ISSN : 0970-5120

THE JOURNAL of the INDIAN ACADEMY of MATHEMATICS

Volume 46 2024 No. 1

INDIAN ACADEMY OF MATHEMATICS
IAM
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PUBLISHED BY THE ACADEMY

2024

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Journal of Indian Acad. Math. ISSN: 0970-5120
Vol. 46, No. 1 (2024) pp. 1-11

Thomas Koshy

SUMS INVOLVING EXTENDED GIBONACCI POLYNOMIALS

Abstract: We explore three infinite gibonacci polynomial sums, and their Pell and Jacobsthal versions.

Keywords: Extended Gibonacci Polynomials, Fibonacci Polynomial, Lucas Polynomial, Binet-Like Formulas, Jacobsthal, and Jacobsthal-Lucas Polynomial.

Mathematical Subject Classification No.: Primary 11B37, 11B39, 11C08.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*.

Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 4,].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$ [3, 4].

Gibonacci and Jacobsthal polynomials are linked by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $c_n = J_n$ or j_n, $\Delta = \sqrt{x^2 + 4}$, $E = \sqrt{x^2 + 1}$, and $D = \sqrt{4x + 1}$, where $c_n = c_n(x)$.

1.1 Fundamental Gibonacci Identities

Using Binet-like formulas, we can establish the following gibonacci identities [4]:

$$f_a l_a = f_{2a}; \tag{1}$$

$$l_{a-1} + l_{a+1} = \Delta^2 f_a; \tag{2}$$

$$l_{a+b} + (-1)^b l_{a-b} = l_a l_b; \tag{3}$$

$$l_{a+b} - (-1)^b l_{a-b} = \Delta^2 f_a f_b. \tag{4}$$

Identities (3) and (4) imply that

$$l_{2a} + 2(-1)^a = l_a^2; \tag{5}$$

$$l_{2a} - 2(-1)^a = \Delta^2 f_a^2. \tag{6}$$

respectively.

These identities play a major role in our discourse.

2. Gibonacci Sums

With these identities at our fingertips, we now begin our exploration with the following result.

Theorem 1: *Let*

$$\nu_n(x) = \begin{cases} x, & if \;\; n = 1 \\ 1, & otherwise. \end{cases}$$

Then
$$\sum_{n=1}^{\infty} \frac{\nu_n f_{2n-1}}{l_{2n} + 1} = \frac{1}{\Delta}. \qquad (7)$$

Proof: Let $m \geq 2$. Using recursion [4], we will first establish that

$$\sum_{n=1}^{m} \frac{\nu_n f_{2n-1}}{l_{2n} + 1} = \frac{f_{2m}}{l_{2m} + 1}. \qquad (8)$$

To this end, we let A_m denote the left side (LHS) of this equation and B its right side (RHS). With $m \geq 2$, identity (5) yields $l_{2m} = l_{2m-1}^2 - 2$. Identity (1), coupled with this result, then gives

$$\begin{aligned} B_m - B_{m-1} &= \frac{f_{2m}}{l_{2m} + 1} - \frac{f_{2m-1}}{l_{2m-1} + 1} = \frac{f_{2m}}{l_{2m} + 1} - \frac{(l_{2m-1} - 1)f_{2m-1}}{(l_{2m-1} - 1)(l_{2m-1} + 1)}; \\ &= \frac{f_{2m}}{l_{2m} + 1} - \frac{(l_{2m-1} - 1)f_{2m-1}}{(l_{2m-1} - 1)(l_{2m-1} + 1)} = \frac{f_{2m-1}}{l_{2m} + 1}; \\ &= A_m - A_{m-1}. \end{aligned}$$

With recursion, this yields

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_2 - B_2 \\ &= \left(\frac{x}{l_2 + 1} + \frac{x}{l_4 + 1} \right) - \frac{f_4}{l_4 + 1} \\ &= 0. \end{aligned}$$

Thus, $A_m = B_m$, as claimed.

