\infty} \frac{2L_{24n-5}+161}{(F_{12n-3}^2+64)^2} = -\frac{5}{384}+\frac{5\sqrt{5}}{576}; \qquad \sum_{n=1}^{\infty} \frac{2L_{24n-5}-161}{(L_{12n-3}^2-320)^2} = \frac{5}{4,608}-\frac{\sqrt{5}}{2,880}.$$

We now encourage gibonacci enthusiasts to explore the corresponding sums with $p, k, r, t \le 3$.

Next, we explore the Pell version of Theorem 1, and its implications.

4. Pell Version and Consequences

With the Pell-gibonacci relationship $b_n(x) = g_n(2x)$ and letting $E = \sqrt{x^2+1}$;

$A^* = 2q_{2(2pn+t-p)k+r} - (-1)^{tk} q_{2pk}q_r$; and $B^* = 2q_{2(2pn+t-p)k+r} + (-1)^{tk} q_{2pk}q_r$.

Theorem 1 yields the following the Pell versions:

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} A^* p_{2pk} p_r}{4E^2[p_{(2pn+t-p)k}^2 - (-1)^{tk} p_{pk}^2]^2} = \frac{p_{tk+r}^2}{p_{tk}^2} - \gamma^{2r};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} 4E^2 B^* p_{2pk} p_r}{[q_{(2pn+t-p)k}^2 + (-1)^{tk} 4E^2 p_{pk}^2]^2} = \frac{q_{tk+r}^2}{q_{tk}^2} - \gamma^{2r},$$

where $\gamma = x + E$.

Consequently, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk}[Q_{2(2pn+t-p)k+r} - (-1)^{tk} Q_{2pk} Q_r] P_{2pk} P_r}{2[P^2_{(2pn+t-p)k} - (-1)^{tk} P^2_{pk}]^2} = \frac{P^2_{tk+r}}{P^2_{tk}} - \gamma^{2r};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} 2[Q_{2(2pn+t-p)k+r} + (-1)^{tk} Q_{2pk} Q_r] P_{2pk} P_r}{[Q^2_{(2pn+t-p)k} + (-1)^{tk} 2P^2_{pk}]^2} = \frac{Q^2_{tk+r}}{Q^2_{tk}} - \gamma^{2r}.$$

With $p \le 2$ and $k = r = t = 1$, it then follows that

$$\sum_{n=1}^{\infty} \frac{Q_{4n+1} + 3}{(P^2_{2n} + 1)^2} = -1 + 2\sqrt{2}; \qquad \sum_{n=1}^{\infty} \frac{Q_{4n+1} - 3}{(Q^2_{2n} - 2)^2} = 3 - \sqrt{2};$$

$$\sum_{n=1}^{\infty} \frac{Q_{8n-1} + 7}{(P^2_{4n-1} + 4)^2} = -\frac{1}{6} + \frac{\sqrt{2}}{3}; \qquad \sum_{n=1}^{\infty} \frac{Q_{8n-1} - 17}{(Q^2_{4n-1} - 8)^2} = \frac{1}{4} - \frac{\sqrt{2}}{12}.$$

We now pursue the Jacobsthal counterpart of Theorem 1 and its consequences.

5. Jacobsthal Version and Implications

Recall from Section 1 that Jacobsthal and Gibonacci polynomials are linked by the relationships

$$f_n(1/\sqrt{x}) = \frac{J_n(x)}{x^{(n-1)/2}} \quad \text{and} \quad l_n(1/\sqrt{x}) = \frac{j_n(x)}{x^{(n/2)}} \ [3, 4].$$

Again, in the interest of conciseness, clarity, and convenience, we let $c_n = J_n$ or j_n, $D = \sqrt{4x+1}$, and $2\omega = 1 + D$. Then $\alpha(1/\sqrt{x}) = \frac{1+D}{\sqrt{x}} = \frac{\omega}{\sqrt{x}}$ [6].

In addition, we let L denote the fractional expression on the left side of the given gibonacci equation and R that on its right side; and LHS and RHS the left-hand side and right-hand side of the corresponding Jacobsthal equation, as in [6, 4].

With this brief background, we now begin our exploration.

Proof: Case 1. Suppose $g_n = f_n$.

We have $L = \dfrac{(-1)^{tk}[2l_{2(2pn+t-p)k+r} - (-1)^{tk} l_{2pk}l_r]f_{2pk}f_r}{\Delta^2[f^2_{(2pn+t-p)k} - (-1)^{tk} f^2_{pk}]^2}$.

Replacing x with $1/\sqrt{x}$, and A as in Subsection 1.1, we get

$$f_{2pk}f_r = \frac{J_{2pk}J_r}{x^{(2pk+r)/2-1}};$$

$$l_{2pk}l_r = \frac{j_{2pk}j_r}{x^{(2pk+r)/2}};$$

$$2l_{2(2pn+t-p)k+r} = \frac{2j_{2(2pn+t-p)k+r}}{x^{(2pn+t-p)k+r/2}};$$

$$Af_{2pk}f_r = [2j_{2(2pn+t-p)k+r} - (-1)^{tk}x^{(2pn+t-2p)k}j_{2pk}j_r]J_{2pk}J_r;$$

$$\Delta^2[f^2_{(2pn+t-p)k} - (-1)^{tk} f^2_{pk}]^2 = \frac{D^2[J^2_{(2pn+t-p)k}] - (-1)^{tk}x^{(2pn+t-2p)k}J^2_{pk}}{x^{2(2pn+t-p)k-1}},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Consequently, we have

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{(-1)^{tk}x^{(2pn+t-2p)k}[2j_{2(2pn+t-p)k+r} - (-1)^{tk}x^{(2pn+t-2p)k}j_{2pk}j_r]J_{2pk}J_r}{D^2x^r[J^2_{(2pn+t-p)k} - (-1)^{tk}x^{(2pn+t-2p)k}J^2_{pk}]^2},$$

where $c_n = c_n(x)$.

Now, turn to the right side; we have $R = \dfrac{f^2_{tk+r}}{f^2_{tk}} - \alpha^{2r}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with x^{tk+r-1}, this yields

$$R = \frac{[x^{(tk+r-1)/2} f_{tk+r}]^2}{x^r [x^{(tk-1)/2} f_{tk}]^2} - \frac{\omega^2 r}{x^r};$$

$$\text{RHS} = \frac{J_{tk+r}^2}{x^r J_{tk}^2} - \frac{\omega^{2r}}{x^r},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Combining the two sides, we get the desired Jacobsthal version of equation (8):

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} x^{(2pn+t-2p)k} [2j_{2(2pn+t-p)k+r} - (-1)^{tk} x^{(2pn+t-2p)k} j_{2pk} j_r] J_{2pk} J_r}{D^2 [J_{(2pn+t-p)k}^2 - (-1)^{tk} x^{(2pn+t-2p)k} J_{pk}^2]^2} = \frac{J_{tk+r}^2}{J_{tk}^2} - \omega^{2r}, \tag{11}$$

where $c_n = c_n(x)$.

Next, we pursue the Jacobsthal-Lucas version of theorem 1.

Case 2: With $g_n = l_n$, we have

$$L = \frac{(-1)^{tk+1} \Delta^2 [2l_{2(2pn+t-p)k+r} + (-1)^{tk} l_{2pk} l_r] f_{2pk} f_r}{[l_{(2pn+t-p)k}^2 + (-1)^{tk} \Delta^2 f_{pk}^2]^2}.$$

Again, replace x with $1/\sqrt{x}$; as above, we then get

$$Bf_{2pk} f_r = \frac{D^2 [2j_{2(2pn+t-p)k+r} + (-1)^{tk} x^{(2pn+t-2p)k} j_{2pk} j_r] J_{2pk} J_r}{x^{(2pn+t)k+r}};$$

$$[l_{(2pn+t-p)k}^2 + (-1)^{tk} \Delta^2 f_{pk}^2]^2 = \frac{[j_{(2pn+t-p)k}^2 + (-1)^{tk} D^2 x^{(2pn+t-2p)k} J_{pk}^2]^2}{x^{2(2pn+t-p)k}},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$. Thus,

$$\text{LHS} = \frac{(-1)^{tk+1} D^2 x^{(2pn+t-2p)k}[2j_{2(2pn+t-p)k+r} + (-1)^{tk} x^{(2pn+t-2p)k} j_{2pk} j_r] J_{2pk} J_r}{x^r [j^2_{(2pn+t-p)k} + (-1)^{tk} D^2 x^{(2pn+t-2p)k} J^2_{pk}]^2},$$

where $c_n = c_n(x)$.

Turning to the right side, we have $R = \frac{l^2_{tk+r}}{l^2_{tk}} - \alpha^{2r}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with x^{tk+r} yields

$$R = \frac{[x^{(tk+r)/2} l_{tk+r}]^2}{x^r [x^{tk/2} l_{tk}]^2} - \frac{\omega^{2r}}{x^r};$$

$$\text{RHS} = \frac{j^2_{tk+r}}{x^r j^2_{tk}} - \frac{\omega^{2r}}{x^r},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Merging the two sides gives the Jacobsthal-Lucas version of equation (8):

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} D^2 x^{(2pn+t-2p)k}[2j_{2(2pn+t-p)k+r} + (-1)^{tk} x^{(2pn+t-2p)k} j_{2pk} j_r] J_{2pk} J_r}{[j^2_{(2pn+t-p)k} + (-1)^{tk} D^2 x^{(2pn+t-2p)k} J^2_{pk}]^2} = \frac{j^2_{tk+r}}{j^2_{tk}} - \omega^{2r}, \tag{12}$$

where $c_n = c_n(x)$.

This equation, coupled with formula (11), yields the desired Jacobsthal version of Theorem 1, as in the next theorem. To this end, we modify the definitions of μ, μ^*, ν and ν^* to fit the context:

$$\mu = \begin{cases} 1, & \text{if } c_n = J_n \\ D^2, & \text{otherwise;} \end{cases} \qquad \mu^* = \begin{cases} \frac{1}{D^2} & \text{if } c_n = J_n \\ D^2, & \text{otherwise;} \end{cases}$$

$$\nu = \begin{cases} -1, & \text{if } c_n = J_n \\ 1, & \text{otherwise;} \end{cases} \quad \text{and} \quad \nu = \begin{cases} 1, & \text{if } c_n = J_n \\ -1, & \text{otherwise.} \end{cases}$$

We are now ready to showcase the desired Jacobsthal version.

Theorem 2: *Let k, p, r, and t be positive integers, where $t \leq 2p$. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1}\mu^*\nu^* x^{(2pn+t-2p)k}[2j_{2(2pn+t-p)k+r} + (-1)^{tk}\nu x^{(2pn+t-2p)k} j_{2pk}j_r]J_{2pk}J_r}{[c^2_{(2pn+t-p)k} + (-1)^{tk}\mu\nu x^{(2pn+t-2p)k}J^2_{pk}]^2} = \frac{j^2_{tk+r}}{j^2_{tk}} - \omega^{2r}, \tag{13}$$

□

In particular, equations (9) and (10) follow from this. Additionally, with $p, k \leq 2$ and $r = 1 = t$, we get:

$$\sum_{n=1}^{\infty} \frac{2^n(2j_{4n+1} + 5 \cdot 2^{n-1})}{(J^2_{2n} + 2^{n-1})^2} = 27; \qquad \sum_{n=1}^{\infty} \frac{2^n(2j_{4n+1} - 5 \cdot 2^{n-1})}{(j^2_{2n} - 9 \cdot 2^{n-1})^2} = \frac{7}{3};$$

$$\sum_{n=1}^{\infty} \frac{2^{4n-2}(2j_{8n+1} - 17 \cdot 24^{n-2})}{(J^2_{4n} - 2^{4n-2})^2} = 9; \qquad \sum_{n=1}^{\infty} \frac{2^{4n-2}(2j_{8n+1} + 17 \cdot 2^{4n-2})}{(j^2_{4n} + 9 \cdot 2^{4n-2})^2} = \frac{51}{1,125};$$

$$\sum_{n=1}^{\infty} \frac{2^{4n-3}(2j_{8n-1} + 17 \cdot 2^{4n-3})}{(J^2_{4n-1} + 2^{4n-3})^2} = \frac{27}{5}; \qquad \sum_{n=1}^{\infty} \frac{2^{4n-3}(2j_{8n-1} - 17 \cdot 2^{4n-3})}{(j^2_{4n-1} - 9 \cdot 2^{4n-3})^2} = \frac{7}{15};$$

$$\sum_{n=1}^{\infty} \frac{2^{8n-6}(2j_{16n-3} - 257 \cdot 2^{8n-6})}{(J^2_{8n-2} - 25 \cdot 2^{8n-6})^2} = \frac{9}{17}; \qquad \sum_{n=1}^{\infty} \frac{2^{8n-6}(2j_{16n-3} + 257 \cdot 2^{8n-6})}{(j^2_{8n-2} + 225 \cdot 2^{8n-6})^2} = \frac{51}{19,125}.$$

Finally, we encourage giboancci enthusiasts to compute the Jacobsthal sums with $p, k, r, t \in \{2, 3\}$.

6. Acknowledgment

The author would like to thank Z. Gao for a careful reading of the article and for his computational confirmation.

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(*Received*, September 25, 2024)

Journal of Indian Acad. Math.
Vol. 46, No. 2 (2024) pp. 207-223
ISSN: 0970-5120

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HARDY-ROGERS-TYPE MAPPINGS IN RELATIONAL METRIC SPACES AND FIXED POINT THEOREMS

Abstract: This work aims to present an analogous version of the fixed point theorem of Hardy and Rogers in relational metric spaces. We extend the notions of several mappings and some well-known classical fixed point theorems in the settings of relational metric spaces. Some examples are also provided which illustrate the results. This investigation offers generalization and unification of several known results.

Keywords: Metric Space; Relation-Theoretic Contraction Principle; Fixed Point.

Mathematics Subject Classification (2000) No.: 47H10, 54H25.

1. Introduction

Fixed point theory, of course, entails the search for a combination of conditions on a set X and a mapping $T : X \to X$ which in turn assures that T leaves at least one point of X fixed, i.e. $x = Tx$ for some $x \in X$. Banach [2] proved the famous Banach contraction principle or Banach's fixed point theorem in 1922. Banach's fixed point result has laid down the foundation of modern fixed point theory for Banach contractions in the metric and Banach spaces. A drawback of Banach contraction is that it is continuous (indeed, uniformly continuous) and cannot characterize the completeness of the space. These drawbacks were considered by Kannan [6, 7], and he introduced a new type of mapping known as Kannan type mapping. Such mappings are not necessarily continuous as shown by Kannan. Subrahmanyam [16] showed that the completeness of metric spaces can be characterized by Kannan mappings. Reich [13, 14] introduced a new type of mappings called the Reich type mapping and generalized and unified the notions of Banach contractions and Kannan type mappings. Chatterjea [4] introduced another type of mapping which is independent of the mappings considered by Banach [2] and Kannan [6, 7]. Hardy and Rogers [5] established a generalization of the fixed point theorem of Reich. The mappings introduced by Hardy and Rogers [5] were a unification and generalization of the classes of mappings considered by Banach [2], Kannan [6, 7], Reich [13, 14], and Chatterjea [4].

In 2003, Ran and Reurings [12] introduced the fixed point theory in metric spaces endowed with a partial order and proved an analogue of Banach's fixed point theorem in partially ordered sets. They found uses of such results in finding the solutions of linear and nonlinear matrix equations. Very recently, Alam and Imdad [1] introduced a novel variant of Banach's fixed point theorem on a complete metric space endowed with a binary relation which under universal relation reduces into the Banach contraction principle. Further, they derived the results of Ran and Reurings [12], Nieto, R. Rodríguez-López [10, 11], and Turinici [17, 19] from the relation-theoretic contraction principle.

In this work, we have introduced the notion of relation theoretic Hardy-Rogers- type contractions and proved some fixed point results for such mappings. These results generalize, unify, and extend the results of Banach [2], Kannan [6, 7], Chatterjea [4], Reich [13, 14], Hardy and Rogers [5] and Alam and Imdad [1] in relational metric spaces.

2. Preliminaries

This section presents basic definitions, propositions, and relevant necessary relation theoretic analogues of standard metrical notions.

Definition 1 (Hardy-Rogers-type mapping): Let (X, d) be a metric space and $T : X \to X$ be a mapping. The mapping T is called a Hardy-Rogers-type mapping, if there exist nonnegative constants a_i satisfying $\sum_{i=1}^{5} a_i < 1$ such that:

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx) \qquad (1)$$

for all $x, y \in X$. Without loss of generality, we may assume that $a_2 = a_3$ and $a_4 = a_5$ (see [5]). Therefore, the condition (1) is equivalent to the following condition:

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2[d(x, Tx) + d(y, Ty)] + a_3[d(x, Ty) + d(y, Tx)] \quad (2)$$

for all $x, y \in X$, where $a_1 + 2a_2 + 2a_3 < 1$.

Definition 2 (see [9]): Let X be a nonempty set. A subset $\mathcal{R}$ of $X \times X$ is called a binary relation on X. Notice that for each pair $x, y \in X$ one of the following conditions holds:

1. $(x, y) \in \mathcal{R}$ which amounts $\mathcal{R}$ to saying that "x is $\mathcal{R}$-related to y" or "x relates to y under $\mathcal{R}$ ". Sometimes, we write $x\mathcal{R}y$ instead of $(x, y) \in \mathcal{R}$.

2. $(x, y) \notin \mathcal{R}$ which means that "x is not $\mathcal{R}$-related to y" or "x does not relate to y under $\mathcal{R}$ ".

Trivially, $X \times X$ and being subsets of $X \times X$ are binary relations on X, which are respectively called the universal relation (or full relation) and empty relation. Another important relation of this kind is the relation:

$$\Delta_X = \{(x,x) : x \in X\}$$

and called the identity relation or the diagonal relation on X. Throughout the discussion, $\mathcal{R}$ stands for a nonempty binary relation, but for the sake of simplicity, we write only "binary relation" instead of "nonempty binary relation".

Definition 3 (see [1]): Let $\mathcal{R}$ be a binary relation defined on a nonempty set X and $x, y \in X$. We say that x and y are $\mathcal{R}$-comparative if either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$. We denote it by $[x, y] \in \mathcal{R}$.

Definition 4 (see [9]): Let X be a nonempty set and $\mathcal{R}$ a binary relation on X.

1. The inverse, transpose, or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is defined by

$$\mathcal{R}^{-1} = \{(x, y) \in X \times X : (y, x) \in \mathcal{R}\}.$$

2. The reflexive closure of $\mathcal{R}$, denoted by $\mathcal{R}^{\#}$, is defined to be the set $\mathcal{R} \cup \Delta_X$ (i.e., $\mathcal{R}^{\#} := \mathcal{R} \cup \Delta_X$). Indeed $\mathcal{R}^{\#}$ is the smallest reflexive relation on X containing $\mathcal{R}$.

3. The symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}_s$, is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$ (i.e., $\mathcal{R}_s := \mathcal{R} \cup \mathcal{R}^{-1}$). Indeed $\mathcal{R}_s$ is the smallest symmetric relation on X containing $\mathcal{R}$.

Proposition 1 (see [1]): For a binary relation $\mathcal{R}$ defined on a nonempty set X

$$(x, y) \in \mathcal{R}_s \Leftrightarrow [x, y] \in \mathcal{R}.$$

Definition 5 (see [1]): Let X be a nonempty set and $\mathcal{R}$ a binary relation on X.

(a) A sequence $\{x_n\}$ in X is called $\mathcal{R}$-preserving if $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}$.

(b) If $T : X \to X$ is a mapping, then $\mathcal{R}$ is called T-closed if for every $(x, y) \in \mathcal{R}$, we have $(Tx, Ty) \in \mathcal{R}$.

Proposition 2 (see [1]): Let X, T, and $\mathcal{R}$ be the same as in Definition 5. If $\mathcal{R}$ is T-closed, then $\mathcal{R}_s$ is also T-closed.

If (X, d) is a metric space such that $\mathcal{R}$ is a binary relation on X, then we say that the space X is a relational space with distance d and relation $\mathcal{R}$.

Definition 6 (see [1]): Let (X, d) be a metric space with a binary relation $\mathcal{R}$. Then (X, d) is called $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in X, converges to a point in X.

Every complete metric space is $\mathcal{R}$-complete for any binary relation

$\mathcal{R}$. Particularly, under the universal relation the notion of $\mathcal{R}$-completeness coincides with the usual completeness.

Definition 7 (see [1]): Let (X, d) be a metric space with a binary relation $\mathcal{R}$ and $x \in X$. A self-mapping T on X is called $\mathcal{R}$-continuous at x if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n \to x$, we have $Tx_n \to Tx$. Moreover, T is called $\mathcal{R}$-continuous on X if it is $\mathcal{R}$-continuous at each point of X. The relation $\mathcal{R}$ is called d-self-closed if whenever $\{x_n\}$ is an $\mathcal{R}$-preserving sequence and $x_n \to x$ then there exists a subsequence $\{x_{n_k}\}$ such that $[x_{n_k}, x] \in \mathcal{R}$ for all $k \in \mathbb{N}$.

The sequence $\{x_n\}$ is called $\mathcal{R}$-convergent to x if $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}$ and $x_n \to x$ implies that $(x_n, x) \in \mathcal{R}_s$.

Every continuous mapping is $\mathcal{R}$-continuous for any binary relation $\mathcal{R}$. Particularly, under the universal relation the notion of $\mathcal{R}$-continuity coincides with the usual continuity.

Definition 8 (see [8, 15]): Let X be a nonempty set and $\mathcal{R}$ a binary relation on X.

(a) A subset E of X is called $\mathcal{R}$-directed if for each $x, y \in E$, there exists $z \in X$ such that $(x, z) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$.

(b) For $x, y \in X$, a path of length r (where r is a natural number) in $\mathcal{R}$ from x to y is a finite sequence $\{z_i\}_{i=0}^{r} \subset X$ satisfying the following conditions:

(i) $z_0 = x$ and $z_r = y$;

(ii) $(z_i, z_{i+1}) \in \mathcal{R}$ for each $0 \le i \le r-1$.

(c) A subset E of X is called $\mathcal{R}$-connected if for each pair $x, y \in E$, there exists a path in $\mathcal{R}$ from x to y.

Following [1] we use the following notation:

(i) $F(T) =$ the set of all fixed points of T;

(ii) $X(T,\mathcal{R}) := \{x \in X : (x,Tx) \in \mathcal{R}\}$.

In the next section, we state the findings of this work.

3. Main results

In this section, first, we define the generalized relation-theoretic Hardy-Rogers-type mappings and prove the existence and uniqueness of fixed points of such types of mappings in relational metric spaces.

Definition 9 (Relation theoretic Hardy-Rogers-type mapping): Let X denote a relational metric space with distance function d and relation $\mathcal{R}$, and $T : X \to X$ be a mapping. The mapping T is called a relation theoretic Hardy-Rogers-type mapping if there exist nonnegative constants a_i satisfying $a_1 + 2a_2 + 2a_3 < 1$ such that

$$d(Tx, Ty) \le a_1 d(x,y) + a_2[d(x,Tx) + d(y,Ty)] + a_3[d(x,Ty) + d(y,Tx)] \quad (3)$$

for all $x, y \in X$ with $(x,y) \in \mathcal{R}$.