Since, $\lim_{m\to\infty} \frac{f_m}{l_m + 1} = \frac{1}{\Delta}$, the given result now follows from equation (8), as desired. □

It follows from equation (7) that [2, 4, 6],

$$\sum_{n=1}^{\infty} \frac{F_{2^{n-1}}}{L_{2^n} + 1} = \frac{\sqrt{5}}{5}. \tag{9}$$

The next result invokes identities (1), (3), and (5).

Theorem 2:

$$\sum_{n=1}^{\infty} \frac{l_{2^{n+1}}}{f_{3.2^n}} = \frac{\Delta^2}{l_3}. \tag{10}$$

Proof: With $a = 3{\cdot}2^{m-1}$ and $b = 2^{m-1}$, it follows from identity (3) that

$$l_{2^{m+1}} + l_{2^m} = l_{3\cdot 2^{m-1}} l_{2^{m-1}}. \tag{11}$$

Using recursion [4] and identities (1), (3), and (11), we will now confirm that

$$\sum_{n=1}^{m} \frac{l_{2^{n+1}}}{f_{3.2^n}} = \frac{l_4}{f_6} + \frac{l_2}{f_6} - \frac{l_{2^m}}{f_{3.2^m}}, \tag{12}$$

where $m \geq 2$, Letting $A_m =$ LHS and $B =$ RHS of this equation, we then get

$$B_m - B_{m-1} = \frac{l_{2^{m-1}}}{f_{3.2^{m-1}}} - \frac{l_{2^m}}{f_{3.2^m}} = \frac{l_{3.2^{m-1}} l_{2^{m-1}}}{f_{3.2^m}} - \frac{l_{2^m}}{f_{3.2^m}};$$

$$= \frac{l_{2^{m+1}} + l_{2^m}}{f_{3.2^m}} - \frac{l_{2^m}}{f_{3.2^m}} = \frac{l_{2^{m+1}}}{f_{3.2^m}};$$

$$= A_m - A_{m-1}.$$

Recursively, this yields

$$A_m - B_m = A_{m-1} - B_{m-1} = \cdots = A_2 - B_2$$

$$= \left(\frac{l_4}{f_6} + \frac{l_8}{f_{12}}\right) - \left(\frac{l_4}{f_6} + \frac{l_2}{f_6} - \frac{l_4}{f_{12}}\right) = \left(\frac{l_4}{f_6} + \frac{l_8}{f_{12}}\right) - \left(\frac{l_4}{f_6} + \frac{l_8}{f_{12}}\right)$$

$$= 0,$$

confirming the validity of equation (12).

Since, $\lim_{m\to\infty} \frac{l_m}{f_{3m}} = 0$, equation (12) yields the desired result. □

It then follows that

$$\sum_{n=1}^{\infty} \frac{l_{2n+1}}{f_{3.2^n}} = \frac{l_4}{f_6} + \frac{l_2}{f_6} = \left(\frac{l_4}{f_6} + \frac{l_8}{f_{12}}\right) - \left(\frac{l_4}{f_6} + \frac{l_2}{f_6} - \frac{l_4}{f_{12}}\right) = \frac{\Delta^2}{l_3}$$

$$\sum_{n=0}^{\infty} \frac{l_{2n+1}}{f_{3.2^n}} = \frac{l_2}{f_3} + \frac{\Delta^2}{l_3}.$$

Consequently, we have [5, 7]

$$\sum_{n=1}^{\infty} \frac{L_{2n+1}}{F_{3.2^n}} = \frac{5}{4}; \qquad \sum_{n=0}^{\infty} \frac{L_{2n+1}}{F_{3.2^n}} = \frac{11}{4},$$

respectively.

The following result showcases an application of identity (5).