Relation theoretic Hardy-Rogers-type mappings are weaker than the Hardy-Rogers-type mappings in the sense that relation theoretic Hardy-Rogers-type mappings satisfies the contractive condition (2) for only those pairs $(x,y) \in X \times X$ which are members of $\mathcal{R}$. In contrast, Hardy-Rogers-type mapping must satisfy the contractive condition for all the pairs $(x,y) \in X \times X$.

Example 1: Let $X = \mathbb{R}$ be the metric space with ordinary distance and define a binary relation $\mathcal{R}$ on X by $\mathcal{R} = \{(x,y) : x, y \in \mathbb{Q} \cap [0,\infty)\}$, where $\mathbb{Q}$ is the set of all rational numbers. Define a mapping $T : X \to X$ by:

$$Tx = \begin{cases} \dfrac{x}{2} & \text{if} \quad x \in \mathbb{Q} \cap [0,\infty); \\ 1, & \text{otherwise.} \end{cases}$$

Then T is not a Hardy-Rogers-type mapping, for example at $x = 1$ and $y = 1 + \frac{\pi}{n}$ (where $n \in \mathbb{N}$) and for large enough n the contractive condition (2) fails. At the same time, it is easy to see that T is relation theoretic Hardy-Rogers-type mapping with $a_1 = \frac{1}{2}$ and arbitrary a_2, a_3 such that $0 < a_2 + a_3 < \frac{1}{4}$.

The following theorem is an existence result for a relation theoretic Hardy-Roger-type mapping.

Theorem 1: Let (X, d) be a metric space, $\mathcal{R}$ a binary relation on X such that (X, d) is $\mathcal{R}$-complete. Let $T : X \to X$ be a relation theoretic Hardy-Rogers-type mapping and the following conditions hold:

1. $X(T, \mathcal{R})$ is nonempty;
2. is T-closed;
3. T is $\mathcal{R}$-continuous on X; or $\mathcal{R}$ is d-self-closed.

Then T has a fixed point, i.e., there exists $z \in X$ such that $Tz = z$.

Proof: Because $X(T, \mathcal{R}) \neq \varnothing$, there exists $x_0 \in X(T, \mathcal{R})$, i.e., $(x_0, Tx_0) \in \mathcal{R}$. Define a sequence $\{x_n\}$ of Picard iterates, i.e.

$$x_n = T^n x_0 \text{ for all } n \in \mathbb{N}_0 \tag{4}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the set of whole numbers. Since, $(x_0, Tx_0) \in \mathcal{R}$ using T-closedness of $\mathcal{R}$, we have:

$$(T^n x_0, T^{n+1} x_0) \in \mathcal{R}, \text{ i.e. } (x_n, x_{n+1}) \in \mathcal{R} \text{ for all } n \in \mathbb{N}_0. \tag{5}$$

Thus, the sequence $\{x_n\}$ is $\mathcal{R}$-preserving.

Therefore, using (3) with $x = x_{n-1}$, $y = x_n$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1},\ Tx_n), \\ &\leq a_1 d(x_{n-1}, x_n) + a_2[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \\ &\quad + a_3[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})] \\ &= a_1 d(x_{n-1}, x_n) + a_2[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &\quad + a_3[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]. \end{aligned}$$

This shows that;

$$d(x_n, x_{n+1}) \leq r d(x_{n-1}, x_n) \tag{6}$$

where $r = \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3}$. Since $a_1 + 2a_2 + 2a_3 < 1$, hence, $r < 1$. By induction process:

$$d(x_n, x_{n+1}) \leq r^n d(x_0, x_1) \text{ for all } n \in \mathbb{N}. \tag{7}$$

So, for any positive integers $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq (r^n + r^{n-1} + \cdots + r^{m-1}) d(x_0, x_1) \\ &\leq \frac{r^n}{1 - r}(x_0, x_1). \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is $\mathcal{R}$-complete metric space, there exists $z \in X$ such that:

$$\lim_{n \to \infty} x_n = z. \tag{8}$$

First suppose that T is $\mathcal{R}$-continuous. Then by (4) and (8), we have $x_{n+1} = Tx_n \to Tz$. By uniqueness of the limit, we must have $Tz = z$, i.e. z is a fixed point of T.

Now suppose that $\mathcal{R}$ is d-self-closed, then there exists a subsequence $\{x_{n_k}\}$ such that $[x_{n_k}, x] \in \mathcal{R}$ for all $k \in \mathbb{N}$. If $[x_{n_k}, z] \in \mathcal{R}$ (the proof for the case $(z, x_{n_k}) \in \mathcal{R}$ will be the same), then as T is Hardy-Rogers-type mapping we have:

$$d(x_{n_k+1}, Tz) = d(Tx_{n_k}, Tz)$$

$$\leq a_1 d(x_{n_k}, z) + a_2[d(x_{n_k}, Tx_{n_k}) + d(z, Tz)] + a_3[d(x_{n_k}, Tz) + a_5 d(z, Tx_{n_k})]$$

$$= a_1 d(x_{n_k}, z) + a_2[d(x_{n_k}, x_{n_k+1}) + d(z, Tz)] + a_3[d(x_{n_k}, Tz) + a_5 d(z, x_{n_k+1})].$$

Letting $k \to \infty$ we get:

$$\begin{aligned} d(z, Tz) &\leq a_1 \cdot 0 + a_2[0 + d(z, Tz)] + a_3[d(z, Tz) + 0] \\ &= (a_2 + a_3) d(z, Tz). \end{aligned}$$

Since, $a_2 + a_3 < 1$, we must have $d(z, Tz) = 0$, i.e. $Tz = z$. This proves the result.

Example 2: Let $X_1 = [0, 1/2)$ and $X_2 = [1/2, 1]$, $X = X_1 \cup X_2$ and d is the usual distance defined on X. Define a relation $\mathcal{R}$ on X such that:

$$\mathcal{R} = \{(x, y) : x, y \in X_1 \text{ with } y < x\} \cup \{(x, y) : x, y \in X_2\}.$$

Define a mapping $T : X \to X$ by:

$$Tx = \begin{cases} \dfrac{x}{2} & \text{if } x \in X_1 \\ \dfrac{1}{2} & \text{if } x \in X_2 \end{cases}$$

Then, we note that T is not a Hardy-Rogers-type mapping on X. Indeed, for $x = 0$ and $y = 1/2$ the condition (2) is not satisfied. On the other hand, it is easy to verify that T is a relation theoretic Hardy-Rogers-type mapping on X with

$a_1 = \frac{1}{2}, a_2 = a_3 = \frac{1}{9}$. Also, for any $x_0 \in X_1$ we have $Tx_0 = \frac{x_0}{2} \in X_1$, hence, $(x_0, Tx_0) \in \mathcal{R}$ for all $x_0 \in X_1$ so $X(T,\mathcal{R}) \neq \varnothing$. If $(x,y) \in \mathcal{R}$, then we must have $(x,y) \in X_1 \times X_1, y < x$ or $(x,y) \in X_2 \times X_2$. If $(x,y) \in X_1 \times X_1, y < x$, then by definition of T we have $Tx, Ty \in X_1, Ty < Tx$, i.e. $(Tx, Ty) \in \mathcal{R}$. If $(x,y) \in X_2 \times X_2$, then we have $Tx = Ty = \frac{1}{2}$, so $(Tx, Ty) = (\frac{1}{2}, \frac{1}{2}) \in \mathcal{R}$. Therefore, $\mathcal{R}$ is T-closed. Finally, it is easy to see that T is $\mathcal{R}$-continuous on X. Hence, by Theorem 1, T must have a fixed point. Indeed, T has two fixed points, namely 0 and 1/2.

In the next theorem, we provide a sufficient condition for the uniqueness of the fixed point of a relation theoretic Hardy-Rogers-type mapping.

Theorem 2: Suppose that all the conditions of Theorem 1 are satisfied and that $\Delta_X \subseteq \mathcal{R}$ and $X(T,\mathcal{R})$ is $\mathcal{R}_s$-connected. Then T has a unique fixed point.

Proof: By Theorem 1 we have $F(T) \neq \varnothing$. Let x and y be two fixed points of T, i.e. $x, y \in F(T)$, then for all $n \in \mathbb{N}_0$, we have:

$$T^n x = x, T^n y = y. \tag{9}$$

Since, $\Delta_X \subseteq \mathcal{R}$, hence, $x, y \in X(T,\mathcal{R})$. By $\mathcal{R}_s$-connectedness of $X(T,\mathcal{R})$, there exists a path (say $z_0, z_1, \ldots z_k$) of some finite length k in $\mathcal{R}_s$ from x to y so that:

$$z_0 = x, z_k = y \text{ and } [z_i, z_{i+1}] \in \mathcal{R} \text{ for each } i \in \{0, 1, \ldots, k-1\}. \tag{10}$$

By T-closedness of $\mathcal{R}$ we have

$$[T^n z_i, T^n z_{i+1}] \in \mathcal{R} \text{ for each } i \in \{0, 1, \ldots, k-1\} \text{ and for each } n \in \mathbb{N} \cup \{0\}. \tag{11}$$

Hence, using (3), for each $i \in \{0, 1, \ldots, k\}$ and for all $n \in \mathbb{N}$ we get

$$d(T^n z_i, T^{n+1} z_i) \leq a_1 d(T^{n-1} z_i, T^n z_i) + a_2 [d(T^{n-1} z_i, T^n z_i) + d(T^n z_i, T^{n+1} z_i)]$$

$$+ a_3[d(T^{n-1}z_i, T^{n+1}z_i) + d(T^n z_i, T^n z_i)]$$

$$\leq (a_1 + a_2 + a_3)d(T^{n-1}z_i, T^n z_i) + (a_2 + a_3)d(T^n z_i, T^{n+1}z_i).$$

Hence,

$$d(T^n z_i, T^{n+1}z_i) \leq r d(T^{n-1}z_i, T^n z_i)$$

Where $r = \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} < 1$ (as $a_1 + a_2 + a_3 < 1$). Repetition of the above process gives:

$$d(T^n z_i, T^{n+1}z_i) \leq r^n d(z_i, Tz_i) \tag{12}$$

for all $n \in \mathbb{N}$ and for each $i \in \{0, 1, \ldots, k\}$. Again

$$d(T^n z_i, T^n z_{i+1}) \leq a_1 d(T^{n-1}z_i, T^{n-1}z_{i+1}) + a_2[d(T^{n-1}z_i, T^n z_i) + d(T^{n-1}z_{i+1}, T^n z_{i+1})]$$

$$+ a_3[d(T^{n-1}z_i, T^n z_{i+1}) + d(T^{n-1}z_{i+1}, T^n z_i)]$$

$$\leq a_1 d(T^{n-1}z_i, T^n z_i) + a_1 d(T^n z_i, T^n z_{i+1}) + a_1 d(T^n z_{i+1}, T^{n-1}z_{i+1})$$

$$+ a_2[d(T^{n-1}z_i, T^n z_i) + a(T^{n-1}z_{i+1}, T^n z_{i+1})] + a_3[d(T^{n-1}z_i, T^n z_i)$$

$$+ d(T^n z_i, T^n z_{i+1}) + d(T^{n-1}z_{i+1}, T^n z_{i+1}) + d(T^n z_{i+1}, T^n z_i)]$$

so that

$$(1 - a_1 - 2a_3)d(T^n z_i, T^n z_{i+1})$$

$$\leq (a_1 + a_2 + a_3)d(T^{n-1}z_i, T^n z_i) + (a_1 + a_2 + a_3)d(T^n z_{i+1}, T^{n-1}z_{i+1}).$$

Since, $1 - a_1 - 2a_2 > 0$, hence using (12) in the above inequality we obtain:

$$\lim_{n \to \infty} d(T^n z_i, T^n z_{i+1}) = 0 \text{ for each } i \in \{0, 1, \ldots, k-1\}.$$

Finally, making use of triangular inequality and the above equation we obtain

$$d(x,y) = d(T^n z_0, T^n z_k) \leq \sum_{i=0}^{k-1} d(T^n z_i, T^n z_{i+1}) \to 0 \text{ as } n \to \infty .$$

Hence, T has a unique fixed point.

In the next theorem, we replace the $\mathcal{R}$ -completeness of the relational space by imposing another condition on the mapping T.

Theorem 3: Let (X,d) be a metric space, $\mathcal{R}$ a binary relation on X and $T : X \to X$ be a relation theoretic Hardy-Rogers-type mapping and the following conditions hold:

1. $X(T,\mathcal{R})$ is nonempty;
2. $\mathcal{R}$ is T-closed;
3. there exists $u \in X(T,\mathcal{R})$ such that the following condition holds:

$$d(u,Tu) \leq d(x,Tx) \text{ for all } x \in X(T,\mathcal{R}). \tag{13}$$

Then T has a fixed point. If $\Delta_X \subseteq \mathcal{R}$ and $X(T,\mathcal{R})$ is $\mathcal{R}_s$ -connected, then T has a unique fixed point.

Proof: Let us denote:

$$D(x) = d(x,Tx), x \in X .$$

Since, $X(T,\mathcal{R}) \neq \varnothing$, by (13) we have:

$$D(u) \leq D(x) \text{ for all } x \in X(T,\mathcal{R}). \tag{14}$$

We claim that $Tu = u$. On the contrary, suppose that $Tu \neq u$. Since, $u \in X(T,\mathcal{R})$, by T-closedness of $\mathcal{R}$ we have $Tu \in X(T,\mathcal{R})$. As T is a relational theoretic Hardy-Rogers- type mapping we have:

$$D(Tu) = d(Tu, TTu)$$

$$\leq a_1 d(u, Tu) + a_2[d(u, Tu) + d(Tu, TTu)] + a_3[d(u, TTu) + d(Tu, Tu)]$$

$$= a_1 D(u) + a_2[D(u) + D(Tu)] + a_3[d(u, TTu) + 0]$$

$$= a_1 D(u) + a_2[D(u) + D(Tu)] + a_3[d(u, Tu) + d(Tu, TTu)]$$

$$= (a_1 + a_2 + a_3)D(u) + (a_2 + a_3)D(Tu) .$$

This shows that $(1 - a_2 - a_3)D(Tu) \leq (a_1 + a_2 + a_3)D(u)$, i.e.

$$D(Tu) \leq \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} D(u) = rD(u)$$

where $r = \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} < 1$. Hence, we must have $D(Tu) < D(u)$ and $Tu \in X(T, \mathcal{R})$.

This contradicts inequality (14). Hence, we must $Tu = u$, i.e. u is a fixed point of T.

The uniqueness of the fixed point can be proved by following the same process used in the proof of Theorem 2.

4. Consequences

Let X denotes a relational metric space with distance function d and relation $\mathcal{R}$, and $T : X \to X$ be a mapping. The mapping T is called a relation theoretic contraction, if there exists nonnegative constant a satisfying $a < 1$ such that

$$d(Tx, Ty) \leq ad(x, y)$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$.

The mapping T is called a relation theoretic Kannan-type mapping, if there exists nonnegative constant a satisfying $a < \frac{1}{2}$ such that

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$.

The mapping T is called a relation theoretic Reich-type mapping, if there exist nonnegative constants a_1, a_2 satisfying $a_1 + 2a_2 < 1$ such that

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$.

The mapping T is called a relation theoretic Chaterjea-type mapping, if there exists nonnegative constant a satisfying $a < \frac{1}{2}$ such that:

$$d(Tx, Ty) \leq a[d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$.

Assuming the following values of constants a_1, a_2 and a_3 in Theorem 2:

(A) $a_1 = a, a_2 = a_3 = 0$;

(B) $a_1 = a_3 = 0, a_2 = a$;

(C) $a_3 = 0$;

(D) $a_1 = a_2 = 0, a_3 = a$;

we obtain the following fixed point results respectively:

Corollary 4: Let (X, d) be a metric space, $\mathcal{R}$ a binary relation on X such that (X, d) is $\mathcal{R}$-complete. Let $T : X \to X$ be at least one of the following:

(A) a relation theoretic contraction;

(B) a relation theoretic Kannan-type mapping;

(C) a relation theoretic Reich-type mapping;

(D) a relation theoretic Chterjea-type mapping;

Suppose, the following conditions hold:

1. $X(T, \mathcal{R})$ is nonempty;

2. $\mathcal{R}$ is T-closed;

3. T is $\mathcal{R}$-continuous on X; or $\mathcal{R}$ is d-self-closed.

Then T has a fixed point, i.e., there exists $x \in X$ such that $Tz = z$. In addition, suppose that $\Delta_X \subseteq \mathcal{R}$ and $X(T, \mathcal{R})$ is $\mathcal{R}_s$-connected, then T has a unique fixed point.

We have used a general binary relation $\mathcal{R}$ and proved the existence of relation theoretic fixed point of Hardy-Rogers-type mappings. Hence, by assuming $\mathcal{R}$ with some properties, we can obtain several fixed point results and their generalizations.

1. With $\mathcal{R} = X \times X$ (universal relation) in Theorem 2, we get the fixed point result for Hardy-Rogers-type mapping [5] in a metric space.

2. Assuming $\mathcal{R}$ to be a partial order relation, we can get a generalized and improved version of the results of Nieto and Rodriguez-Lopez [10].

3. If we assume $\mathcal{R}$ as a preorder (i.e., reflexive and transitive) in Theorem 2, we get a generalized and improved version of Theorem 1 of Turinici [18].

4. If $\mathcal{R}$ is assumed to be transitive in Theorem 2, we obtain a generalized and improved version of the results of Ben-El-Mechaiekh [3].

5. With $\mathcal{R}$ as a tolerance relation (see, [1]) in Theorem 2, we obtain a generalized and improved version of the theorems of Turinici [17, 18] and Ran and Reurings [12].

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(*Received*, September 30, 2024)
(*Revised*, October 19, 2024)

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Journal of Indian Acad. Math. ISSN: 0970-5120
Vol. 46, No. 2 (2024) pp. 247-256

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ALMOST NORMALITY VIA SOFT IDEAL

Abstract. The present paper introduces a new soft separation axiom called soft almost -normality which is strictly weaker than the axiom of soft -normality due to Guler and Kale [6]. Some of the properties and characterizations of soft almost -normal ideal topological spaces have been studied.

Keywords: Soft Sets, Soft Topology, Soft Ideal, Soft -normaland Soft Almost -normal Spaces.

Mathematics Subject Classification No.: 54C50, 26A21.

1. Introduction

In 1999, Molodtsov [13] introduced the concept of soft sets. Since the inception of soft sets many authors such as Ali *et al*. [1], Maji *et al*. [11], Pie and Miao [15] contributed in the development of soft set theory. In 2011, Shabirand Naz [18] introduced the concept of soft topological spaces as an extension of topological spaces. In the same paper they studied the concept of soft closure, soft interior, soft neighbourhood, soft subspaces and soft separation axioms. The soft separation axioms further studied by Georgiou, Megaritis and Petropoulos [5] using the notion of soft point. The study of soft ideal topological spaces was initiated by Kandil *et al*. [9] in the year 2014. In 2015, Gulerand Kale [6] introduced and studied soft -normal spaces.The purpose of thispaper is to introduceand study the axiom of soft almost -normality in soft ideal topological spaces.

2. Preliminaries

Throughout this paper X denotes a nonempty set, Ω denotes the set of parameters and $S(X, \Omega)$ refers to the family of all soft sets of X relative to Ω. For the basic notions on soft setsthe readers should refer [2, 5, 10, 11, 18, 19,].

A non-empty collection Γ of subsets of a non-empty set X is said to be topology on X if it contains empty soft set ϕ and whole soft set X andclosed with respect to arbitrary union and finite intersection. If Γ is a soft topology on X then the triplet (X, Γ, Ω) is called a soft topological space and members of Γ are called as soft open sets in X. The complement of a soft open sets is called soft closed sets in X. The family of all soft closed sets of a soft topological space is denoted by $SC(X, \Omega)$.

Definition 2.1 [18, 20]: Let (ξ, Ω) be a soft set of a soft topological space (X, Γ, Ω). Then the closure and interior of (ξ, Ω) are defined as follows:

$$\mathrm{Cl}(\xi, \Omega) = \cap \{(\sigma, \Omega) : (\sigma, \Omega) \in SC(X, \Omega) \text{ and } (\xi, \Omega) \subseteq (\sigma, \Omega)\}.$$

$$\mathrm{Int}(\xi, \Omega) = \cup \{(\sigma, \Omega) : (\sigma, \Omega) \in \Gamma \text{ and } (\sigma, \Omega) \subseteq (\xi, \Omega)\}.$$

Lemma 2.1 [3, 18, 20]: Let (X, Γ, Ω) be a soft topological space and let $(\xi, \Omega), (\sigma, \Omega) \in S(X, \Omega)$. Then:

(i) $(\xi, \Omega) \in SC(X, \Omega) \Leftrightarrow (\xi, \Omega) = \mathrm{Cl}(\xi, \Omega)$

(ii) $(\xi, \Omega) \subseteq (\sigma, \Omega) \Rightarrow \mathrm{Cl}\,(\xi, \Omega) \subseteq \mathrm{Cl}(\sigma, \Omega)$.

(iii) $(\xi, \Omega) \in \Gamma \Leftrightarrow (\xi, \Omega) = \mathrm{Int}\,(\xi, \Omega)$.

(iv) $(\xi, \Omega) \subseteq (\sigma, \Omega) \Rightarrow \mathrm{Int}\,(\xi, \Omega) \subseteq \mathrm{Int}\,(\sigma, \Omega)$.

(v) $(\mathrm{Cl}\,(\xi, \Omega))^c = \mathrm{Int}\,((\xi, \Omega)^c)$.

(vi) $\left(\mathrm{Int}\,(\xi, \Omega)\right)^c = \mathrm{Cl}\,((\xi, \Omega)^c)$.

Definition 2.2 [8]: Let Y be a non-empty subset of a soft topological space (X, Γ, Ω).Then the family $\Gamma_Y = \{(\xi_Y, \Omega) : (\xi, \Omega) \in \Gamma\}$ is called soft relative topology on Y and (Y, Γ_Y, Ω) is called a soft subspace of (X, Γ, Ω).

Lemma 2.2 [18]: Let (Y, Γ_Y, Ω) be a soft subspace of a soft topological space (X, Γ, Ω) and $(\xi, \Omega) \in \Gamma_Y$. If $\widetilde{Y} \in \Gamma$ then $(\xi, \Omega) \in \Gamma$.

Lemma 2.3 [18]: Let (Y,Γ_Y,Ω) be a soft subspace of a soft topological space (X,Γ,Ω) and (ξ,Ω) be a soft set over X, then:

(i) $(\xi,\Omega) \in \Gamma_Y \quad (\xi,\Omega) = \widetilde{Y} \cap (\sigma,\Omega)$ for some $(\sigma,\Omega) \in \Gamma$.

(ii) $(\xi,\Omega) \in SC(Y,\Omega) \quad (\xi,\Omega) = \widetilde{Y} \cap (\sigma,\Omega)$ for some $(\sigma,\Omega) \in SC(X,\Omega)$.

Lemma 2.4 [17]: Let (Y,Γ_Y,Ω) be a soft subspace of a soft topological space (X,Γ,Ω). Then a soft set $(\xi_Y,\Omega) \in SC(Y,\Omega) \Rightarrow (\xi_Y,\Omega) \in SC(X,\Omega) \quad \widetilde{Y} \in SC(X,\Omega)$,

Definition 2.3 [7]: A soft set (ξ,Ω) in a soft topological space (X,Γ,Ω) is said to be:

(i) Soft regular open if $(\xi,\Omega) = \mathrm{Int}\,(\mathrm{Cl}(\xi,\Omega))$.

(ii) Soft regular closed if $(\xi,\Omega) = \mathrm{Cl}(\mathrm{Int}(\xi,\Omega))$.

Remark 2.1 [8]: Every soft regular open (resp. soft regular closed) set is soft open (resp. soft closed). But a soft regular open (resp. soft regular closed) set may fails to be soft open (resp. soft closed)

Definition 2.4 [10]: Let $S(X,\Omega)$ and $S(Y,\Psi)$ be families of soft sets over X and Y respectively. Let $u: X \rightarrow Y$ and $p: \Omega \rightarrow \Psi$ be mappings. Then a mapping $f_{pu}: S(X,\Omega) \rightarrow S(Y,\Psi)$ is defined as:

(i) Let $(\xi,\Omega) \in S(X,\Omega)$. Then the image of (ξ,Ω) under f_{pu} is written as $f_{pu}(\xi,\Omega) = (f_{pu}(\xi), p\,(\Omega)) \in S(Y,\Psi)$ such that

$$f_{pu}(\xi)(k) = \begin{cases} {}_{e\in p^{-1}(k)\cap\Omega}\, u(\xi(e)), & p^{-1}(k) \cap \Omega \neq \phi \\ \phi, & p^{-1}(k) \cap \Omega = \phi. \end{cases}$$

For all $k \in \Psi$.

(ii) Let $(\mu,\Psi) \in S(Y,\Psi)$. Then the inverse image of (μ,Ψ) under f_{pu} is given by

$$f_{pu}^{-1}(\mu)(e) = \begin{cases} u^{-1}\mu\big(p(e)\big), & p(e) \in \Psi \\ \phi, & \text{otherwise} \end{cases}$$

For all $e \in \Omega$.

Definition 2.5 [12]**:** A mapping f_{pu}:$S(X, \Omega) \rightarrow S(Y, \Psi)$ is said to be soft injective (respectively surjective, bijective) if the mappings $u : X \rightarrow Y$ and $p : \Omega \rightarrow \Psi$ are injective (respectively, surjective, bijective).

Definition 2.6 [6, 20, 21]: A soft mapping f_{pu}: $(X, \Gamma, \Omega) \rightarrow (Y, \eta, \Psi)$ is called:

(i) Soft continuous if $f_{pu}^{-1}(\mu, \Psi) \in \Gamma$ for all soft sets $(\mu, \Psi) \in \eta$.

(ii) Softopen if $f_{pu}(\xi, \Omega) \in \eta$ for all soft sets $(\xi, \Omega) \in \Gamma$.

(iii) Soft homeomorphism if f_{pu} is bijection, soft continuous and soft open.

Definition 2.7 [20]: A soft set $(\xi, \Omega) \in S(X, \Omega)$ is called a soft point if there exists a point $x \in X$ and $\beta \in \Omega$ such that $\xi(\beta) = \{x\}$ and $\xi(\beta^c) = \phi$ for each $\beta^c \in \Omega \quad \{\beta\}$ and the soft point (ξ, Ω) is denoted by x_β. The family of all soft points over X is denoted by $SP(X, \Omega)$.

Definition 2.8 [20]: A soft point $x_\beta \in (\sigma, \Omega)$ if $x_\beta \sqsubseteq (\sigma, \Omega)$.

Lemma 2.5 [4,14]: Let (ξ, Ω), $(\sigma, \Omega) \in S(X, \Omega)$ and $x_\beta \in SP(X, \Omega)$. Then we have:

(i) $x_\beta \in (\xi, \Omega) \qquad x_\beta \quad (\xi, \Omega)^c$.

(ii) $x_\beta \in (\xi, \Omega) \cup (\sigma, \Omega) \qquad x_\beta \in (\xi, \Omega)$ or $x_\beta \in (\sigma, \Omega)$.

(iii) $x_\beta \in (\xi, \Omega) \cap (\sigma, \Omega) \qquad x_\beta \in (\xi, \Omega)$ and $x_\beta \in (\sigma, \Omega)$.

Definition 2.9 [9]: A non-empty family of soft sets over (X,Ω) is said to be a soft ideal on X if:

(i) $(\xi, \Omega), (\sigma, \Omega) \in \qquad (\xi, \Omega) \cup (\sigma, \Omega) \in$.

(ii) $(\xi, \Omega) \in$ and $(\sigma, \Omega) \sqsubseteq (\xi, \Omega) \qquad (\sigma, \Omega) \in$.

A soft topological space (X, Γ, Ω) with a soft ideal is called soft ideal topological space and is denoted by $SITS(X, \Gamma, \Omega, \ )$.

Definition 2.10 [6]: A $SITS(X, \Gamma, \Omega, \ )$ is said to be a soft -normal if for each pair of soft sets (ξ, Ω),$(\sigma, \Omega) \in SC(X, \Omega)$ such that $(\xi, \Omega) \cap (\sigma, \Omega) = \phi$, $\exists$ soft sets $(\mu,\Omega), (\lambda,\Omega) \in \Gamma$ such that (ξ,Ω) $(\mu,\Omega) \in$, (σ,Ω) $(\lambda,\Omega) \in$ and $(\mu,\Omega) \cap (\lambda,\Omega) = \phi$.

3. Almost $\mathcal{I}$-Normal Spaces

Definition 3.1: A SITS $(X,\Gamma,\Omega,\ )$ is soft almost -normal, $\forall$ pair of soft sets $(\xi,\Omega) \in SC(X,\Omega)$ and $(\sigma,\Omega) \in SRC(X,\Omega)$ such that $(\xi,\Omega) \cap (\sigma,\Omega) = \phi$, $\exists$ soft sets $(\mu,\Omega), (\lambda,\Omega) \in \Gamma$ such that (ξ,Ω) $(\mu,\Omega) \in$,(σ,Ω) $(\lambda,\Omega) \in$ and $(\mu,\Omega) \cap (\lambda,\Omega) = \phi$.

Theorem 3.1: If a SITS $(X,\Gamma,\Omega,\ )$ is soft -normal then it is soft almost -normal.

Proof: Let $(\xi,\Omega) \in SC(X,\Omega)$ and $(\sigma,\Omega) \in SRC(X,\Omega)$ such that $(\xi,\Omega) \cap (\sigma,\Omega) = \phi$. Then $(\sigma,\Omega) \in SC(X,\Omega)$ because every soft regular closed set is soft closed. Therefore by definition 2.10, $\exists$ soft sets $(\mu,\Omega), (\lambda,\Omega) \in \Gamma$ such that (ξ,Ω) $(\mu,\Omega) \in$, (σ,Ω) $(\lambda,\Omega) \in$ and $(\mu,\Omega) \cap (\lambda,\Omega) = \phi$. Hence, by Definition 3.1, SITS $(X,\Gamma,\Omega,\ )$ is soft almost -normal.

Remark 3.1: The following example shows that a soft almost -normal space may fails to be soft -normal.

Example 3.1: Let $X = \{x_1, x_2, x_2\}$, $\Omega = \{\beta_1, \beta_2\}$ and $(\lambda,\Omega), (\mu,\Omega)$ and (ν,Ω) are soft sets defined as follows:

$(\lambda,\Omega) = \{(\beta_1,\{x_1\}),(\beta_2,\{x_1\}),$

$(\mu,\Omega) = \{(\beta_1,\{x_1,x_2\}),(\beta_2,\{x_1,x_2\})\},$

$(\nu,\Omega) = \{(\beta_1,\{x_1,x_3\}),(\beta_2,\{x_1,x_3\})\}.$

Let $= \{\phi, \{(\beta_1,\{x_2\}),(\beta_2,\phi)\}, \{(\beta_1,\phi),(\beta_2,\{x_1\})\}, \{(\beta_1,\{x_2\}), \beta_2,\{x_1\})\}\}$ be a soft ideal on (X,Ω). Then the SITS$(X,\Gamma,\Omega,\ )$, where $\Gamma = \{\phi, X, (\lambda,\Omega), (\mu,\Omega), (\nu,\Omega)\}$ is soft almost -normal but not soft -normal.

The following theorem summarizes the properties of soft almost -normal spaces.

Theorem 3.2: For a SITS$(X,\Gamma,\Omega,\ )$ we have the following:

(i) $(X,\Gamma,\Omega,\ )$is soft almost -normal.

(ii) For each soft set $(\xi,\Omega) \in \mathrm{SC}(X,\Omega)$ and $\forall, (\sigma,\Omega) \in \mathrm{SRC}(X,\Omega)$ containing (ξ,Ω), $\exists$ a soft set $(\mu,\Omega) \in \Gamma$ such that $(\xi,\Omega) \quad (\mu,\Omega) \in$ and $\mathrm{Cl}(\mu,\Omega) \quad (\sigma,\Omega) \in$.

(iii) For every $(\xi,\Omega) \in \mathrm{SRC}(X,\Omega)$ and each soft set $(\sigma,\Omega) \in \Gamma$ containing (ξ,Ω), $\exists$ $(\mu,\Omega) \in \Gamma$ such that $(\xi,\Omega) \quad (\mu,\Omega) \in$ and $\mathrm{Cl}(\mu,\Omega) \quad (\sigma,\Omega) \in$.

(iv) For every $(\xi,\Omega) \in \mathrm{SC}(X,\Omega)$ and $(\sigma,\Omega) \in \mathrm{SRC}(X,\Omega)$ such that $(\xi,\Omega) \cap (\sigma,\Omega) = \phi$, $\exists$ soft sets $(\mu,\Omega), (\lambda,\Omega) \in \Gamma$ such that $(\xi,\Omega) \quad (\mu,\Omega) \in$, $(\sigma,\Omega) \quad (\lambda,\Omega) \in$ and $\mathrm{Cl}(\mu,\Omega) \,\tilde{\cap}\, \mathrm{Cl}(\lambda,\Omega) = \phi$.

Proof: (i) $\dot{\Rightarrow}$ (ii) Let $(\xi,\Omega) \in \mathrm{SC}(X,\Omega)$ and $(\sigma,\Omega) \in \mathrm{SRO}(X,\Omega)$ such that $(\xi,\Omega) \quad (\sigma,\Omega) \in$. Then, $(\sigma,\Omega)^c \in \mathrm{SRC}(X,\Omega)$ such that $(\xi,\Omega) \,\tilde{\cap}\, (\sigma,\Omega)^c \phi$. Therefore, $\exists$ $(\mu,\Omega), (\lambda,\Omega) \in \Gamma$ such that $(\xi,\Omega) \quad (\mu,\Omega) \in$, $(\sigma,\Omega)^c \quad (\lambda,\Omega) \in$ and $(\mu,\Omega) \,\tilde{\cap}\, (\lambda,\Omega) = \phi$. Thus, $(\xi,\Omega) \quad (\mu,\Omega) \in$, $(\lambda,\Omega)^c \quad (\sigma,\Omega) \in$ and hence $(\xi,\Omega) \quad (\mu,\Omega) \in$ and $\mathrm{Cl}(\mu,\Omega) \quad (\sigma,\Omega) \in$ because $(\lambda,\Omega)^c \in \mathrm{SC}(X,\Omega)$.

(ii) $\dot{\Rightarrow}$ (iii) Let $(\xi,\Omega) \in \mathrm{SRC}(X,\Omega)$ and $(\sigma,\Omega) \in \Gamma$ containing (ξ,Ω). Then, $(\xi,\Omega)^c \in \mathrm{SRO}(X,\Omega)$ and $(\sigma,\Omega)^c \in \mathrm{SC}(X,\Omega)$ such that $(\sigma,\Omega)^c \quad (\xi,\Omega)^c \in$. Therefore by (ii), $\exists$ a soft set $(\delta,\Omega) \in \Gamma$ such that $(\sigma,\Omega)^c \quad (\delta,\Omega) \in$ and $\mathrm{Cl}(\delta,\Omega) \quad (\xi,\Omega)^c \in$. Then $(\xi,\Omega) \quad (\mathrm{Cl}(\delta,\Omega))^c \in$ and $(\delta,\Omega)^c \quad (\sigma,\Omega) \in$. Put $(\mu,\Omega) = (\mathrm{Cl}(\delta,\Omega))^c$. Then $(\mu,\Omega) \in \Gamma, (\xi,\Omega) \quad (\mu,\Omega) \in$ and $\mathrm{Cl}(\mu,\Omega) \quad (\sigma,\Omega) \in$.

(iii) $\dot{\Rightarrow}$(iv) Let $(\xi,\Omega) \in \mathrm{SRC}(X,\Omega)$ and $(\sigma,\Omega) \in \mathrm{SC}(X,\Omega)$ such that $(\xi,\Omega) \,\tilde{\cap}\, (\sigma,\Omega) = \phi$. Then $(\sigma,\Omega)^c \in \Gamma$ such that $(\xi,\Omega) \quad (\sigma,\Omega)^c \in$. Therefore, $\exists$ $(\delta,\Omega) \in \Gamma$ such that $(\xi,\Omega) \quad (\delta,\Omega) \in$ and $\mathrm{Cl}(\delta,\Omega) \quad (\sigma,\Omega)^c \in$. Now the soft set $(\delta,\Omega) \in \Gamma$ containing $(\xi,\Omega) \in \mathrm{SRC}(X,\Omega)$, $\exists$ a soft set $(\mu,\Omega) \in \Gamma$ such that $(\xi,\Omega) \quad (\mu,\Omega) \in$ and $\mathrm{Cl}(\mu,\Omega) \quad (\delta,\Omega) \in$. Let $(\mathrm{Cl}(\delta,\Omega))^c = (\lambda,\Omega)$. Then $\mathrm{Cl}(\mu,\Omega) \,\tilde{\cap}\, \mathrm{Cl}(\lambda,\Omega) = \phi$ and $(\xi,\Omega) \quad (\mu,\Omega) \in$, $(\sigma,\Omega) \quad (\lambda,\Omega) \in$.

(iv) $\dot{\Rightarrow}$ (i) Obvious.

Lemma 3.1 [6]: Let $(\xi,\Omega), (\sigma,\Omega) \in \mathrm{S}(X,\Omega)$ and $f_{pu}: \mathrm{S}(X,\Omega) \to \mathrm{S}(Y,\Psi)$ is a injective mapping. Then $f_{pu}((\xi,\Omega) \quad (\sigma,\Omega)) = f_{pu}(\xi,\Omega) \quad f_{pu}(\sigma,\Omega)$.

Theorem 3.3: Soft almost- normality is preserved when mappings are soft closed, soft continuous and soft open.

Proof: Let f_{pu} be a soft closed, soft open and soft continuous mapping from a soft almost -normal space $(X,\Gamma,\Omega,\ )$ onto a $\mathrm{SITS}(Y,\eta,\Psi,J)$. Let $(\xi,\Psi) \in \mathrm{SC}(Y,\Omega)$ and $(\sigma,\Psi) \in \mathrm{SRO}(Y,\Omega)$ containing (ξ,Ψ). Put $f_{pu}^{-1}(\xi,\Psi) = (\delta,\Omega)$ and

$f_{pu}^{-1}(\delta, \Psi) = (£, \Omega)$. Then $(\delta, \Omega) \in SC(X, \Omega)$ such that $(\delta, \Omega) \subseteq (£, \Omega)$ because f_{pu} is soft continuous. Now, $(£, \Omega)$ $Int\,(Cl(£, \Omega)) \in$, $Int\,(Cl(£\,, \Omega)) \in SRO(X, \Omega)$ containing $(\delta, \Omega) \in SC(X, \Omega)$ because $(£, \Omega) \in \Gamma$. Therefore by Theorem 3.2 (iii), $\exists\ (\mu, \Omega) \in \Gamma$ such that (δ, Ω) $(\mu, \Omega) \in$ and $Cl(\mu, \Omega)$ $Int\left(Cl\,((£, \Omega)\right) \in$ because $(X, \Gamma, \Omega,\ )$ is soft almost -normal. By Lemma 3.1, $f_{pu}(\delta, \Omega)$ $f_{pu}(\mu, \Omega) \in J$ and $f_{pu}\left(Cl\,(\mu, \Omega)\right)$ $f_{pu}\left(Int\left(Cl((£, \Omega))\right)\right) \in J$.

Consequently, $Cl((£, \Omega) = Cl(f_{pu}^{-1}(\sigma, \Psi)) = f_{pu}^{-1}(Cl(\sigma, \Psi))$ because f_{pu} is soft continuous and soft open. Also, soft openness and soft continuity of f_{pu} implies $Int\,(f_{pu}^{-1}(Cl(\sigma, \Psi)) = f_{pu}^{-1}(Int(Cl(\sigma, \Psi)))$.

Thus, $f_{pu}\left(Int\left(Cl(£, \Omega)\right)\right) = Int\left(Cl(\sigma, \Psi)\right)$. Again since f_{pu} is soft open, $f_{pu}(\mu, \Omega) \in \eta$ and f_{pu} is soft closed and soft continuous, $f_{pu}\left(Cl(\mu, \Omega)\right) = (Cl\left(f_{pu}(\mu, \Omega)\right)$. Thus, $f_{pu}(\mu, \Omega) \in \eta$ such that (ξ, Ψ) $f_{pu}(\mu, \Omega) \in J$ and $(Cl(f_{pu}(\mu, \Omega))$ $(\sigma, \Psi) \in J$. Hence, (Y, η, Ψ, J) is soft almost J-normal.

Theorem 3.4: If a $SITS(X, \Gamma, \Omega,\ )$ is soft almost -normal and $(Y, \Omega) \in \Gamma \cap SC(X, \Omega)$. Then the soft subspace $(Y, \Gamma_Y, \Omega,\ _Y)$ is soft almost -normal.

Proof: Let $(\xi, \Omega) \in SC(Y, \Omega)$ and $(\sigma, \Omega) \in SRO(Y, \Omega)$ such that (ξ, Ω) $(\sigma, \Omega) \in$. Now $(Y, \Omega) \in SC(Y, \Omega)$ $(\xi, \Omega) \in SC(X, \Omega)$ and $(Y, \Omega) \in \Gamma$, we have (σ, Ω) $Int\left(Cl(\sigma, \Omega)\right) \in$. Since, $(\sigma, \Omega) \in SRO(Y, \Omega)$, we have $(\sigma, \Omega) = Int_Y(Cl_Y(\sigma, \Omega)) \supseteq Int(Cl(\sigma, \Omega)) \cap (Y, \Omega)$. Also, $Int\,(Cl(\sigma, \Omega))$ $Int(Cl(Y, \Omega)) = (Y, \Omega) \in$ and therefore $Int\left(Cl(\sigma, \Omega)\right)$ $(\sigma, \Omega) \in$. Hence, $(\sigma, \Omega) = Int\left(Cl(\sigma, \Omega)\right)$ and $(\sigma, \Omega) \in SRO(X, \Omega)$. Thus, $(\xi, \Omega) \in SC(X, \Omega)$ contained in a soft set $(\sigma, \Omega) \in SRO(X, \Omega)$. Since, $(X, \Gamma, \Omega,\ )$is soft almost -normal, $\exists$ a soft set $(\mu, \Omega) \in \Gamma$ such that (ξ, Ω) $(\mu, \Omega) \in$ and $Cl(\mu, \Omega)$ $(\sigma, \Omega) \in$. Thus, $(\mu, \Omega) \in \Gamma_Y$ such that (ξ, Ω) $(\mu, \Omega) \in$ and $(Cl_Y(\mu, \Omega))$ $(\sigma, \Omega) \in$. Hence,$(Y, \Gamma_Y, \Omega,\ _Y)$ is soft almost -normal.

Lemma 3.2: If (Y, Ω) is a soft regular closed subset of a $STS(X, \Gamma, \Omega)$ and (ξ, Ω) is a soft regular closed in soft subspace (Y, Γ_Y, Ω), then (ξ, Ω) is soft regular closed in X.

Theorem 3.5: If a $SITS(X, \Gamma, \Omega,\ )$is soft almost -normal and $(Y, \Omega) \in SRC(X, \Omega)$. Then the subspace $(Y, \Gamma_Y, \Omega,\ _Y)$ is soft almost -normal.

Proof: Suppose $(\mathrm{Y}, \Gamma_{\mathrm{Y}}, \Omega, \ _{\mathrm{Y}})$ be a soft subspace of a soft almost -normal space $(\mathrm{X}, \Gamma, \Omega, \ )$ and $(\mathrm{Y}, \Omega) \in \mathrm{SRC}(\mathrm{X}, \Omega)$. Let $(\xi, \Omega) \in \mathrm{SC}(\mathrm{Y}, \Omega)$ and $(\sigma, \Omega) \in \mathrm{SRC}(\mathrm{Y}, \Omega)$ such that $(\xi, \Omega) \cap (\sigma, \Omega) = \phi$. Since, $(\mathrm{Y}, \Omega) \in \mathrm{SRC}(\mathrm{X}, \Omega)$ and $(\sigma, \Omega) \in \mathrm{SRC}(\mathrm{Y}, \Omega)$ by Lemma 3.2, $(\sigma, \Omega) \in \mathrm{SRC}(\mathrm{X}, \Omega)$. Since, $(\mathrm{X}, \Gamma, \Omega, \ )$ is soft almost -normal $\exists$ soft sets $(\mu, \Omega), (\lambda, \Omega) \in \Gamma$ such that $(\xi, \Omega) \quad (\mu, \Omega) \in \ ,$ $(\sigma, \Omega) \quad (\lambda, \Omega) \in$ and $(\mu, \Omega) \cap (\lambda, \Omega) = \phi$. And so, $(\mu, \Omega) \cap \widetilde{\mathrm{Y}} \in \Gamma_{\mathrm{Y}}$ and $(\lambda, \Omega) \cap \widetilde{\mathrm{Y}} \in \Gamma_{\mathrm{Y}}$ such that $(\xi, \Omega) \quad ((\mu, \Omega) \cap \widetilde{\mathrm{Y}}) \in$ and $(\sigma, \Omega) \quad ((\lambda, \Omega) \cap \widetilde{\mathrm{Y}}) \in$ and $\left((\mu, \Omega) \cap \widetilde{\mathrm{Y}}\right) \cap \left((\lambda, \Omega) \cap \widetilde{\mathrm{Y}}\right) = \left((\mu, \Omega) \cap (\lambda, \Omega)\right) \cap \widetilde{\mathrm{Y}} = \phi$. Hence, $(\mathrm{Y}, \Gamma_{\mathrm{Y}}, \Omega, \ _{\mathrm{Y}})$ is soft almost -normal.

Definition 3.2: A SITS $(\mathrm{X}, \Gamma, \Omega, \ )$ is called soft semi -normal, if $\forall\, (\xi, \Omega) \in \mathrm{SC}(\mathrm{X}, \Omega)$ and if $\forall$ soft sets $(\sigma, \Omega) \supseteq (\xi, \Omega), \exists\, (\mu, \Omega) \in \Gamma$ such that $(\xi, \Omega) \quad (\mu, \Omega) \in \quad \mathrm{Int}(\mathrm{Cl}(\mu, \Omega)) \quad (\sigma, \Omega) \in \ .$

Remark 3.2: **S**oft -normality implies soft semi -normality, but the converse may be false.

Corollary 3.1: A soft almost -normal space is soft -normal if and only if it is soft semi -normal.