Theorem 3:

$$\sum_{n=1}^{\infty} \frac{f_{2n-1}^2}{l_{2n}^2 - 1} = \frac{\Delta^2 + 1}{\Delta^2(l_2^2 - 1)}. \tag{13}$$

Proof: Using recursion [4], we will first establish that

$$\sum_{n=1}^{m} \frac{f_{2^{n-1}}^2}{l_{2^n}^2 - 1} = \frac{1}{l_2^2 - 1} + \frac{1}{\Delta^2}\left(\frac{1}{l_2^2 - 1} - \frac{1}{l_{2^m}^2}\right), \tag{14}$$

where $m \geq 2$.

With $m \geq 2$ and $a = 2^{m-1}$, it follows by identities (5) and (6) that $l_{2^m} - 2 = \Delta^2 f_{2^{m-1}}^2$ and $l_{2^m} + 2 = f_{2^{m-1}}^2$, respectively. Then

$$\Delta^2 f_{2^{m-1}}^2 (l_{2^{m-1}}^2 - 1) = (l_{2^m} - 2)(l_{2^m} + 1) = l_{2^m}^2 - l_{2^m} - 2 = l_{2^m}^2 - l_{2^{m-1}}.$$

Let A_m = LHS and B_m = RHS of equation (13). Then

$$B_m - B_{m-1} = \frac{1}{\Delta^2}\left(\frac{1}{l_{2^{m-1}}^2 - 1} - \frac{1}{l_{2^m}^2 - 1}\right) = \frac{l_{2^m}^2 - l_{2^{m-1}}^2}{\Delta^2 (l_{2^{m-1}}^2 - 1)(l_{2^m}^2 - 1)}$$

$$= \frac{\Delta^2 f_{2^{m-1}}^2 (l_{2^{m-1}}^2 - 1)}{\Delta^2 (l_{2^{m-1}}^2 - 1)(l_{2^m}^2 - 1)} = \frac{l_{2^{m-1}}^2}{l_{2^m}^2 - 1}$$

$$= A_m - A_{m-1}.$$

Recursively, this yields

$$A_m - B_m = A_{m-1} - B_{m-1} = \cdots = A_2 - B_2$$

$$= \left(\frac{f_1^2}{l_2^2 - 1} + \frac{f_2^2}{l_4^2 - 1}\right) - \left[\frac{1}{l_2^2 - 1} + \frac{1}{\Delta^2}\left(\frac{1}{l_2^2 - 1} - \frac{1}{l_4^2 - 1}\right)\right]$$

$$= \left(\frac{f_1^2}{l_2^2 - 1} + \frac{f_2^2}{l_4^2 - 1}\right) - \left[\frac{1}{l_2^2 - 1} + \frac{l_4^2 - l_2^2}{\Delta^2 (l_2^2 - 1)(l_4^2 - 1)}\right]$$

$$= \left(\frac{f_1^2}{l_2^2 - 1} + \frac{f_2^2}{l_4^2 - 1}\right) - \left[\frac{1}{l_2^2 - 1} + \frac{\Delta^2 f_2^2 (l_2^2 - 1)}{\Delta^2 (l_2^2 - 1)(l_4^2 - 1)}\right]$$

$$= \left(\frac{f_1^2}{l_2^2 - 1} + \frac{f_2^2}{l_4^2 - 1} \right) - \left(\frac{1}{l_2^2 - 1} + \frac{f_2^2}{l_4^2 - 1} \right)$$

$$= 0,$$

confirming the validity of equation (14), as desired.

Since $\lim_{m \to \infty} \frac{1}{l_{2^m}^2 - 1} = 0$, the given result now follows from it. □

This theorem implies [4, 5, 7],

$$\sum_{n=1}^{\infty} \frac{F_{2^n-1}^2}{L_{2^n}^2 - 1} = \frac{3}{20}.$$

We now explore the Pell versions of equations (7), (10), and (13).