Theorem 3.6: Every soft semi -normal, soft almost -normal space is soft -normal.

Proof: Let $(\sigma, \Omega) \in \Gamma$ be a soft set such that $(\xi, \Omega) \subseteq (\sigma, \Omega)$ where $(\xi, \Omega) \in \mathrm{SC}(\mathrm{X}, \Omega)$. Then by soft semi -normality, $\exists$ a soft set $(\mu, \Omega) \in \Gamma$ such that $(\xi, \Omega) \quad (\mu, \Omega) \in$ and $\mathrm{Int}(\mathrm{Cl}(\mu, \Omega)) \quad (\sigma, \Omega) \in \ .$ Put $(\chi, \Omega) = \mathrm{Int}(\mathrm{Cl}(\mu, \Omega))$. Then $(\chi, \Omega) \in \mathrm{SRO}(\mathrm{X}, \Omega)$ contained in $(\xi, \Omega) \in \mathrm{SC}(\mathrm{X}, \Omega)$. Therefore by Theorem 3.2(ii), $\exists\ (\lambda, \Omega) \in \Gamma$ such that $(\xi, \Omega) \quad (\lambda, \Omega) \in$ and $\mathrm{Cl}(\lambda, \Omega) \quad (\chi, \Omega) \in \ .$ Consequently $(\lambda, \Omega) \in \Gamma$ such that $(\xi, \Omega) \quad (\lambda, \Omega) \in$ and $\mathrm{Cl}(\lambda, \Omega) \quad (\sigma, \Omega) \in \ .$ Hence, $(\mathrm{X}, \Gamma, \Omega, \ )$ is soft -normal.

Conclusions

In this paper we have introduced the notion of soft almost -normal spaces. An example is given to show that the soft almost -normal space may fails to be soft -normal.It has been shown that soft almost -normal space is soft -normal if and only if it is soft semi -normal.We studied that soft almost normality is preserved under soft closed, soft continuous and soft open mappings. Two theorems of soft subspaces with reference to soft almost -normal spaces has also been obtained. Several characterizations of soft almost -normal spaces have been obtained. We believe that these results will contribute to the study of separation axioms in soft ideal topological spaces and will help the researcher for further studies.

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(*Received*, October 23, 2024)

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Journal of Indian Acad. Math. ISSN: 0970-5120
Vol. 46, No. 2 (2024) pp. 257-264

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SOLUTION OF KAPLER'S EQUATION

Abstract: The profound contributions of Johannes Kepler to the field of astronomy and mathematics continue to shape our understanding of the universe. The amalgamation of these derivations establishes *Kapler's* planetary motion *Equations* collectively known as his "Revelations on Planetary Motion," stand as a testament to his meticulous observations, innovative thinking, and unwavering dedication to unraveling the mysteries of the cosmos. *In this paper we solve the Kapler's* planetary motion *Equations.*

Key words: Kepler Equation, Celestial Mechanics, Iteration Method.

Mathematics Subject Classification (2010) No.: 83C40, 83C45.

1. Introduction: Unraveling Heavens

Throughout history, humanity's fascination with the night sky has driven the pursuit of understanding the motions of celestial bodies. In the early 17th century, Johannes Kepler emerged as a key figure in this quest, overturning traditional cosmological beliefs and introducing a new paradigm for comprehending planetary orbits. Kepler's groundbreaking laws provided an illustrious confirmation of his revolutionary ideas.

2. Kepler's Laws: The Cornerstones of Celestial Mechanics

2.1 The Law of Ellipses: A Paradigm Shift: Kepler's first law, the Law of Ellipses, shattered the prevailing notion of circular planetary orbits by revealing that planets trace elliptical paths around the Sun. This insight not only corrected inaccuracies in earlier models, but also laid the foundation for a more accurate representation of planetary motion. The law's formulation established the Sun as a focal point within these elliptical paths, transforming the understanding of the solar system's geometry.

2.2 The Law of Equal Areas: Dynamic Orbits: Kepler's second law, the Law of Equal Areas, introduced the concept of equal areas being swept out in equal times by a line joining a planet and the Sun. This law highlighted the non-uniform nature of planetary motion, dispelling the notion of constant planet speed along its orbit. Instead, Kepler's insight revealed that planets experience acceleration and deceleration in response to their varying distances from the Sun.

2.3 The Harmonic Law: A Cosmic Symphony: Kepler's third law, The Harmonic Law, often referred to as the Harmonic Law, presented a mathematical expression between a planet's orbital period and its distance from the Sun. This elegant formula, which expressed the square of a planet's orbital period as proportional to the cube of its semi-major axis, illuminated the underlying harmony in the solar system. This law not only solidified Kepler's findings but also laid the groundwork for Newton's later work on universal gravitation.

Newton's Proclamation of Motion: A Trilogy of Fundamental Laws

In an epoch-defining moment, Sir Isaac Newton unveiled a triumvirate of elegantly straightforward laws of motion, whose harmonious interplay, when coupled with his revolutionary Law of Universal Gravitation, constructs the very edifice of theoretical mechanics.

The First Law: The Pinnacle of Inertia

Newton's inaugural decree posits that an object, unswervingly immersed in a state of tranquility or ceaseless uniform motion along a linear trajectory, remains ensconced in that very state unless coerced into metamorphosis by the imposing hands of external forces.

The Second Law: The Dance of Momentum and Force

Casting his gaze upon momentum's metamorphosis, Newton's second tenet divulges that the metamorphosis rate of momentum, gauged relative to an inertial vantage, stands in perfect proportion to the vigour of the force that imposes itself, unfurling as if in choreographed synchrony, along the identical compass of that force.

The Third Law: The Balancing Act of Action and Reaction

Emanating wisdom akin to a cosmic riddle, Newton's third dictum entwines every action with an equal and opposite reaction. A symphony of harmonious counterbalance governs the interactions of the universe, elegantly ensuring equilibrium in every altercation.

Mathematically embodied, the second law assumes the following form:

$$F = m\frac{d^2\mathbf{r}}{dt^2}, \tag{1}$$

Where, F = Vector sum of all forces,
m = mass,

Newton's Universal Gravitation law

In the illustrious tapestry of scientific revelations, Newton's indelible contribution shines brightly through his formulation of the Law of Universal Gravitation. An exquisite cosmic symphony unfolds as this law proclaims that any two celestial bodies share an intimate dance of attraction, a force that waltzes in direct proportion to the product of their masses while gracefully waltzing inversely to the square of the distance that separates them.

Mathematically embodied, this celestial pact is elegantly inscribed as:

$$F = \frac{Gm_1m_2}{r^3}\mathbf{r} \tag{2}$$

Here:

F = gravitational force between the two objects,
G = gravitational constant (approximately 6.67430 x 10^ 11 N(m/kg)^2),
m1 and m2 = masses of the two objects,
r = distance between the centers of the two objects.

Solution of Kapler's Equations

Newton stood as the vanguard in articulating and resolving the intricate enigma of the two-body problem. This celestial quandary boasts significance twofold: Primarily, it addresses the gravitational interactions within spherical entities whose masses nestle within their spherical shells-an exclusive predicament in gravitational studies amenable to rigorous, and somewhat uncomplicated, resolutions. Secondly, it serves as the bedrock for practical quandaries concerning orbital movements, allowing them to be approximated as two-body problems. In essence, the two-body solution emerges as the cardinal cornerstone, from which the edifice of more precise calculations unfurls.

Equations (1) and (2) combine well to provide the two-body problem's mathematical representation. The radial vector r embraces a positive connotation as it sweeps away from the point of origin since the earth's core serves as the origin of our coordinate system. Equation (1) and (2) undergo a transmutation, orchestrating them to articulate the force that caresses the satellite endowed with mass m—a force that serenades in response to the gravitational allure cast by the colossal mass M of the earth

$$F_m = m\frac{d^2\mathbf{r}}{dt^2}$$

$$F_m = -GmM\frac{\mathbf{r}}{r^3} \tag{1}$$

$$\frac{d^2\mathbf{r}}{dt^2} + GM\frac{\mathbf{r}}{r^3} = \frac{d^2}{dt^2}\mathbf{r} + \mu\frac{\mathbf{r}}{r^3} = 0 \tag{3}$$

The process of entering the basic equation, known as the "process of integration of two equations," is simpler for individuals with a basic understanding of vector analysis. This process is completed by determining the validity of Kepler's law.

Verification of Invariable Vector

Performing the operation of cross-multiplication on equation (3) with respect to the factor r:

$$\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2} + \frac{\mu}{r^3}\mathbf{r}\times\mathbf{r} = 0$$

The second term evaluates to zero due to the property that the cross product of a vector with itself, i.e., r × r, results in zero. Consequently,

$$\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2} = 0 \tag{4}$$

Calculate the derivative of the angular momentum L:

$$\frac{dL}{dt^2} = \frac{d}{dt^2}\left(\mathbf{r} \times \frac{d\mathbf{r}}{dt^2}\right) = \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} + \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} = 0 \tag{5}$$

The initial term on the right-hand side becomes zero as a result of equation (4). Similarly, the second term is effective because the cross product of two parallel vectors is zero. Thus, $\frac{dL}{dt} = 0,$ we arrive at the conclusion: or, in alternative words, the vector L remains constant.

Solution of Kepler's 1st Law equation

Cross-multiplying equation (3) with the angular momentum L:

$$\frac{d^2\mathbf{r}}{dt^2} \times L = -\frac{\mu}{r^3}\mathbf{r} \times L = -\frac{\mu}{r^3}\mathbf{r} \times \left(\mathbf{r} \times \frac{d\mathbf{r}}{dt^2}\right)$$

By triple vector product, $x \times (y \times z) = (x \cdot z)y - (x \cdot y)z$, the resulting expression takes the form:

$$-\frac{\mu}{r^3}\left[\left(\mathbf{r} \cdot \frac{d\mathbf{r}}{dt}\right)\mathbf{r} - (\mathbf{r} \cdot \mathbf{r})\frac{d\mathbf{r}}{dt}\right] = -\frac{\mu}{r^3}\left(r\frac{d\mathbf{r}}{dt}\mathbf{r} - r^2\frac{d\mathbf{r}}{dt}\right) = \mu\left(\frac{1}{r}\frac{d\mathbf{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\mathbf{r}\right) = \mu\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right),$$

Hence, $$\frac{d^2\mathbf{r}}{dt^2} \times L = \mu\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right)$$

This equation is amenable to direct integration due to the constancy of L:

$$\frac{d\mathbf{r}}{dt} \times L = \mu\left(\frac{\mathbf{r}}{r}\right) + c,$$

where c is a constant vector of integration. To clarify matters that will be elucidated subsequently, we represent c as μe. Consequently, the last expression transforms into:

$$\frac{d\mathbf{r}}{dt} \times L = \frac{\mu}{r}(\mathbf{r} + er) \tag{6}$$

Finally, compute the dot product of equation (6) with vector r, utilizing the property:

$$x \,.\, y \times z = z \,.\, x \times y$$

$$\mathbf{r}\cdot\left(\frac{d\mathbf{r}}{dt}\times L\right)=\frac{\mu}{r}\left(\mathbf{r}\cdot\mathbf{r}+e\cdot\mathbf{r}r\right),$$

$$\text{or, } L\cdot\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right)=L\cdot L=L^2=\frac{\mu}{r}\left(r^2+er^2\cos v\right),$$

where v denotes the angle between vectors e and r-

Therefore ,

$$r=\frac{L^2/\mu}{1+e\cos v} \tag{7}$$

Equation (7) serves as the all-encompassing polar coordinate representation describing a conic section, where the origin of coordinates is located at one of the focal points.

If $0 \le e \le 1$, the orbit coincides with an ellipse. Thus, we get Kepler's 1st equation of planetary motion.

Solution of Kepler's 2nd Law equation

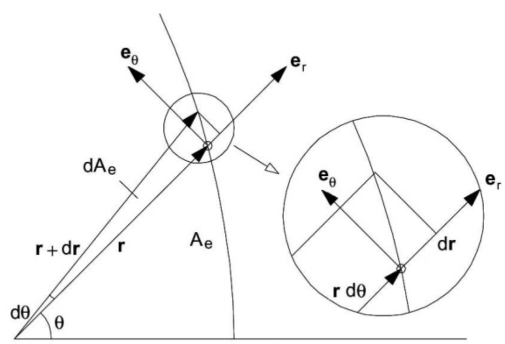

Fig. 1

Reformulate the equation in term of L,

$$\mathbf{r}\cdot(\times L)\frac{d\mathbf{r}}{dt}=\frac{dr}{dt}e_r+r\frac{d\theta}{dt}e_\theta \quad \text{(as depicted in Figure 1)} \tag{8}$$

$$L = \mathbf{r} \times \frac{d\mathbf{r}}{dt} = \mathbf{r} \times \left(\frac{dr}{dt} e_r + r \frac{d\theta}{dt} e_\theta \right) = \frac{dr}{dt} \mathbf{r} \times e_r + r^2 \frac{d\theta}{dt} e_r \times e_\theta \tag{9}$$

In the provided equation, the first term on the right-hand side equates to zero due to the collinearity of vectors r and er . Consequently, Equation (8) can be reformulated as follows:

$$L = r^2 \frac{d\theta}{dt} \tag{10}$$

A visual examination of Figure 1 reveals that.

$$r^2 d\theta = 2dA_e, \tag{11}$$

In other words, it signifies twice the area covered by the radius vector within a specific timeframe. Given the constancy of L, it can be deduced that the satellite traverses equivalent areas over equal time intervals. Thus, we get Kepler's 2^{nd} equation of planetary motion.

Solution of Kepler's 3^{rd} Law equation

Equation (7) amount of semi-latus rectus p. So,

$$\frac{L^2}{\mu} = p = a\left(1 - e^2\right),$$

or,

$$L = \left[\mu a \left(1 - e^2\right) \right]^{\frac{1}{2}} \tag{12}$$

By combining equation (10) and (11), the following conclusion can be drawn

$$L = 2 \frac{dA_e}{dt}$$

Since L is constant, it follows that

$$A_e(t) = \frac{L}{2} t = \frac{\left[\mu a \left(1 - e^2\right) \right]^{\frac{1}{2}}}{2} t \tag{13}$$

When t = T, or one orbital period,

$$A_e = \pi ab = \pi a^2 \left(1-e^2\right)^{\frac{1}{2}} \tag{14}$$

Combining eqn. (13) and (14), it follows that

$$\frac{\left[\mu a\left(1-e^2\right)\right]^{\frac{1}{2}}}{2} T = \pi a^2 \left(1-e^2\right)^{\frac{1}{2}}$$

or,

$$T = 2\pi \frac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}} \tag{15}$$

Thus, we get Kepler's 3rd equation of planetary motion.

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(*Received*, August 3, 2024)

Journal of Indian Acad. Math. ISSN: 0970-5120
Vol. 46, No. 2 (2024) pp. 265-285

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Γ_2 GRAPH OVER THE RING OF GAUSSIAN INTEGERS MODULO n

Abstract: This work is an investigation on some properties of Γ_2 graph, first introduced by R. Sen Gupta. Sen Gupta primarily focused on the Γ_2 graph, with particular attention to Z_n. We have extended the same to the ring of Gaussian integers modulo *n* and generalised the structure of this graph for different prime power factorizations of *n*. Our work also determines the connectedness and various properties of this graphs including diameter, girth, independence number, domination number, completeness, eulerian characterstics and planarity.

Keywords: Gaussian Integers, Ideal, Bipartite Graph, Complete Graph.

Mathematics Subject Classification (2020) No.: 05C25, 13A99.

1. Introduction

Zero divisor graph holds an important place among the graphs associated to a ring because it unveiled several critical and intriguing aspects of the set of zero divisors. This inspired various scholars to define and investigate different types of graphs over rings altering the adjacency relations. The concept of zero divisor graph was first introduced by Beck [6], whose primary interest was coloring of a commutative ring. Later in [4], Anderson and Livingston defined the graph $\Gamma(R)$ taking the set of non-zero zero divisors as it's vertex set, as the undirected graph where, x and y are adjacent if and only if $xy = 0$. In [3], Anderson and Badawi defined the total graph over a commutative ring, denoted by $T(\Gamma(R))$, where they used the addition operation involving the zero divisors. To have some more insights on zero divisor graphs, one can refer to [1, 2, 5, 10, 14].

In [13], Sengupta *et al*. presented a variant of the zero divisor graph, represented by Γ_1, that modifies the conventional zero divisor graph by adding multiplicative and additive operations on , as the undirected graph (V, E) in which $V = R \setminus \{0\}$ and for any $a, b \in V$, $ab \in E$ if and only if $a \neq b$ and either $a.b = 0$ or $b.a = 0$ or $a + b$ is a unit. Later in [12], Sengupta introduced another variant Γ_2, with the same set of vertices, altering the last segment of the adjacency from Γ_1.

The inspiration of this study comes from Sengupta's Γ_2 graph, which was first introduced and studied giving particular attention to Z_n in [12]. Our work is an extension to the ring of Gaussian integers modulo n. The ring of gaussian integers modulo n gives rise to distinctive graph representations based on varying adjacency criteria, providing valuable insights to symmetry, connectivity and other graph properties that reflect the behavior of modular arithmetic. First, we will consider the scenario in which n is a prime or a power of a prime. The general case is next examined, and characteristics such as diameter, girth, independence number, domination number, completeness, eulerian characteristics, and planarity are determined.

2. Preliminaries

Sengupta defined the graph Γ_2 in the following way:

Definition 2.1 (Sen Gupta, [12]): *For a ring R, Γ_2 is an undirected graph (V, E) in which $V = R \setminus \{0\}$ and for any $a, b \in V$, $ab \in E$ if and only if $a \neq b$ and either $a.b = 0$ or $b.a = 0$ or $a + b$ is a zero-divisor (including 0).*

Before delving into the main results, we first outline some essential notations, definitions and theorems that were instrumental in analyzing the graph Γ_2 for $\mathbb{Z}_n[i]$.

Let R be a ring with unity and Reg(R) denotes the set of units of the ring R, whereas $Z(R)$ denotes the set of zero divisors of R. Throughout this paper, we will consider $\mathrm{Reg}(\Gamma_2(R))$ as the (induced) subgraph of $\Gamma_2(R)$ with the vertices Reg(R) and $Z(\Gamma_2(R))$ as the (induced) subgraph of $\Gamma_2(R)$ with the vertices $Z(R)$.

With the usual complex number operations and norm $N(a+ib) = a^2 + b^2$, the set of complex numbers $a+ib$, where a and b are integers, forms an Euclidean domain. This domain is called the ring of Gaussian integers and is denoted by $\mathbb{Z}[i]$. For a natural number n, let $< n >$ be the principal ideal generated by n in $\mathbb{Z}[i]$, and let $\mathbb{Z}_n$ be the ring of integers modulo n. In [8], Dresden and Dymacek obtained the following result:

Theorem 2.1 (Dresden and Dymacek, [8]):

$$\mathbb{Z}[i]/< n > \cong \mathbb{Z}_n[i] = \{\overline{a} + i\overline{b} : \overline{a}, \overline{b} \in \mathbb{Z}_n\}$$

The above result directly implies that $\mathbb{Z}_n[i]$ is a principal ideal ring. This ring is called the ring of Gaussian integers modulo n.

In this paper, p and q will denote prime integers such that $p \equiv 1$ (mod 4) and $q \equiv 3$ (mod 4). It is known that, $p \equiv 1$ (mod 4) if and only if there are integers a, b such that $p = a^2 + b^2$. In addition to the results already available on $\mathbb{Z}_n[i]$, Osba *et al*. in [11] further added the following:

Lemma 2.1 (Osba *et al*., [11]): *$\overline{a} + i\overline{b}$ is a unit in $\mathbb{Z}_n[i]$ if and only if $a^2 + b^2$ is a unit of $\mathbb{Z}_n$.*

Corollary 2.1 (Osba *et al*., [11]): *If $n = \prod_{j=1}^{s} a_j^{k_j}$ is the prime power factorization of the positive integer n, then $\mathbb{Z}_n[i]$ is the direct product of the rings $\mathbb{Z}_{a_j^{k_j}}[i]$.*

While $\mathbb{Z}_n$ is a local ring if n is a power of a prime, this is not the case for $Z_n[i]$. The finding of Osba *et al*. regarding this is as follows:

Theorem 2.2 (Osba *et al*., [11]): *If* $m = p^k$ *for some prime p and positive integer k, then* $\mathbb{Z}_m[i]$ *is a local ring if and only if* $p = 2$ *or* $p \equiv 3$ (mod 4).

For usual graph-theoretic definitions and notations, one can look at [9]. Let nbd (v) be the set of vertices adjacent to v. A graph is called planar if it can be drawn in the plane so that its edges intersect only at their ends. A connected graph is called eulerian if there exists a closed trail containing every edge of G. A dominating set for a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is adjacent to atleast one member of D. The domination number $\delta(G)$ is the minimum size of a dominating set in G. An independent vertex set of a graph G is a subset of the vertices of G such that no two vertices in the subset are adjacent to each other. The size of the maximum independent vertex set is called the independence number and is denoted by $\alpha(G)$.

3. Γ_2 Graph for $\mathbb{Z}_{2n}[i]$:

In this section, we consider the Γ_2 graph for $\mathbb{Z}_{2n}[i]$. As $2 = (1+i)(1-i) = -i(1+i)^2$, so $\mathbb{Z}_2[i]$ is isomorphic to $\mathbb{Z}[i]/< (1+i)^2 >$, where $\{0, 1+i\}$ is its only maximal ideal. As both $1+i$ and $1-i$ will generate the same maximal ideal in $\mathbb{Z}_2[i]$, therefore $Z(\mathbb{Z}_2[i]) = < 1+i >$, which implies that the subgraph induced by the zero divisors of $\mathbb{Z}_2[i]$ is the graph with one vertex only and no edge.

Now if n is an integer greater than 1, then $2^n = (-i)^n (1+i)^{2n}$ and $\mathbb{Z}_{2^n}[i] = \mathbb{Z}[i]/< 2^n > = \mathbb{Z}[i]/< \left(1+i\right)^{2n} >$. Therefore $\mathbb{Z}_2[i]$ is local with its only maximal ideal $M = < 1+i >$ and $Z(\mathbb{Z}_{2^n}[i]) = < 1+i > = \{a + ib : a, b \in \mathbb{Z}_{2^n}$ and a, b are both even or both odd$\}$. Hence, set of zero divisors are basically all those elements $a + ib$ in $\mathbb{Z}_2[i]$ where a, b are both even or both odd. The number of units in $\mathbb{Z}_2[i]$ is 2^{2n-1}, see [7]. Therefore, number of zero divisors in $\Gamma_2(\mathbb{Z}_{2^n}[i])$ is $2^{2n} - 2^{2n-1} - 1 = 2^{2n-1} - 1$.

Theorem 3.1: $\Gamma_2(\mathbb{Z}_{2^n}[i])$ *is not connected and it has exactly two components consisting of the zero divisors and units of* $\mathbb{Z}_{2n}[i]$ *respectively. The first is* $K_{2^{2n-1}-1}$, *while the second is* $K_{2^{2n-1}}$.

Proof: If u is a unit and y is a zero divisor, $u + y$ is always of the form $a + ib$, where one is odd and the other is even, hence not a zero divisor. Therefore a unit and a zero divisor is never adjacent making the graph disconnected. Since, $Z(\mathbb{Z}_{2^n}[i])$ is closed under addition, thus $Z(\Gamma_2(\mathbb{Z}_{2^n}[i]))$ is the complete graph $K_{2^{2n-1}-1}$. For any $u, v \in \mathrm{Reg}(\mathbb{Z}_{2^n}[i])$, obviously $uv \neq 0$. But $u + v$ is always of the form $a + ib$ where a and b are both even or both odd, which implies that $u + v \in Z(\mathbb{Z}_{2^n}[i])$. Therefore $\mathrm{Reg}(\Gamma_2(\mathbb{Z}_{2^n}[i]))$ is the complete graph $K_{2^{2n-1}}$. □

Example 3.1: *Here two examples of* Γ_2 *graph of* $\mathbb{Z}_{2^n}[i]$ *are displayed in* Figure 1 *and* Figure 2.

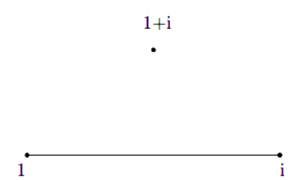

Figure 1: $\Gamma_2\ (\mathbb{Z}_{2^n}[i])$

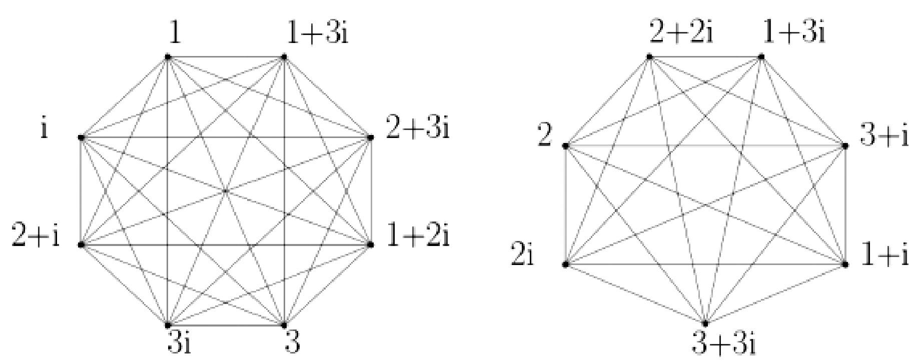

Figure 2: $\Gamma_2\ (\mathbb{Z}_4[i])$

Following the results from Theorem 3.1, we determine the diameter (diam) and girth (gr) of $\Gamma_2(\mathbb{Z}_{2^n}[i])$.

Corollary 3.1:

1. $\text{diam}(\Gamma_2(\mathbb{Z}_{2^n}[i])) = \infty$.

2. $\text{diam}(Z(\Gamma_2(\mathbb{Z}_{2^n}[i]))) = 1$.

3. $\text{diam}(\text{Reg}(\Gamma_2(\mathbb{Z}_{2^n}[i]))) = 1$.

Corollary 3.2: *If* $n \geq 2$, $\text{gr}(\Gamma_2(\mathbb{Z}_{2^n}[i]) = 3$.

4. Γ_2 Graph for $\mathbb{Z}_{q^n}[i], q \equiv 3 \pmod 4$

For a prime q congruent to 3 (mod 4), as $\mathbb{Z}_q[i]$ is isomorphic to the field $Z[i]/<q>$, therefore $\mathbb{Z}_q[i]$ has no nonzero zero divisors. If $n > 1$, then $Z_{q^n}[i] \cong \mathbb{Z}[i]/<q^n>$ is a local ring with the maximal ideal $<q>$. The number of units in $\mathbb{Z}_{q^n}[i]$ is $q^{2n}q^{2n-2}$, see [7]. Therefore the number of zero divisors in $\Gamma_2(\mathbb{Z}_{q^n}[i])$ is $q^{2n-2} - 1$. We now look at the structure of Γ_2 by moving to the following results.

Theorem 4.1: $\Gamma_2(\mathbb{Z}_q[i])$ *is always disconnected having* $(q^2 - 1)/2$ *components, each being equal to* K_2.

Proof: Since, $\mathbb{Z}_q[i]$ has no nonzero zero divisors, no criterias of adjacency come into play except whenever the usual sum results to 0, which implies that every pair of additive inverses will form K_2. As the number of vertices is $q^2 - 1$, hence, there are $(q^2 - 1)/2$ components, each K_2 in $\Gamma_2(\mathbb{Z}_q[i])$. □

Example 4.1: *The graph* $\Gamma_2(\mathbb{Z}_3[i])$ *is given in* Figure 3.

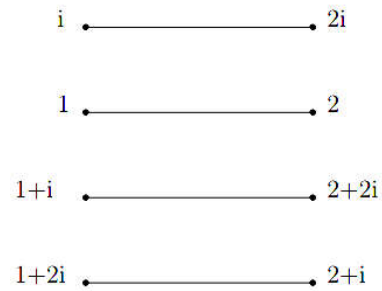

Figure 3: $\Gamma_2(\mathbb{Z}_3[i])$

Theorem 4.2: *For* $n \geq 2$, $\Gamma_2(\mathbb{Z}_{q^n}[i])$ *is disconnected with* $(q^2+1)/2$ *components. The component of the graph* $\Gamma_2(\mathbb{Z}_{q^n}[i])$; *having the zero divisors as the vertices is complete* $K_{q^{2n-2}-1}$. *Also the set of units form* $(q^2-1)/2$ *number of complete bipartite graphs each* $K_{q^{2n-2},\, q^{2n-2}}$.

Proof: Suppose there exists at least one zero divisor x which is adjacent to a unit v, then $x+v \in Z(\mathbb{Z}_{q^n}[i])$. Now the fact that $\mathbb{Z}_{q^n}[i]$ is a local ring with maximal ideal $< q >$, implies that the zero divisors are precisely the elements of the ideal $< q >$. So $x+v \in < q > \Rightarrow v = (x+v)+(-x) \in < q >$, hence this is a contradiction. Thus, $\Gamma_2(\mathbb{Z}_{q^n}[i])$ is never connected as zero divisors and units are not connected.

Any x, y belonging to the set of zero divisors will result in $x+y$ being a zero divisor. Therefore, the component of the graph $\Gamma_2(\mathbb{Z}_{q^n}[i])$ having the zero divisors as the vertices is complete and it is $K_{q^{2n-2}-1}$.

As $Z(\mathbb{Z}_{q^n}[i])$ is an ideal of $\mathbb{Z}_{q^n}[i]$, for two distinct elements in $u+Z(\mathbb{Z}_{q^n}[i])$, say $u+z_1$ and $u+z_2$, where $u \in \mathrm{Reg}(\mathbb{Z}_{q^n}[i])$; the adjacency of these two elements will imply; $(u+z_1)+(u+z_2) \in Z(\mathbb{Z}_{q^n}[i]) \Rightarrow 2u \in Z(\mathbb{Z}_{q^n}[i])$ $\Rightarrow 2 \in Z(\mathbb{Z}_{q^n}[i])$, hence this is a contradiction. So no two elements of

$u + Z(\mathbb{Z}_{q^n}[i])$ are adjacent to each other. Again, $u + Z(\mathbb{Z}_{q^n}[i])$ and $-u + Z(\mathbb{Z}_{q^n}[i])$ are disjoint. We prove this by method of contradiction. Suppose $u + Z(\mathbb{Z}_{q^n}[i])$ and $-u + Z(\mathbb{Z}_{q^n}[i])$ are not disjoint. Then for some $z_1, z_2 \in Z(\mathbb{Z}_{q^n}[i])$ and $v \in \text{Reg}(\mathbb{Z}_{q^n}[i])$, $u + z_1 = v = -u + z_2 \Rightarrow u = v - z_1$. Thus, $-v + z_1 + z_2 = v$, $\Rightarrow z_1 + z_2 = 2v$ a zero divisor $\Rightarrow v$ is a zero divisor, which is a contradiction. Therefore $u + Z(\mathbb{Z}_{q^n}[i])$ and $-u + Z(\mathbb{Z}_{q^n}[i])$ are disjoint. But since each element of $u + Z(\mathbb{Z}_{q^n}[i])$ is adjacent to $-u + Z(\mathbb{Z}_{q^n}[i])$, $u + Z(\mathbb{Z}_{q^n}[i]) \cup -u + Z(\mathbb{Z}_{q^n}[i])$ is a complete bipartite subgraph of $\text{Reg}(\mathbb{Z}_{q^n}[i])$. Also for $v + z_1$ being adjacent to $u + z_2$ where $u, v \in \text{Reg}(\mathbb{Z}_{q^n}[i])$ and $z_1, z_2 \in Z(\mathbb{Z}_{q^n}[i]), u + v \in Z(\mathbb{Z}_{q^n}[i]) \Rightarrow v + Z(\mathbb{Z}_{q^n}[i]) = -u + Z(\mathbb{Z}_{q^n}[i])$; implying anything connected to $u + Z(\mathbb{Z}_{q^n}[i])$ is basically $-u + Z(\mathbb{Z}_{q^n}[i])$.

Now, $\left|u + Z(\mathbb{Z}_{q^n}[i])\right| = \left|Z(\mathbb{Z}_{q^n}[i])\right| = q^{2n-2}$. Hence regular elements form $K_{q^{2n-2}, q^{2n-2}}$. Number of units being $q^{2n}q^{2n-2}$, there are number of complete bipartite graphs each $K_{q^{2n-2}, q^{2n-2}}$. □

Following two results ensue directly from Theorem 4.1 and Theorem 4.2.

Corollary 4: 1. $\text{diam}(\Gamma_2((\mathbb{Z}_{q^n}[i]))) = \infty$.

2. $\text{diam}(Z(\Gamma_2((\mathbb{Z}_{q^n}[i]))) = 1, \; n \geq 2$.

3. $\text{diam}(\text{Reg}(\Gamma_2((\mathbb{Z}_{q^n}[i]))) = \infty$.

Corollary 4.2: $\text{gr}(\Gamma_2(\mathbb{Z}_{q^n}[i]) = 3, \; n \geq 2$.

5. $\Gamma 2$ **Graph for** $\mathbb{Z}_{p^n}[i], p \equiv 1$ (mod 4)

If p is a prime congruent to 1 modulo 4, there exists $a,\ b \in Z$ such that $p = a^2 + b^2 = (a + ib)(aib)$. Since $\mathbb{Z}[i]$ is an unique factorization domain and $a + ib$ and $a - ib$ are gaussian primes in $\mathbb{Z}[i], < a + ib >$ and $< aib >$ are the only maximal ideals in $\mathbb{Z}[i]$ containing p. Thus, we have $\mathbb{Z}_p[i] \cong \mathbb{Z}[i] / < p > \cong (\mathbb{Z}[i] / < a + ib >) \times (\mathbb{Z}[i] / < a - ib >)$. It is trivial to see that in $\mathbb{Z}_p[i]$, the intersection of the maximal ideals is the singleton set $\{0\}$. Similarly, $p^n = (a^2 + b^2)^n = (a + ib)^n (aib)^n$. Therefore p^n is contained in only two maximal ideals in $\mathbb{Z}[i]$, $< a + ib >$ and $< a - ib >$. Hence, $Z(\mathbb{Z}_{p^n}[i]) = < a + ib > \cup < a - ib >$ and $Z(\mathbb{Z}_{p^n}[i])$ is closed under multiplication by elements of the ring as $Z(\mathbb{Z}_{p^n}[i])$ is union of prime ideals. The number of units in $Z(\mathbb{Z}_{p^n}[i])$ is $(p^n p^{n-1})^2$, see [7]. Hence number of zero divisors in $\Gamma_2(\mathbb{Z}_{p^n}[i])$ is $2p^{2n-1}p^{2n-2} - 1$. In the ring $\mathbb{Z}_{p^n}[i]$, the cardinality of both maximal ideals is the same. Additionally, the set $W = \{a_1 p + b_1 pi : a_1,\ b_1 \in \mathbb{Z}_{p^n}\}$, which is the intersection of the ideals $< a + ib >$ and $< a - ib >$, forms an ideal itself, with a cardinality of p^{2n-2}. Furthermore, if two elements If $x \in < a + ib > \setminus W, y \in < a + ib > \setminus W$, then $x + y$ is a unit.

Theorem 5.1: *For every unit u, there exists a pair x, y, where* $x \in < a + ib > \setminus W$ *and* $y \in < a - ib > \setminus W$ *such that* $u = x + y$. *In* $\mathbb{Z}_p[i]$, *this pair* $x, y \in Z(\mathbb{Z}_p[i])$ *is unique.*

Proof: Existence: Suppose, $u = c + id \in \mathrm{Reg}(\mathbb{Z}_{p^n}[i])$. Then, $u = c + id = [d.(2b)^{-1} i.c.(2b)^{-1}](a + ib) + [-d.(2b)^{-1} + i.c.(2b)^{-1}](a - ib)$. Now $(2b)^{-1}$ exists as $b \in \mathbb{Z}_{p^n}$ and $b \neq 0$, $gcd(b, p) = 1 \Rightarrow gcd(2b, p) = 1 \Rightarrow gcd(2b, p^n) = 1$. Hence, there exists a pair $x, y \in Z(\mathbb{Z}_{p^n}[i])$ such that, $u = x + y$. Also it is clear that $x \in < a + ib > \setminus W$, $y \in < a + ib > \setminus W$.

Uniqueness: Suppose, there are two distinct pairs of zero divisors $x,\ y$ and $m,\ n$ such that $x + y = u = m + n$ where $x, m \in < a + ib > \setminus W$ and $y, n \in < a - ib > \setminus W$, also $x,\ m$ and $y,\ n$ are different.

Then for some $\alpha \in \mathbb{Z}_p[i], x = m + \alpha(a+ib) \Rightarrow m + \alpha(a+ib) + y = u$. Again, $m + n = u \Rightarrow \alpha(a+ib) + y = n \Rightarrow \alpha(a+ib) = n - y \in \langle a - ib \rangle$, which is a contradiction as y and n are different, so $n - y \neq 0$ and the only possibility here is 0. If one of these two pairs is identical, suppose $n = y$, then for $x = m + \alpha(a+ib), \alpha(a+ib) = n - y = 0 \Rightarrow x = m$, which is a contradiction. Hence, for every unit u, there exists a unique pair $x, \ y \in Z(\mathbb{Z}_p[i])$ such that, $u = x + y$.

If a unit u can be expressed as sum of two zero divisors, we call them associated zero divisors of u. In $\mathbb{Z}_{p^n}[i]$, if x and y are associated zero divisors of a unit u, then the other associated pairs are $x + w_1$ and $y + w_2$ such that $w_1, w_2 \in W$ and $w_1 + w_2 = 0$. If for a unit u, the associated zero divisors are x and y, then u is adjacent to $-x$ and $-y$.

Theorem 5.2: *For a unit* $u, nbd(u) = \{-x + \langle a - ib \rangle\} \cup$, $\{-y + \langle a + ib \rangle\}$ *where* $x \in \langle a + ib \rangle \backslash W$ *and* $y \in \langle a - ib \rangle \backslash W$ *such that* $x + y = u$.

Proof: It is clear from Theorem 5.1 that for every unit u, there exists a pair $x, \ y \in Z(\mathbb{Z}_{p^n}[i])$ such that, $u = x + y$. Therefore every term in the coset $\{-x + \langle a - ib \rangle\}$ is adjacent to u as for $z' \in \langle a - ib \rangle, (-x + z')$ $+ u = (-x + z') + (x + y) = y + z' \in \langle a - ib \rangle$. Similarly, every term in $\{-y + \langle a + ib \rangle\}$ is adjacent to u and hence the result follows.

Corollary 5.1: *For any two distinct units* u_1 *and* u_2, *there exists atleast two units* u' *and* v' *such that* $u', v' \in \mathrm{nbd}(u_1) \cap \mathrm{nbd}(u_2)$.

Proof: Suppose the associated zero divisors of u_1 and u_2 are respectively x, y and x', y'. Then $\mathrm{nbd}(u_1) = \{-x + \langle a - ib \rangle\} \cup \{-y + \langle a + ib \rangle\}$ and similarly $\mathrm{nbd}(u_2) = \{-x' + \langle a - ib \rangle\} \cup \{-y' + \langle a - ib \rangle\}$. Hence, $-x - y'$ and $-x' - y$ are in the intersection of these two neighbourhoods. □

Theorem 5.3: $\Gamma_2(\mathbb{Z}_p[i])$ *is connected. Also the subgraph induced by the zero divisors is the complete graph* K_{2p-2}.

Proof: Assume that $x, y \in Z(\mathbb{Z}_p[i])$ and both belong to the same ideal. Then they are adjacent. But if x, y belong to different ideals, then $x = \alpha(a + ib)$ and $y = \beta(a - ib)$, where $\alpha, \beta \in \mathbb{Z}_p[i]$. Thus, $xy = \alpha(a + ib)\beta(a - ib) = \alpha\beta(a^2 + b^2) = \alpha\beta p = 0$, which implies that x and y are adjacent. Hence, the subgraph induced by the zero divisors is the complete graph K_{2p-2}.

Again for any unit u, there exists unique pairs of associated zero divisors $x, y \in Z(\mathbb{Z}_p[i])$ such that, $u = x + y$. Then for some $m \in Z(\mathbb{Z}_p[i])$ not adjacent to u, there will be a path $u - (x) - m$ or $u - (y) - m$.

Now from Corollary 5.1, for two non adjacent units u and v, there exists units u' and v' such that $u', v' \in nbd(u) \cap nbd(v)$. Thus, there will be paths $u - u' - v$ or $u - v' - v$. Hence, $\Gamma_2(\mathbb{Z}_p[i])$ is connected. $\square$

Example 5.1: *The connectedness property of* $\Gamma_2(\mathbb{Z}_5[i])$ *can be seen in* Figure 4.

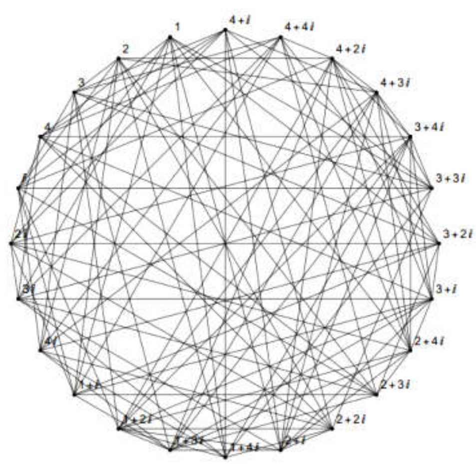

Figure 4: $\Gamma_2(\mathbb{Z}_5[i])$

Corollary 5.2: *In* $\Gamma_2(\mathbb{Z}_p[i])$, $\deg(u) = 2p-1$, *where* u *is a unit. On the other hand* $\deg(z) = 3p-4$, *where* z *is a zero divisor.*

Proof: Since $Z(\Gamma_2(\mathbb{Z}_p[i]))$ is complete with $2p-2$ vertices, number of adjacent zero divisors to a zero divisor is $2p-3$. Also a zero divisor is adjacent to $p-1$ units. Hence, degree of a zero divisor is $3p-4$. On the other hand, there are total $2p-1$ zero divisors including 0. As a unit is adjacent to a vertex if only their sum results in a zero divisor (including 0), hence degree of a unit is $2p-1$. □

Theorem 5.4: *For* $n \geq 2$, $Z(\Gamma_2(\mathbb{Z}_{p^n}[i]))$ *is* p^{2n-2}*-connected and* $\Gamma_2(\mathbb{Z}_{p^n}[i])$ *is connected.*

Proof: If two zero divisors x, y belong to the same ideal, then $x-y$. Again if x, y belong to different ideals and $m \in W$, then there is a path $x-m-y$. Hence, $Z(\Gamma_2(\mathbb{Z}_{p^n}[i]))$ is connected with $\text{diam}(Z(\Gamma_2(\mathbb{Z}_p[i]))) = 2$. Note that deleting every vertex from the vertex set W will disconnect the subgraph induced by the zero divisors, $Z(\Gamma_2(\mathbb{Z}_{p^n}[i]))$ in two components. Hence, $Z(\Gamma_2(\mathbb{Z}_p[i]))$ is p^{2n-2} -connected.

Next assume that u is any unit and z_1 is a zero divisor such that they are not adjacent. Since there exists unique pairs of associated zero divisors $x, y \in Z(\mathbb{Z}_{p^n}[i])$ such that $u = x + y$, thus, there will be a path $u-(x)-z_1$ or $u-(y)-z_1$ depending on whether z_1 belongs to $< a+ib >$ or $< a-ib >$. Moreover from Corollary 5.1, for any two non adjacent units u and v, there exists units u' and v' such that $u', v' \in \text{nbd}(u) \cap \text{nbd}(v)$, thus having paths $u-u'-v$ or $u-v'-v$. Hence, $\Gamma_2(\mathbb{Z}_{p^n}[i])$ is connected.

We conclude this section with the following corollaries.

Corollary 5.3: $\text{diam}(\Gamma_2(\mathbb{Z}_{p^n}[i])) = 2$.

Corollary 5.4: $\mathrm{gr}(\Gamma_2(\mathbb{Z}_{p^n}[i])) = 3$.

Corollary 5.5: *In* $\Gamma_2(\mathbb{Z}_{p^n}[i]), n \geq 2; \deg(u) = 2p^{2n-1} - p^{2n-2}$, *where* u *is a unit. On the other hand,* $\deg(z) = 2p^{2n-1} - p^{2n-2} - 2$, *where* z *is a zero divisor.*

Proof: A unit is adjacent to a vertex if only their sum is a zero divisor (including 0). Hence degree of a unit is $2p^{2n-1} - p^{2n-2}$. Since any $w \in W$ is adjacent to all the zero divisors in $\Gamma_2(\mathbb{Z}_{p^n}[i])$, therefore $\deg(w) = 2p^{2n-1} - p^{2n-2}$. Furthermore if $x \in \langle a + ib \rangle \setminus W$, then x is adjacent to all the zero divisors of $\langle a + ib \rangle$. Also, x is adjacent to $p^{2n-1} - p^{2n-2}$ units since $x + y$ is a unit for any $y \in \langle a - ib \rangle \setminus W$. Therefore,

$$\deg(x) = [p^{2n-1} - 2] + [p^{2n-1} - p^{2n-2}] = 2p^{2n-1} - p^{2n-2} - 2.$$

6. Γ2 Graph for $Z_n[i]$

In this section, we study the general case where n is any prime power factorization of distinct primes. Note that the set of zero divisors is nothing but the union of all the maximal ideals in $\mathbb{Z}_n[i]$. If $n = 2^m \prod_{j=1}^{s} q_j^{r_j} \prod_{i=1}^{t} p_i^{l_i}$, where $p_i \equiv 1(mod 4)$ and $q_j \equiv 3(mod 4)$, number of units in $\mathbb{Z}_n[i]$ is $2^{2m-1} \prod_{j=1}^{s} (q_j^{2r_j} - q_j^{2r_j - 2}) \prod_{i=1}^{t} (p_i^{l_i} - p_i^{l_i - 1})^2$. Thus number of zero divisors is $n^2 - 2^{2m-1} \prod_{j=1}^{s} (q_j^{2r_j} - q_j^{2r_j - 2}) \prod_{i=1}^{t} (p_i^{l_i} - p_i^{l_i - 1})^2$. The main goal of this section is to generalise the structure of $\Gamma_2(\mathbb{Z}_n[i])$ and to investigate the various properties of this graph like planarity, Eulerian characteristics, domination number and independence number.