3. Pell Consequences

Using the Pell-gibonacci relationship $b_n(x) = g_n(2x)$ [4], we get the desired counterparts and their numeric versions:

$$\sum_{n=1}^{\infty} \frac{\nu_n(2x) p_{2^n-1}}{q_{2^n} + 1} = \frac{1}{2E}; \qquad \sum_{n=1}^{\infty} \frac{\nu_n(2) P_{2^n-1}}{2Q_{2^n} + 1} = \frac{\sqrt{2}}{4};$$

$$\sum_{n=1}^{\infty} \frac{q_{2^n+1}}{p_{3 \cdot 2^n}} = \frac{4E^2}{q_3}; \qquad \sum_{n=1}^{\infty} \frac{Q_{2^n+1}}{P_{3 \cdot 2^n}} = \frac{2}{7};$$

$$\sum_{n=1}^{\infty} \frac{p_{2^n-1}^2}{q_{2^n}^2 - 1} = \frac{4E^2 + 1}{4E^2(q_2^2 - 1)}; \qquad \sum_{n=1}^{\infty} \frac{P_{2^n-1}^2}{4Q_{2^n}^2 - 1} = \frac{9}{64}.$$

respectively.

Next, we explore the Jacobsthal versions of the gibonacci sums (7), (10), and (13) using the Jacobsthal-gibonacci relationships in Section 1.

4. Jacobsthal Consequences

In the interest of brevity and clarity, we let A denote the left-hand side of each equation and B its right-hand side, and LHS and RHS those of the corresponding Jacobsthal equation, respectively.

4.1 Jacobsthal Version of Equation (7): Proof: Let $A = \frac{\nu_n f_{2n-1}}{l_{2n} + 1}$. Replacing x with $1/\sqrt{x}$, and multiplying the numerator and denominator of the resulting expression with $x^{2^{n-1}}$, we get

$$A = \frac{\nu_n(1/\sqrt{x})x(x^{2^{n-1}-1}f_{2n-1}}{x^{2^{n-1}}l_{2n} + x^{2^{n-1}}};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{\nu_n(1/\sqrt{x})xJ_{2n-1}}{j_{2n} + x^{2^{n-1}}},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Now, let $B = \frac{1}{\Delta}$. Replacing x with $1/\sqrt{x}$, we get

$$\text{RHS} = \frac{\sqrt{x}}{D}.$$

Equating the two sides, we get the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{\nu_n(1/\sqrt{x})xJ_{2n-1}}{j_{2n} + x^{2^{n-1}}} = \frac{\sqrt{x}}{D}. \tag{15}$$

□

In particular, this yields [2, 4, 6]

$$\sum_{n=1}^{\infty} \frac{F_{2n-1}}{L_{2n} + 1} = \frac{\sqrt{5}}{5},$$

as in equation (9).

4.2 Jacobsthal Version of Equation (10): Proof: Let $A = \frac{l_{2n+1}}{f_{3.2^n}}$. Now, replace x with $1/\sqrt{x}$, and multiply the numerator and denominator of the resulting expression with $x^{(3.2^n-1)/2}$. Then

$$A = \frac{x^{(2^n-1)/2}[x^{2^{(n+1)/2}} l_{2n+1}}{x^{(3.2^n-1)/2} f_{3.2^n}};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{x^{(2^n-1)/2} j_{2n+1}}{J_{3.2^n}},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next, we let $B = \frac{\Delta^2}{l_3}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with $\sqrt{x}$, we get

$$B = \frac{D^2}{xl_3(1/\sqrt{x})};$$

$$\text{RHS} = \frac{D^2\sqrt{x}}{j_3}.$$

Equating the two sides yields the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{x^{(2^n-1)/2} j_{2n+1}}{J_{3.2^n}} = \frac{D^2\sqrt{x}}{j_3}. \tag{16}$$

□

This implies [5, 7],

$$\sum_{n=1}^{\infty} \frac{L_{2n+1}}{F_{3.2^n}} = \frac{5}{4},$$

as we found earlier.