Theorem 6.1: $\Gamma_2(\mathbb{Z}_n[i])$ *is not a complete graph for any n.*

Proof: $\Gamma_2(\mathbb{Z}_{2^r}[i])$ and $\Gamma_2(\mathbb{Z}_{q^r}[i])$ are always disconnected. For all other

n, $\mathbb{Z}_n[i]$ has atleast two units 1 and -1. Hence 1 and -2 are never adjacent. □

Theorem 6.2: $\Gamma_2(\mathbb{Z}_n[i])$ *is always connected if and only if* $n = p^r$ *or the prime power factorization of n has more than one prime factor and in the latter case* $\text{diam}(\Gamma_2(\mathbb{Z}_n[i])) = 2$ *and* $\text{gr}(\Gamma_2(\mathbb{Z}_n[i])) = 3$.

Proof: From the previous results, it is clear that $\Gamma_2(\mathbb{Z}_{2^r}[i])$ and $\Gamma_2(\mathbb{Z}_{q^r}[i])$ are always disconnected. Again $\Gamma_2(\mathbb{Z}_{p^n}[i])$ is connected with diameter 2 and girth 3. Now for the next part, from Theorem 2.8 of [12], we have, for $R = R_1 \times R_2 \times R_3 \ldots \times R_n$, where R_i's are rings with unity and $n > 1, n \in N$, $\Gamma_2(R)$ is connected with $\text{diam}(\Gamma_2(R)) \leq 2$ and $\text{gr}(\Gamma_2(R)) = 3$. If $n = \prod_{j=1}^{s} a_j^{k_j}$ is the prime power factorization of the positive integer n, then $\mathbb{Z}_n[i]$ is the direct product of the rings $\mathbb{Z}_{a_j^{k_j}}[i]$. But $\Gamma_2(\mathbb{Z}_n[i])$ is never a complete graph. Hence, $\Gamma_2(\mathbb{Z}_n[i])$ is always connected with $\text{diam}(\Gamma_2(\mathbb{Z}_n[i])) = 2$ and $\text{gr}(\Gamma_2(\mathbb{Z}_n[i])) = 3$. □

Corollary 6.1: $\Gamma_2(\mathbb{Z}_{q_1 q_2}[i])$ *is connected with diameter 2 and girth 3. The subgraph induced by the zero divisors is* $K_{q_1^2 + q_2^2 - 2}$.

Proof: The first part is trivial. Now for the second part, as the set of zero divisors is the union of the ideals $< q_1 >$ and $< q_2 >$, therefore zero divisors belonging to the same ideal will be adjacent to each other. If x and y belong to different ideals, then $xy = 0$. Hence, zero divisors belonging to different ideals will also be adjacent, thus, forming a complete graph. Since the number of non-zero zero divisors in $\mathbb{Z}_{q_1 q_2}[i]$ is $(q_1 q_2)^2 - [(q_1^2 - 1)(q_2^2 - 1)] - 1 = q_1^2 + q_2^2 - 2$, hence, the subgraph induced by the zero divisors is $K_{q_1^2 + q_2^2 - 2}$. □

6.1 Planarity of $\Gamma_2(\mathbb{Z}_n[i])$

Theorem 6.3 (Kuratowski, [9]): *A graph is planar if and only if it contains no subgraph homeomorphic to* K_5 *or* $K_{3.3}$.

We also state the following result analogous to Kuratowski's theorem.

Theorem 6.4: *$\Gamma_2(\mathbb{Z}_n[i])$ is planar iff $n = 2$ or n is a prime $q \equiv 3 \pmod 4$.*

Proof: Since, $\Gamma_2(\mathbb{Z}_2[i])$ is disconnected with the two components K_1 and K_2, the graph is planar. When n is even, say $n = 2m$, the number of zero divisors in the ideal $< 1 + i >$ is $2m^2 \geq 8$. So, the subgraph of $\Gamma_2(\mathbb{Z}_n[i])$ induced by the elements of this ideal will be the complete graph K_{2m^2-1}. Hence, it will contain a subgraph homeomorphic to K_5.

When n is a prime $q \equiv 3 \pmod 4$, $\Gamma_2(\mathbb{Z}_n[i])$ is disconnected having $(q^2-1)/2$ components, each being equal to K_2, resulting the graph to be planar. But when $n = q^r, r > 1$, the component $Z(\Gamma_2(\mathbb{Z}_n[i]))$ is the complete graph $K_{q^{2r-2}-1}$, while the set of units form $(q^2-1)/2$ number of complete bipartite graphs each $K_{q^{2r-2},q^{2r-2}}$. Therefore it contains subgraph homeomorphic to both K_5 and $K_{3,3}$. When n is a prime $p \equiv 1 \pmod 4$, $Z(\Gamma_2(\mathbb{Z}_n[i]))$ is the complete graph K_{2p-2}. Again when $n = p^r, r > 1$, $\Gamma_2(\mathbb{Z}_n[i])$ has a complete subgraph $K_{q^{2r-2}-1}$ with the vertices of *W*. So $\Gamma_2(\mathbb{Z}_{p^r}[i])$ will always contain a subgraph homemorphic to K_5 for any *r*. Therefore whenever a prime $p \equiv 1 \pmod 4$ occurs in the prime power factorization of *n*, $\Gamma_2(\mathbb{Z}_n[i])$ will always contain a subgraph homemorphic to K_5. Hence, $\Gamma_2(\mathbb{Z}_n[i])$ is planar *iff* $n = 2$ or n is a prime $q \equiv 3 \pmod 4$ □

6.2 Non-eulerian characteristic of $\Gamma_2(\mathbb{Z}_n[i])$

Theorem 6.5 (Euler, [9]): *A connected graph G is eulerian if and only if the degree of each vertex of G is even.*

We state the following result now.

Theorem 6.6: $\Gamma_2(\mathbb{Z}_n[i])$ *is not eulerian for any* n.

Proof: Since, $\Gamma_2(\mathbb{Z}_{2^r}[i])$ and $\Gamma_2(\mathbb{Z}_{q^r}[i])$ are disconnected, they are not eulerian. Note that for any unit u in $\mathbb{Z}_n[i]$, u is adjacent to the vertices of the form $(z-u)$, where z is any zero divisor. But if n is even, then $2u \in Z(\mathbb{Z}_n[i])$ implying $u \in \{z-u : z \in Z(\mathbb{Z}_n[i])\}$. Thus, $\deg(u) = |\{z-u : z \in Z(\mathbb{Z}_n[i])\}| - 1$. Otherwise for the case where n is odd, $\deg(u) = |\{z-u : z \in Z(\mathbb{Z}_n[i])\}|$. When $n = 2^m \prod_{j=1}^{s} q_j^{r_j} \prod_{i=1}^{t} p_i^{l_i}, \deg(u) = n^2 - 2^{2m-1} \prod_{j=1}^{s} (q_j^{2r_j} - q_j^{2r_j-2}) \prod_{i=1}^{t} (p_i^{l_i} - p_i^{l_i-1})^2 - 1$ which is odd. But if $n = \prod_{j=1}^{s} q_j^{r_j} \prod_{i=1}^{t} p_i^{l_i}$, then $\deg(u) = n^2 - \prod_{j=1}^{s} (q_j^{2r_j} - q_j^{2r_j-2}) \prod_{i=1}^{t} (p_i^{l_i} - p_i^{l_i-1})^2$ which is again odd.

Hence, $\Gamma_2(\mathbb{Z}_n[i])$ is not eulerian for any n. □

6.3 The Domination Number for $\Gamma_2(\mathbb{Z}_n[i])$

Theorem 6.7: *The domination number* $\delta(\Gamma_2(\mathbb{Z}_n[i])) = 2$ *if* $n = 2^r$.

Proof: Since $\Gamma_2(\mathbb{Z}_{2^r}[i])$ has two components $K_{2^{2r-1}-1}$ and $K_{2^{2r-1}}$, we can include one vertex from each of these components in a set so that every vertex outside this set are adjacent to atleast one vertex in the set. □

Theorem 6.8: *The domination number*

$$\delta(\Gamma_2(\mathbb{Z}_n[i])) = \begin{cases} (q^2-1)/2, & n = q \\ q^2, & n = q^r, r > 1. \end{cases}$$

Proof: Since $\Gamma_2(\mathbb{Z}_q[i])$ is disconnected having $(q^2-1)/2$ components of K_2, therefore each component will contribute one vertex each to the dominating set. Thus, $\delta(\Gamma_2(\mathbb{Z}_q[i])) = (q^2-1)/2$.

For $r \geq 2$, $\Gamma_2(\mathbb{Z}_{q^r}[i])$ is disconnected with $(q^2+1)/2$ components. Since, the component of the graph $\Gamma_2(\mathbb{Z}_{q^r}[i])$ having the zero divisors as the vertices is complete $K_{q^{2r-2}-1}$, we can take only one vertex from this component. As the set of units form $(q^2-1)/2$ number of complete bipartite graphs each $K_{q^{2r-2},\ q^{2r-2}}$, we can include two vertices from each of these components, both from different partitions. Hence, $\delta(\Gamma_2(\mathbb{Z}_{q^r}[i])) = 1 + 2 \times [(q^2-1)/2] = 1 + q^2 - 1 = q^2$. □

Theorem 6.9: *The domination number*

$$\delta(\Gamma_2(\mathbb{Z}_n[i])) = \begin{cases} (p-1), & n = p \\ p, & n = p^r, r > 1. \end{cases}$$

Proof: In $\Gamma_2(\mathbb{Z}_p[i])$, zero divisors form a complete graph K_{2p-2}. If $x \in \langle a+ib \rangle \backslash W$, then x is adjacent to $p-1$ number of unique units and the set of units adjacent to x is $-x + \{\langle a-ib \rangle \backslash W\}$. Note that, inclusion of each element from $\langle a+ib \rangle \backslash W$ in a set will result in all the outside vertices adjacent to atleast one vertex of this set. Hence, $\delta(\Gamma_2(\mathbb{Z}_p[i]) =$ Number of zero divisors in $\langle a-ib \rangle \backslash W = p-1$.

On the other hand, in $\Gamma_2(\mathbb{Z}_{p^r}[i]), r > 1$, the zero divisors from $\{\langle a+ib \rangle \backslash W\}$ and $\{\langle a-ib \rangle \backslash W\}$ are not adjacent. Vertices from W are adjacent to elements of both the ideals, but no units. So first we include one element (not 0) from W in the dominating set. For $x \in \langle a+ib \rangle \backslash W$, each element of the set $\{x + w_i : w_i \in W\}$ has the same set of units adjacent to them. Thus, for any $x \in \langle a+ib \rangle \backslash W$, there are p^{2n-2} zero divisors having the same set of adjacent units. If some $x' \notin \{x + w_i : w_i \in W\}$, then x' has unique set of units adjacent to it. Therefore total number of unique sets of units $= [p^{2n-1} - p^{2n-2}] / p^{2n-2} = p-1$. Adding only those zero divisors from $\langle a+ib \rangle \backslash W$ which have different set of units adjacent to it, we get the the least dominating set. □

Hence, $\delta(\Gamma_2(\mathbb{Z}_n[i])) = 1 + (p-1) = p$. □

6.4 The Independence Number for $\Gamma_2(\mathbb{Z}_n[i])$

Theorem 6.10: *The independence number,* $\alpha(\Gamma_2(\mathbb{Z}_n[i])) = 2$ *if* $n = 2^r$.

Proof: Since in $\Gamma_2(\mathbb{Z}_{2^r}[i])$, both the components are complete, we can include only one vertex from each of these components in the independent set so that no two vertices inside this set are adjacent. □

Theorem 6.11: *The independence number*

$$\alpha(\Gamma_2(\mathbb{Z}_n[i])) = \begin{cases} \frac{q^2-1}{2} & n = q \\ 1 + \frac{q^{2r}-q^{2r-2}}{2}, & n = q^r,\ r > 1. \end{cases}$$

Proof: As $\Gamma_2(\mathbb{Z}_q[i])$ has $(q^2-1)/2$ components each being equal to K_2, so only one vertex from each component can be taken into the independent set. Hence, $\alpha(\Gamma_2(\mathbb{Z}_q[i])) = (q^2-1)/2$.

For $r \geq 2$, $\Gamma_2(\mathbb{Z}_{q^r}[i])$ has a component of the graph having the zero divisors, which is $K_{q^{2r-2}-1}$. Therefore we can only take one vertex from this component. Since the set of units form $(q^2-1)/2$ number of complete bipartite graphs each $K_{q^{2r-2},\ q^{2r-2}}$, we can take q^{2r-2} vertices from one partition of a component as they are not adjacent to one another. Hence, number of elements in the largest independent set that is $\alpha(\Gamma_2(\mathbb{Z}_{q^r}[i])) = 1 + [(q^2-1)/2] \times q^{2r-2} = 1 + [\frac{q^{2r}-q^{2r-2}}{2}]$. □

In $\mathbb{Z}_{p^r}[i]$, for any zero divisor $x \notin W$, we defne the elements of $-x + W$ as the W-inverses of x. Note that for $W = \{0\}$, they are simply the additive inverses of each other. It is trivial to see that, elements from $x + W$ has the same set of

W-inverses. For some $x' \notin x+W$, $x'+W$ will have di erent W-inverses than the elements from $x+W$. The set of W-inverses of the elements of $-x+W$ is nothing but the set $x+W$.

Theorem 6.12: *The independence number*

$$\alpha(\Gamma_2(\mathbb{Z}_n[i])) = \begin{cases} 1+(\frac{p-1}{2})^2, & n = p \\ 2+p^{2n-2}(\frac{p-1}{2})^2, & n = p^r \ for \ r > 1. \end{cases}$$

Proof: In $\Gamma_2(\mathbb{Z}_{p^r}[i])$, a unit is of the form $x+y$, where $x \in <a+ib>\backslash W$ and $y \in <a-ib>\backslash W$. Any two units $x+y$ and $m+n$ will be adjacent if either $x+m \in W$ or $y+n \in W$ or both, simply if $m \in -x+W$ or $n \in -y+W$. Therefore only those units of the form $\{x_i+y_j\}$ can be included in the independent set so that no two elements from $\{x_i\}$ or no two elements from $\{y_j\}$ are W-inverses of each other.

In $<a+ib>\backslash W$, number of cosets of the form $\{x_i+W\}$ such that no two elements from these cosets are W-inverses of each other is $\frac{p-1}{2}$. Similarly in $<a-ib>\backslash W$, number of cosets of the form $\{y_i+W\}$ such that no two elements from these cosets are W-inverses of each other is $\frac{p-1}{2}$. These two selected set of zero divisors after addition will result in mutually non-adjacent units. Note that for $m,\ n \in x+W$, the sets $m+y+W$ and $n+y+W$ are same. Therefore taking only one element each from the $\frac{p-1}{2}$ cosets of the form $\{x_i+W\}$ and all the elements from the selected $\frac{p-1}{2}$ cosets of the form $\{y_j+W\}$, we get the unique units that are mutualy non-adjacent. Thus, total units that can be included in the independent set $= \frac{p-1}{2}p^{2n-2} + \frac{p-1}{2}p^{2n-2} + \frac{p-1}{2}p^{2n-2} + \ldots$ upto $\frac{p-1}{2}$ terms $= (\frac{p-1}{2})^2 p^{2n-2}$.

In $\Gamma_2(\mathbb{Z}_p[i])$, since $W=\{0\}$, every unit of the form $\{x_i+y_j\}$ will be unique. Also only one zero divisor that too from the above selected sets $\{x_i\}$ or $\{y_j\}$

can be included in the independent set since the subgraph induced by the zero divisors is K_{2p-2}. Therefore, $\alpha(\Gamma_2(\mathbb{Z}_p[i])) = 1 + (\frac{p-1}{2})^2$.

In $\Gamma_2(\mathbb{Z}_{p^r}[i])$, $r \geq 2$; since elements from $< a + ib > \backslash W$ and $< a - ib > \backslash W$ are not adjacent, two zero divisors each from the above selected sets $\{x_i\}$ and $\{y_j\}$ can be included in the independent set. Hence, $\alpha(\Gamma_2(\mathbb{Z}_{p^r}[i])) = 2 + p^{2n-2}(\frac{p-1}{2})^2$.

We conclude this paper with the following example:

Example 6.1: *An independent set of* $\Gamma_2(\mathbb{Z}_5[i])$ *is* $\{1, 2, 1+2i, 4+4i, 4+i\}$.

Acknowledgement

The author would like to express sincere gratitude to the reviewers for their insightful comments and constructive suggestions, which have greatly enhanced the quality of this paper.

Declarations

The authors declare that they have no conflict of interest.

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(*Received*, October 28, 2024)
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Journal of Indian Acad. Math. ISSN: 0970-5120
Vol. 46, No. 2 (2024) pp. 287-302

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RELIABILITY MEASURES OF A TWO DISSIMILAR ELECTRIC VEHICLES USING LINEAR DIFFERENTIAL EQUATION TECHNIQUE

Abstract: The reliability of Hybrid Electric Vehicles (HEVs as System 1) and Plug-in Hybrid Electric Vehicles (PHEVs as System 2) is a critical factor influencing their adoption, performance, and long-term sustainability. As the global shift towards cleaner, more energy-efficient transportation intensifies, understanding the reliability of these vehicles becomes essential to ensure their effectiveness in real-world applications. This paper investigates various reliability measures of a two dissimilar electrical vehicles HEVs and PHEVs, focusing on key performance reliability indicators such as Mean Time to System Failure (MTSF), availability, and failure rates using Linear Differential Equations (LDEs) technique. It is assumed that the distribution of repair rates and failure rates are exponential. A comprehensive analysis is conducted of both systems and examined how the failure rates affected the system performance measures. For different values of failure rates were used to examine a few exceptional instances and results are also presented graphically using MATLAB software.

Keywords: Linear Differential Equation Reliability, Hybrid Electric Vehicles, Plug in Hybrid Electric Vehicle, MTSF, Steady-State Availability.

Mathematics Subject Classification (2010) No.: 60K10, 62N05, 96B25.

1. Introduction

In the subject of system reliability, number of author's analyzed and assessed reliability metrics, including MTSF, steady-state availability analysis, busy period of repairmen, and cost analysis with the Markov renewal process using the regeneration point approach. In this work, dependability measures are assessed using linear differential equations technique. This approach is less complicated than the others, and MATLAB software can be used to carry out computations. Many authors previously employed linear differential equations techniques to evaluate reliability measures for various systems.

Yusuf, I. *et al*. [12] used a mathematical modeling technique to analyze the reliability measure of a two-unit system in cold standby mode. The system operates using a supporting device and failed unit is repaired by repairable service station. Using supplementary variable approaches and Laplace transformation, El. Sherbeny *et al*. [2] examined the dependability and sensitivity analysis as well as the administrative delay in repair of several industrial system cost free warranty policies.Gandoman, F. H. *et al*. [4] discussed the role, mechanism and outcomes of the different failures for evaluating reliability and safety of Li-ion batteries in electrical vehicles. They looked into safety and reliability assessment models for the primary failure modes and current issues with Li-ion batteries in electric vehicles, as well as degradation under normal and abnormal conditions.

Using a first order linear differential equation, Gupta, P., and Kaur, G. [5] evaluated the availability and reliability of a system having two-unitsin standby system that, in the event of a complete failure, switches to a comparable, replacement, or duplicate system that is guaranteed to operate without interruption. Kadyan, M. S. and Kumar, R. [7] commented about the sugar industry's feeding system, which is made up of six subsystems that were examined utilizing the supplementary variable approach. To illustrate the behavior of availability and profit in the feeding system, they also took into consideration a particular instance. ShafiqS. *et al*. [9] derived a probabilistic model that evaluate the significant impact on reliability measures of proposed composite power systems Full Electric Vehicles (FEVS) and Plug-in Hybrid Electric Vechicles (PHEVs) in different atmospheric conditions summer and winter seasons. Gao *et al*. [3] used the Laplace transform approach to study the MTTF and reliability function of the K-out-of M+W+C: G mixed standby system with an unstable repair facility.In a study conducted by Danjuma, M. U. *et al*. [1] the Markov birth-death process was used to calculate the steady-state availability and MTTF of a system consisting of three components connected in series and parallel with a cold standby unit. Using a Markov model, Zaidi, Z. [13] explored several approaches and methodologies for transiently analyzing the dependability of a three-component system.

Lane *et al*. examined information obtained from a survey conducted in late 2013 among 1080 drivers in United State to evaluate the factors that affect the inclination of potential car buyers toward two distinct types of plug-in electric vehicles (PEVs): plug-in hybrid electric vehicles (PHEVs) and battery electric vehicles (BEVs). Thakur H. *et al*. [10] looked at availability analysis and meantime to system failure for a system with two subsystems subsystem 1 and subsystem 2 each with a different configuration in their study and used linear differential equation method to evaluate reliability measures using reasonable assumptions. Yusuf, I., and Bala, S. I. [11] investigated a redundant air conditioning system with a warm standby reserve unit and a primary unit that operated in various weather conditions. They applied Kolmogorov's forward equations technique, several system performance metrics were derived, including profit function, availability, busy period, mean time to system failure (MTSF), and frequency of preventive, minor, and major maintenance. Joshi P. K. *et al*. [6] were examined the availability and dependability of a two-unit standby redundancy system by using the linear differential equation solution approach.

The goal of this study is to investigate the dependability metrics, such as MTSF and steady state availability analysis, of a hybrid four-wheeler (system 1) and plug-in hybrid four-wheeler (system 2) using linear differential equation technique. Plug-in hybrid four-wheeler (System 2) is superior to hybrid four-wheeler (System 1) based on the computation presented in the study regarding the influence of the battery charging option and switching. A graphical representation of measures of system effectiveness of both the system is also explored.

2. Model Description and Assumptions

2.1 Hybrid Electric Vehicle (HEV): Hybrid Electric Automobiles are power-driven by an internal combustion engine and an electric motor, with the electric motor using the energy stored in batteries. It is important to note that hybrid electric vehicles lack the ability to directly connect to an external power source in order to recharge their batteries.

2.2 Plug in Hybrid Electric Vehicle (PHEV): A Plug-in Hybrid Electric Vehicle harnesses the power of batteries to run an electric motor, and an alternative fuel such as diesel or gasoline to run an internal combustion engine or other propulsion methods. PHEV can replenish their batteries using charging devices and regenerative braking.Throughout the study of research paper, the following assumptions has been made:

- System 1 (HEV) can generate electricity through regenerative braking rather than by plugging into a charging station to recharge the vehicle's battery.
- The hybrid four-wheeler in system 1 continues to run even if the battery failed.

- System 2 (PHEV) will only use its internal combustion engine as a backup and will be primarily driven by an electric motor.
- The switch in system 2 (PHEV) is utilized to turn on the petrol supply.
- System 2 (PHEV) uses an automatic switch to start the engine immediately when the battery dies (failed), providing the switch is in working order at the time of need; otherwise, the engine won't start until the switch is repaired.
- System 1 has only two modes, namely failure and normal.
- System 2 has three operating modes: normal, partial, and failure.
- The repair is perfect-almost like new.
- Only one change may be made at a time in a single state.
- All failurerates and repair ratesare constant and follows the exponential distribution.

3. Notations and Symbol

S_i -Transition state of the system, i = 0, 1, 2, 3, 4

P_N -Petrol supply Normal

B_N -Battery fully charged

B_{NP} -Battery partially charged

P_{NP} -Petrol supply is partially

B_F -Battery failed.

P_F -Petrol supply failed.

α-ailure rate of petrol supply

β-failure rate of battery

α'-failure rate of petrol when battery already failed

β'-failure rate of battery when petrol already failed

δ-Repair rate of Petrol supply

λ-Repair rate of battery

δ_1-rate of charging battery

α_2-rate of completion of battery charging

α_1-rate of filling petrol

μ_1-rate of completion of filling petrol

θ-replacement rate of both petrol &battery

4. Transition probability of Hybrid Electric Vehicle (System 1)

Figure 1 shows the transition probability of different states of system 1.

Up State; $S_0 \equiv (B_N, P_N)$, $S_1 \equiv (B_N, P_F)$, $S_2 \equiv (B_F, P_N)$

Down State; $S_3 \equiv (B_F, P_F)$

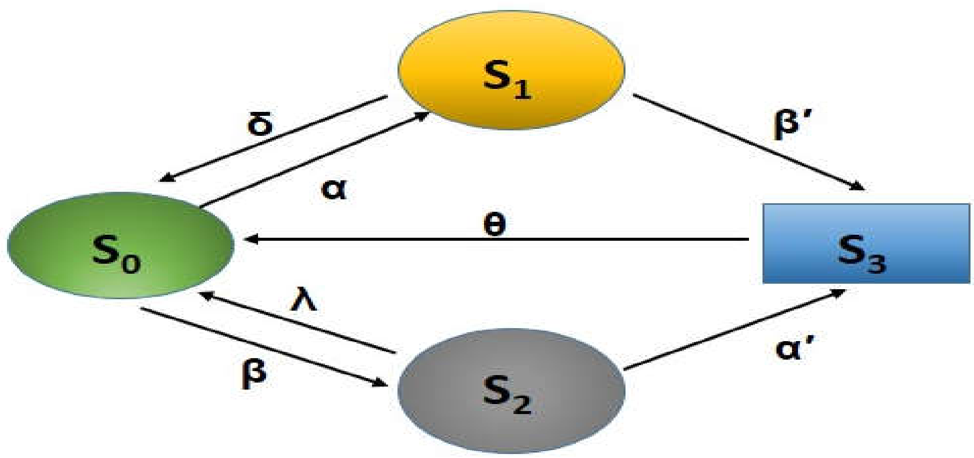

Figure 1: Diagram illustrating the Second System's state transition

5. Measures of System Effectiveness of System 1

5.1 Mean Time to System Failure: Using above aforementioned presumptions and first order differential equation approach MTSF of the suggested system is analysed. Pi(t) is defined, using Figure 1, as the probability that the system is in state Si at time t. The probability row vector at time t is represented by P(t). Take into consideration the initial conditions as

$$P(0) = [P_0(0), P_1(0), P_2(0), P_3(0)], = [1, 0, 0, 0] \quad (1)$$

The derived system of differential equations is as follows:

$$\frac{dp\ (t)}{dt} = (\alpha + \beta)\ P_0(t) + \delta P_1(t) + \lambda\ P_2(t) + \theta P_3(t)$$

$$\frac{dp_1(t)}{dt} = (\beta' + \delta)P_1(t) + \alpha\ P_0(t)$$

$$\frac{dp_2(t)}{dt} = (\alpha' + \lambda\ )P_2(t) + \beta P_0(t)$$

$$\frac{dp_3(t)}{dt} = \theta P_3(t) + \beta' P_1(t) + \alpha' P_2(t)$$

This can be expressed as in the form of system of differential equation:

$$\frac{dp(t)}{dt} = AP(2)$$

In equation (2), matrix A is defined as

$$A = \begin{matrix} (\alpha+\beta) & \delta & \lambda & \theta \\ \alpha & (\beta'+\delta) & 0 & 0 \\ \beta & 0 & (\alpha'+\lambda) & 0 \\ 0 & \beta' & \alpha' & \theta \end{matrix}$$

In order to streamline the assessment of the transition solution eliminating the rows and columns in matrix A that correspond to the absorption state.

This modified matrix, named Q, is created by transposing A. The expected time needed to attain an absorbing condition is derived from

$$E[T_{P(0)\rightarrow(absorbing)}] = P(0)\int_0^\infty e^{Qt}dt$$

and $\int_0^\infty e^{Qt}dt = \quad Q^{-1}$, since $Q\ ^{1} < 0$

where $Q = \begin{bmatrix} (\alpha+\beta) & \alpha & \beta \\ \delta & (\beta'+\delta) & 0 \\ \lambda & 0 & (\alpha'+\lambda) \end{bmatrix}$

The MTSF can be expressed explicitly as:

$$\text{MTSF}_1 = E[TP(0) \rightarrow (absorbing)] = P(0)(\quad Q^{-1})\begin{bmatrix}1\\1\\1\end{bmatrix} \tag{3}$$

$$\text{MTSF}_1 = \frac{(\alpha+\beta_1^{'})(\lambda+\alpha^{'})+\delta(\lambda+\alpha^{'})+\beta(\delta+\beta^{'})}{\alpha\beta^{'}(\lambda+\alpha^{'})+\beta\alpha^{'}(\delta+\beta^{'})} \tag{4}$$

5.2 Steady-State Availability Analysis: For steady state availability analysis initial condition is same as given in equation (1).From equation (1), this can be expressed as in the form of system of differential equation:

$P = A\ P$

$$\begin{bmatrix}\dot{p}_0\\\dot{p}_1\\\dot{p}_2\\\dot{p}_3\end{bmatrix} = \begin{matrix}(\alpha+\beta) & \delta & \lambda & \theta\\ \alpha & (\beta^{'}+\delta) & 0 & 0\\ \beta & 0 & (\alpha^{'}+\lambda) & 0\\ 0 & \beta^{'} & \alpha^{'} & \theta\end{matrix}\begin{bmatrix}P_0\\p_1\\p_2\\P_3\end{bmatrix}$$

The expression for steady-state availability is expressed by

$$A_{T_1}(\infty) = 1 \quad P_3(\infty) \tag{5}$$

In the context of steady-state availability, the state probabilities' derivatives reach a point of equilibrium where they become zero. This implies that the product of matrix A and the probability at infinity, denoted as P(∞), is equal to zero.

i.e., $$A\ P(\infty) = 0 \tag{6}$$

$$\begin{matrix}(\alpha+\beta) & \delta & \lambda & \theta\\ \alpha & (\beta^{'}+\delta) & 0 & 0\\ \beta & 0 & (\alpha^{'}+\alpha) & 0\\ 0 & \beta^{'} & \alpha^{'} & \theta\end{matrix}\begin{bmatrix}P_0(\infty)\\P_1(\infty)\\P_2(\infty)\\P_3(\infty)\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \tag{7}$$

$$P_0(\infty) + p_1(\infty) + P_2(\infty) + p_3(\infty) = 1 \tag{8}$$

To obtain $P_5(\infty)$, we replace (8) in one of the redundant rows of (7) and then using MATLAB software to get the matrix-form solution to the following linear equation system.

$$\begin{bmatrix} (\alpha+\beta) & \delta & \lambda & \theta \\ \alpha & (\beta'+\delta) & 0 & 0 \\ \beta & 0 & (\alpha+1) & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} P_0(\infty) \\ P_1(\infty) \\ P_2(\infty) \\ P_3(\infty) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (9)$$

The steady state probabilities for Figure1 are found from solution of equation (9). For $AT_2(\infty)$, the explicit expression is:

$$A_{T_1}(\infty) = \frac{\alpha(\alpha'+\lambda)(\beta'+\theta)+(\beta'+\delta)(\alpha'\beta+\theta\lambda)+\theta(\alpha'\beta'+\alpha\delta)}{\alpha(\alpha'+\lambda)(\beta'+\theta)+(\beta'+\delta)[\beta(\alpha'+\theta)+\theta(\alpha'+\lambda)]} \quad (10)$$

6. Transition probability of Plug in hybrid electric vehicle (System 2)

Figure 2 shows the transition probability of different states of system 2.

Up State: $S_0 \equiv (B_N,P_N)$, $S_1 \equiv (B_{NP},P_N)$, $S_2 \equiv (B_N,P_{NP})$, $S_3 \equiv (B_N,P_F)$, $S_4 \equiv (B_F,P_N)$
Down State: $S_5 \equiv (B_F, P_F)$

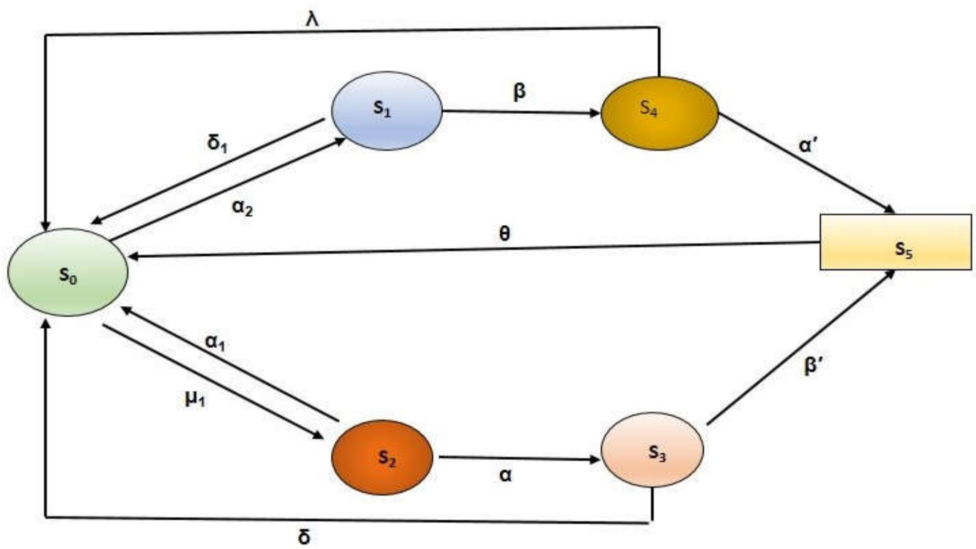

Figure 2: Diagram illustrating the Second System's state transition

7. Measures of System Effectiveness of System 2

7.1 Mean Time to System Failure: Using above aforementioned presumptions and first order differential equation approach MTSF of the suggested system is analysed. Pi(t) is defined, using Figure 2, as the probability that the system is in state Si at time t. The probability row vector at time t is represented by P(t).

Take into consideration the initial conditions as

$$P(0) = [P_0(0), P_1(0), P_2(0), P_3(0), P_4(0), P_5(0)] = [1,0,0,0,0,0] \tag{11}$$

Here is the differential equation system that was derived:

$$\frac{dp\ (t)}{dt} = (\alpha_2 + \alpha_1)\ P_0(t) + \delta_1 P_1(t) + \mu_1\ P_2(t) + \delta P_3(t) + \lambda P_4(t) + \theta P_5(t)$$

$$\frac{dp_1(t)}{dt} = (\beta + \delta_1)p_1(t) + \alpha_2 P_0(t)$$

$$\frac{dP_2(t)}{dt} = (\alpha + \mu_1)P_2(t) + \alpha_1 P_0(t)$$

$$\frac{dP_3(t)}{dt} = \left(\beta' + \delta\right)P_3(t) + \alpha P_2(t)$$

$$\frac{dP_4(t)}{dt} = (\lambda + \alpha')P_4(t) + \beta P_1(t)$$

$$\frac{dP_5(t)}{dt} = \theta P_5(t) + \beta' P_3(t) + \alpha' P_4(t)$$

This can be expressed as in the form of system of differential equation:

$$P = A\ P \tag{12}$$

where

$$A = \begin{matrix} (\alpha_2 + \alpha_1) & \delta_1 & \mu_1 & \delta & \lambda & 0 \\ \alpha_2 & (\beta + \delta_1) & 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & (\alpha + \mu_1) & 0 & 0 & 0 \\ 0 & 0 & \alpha & \left(\beta' + \delta\right) & 0 & 0 \\ 0 & \beta & 0 & 0 & \left(\alpha' + \lambda\right) & 0 \\ 0 & 0 & 0 & \beta' & \alpha' & \theta \end{matrix}$$

In order to streamline the assessment of the transitions solution, eliminating the row and columns in matrix A that correspond to the absorption state.

This modified matrix, named Q, is created by transposing A. The expected time needed to attain an absorbing condition is derived from

$$E[T_{P(0)\rightarrow(absorbing)}] = P(0)\int_0^{\infty} e^{Qt}dt$$

and

$$\int_0^{\infty} e^{Qt}dt = \quad Q^{\ 1}, \text{ since } Q^{\ 1} < 0$$

where

$$Q = \begin{matrix} (\alpha_2+\alpha_1) & \alpha_2 & \alpha_1 & 0 & 0 \\ \delta_1 & (\beta+\delta_1) & 0 & 0 & \beta \\ \mu_1 & 0 & (\alpha+\mu_1) & \alpha & 0 \\ \delta & 0 & 0 & (\beta'+\delta) & 0 \\ \lambda & 0 & 0 & 0 & (\alpha'+\lambda) \end{matrix}$$

$$\text{MTSF}_2 = E[T_{P(0)\rightarrow(absorbing)}] = P(0)(-Q^{-1}) \begin{matrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{matrix} \tag{13}$$

$$\text{MTSF}_2 = \frac{\alpha_2\beta\alpha'(\alpha+\mu_1)(\delta+\alpha')+\alpha_1\alpha\beta'(\lambda+\alpha')(\beta+\delta_1)}{(\lambda+\alpha')[(\alpha+\mu_1+\alpha_1)(\beta+\delta_1)(\delta+\beta')+\alpha_1\alpha(\beta+s_1)]+\alpha_2(\delta+\beta')(\alpha+\mu_1)(\lambda+\alpha'+\beta)} \tag{14}$$

7.2 Steady-State Availability Analysis: For steady state availability analysis initial condition is same as given in equation (11).

From equation (11), this can be expressed as in the form of system of differential equation:

$$\begin{matrix} \dot{p}_0 \\ \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{p}_4 \\ \dot{p}_5 \end{matrix} = \begin{matrix} (\alpha_2+\alpha_1) & \delta_1 & \mu_1 & \delta & \lambda & 0 \\ \alpha_2 & (\beta+\delta_1) & 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & (\alpha+\mu_1) & 0 & 0 & 0 \\ 0 & 0 & \alpha & (\beta'+\delta) & 0 & 0 \\ 0 & \beta & 0 & 0 & (\alpha'+\lambda) & 0 \\ 0 & 0 & 0 & \beta' & \alpha' & \theta \end{matrix} \begin{matrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{matrix}$$

The expression for steady-state availability is expressed as

$$A_{T_2}(\infty) = 1 \quad P_5(\infty) \tag{15}$$

In the context of steady-state availability, the state probabilities' derivatives reach a point of equilibrium where they become zero. This implies that the product of matrix A and the probability at infinity, denoted as P(∞), is equal to zero. i.e., $A\,P(\infty) = 0$

$$\begin{matrix} (\alpha_2+\alpha_1) & \delta_1 & \mu_1 & \delta & \lambda & 0 \\ \alpha_2 & (\beta+\delta_1) & 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & (\alpha+\mu_1) & 0 & 0 & 0 \\ 0 & 0 & \alpha & (\beta^{'}+s) & 0 & 0 \\ 0 & \beta & 0 & 0 & (\alpha^{'}+\lambda) & 0 \\ 0 & 0 & 0 & \beta^1 & \alpha^1 & \theta \end{matrix} \quad \begin{matrix} p_1(\infty) \\ p_1(\infty) \\ p_2(\infty) \\ p_3(\infty) \\ p_4(\infty) \\ p_5(\infty) \end{matrix} = \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \tag{16}$$

$$P_0(\infty) + p_1(\infty) + P_2(\infty) + p_3(\infty) + p_4(\infty) + p_5(\infty) = 1 \tag{17}$$

To obtain $P_5(\infty)$,replace (17) in one of the redundant rows of (16) and then using MATLAB software to get the matrix-form solution to the following linear equation system.

$$\begin{matrix} (\alpha_2+\alpha_1) & s_1 & \mu_1 & \delta & \lambda & 0 \\ \alpha_2 & (\beta+\delta_1) & 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & (\alpha_1) & 0 & 0 & 0 \\ 0 & 0 & \alpha & (\beta^{'}+\beta) & 0 & 0 \\ 0 & \beta & 0 & 0 & (\alpha^{'}+\lambda) & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{matrix} \quad \begin{matrix} p_0(\infty) \\ p_1(\infty) \\ p_2(\infty) \\ p_3(\infty) \\ p_4(\infty) \\ p_5(\infty) \end{matrix} = \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{matrix} \tag{18}$$

The steady state probabilities for Figure 2 are achieved from solution of equation(18). For $AT_2(\infty)$, the explicit expression is:

$$AT_2(\infty) = \frac{\theta(\lambda+\alpha')(\alpha+\mu_1)(\beta+\delta_1)(\delta+\beta') + \alpha_2\theta(\alpha+\mu_1)(\delta+\beta')(\lambda+\alpha'+\beta) + \alpha_1\theta(\lambda+\alpha')(\beta+\delta_1)(\delta+\beta'+\alpha)}{\begin{gathered} 2\beta(\delta+\beta')(\theta+\alpha')(\alpha+\mu_1)+\alpha_1\beta'(\lambda+\alpha')(\beta+\delta_1)(\alpha+\theta)+ \\ \delta\theta(\alpha_2+\beta)(\lambda+\alpha')(\alpha+\mu_1) + \delta_1\theta(\delta+\beta')(\lambda+\alpha')(\alpha+\mu_1) + \\ \alpha_1\theta(\beta+\delta_1)(\lambda+\alpha')(\alpha+\delta) + \beta'\theta(\alpha_2+\beta)(\alpha+\mu_1)(\lambda+\alpha') \end{gathered}} \tag{19}$$

8. Result and Discussion

Ploting the graph for reliability measures for system 1 and system 2 versus α and β, separately , while keeping the other parameters fixed at $\alpha^{'}=1.5$, $\beta^{'}=1.5$, $\delta = 2.5$, $\lambda = 2$, $\mu_1 = \alpha_2 = 2.5$, $\delta_1 = \alpha_1 = 3$, $\theta = 3.5$ in order to observe the behavior of the system.Taking $\beta = 0.4$ for the curve against α, and we take $\alpha = 0.4$ in addition to other parameters for the curve against β.

Table 1: The relationship between the first and second systems' availability, MTSF, and failure rate "α"

α	$MTSF_1$	$MTSF_2$	$AT_1(\infty)$	$AT_2(\infty)$
0.1	5.44	17.72	0.904	0.984
0.2	4.72	14.26	0.907	0.980
0.3	4.18	12.05	0.910	0.976
0.4	3.78	10.51	0.912	0.973
0.5	3.45	9.39	0.915	0.970
0.6	3.18	8.53	0.917	0.967
0.7	2.97	7.84	0.918	0.965
0.8	2.79	7.30	0.921	0.962
0.9	2.63	6.84	0.923	0.960
1.0	2.50	6.46	0.924	0.957

Table 2: The relationship between the first and second systems' availability, MTSF, and failure rate "β"

β	$MTSF_1$	$MTSF_2$	$AT_1(\infty)$	$AT_2(\infty)$
0.1	5.85	46.8	0.975	0.982
0.2	4.90	29.7	0.953	0.979
0.3	4.25	22.11	0.932	0.976
0.4	3.78	17.81	0.913	0.973
0.5	3.41	15.05	0.894	0.971
0.6	3.12	13.12	0.870	0.968
0.7	2.89	11.83	0.860	0.966
0.8	2.70	10.61	0.840	0.964
0.9	2.53	9.75	0.830	0.962
1.0	2.40	9.01	0.820	0.960

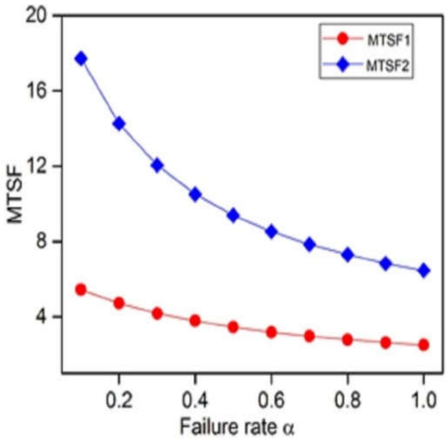

Figure 3: MTSF against α

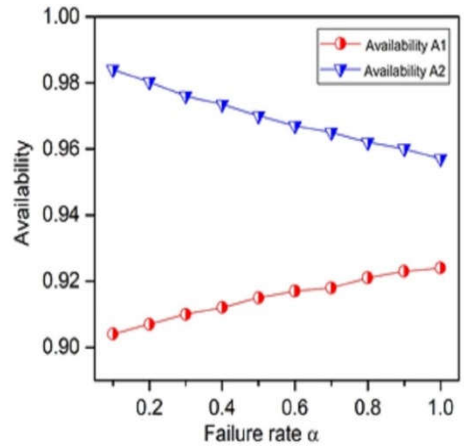

Figure 4: Availability against α

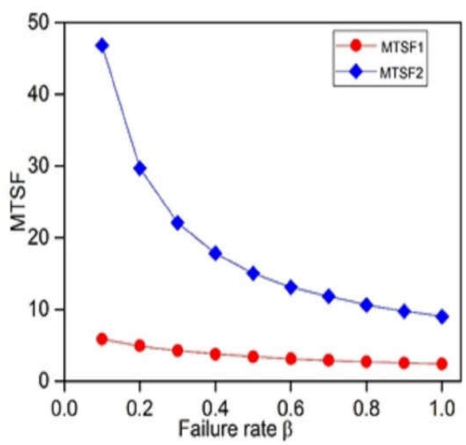

Figure 5: MTSF against β

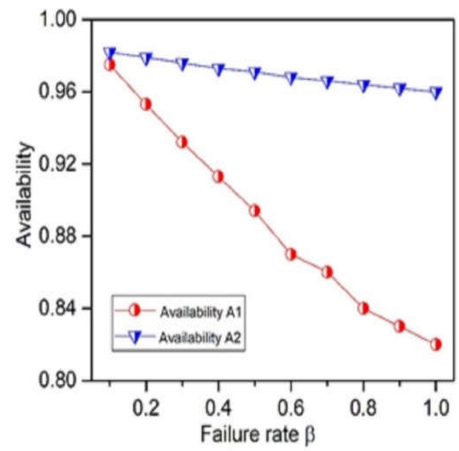

Figure 6: availability against β

As the value of α increases, the MTSF and availability of both systems decline, whereas the availability of system 1 marginally increases, as seen in figures 3 and 4. The graph makes it evident that system 2 has a higher MTSF and availability than system 1.

Figures 5 and 6 show the results of the two systems under study plotted against the failure rate β, representing the MTSF and the steady-state availability. It is evident from the figures that System 2 has a higher MTSF and stable availability than System 1.

9. Conclusion

This research paper explores the reliability measures of two dissimilar electrical vehicles namely Hybrid Electric Vehicles (HEVs) and Plug-in Hybrid Electric Vehicles (PHEVs) through the application of linear differential equations (LDE) to model key performance indicators such as Mean Time to System Failure (MTSF) and steady-state availability. The impact of failure rate on both systems monitored in order to observe system behaviour. The graphical demonstration of both the systems for hypothetical value are shown in figures 3 to 6 and from this conclude that:

- **PHEVs** tend to have higher reliability and better availability due to their ability to use both electric and petrol power sources, unlike HEVs, which rely more on regenerative braking.

- The study quantifies and compares the reliability of both systems, emphasizing that PHEVs generally perform better in terms of failure resistance and availability.

The analysis highlights how hybrid systems with advanced features like recharging and petrol backup provide improved system dependability and offer a significant advantage in terms of real-world usability, especially in cases of battery failure or depletion.

The findings confirm that while current modelling techniques offer valuable insights into the reliability of these vehicles, there is significant potential for further advancements. By integrating machine learning for predictive maintenance, utilizing real-time IoT data for continuous monitoring, and enhancing battery technology with solid-state innovations, the reliability of HEVs and PHEVs can be substantially improved. Additionally, the application of AI-based control systems will further elevate vehicle performance and reliability.

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(*Received*, October 9, 2024)
(*Revised*, January 1, 2025)

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Journal of Indian Acad. Math. ISSN: 0970-5120
Vol. 46, No. 2 (2024) pp. 303-312

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FRACTIONAL PRODUCT CORDIAL LABLEING OF WEB GRAPH, DUMBBELL GRAPH, JELLY FISH GRAPH, JEWEL GRAPH AND TADPOLE GRAPH

Abstract: Let $G = (V, E)$ be a (p, q) graph. Let

$$M = \begin{matrix} 1, 2, \cdots, \frac{p}{2}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{p+1} & \text{if } p \text{ is even} \\ 1, 2, \cdots, \frac{p \quad 1}{2}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{p+3} & \text{if } p \text{ is odd} \end{matrix}$$

Let $\varphi: V(G) \rightarrow M$ be a bijection. For each edge xy assign the label $f(x)f(y)$.φ is called a fractional product cordial labeling (simply called FP-cordial labeling) if $\left| \prod_{\varphi}(0) \quad \prod_{\varphi}(1) \right| \leq 1$, where $\prod_{\varphi}(1)$ and $\prod_{\varphi}(0)$ respectively denotes the number of edges labelled with 1 and not labelled with 1. A graph with a fractional product cordial labeling is called a fractional product cordial graph (Simply FP-cordial graph). In this paper, we investigate the FP-cordial labeling of web, jewel, dumbbell, jelly fish and tadpole graph.

Keywords: Web Graph, Jewel Graph, Dumbbell Graph, Jelly Fish Graph and Tadpole Graph.

Mathematics Subject Classification No.: 05C78.

1. Introduction

We consider only finite, simple and undirected graphs. The number of vertices of a graph G is called the order of G and number of edges is called the size of G. The cordial labeling concept was introduced by I. Cahit [1] and subsequently cordial related concept was studied in [2-8]. The notion of fractional product cordial labeling have been introduced in [9] and fractional product cordial labeling behavior of path, cycle, complete, star, wheel, book with triangle pages, ladder, comb, double comb, bistar, subdivision of the star and subdivision of the bistar have been investigated [9]. Fractional product cordial labeling behavior of subdivision of comb, subdivision of double comb, triangular snake, quadrilateral snake, slanting ladder, triangular ladder, fan graph, flower graph, sunflower graph, helm and closed helm have been investigated [10].

In this paper we investigate the fractional product cordial labeling of some special graphs like web graph, jewel graph, dumbbell graph, jelly fish graph and tadpole graph. x denotes the smallest integer $\geq x$.

2. Preliminaries

Definition 2.1 [1]: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The join $G_1 + G_2$ as $G_1 \cup G_2$ Together with all the edges joining vertices of V_1 to the vertices of V_2.

Definition 2.2 [1]: The graph $W_n = C_n + K_1$is called the wheel graph.

Definition 2.3 [2]: The jewel graph $J_n (n \geq 1)$, is the graph with vertex set $V(J_n) = \{u, v, x, y, u_i : 1 \leq i \leq n\}$ and edge set $E(J_n) = \{uv, xu, xv, yu, yv, xu_i, yu_i : 1 \leq i \leq n\}$. Then the jewel graph$n + 4$ vertices and $2n + 5$ edges.

Definition 2.4 [2]: The tadpole graph $T(m, n)(m \geq 3, n \geq 1)$, is the graph obtained by joining a cycle C_m to a path P_n with a bridge.

Definition 2.5 [2]: The graph obtained by joining two disjoint cycles $x_1 x_2 \dots x_n x_1$ and $y_1 y_2 \dots y_n y_1$ with an edge $x_1 y_1$ is called dumbbell graph and it is denoted by $Db(m, n)(m, n \geq 3)$.

Definition 2.6 [2]: Jelly fish graphs $J(m, n)$ obtained from $u_1 u_2 u_3 u_4 u_1$ by joining u_1 and u_3 with an edge and appending m pendent edges to u_2 and n pendent edges to u_4.

Definition 2.7 [2]: The web graph $Wb_n (n \geq 3)$ is the graph obtained by joining the pendent vertices of a helm H_n to form a cycle and then adding a pendant edge to each vertex of outer cycle.

3. Fractional Product Cordial Labeling

Definition 3.1: Let $G = (V, E)$ be a (p, q) graph.

$$\text{Let} \qquad M = \begin{cases} 1, 2, \cdots, \frac{p}{2}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{p+1} & if\ p\ is\ even \\ 1, 2, \cdots, \frac{p-1}{2}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{p+3} & if\ p\ is\ odd \end{cases}$$

Let $\varphi: V(G) \rightarrow M$ be a bijection.For each edge xy assign the label $f(x)f(y)$. φ is called a fractional product cordial labeling (simply called FP-cordial labeling) if $\left| \prod_{\varphi}(0) \quad \prod_{\varphi}(1) \right| \leq 1$, where $\prod_{\varphi}(1)$ and $\prod_{\varphi}(0)$ respectively denotes the number of edges labelled with 1 and not labelled with 1. A graph with a fractional product cordial labeling is called a fractional product cordial graph (Simply FP-cordial graph).

Theorem 3.2: The web graph Wb_n is FP-cordial for all $n \geq 3$.

Proof: Let $V(Wb_n) = V(W_n) \cup \{z_i': 1 \leq i \leq n\}$ where $W_n = C_n + K_1$, C_n be the cycle $y_1 y_2 \ldots y_n y_1$, $V(K_1) = \{y\}$ and $E(Wb_n) = E(W_n) \quad \{z_i z_i': 1 \leq i \leq n\}$. Then there are $3n + 1$ vertices and $5n$ edges. This proof is divided into two cases:

Case (i): n is odd

Fix the central vertex y is assigned by 1. Assign the labels $2, 3, \ldots, n + 1$ to the vertices $y_1, y_3, \ldots, y_n$ respectively. We assign the label $n + 2$to z_n and assign the labels $n + 2, n + 3, \ldots, \frac{3n+1}{2}$ respectively to the vertices $z_n', z_1', \ldots, z_{\frac{n-5}{2}}$'. We now assign the labels $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$ to the vertices $z_1', z_2', \ldots, z_{n-1}'$ respectively. Also we assign the labels $\frac{1}{n+1}, \frac{1}{n+2}, \ldots, \frac{2}{3n+3}$ respectively to the vertices $z'_{\frac{n-3}{2}}, z'_{\frac{n-1}{2}}, \ldots, z_{n-1}'$.

Hence, $\prod_{\varphi}(0) = \frac{5n+1}{2}$ and $\prod_{\varphi}(1) = \frac{5n-1}{2}$.

Case (ii): n is even

As in Case (i), assign the label to the vertices y, z_n an d$y_i, 1 \leq i \leq n$. We assign the labels $n + 2, n + 3, \ldots, \frac{3n}{2}$ respectively to the vertices $z_n', z_1', \ldots, z_{\frac{n-6}{2}}'$. We now assign the labels $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$ to the vertices $z_1', z_2', \ldots, z_{n-1}'$ respectively. Also we assign the labels -$\frac{1}{n+1}, \frac{1}{n+2}, \ldots, \frac{2}{3n}$ respectively to the vertices $z'_{\frac{n-4}{2}}, z'_{\frac{n-2}{2}}, \ldots, z_{n-1}'$.

Hence, $\prod_{\varphi}(0) = \frac{5n}{2}$ and $\prod_{\varphi}(1) = \frac{5n}{2}$.

Theorem 3.3: The jewel graph J_n is FP-cordialif and only if n {1,3}.

Proof: Let $V(J_n) = \{u, v, x, y, u_i: 1 \leq i \leq n\}$ and $(J_n) = \{uv, xu, xv, yu, yv, xu_i, yu_i: 1 \leq i \leq n\}$. Then there are $n + 4$ vertices and $2n + 5$ edges. This proof is break up into three cases:

Case (i): n is odd and $n \geq 5$

Fix the labels 1,2,4,3 to the vertices x, u, y *and* v respectively. Then we assign the labels $5,6, \dots, \frac{n+3}{2}$ respectively to the vertices $u_1, u_2, \dots, u_{\frac{n-5}{2}}$. We assign the labels $\frac{1}{2}, \frac{1}{3}, \dots, \frac{2}{n+7}$ to the vertices $u_{\frac{n-3}{2}}, u_{\frac{n-1}{2}}, \dots, u_n$ respectively.

Therefore, $\prod_{\varphi}(0) = \frac{2n+6}{2}$ and $\prod_{\varphi}(1) = \frac{2n+4}{2}$.

Case (ii): n is even and $n \geq 2$

Then the following subcases are arises.

Subcase (i): $n = 2$

A FP-cordial labeling J_2 is shown in given table.

n	x	u	y	v	u_1	u_2
2	1	2	$\frac{1}{2}$	3	$\frac{1}{3}$	$\frac{1}{4}$

Subcase (ii): $n \geq 4$

Fix the labels 1,2,4,3 to the vertices x, u, y *and* v respectively. Then we assign the labels $5,6, \dots, \frac{n+4}{2}$ respectively to the vertices $u_1, u_2, \dots, u_{\frac{n-4}{2}}$. We assign the labels $\frac{1}{2}, \frac{1}{3}, \dots, \frac{2}{n+6}$ to the vertices $u_{\frac{n-2}{2}}, u_{\frac{n}{2}}, \dots, u_n$ respectively.

Therefore, $\prod_{\varphi}(0) = \frac{2n+6}{2}$ and $\prod_{\varphi}(1) = \frac{2n+4}{2}$.

Subcase (iii): $\boldsymbol{n \epsilon \{1, 3\}}$

Suppose $J_n \boldsymbol{\epsilon} \Omega_{fpc}$. When $n = 1$, the vertex labels are $1, 2, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. Suppose 1 and 2 the vertex labels are non adjacent vertices then $\prod_{\varphi}(0) = 0$ and $\prod_{\varphi}(1) = 7$, a contradiction.

Suppose 1 and 2 the vertex labels are adjacent vertices then $\prod_{\varphi}(0) = 1$ and $\prod_{\varphi}(1) = 6$, not possible.

In the case $n = 3$, the vertex labels are $1, 2, 3, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. When 1,2 and 3 the vertex labels are non adjacent vertices then $\prod_{\varphi}(0) \leq 1$ a contradiction to the size of J_3 is 11.

When 1,2 and 3 the vertex labels are adjacent vertices then $\prod_{\varphi}(0) \leq 4$ again a contradiction to the size of J_3 is 11.

Theorem 3.4: The dumbbell graph $Db(n, n)$ is FP-cordial for all $n \geq 3$.

Proof: Let $V(Db(n, n)) = \{x_i, y_i : 1 \leq i \leq n\}$ and $\mathrm{E}(Db(n, n)) = \{x_i x_{i+1}, y_i y_{i+1} : 1 \leq i \leq n \quad 1\} \cup \{x_1 x_n, y_1 y_n, x_1 y_1\}$. Then there are $2n$ vertices and $2n + 1$ edges.

Assign the labels $1, 2, \ldots, n$ to the vertices $x_1, x_2, \ldots, x_n$ respectively. We assign the labels $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n+1}$ respectively to the vertices $y_1, y_2, \ldots, y_n$.

Therefore, $\prod_{\varphi}(0) = n$ and $\prod_{\varphi}(1) = n + 1$

Theorem 3.5: The jellyfish graph $J(m, n)$ is FP-cordial for all n.

Proof: Let $V(J(m, n)) = \{x, y, u, v, u_i, v_j : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ and $E(J(n, n)) = \{xy, xu, xv, yu, yv, uu_i, vv_j : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$. Then there are $m + n + 4$ vertices and $m + n + 5$ edges. Then this proof break up into two cases:

Case (i): $\boldsymbol{m + n + 4}$ is even

Fix the labels 1,2,3 to the vertices x, y and u respectively. Then the following subcases are arises:

Subcase (i): $m = n$

We assign the labels $4,5,\dots,n+2$ respectively to the vertices $u_1, u_2, \dots, u_{n-1}$. We now assign the labels $\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n+3}$ to the vertices $u_n, v_1, v_2 \dots, v_n$ respectively.

Therefore, $\prod_{\varphi}(0) = \frac{m+n+6}{2}$and $\prod_{\varphi}(1) = \frac{m+n+4}{2}$.

Subcase (ii): $m > n$

Assign the labels $4,5,\dots,\frac{m+n+4}{2}$ respectively to the vertices $u_1, u_2, \dots, u_{\frac{m+n-2}{2}}$ and assign the labels $\frac{1}{2}, \frac{1}{3}, \dots, \frac{2}{m-n+4}$ to the vertice s$u_{\frac{m+n}{2}}, u_{\frac{m+n+2}{2}}, \dots, u_m$ Now assign the labels $\frac{2}{m-n+6}, \frac{2}{m-n+8}, \dots, \frac{2}{m+n+6}$to the vertices $v, v_1, v_2, \dots, v_n$ respectively.

Hence, $\prod_{\varphi}(0) = \frac{m+n+6}{2}$ and $\prod_{\varphi}(1) = \frac{m+n+4}{2}$.

Subcase (iii): $m < n$

Assign the labels $4,5,\dots,\frac{m+n+4}{2}$ respectively to the vertices $v_1, v_2, \dots, v_{\frac{m+n-2}{2}}$ and assign the labels $\frac{1}{2}, \frac{1}{3}, \dots, \frac{2}{n-m+4}$ to the vertices $v_{\frac{m+n}{2}}, v_{\frac{m+n+2}{2}}, \dots, v_n$ Now assign the labels $\frac{2}{n-m+6}, \frac{2}{n-m+8}, \dots, \frac{2}{m+n+2}$to the vertices $u, u_1, u_2, \dots, u_m$ respectively.

Hence, $\prod_{\varphi}(0) = \frac{m+n+6}{2}$ and $\prod_{\varphi}(1) = \frac{m+n+4}{2}$.

Case (ii): $m + n + 4$ is odd

Fix the labels 1,2,3 to the vertices x, y an du respectively. Then the following subcases are arises:

Subcase (i): $m > n$

Assign the labels $4,5,\dots,\frac{m+n+3}{2}$ respectively to the vertices $u_1, u_2, \dots, u_{\frac{m+n-3}{2}}$ and assign the labels $\frac{1}{2}, \frac{1}{3}, \dots, \frac{2}{m-n+5}$ to the vertices $u_{\frac{m+n-1}{2}}, u_{\frac{m+n+1}{2}}, \dots, u_m$. Now assign the labels $\frac{2}{m-n+7}, \frac{2}{m-n+9}, \dots, \frac{2}{m+n+7}$ to the vertices $v, v_1, v_2, \dots, v_n$ respectively.

Hence, $\prod_\varphi(0) = \frac{m+n+5}{2}$ and $\prod_\varphi(1) = \frac{m+n+5}{2}$.

Subcase (ii): $m < n$

Assign the labels $4,5, \dots, \frac{m+n+3}{2}$ respectively to the vertices $v_1, v_2, \dots, v_{\frac{m+n-3}{2}}$ and assign the labels $\frac{1}{2}, \frac{1}{3}, \dots, \frac{2}{n-m+5}$ to the vertices $v_{\frac{m+n-1}{2}}, v_{\frac{m+n+1}{2}}, \dots, v_n$ Now assign the labels $\frac{2}{n-m+7}, \frac{2}{n-m+9}, \dots, \frac{2}{m+n+7}$ to the vertices $u, u_1, u_2, \dots, u_m$ respectively.

Hence, $\prod_\varphi(0) = \frac{m+n+5}{2}$ and $\prod_\varphi(1) = \frac{m+n+5}{2}$.

Theorem 3.6: The tadpole graph $T(m,n)$ is FP-cordial for all $m \geq 3$ an dfor all n, $T(3,1)$ an d$T(3,2)$ are not FP-cordial.

Proof: Let $V(T(m,n)) = \{u_i, v_j: 1 \leq i \leq m$ an d$1 \leq j \leq n\}$ and $E(T(m,n)) = \{u_i u_{i+1}, v_j v_{j+1} : 1 \leq i \leq m \quad 1 \; and \; 1 \leq j \leq n \quad 1\} \cup \{u_m u_1, u_1 v_1\}$. Then there are $m+n$ vertices and $m+n$ edges. This proof divided into three cases:

Case (i): $m+n$ is even

Then the following subcases are arises:

Subcase (i): $m = n$

Assign the labels $1,2, \dots, \frac{m+n}{2}$ respectively to the vertices $y_n, y_{n-1}, \dots, y_1$ andassign the labels $\frac{1}{2}, \frac{1}{3}, \dots, \frac{2}{m+n+2}$ to the vertices $x_1, x_2, \dots, x_m$ respectively.

Therefore, $\prod_\varphi(0) = \frac{m+n}{2}$ and $\prod_\varphi(1) = \frac{m+n}{2}$.

Subcase (ii): $\boldsymbol{m} < n$

Assign the labels $1,2,\dots,\frac{m+n}{2}$ respectively to the vertices $y_n, y_{n-1}, \dots, y_{\frac{n-m}{2}+1}$ and assign the labels $\frac{1}{2},\frac{1}{3},\dots,\frac{2}{n-m+2}$ to the vertices $y_{\frac{n-m}{2}}, y_{\frac{n-m}{2}-1},\dots,y_1$. Now assign the labels $\frac{2}{n-m+4},\frac{2}{n-m+6},\dots,\frac{2}{m+n+2}$to the vertices $x_1,x_2,\dots,x_m$ respectively.

Hence, $\prod_\varphi(0) = \frac{m+n}{2}$ and $\prod_\varphi(1) = \frac{m+n}{2}$.

Subcase (iii): $\boldsymbol{m} > n$

Assign the labels $1,2,\dots,n$ respectively to the vertices $y_n, y_{n-1},\dots,y_1$ and assign the labels $n+1, n+2,\dots,\frac{m+n}{2} x_1,x_2,\dots,x_{\frac{m-n}{2}}$ and assign the labels $\frac{1}{2},\frac{1}{3},\dots,\frac{2}{m+n+2}$ to the vertices $x_{\frac{m-n}{2}+1}, x_{\frac{m-n}{2}+2},\dots,x_m$ respectively.

Hence, $\prod_\varphi(0) = \frac{m+n}{2}$ and $\prod_\varphi(1) = \frac{m+n}{2}$.

Case (ii): $\boldsymbol{m+n}$ **is odd**

Then the following subcases are arises:

Subcase (i): $\boldsymbol{m < n}$

Assign the labels $1,2,\dots,\frac{m+n-1}{2}$ respectively to the vertices $y_n, y_{n-1},\dots,y_{n-m}$ and assign the labels $\frac{1}{2},\frac{1}{3},\dots,\frac{2}{n-m+3}$ to the verticesy$_{n-m-1}, y_{n-m-2},\dots,y_1$. We now assign the labels $\frac{2}{n-m+5},\frac{2}{n-m+7},\dots,\frac{2}{m+n+3}$ to the vertices $x_1,x_2,\dots,x_m$ respectively.

Hence, $\prod_\varphi(0) = \frac{m+n-1}{2}$ and $\prod_\varphi(1) = \frac{m+n+1}{2}$.

Subcase (ii): $\boldsymbol{m > n}$

Assign the labels $1,2,\dots,n$ respectively to the vertices $y_n, y_{n-1},\dots,y_1$ and assign the labels $n+1,n+2,\dots,\frac{m+n-1}{2}$ to the vertices $x_1,x_2,\dots,x_{\frac{m-n}{2}}$. Now assign the labels $\frac{1}{2},\frac{1}{3},\dots,\frac{2}{m+n+3}$ to the vertices $x_{\frac{m-n+1}{2}}, x_{\frac{m-n+3}{2}},\dots,x_m$ respectively.

Hence, $\prod_{\varphi}(0) = \frac{m+n+1}{2}$ and $\prod_{\varphi}(1) = \frac{m+n-1}{2}$.

Case (iii): $\boldsymbol{m = 3}$ and $\boldsymbol{n \epsilon \{1, 2\}}$

Suppose $T(m,n)\epsilon\Omega_{fpc}$. When $n = 1$, the vertex labels are $1, 2, \frac{1}{2}, \frac{1}{3}$. Suppose 1 and 2 the vertex labels are non adjacent vertices then $\prod_{\varphi}(0) = 0$ and $\prod_{\varphi}(1) = 4$, a contradiction.

Suppose 1 and 2 the vertex labels are adjacent vertices then $\prod_{\varphi}(0) = 1$ and $\prod_{\varphi}(1) = 3$, which is not possible.

When $n = 2$, the vertex labels are $1, 2, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. Suppose 1 and 2 the vertex labels are non adjacent vertices then $\prod_{\varphi}(0) = 0$ and $\prod_{\varphi}(1) = 5$, again a contradiction.

Suppose 1 and 2 the vertex labels are adjacent vertices then $\prod_{\varphi}(0) = 1$ and $\prod_{\varphi}(1) = 4$, not possible.

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(*Received*, October 23, 2024)

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FORM IV
(See Rule 8)

1.	Title of the Journal	Journal of Indian Academy of Mathematics
2.	Place of Publication	5, 1st floor, I. K. Girls School Campus, 14/1 Ushaganj, Near G.P.O. Indore - 452 001 India
3.	Periodicity of Publication	Bi-Annual (Twice in a Year)
4.	Language in which it is published	English
5.	Publisher's name Nationality Address	C. L. Parihar (*Editor*) Indian 5, 1st floor, I. K. Girls School Campus, 14/1 Ushaganj, Near G.P.O. Indore - 452 001 India
6.	Printer's Name Nationality Address	Piyush Gupta Indian 316, Subhash Nagar, Mumfordganj, Prayagraj - 211002
7.	Editor's Name Nationality Address	C. L. Parihar Indian Indore
8.	Name of the Printing Press, where the publication is printed	Radha Krishna Enterprises 6B/4B/9A, Beli Road, Prayagraj - 211002
9.	Name and addresses of the individuals who own the newspaper/journal and partners or shareholders holding more than one per cent of the total capital	No Individual: It is run by Indian Academy of Mathematics 5, 1st floor, I. K. Girls School Campus, 14/1 Ushaganj, Near G.P.O. Indore - 452 001 India

I, C. L. Parihar, hereby declare that the particulars given above are true to the best of my knowledge and belief.

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Dated: December 31, 2021

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Composed & Printed by: Piyush Gupta, Prayagraj-211 002, (U.P.) Mob: 07800682251

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