Finally, we explore the Jacobsthal counterpart of equation (13).

4.3 Jacobsthal Version of Equation (13): Proof: Let $A = \dfrac{f_{2^n+1}^2}{l_{2^n}^2 - 1}$.

Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator of the resulting expression with x^{2^n}, we then get

$$A = \frac{x^{2^{n-1}+1}\left[x^{(2^n-1)/2} f_{2^n-1}\right]^2}{(x^{2^n/2} l_{2^n})^2 - x^{2^n}};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{x^{2^{n-1}+1} j_{2^n-1}^2}{J_{2^n}^2 - x^{2^n}},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next, we let $B = \dfrac{\Delta^2 + 1}{\Delta^2(l_2^2 - 1)}$. Now replace x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with x^2. This gives,

$$B = \frac{x\left(\frac{D^2}{x} + 1\right)x^2}{D^2\left[(x^{2/2} l_2)^2 - x^2\right]};$$

$$\text{RHS} = \frac{(D^2 + x)x^2}{D^2(j_2^2 - x^2)}.$$

Combining the two sides yields the desired Jacobsthal version:

$$\sum_{n-1}^{\infty} \frac{x^{2n-1-1} J_{2^n-1}^2}{j_{2^n}^2 - x^{2^n}} = \frac{(D^2 + x)x^2}{D^2(j_2^{2^n} - x^2)}. \tag{17}$$

□

This yields [4, 5, 7]

$$\sum_{n=1}^{\infty} \frac{F_{2^{n-1}}^2}{L_{2^{n-1}}^2 - 1} = \frac{3}{20},$$

as found earlier.

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(*Received*, November 21, 2023)

Journal of Indian Acad. Math.
Vol. 46, No. 1 (2024) pp. 13-24

ISSN: 0970-5120

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SEVERAL FIXED POINT THEOREM FOR FUZZY MAPPINGS IN COMPLEX VALUED METRIC SPACES

Abstract: The main purpose of this paper is to establish some fixed point results for fuzzy mappings in complete complex valued metric spaces. The derived results generalize some theorems as in the existing literature. Some examples and counter examples are provided to justify the theorem proved here.

Keywords: Fuzzy Mappings, Fixed Point, Hausdorff Distance, Complex Valued Metric Spaces, Fuzzy Fixed Point.

Mathematical Subject Classification No.: 03B52, 03E72, 47H10, 46G20, 58B12.

1. Introduction

Fixed point theory is most interesting and dynamic area of research in functional analysis. This principal plays an important and key role in investigating the existence and uniqueness of solution to various problems in mathematics, physics, engineering, medicines, and social sciences which leads to mathematical models design by system of nonlinear integral equations, functional equations, and differential equations. Banach contraction principal has been generalized in different directions by changing the underlying space.

The concept of fuzzy set was first introduced by Zadeh [10]. Later Weiss [9] obtained many fixed point results for many fixed point theorems for fuzzy mappings in metric spaces. Heilpern [5] initiated the idea of fuzzy contraction mappings and proved a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of Nadler's fixed point theorem for multi-valued mappings [7]. Further work on fuzzy mappings can be seen in [8].

In this paper, we obtain a fixed point theorem and a common fixed point theorem for fuzzy mappings in complex valued complete metric space. An example is also given which supports the obtained results.

Here, the obtained results for fuzzy mapping in metric space, fuzzy mapping in complex valued metric space under certain constrictive conditions are helpful for Hausdorff dimensions computing which are helpful in high energy physics. In high energy physics these results are also helpful for solving the arising geometric problems due to the involvement of fuzzy sets.

2. Preliminaries

Definition 2.1 [3]: *Assume $\mathfrak{C}$ is the set of complex numbers. For $\zeta_1, \zeta_2 \in \mathfrak{C}$ we define a partial order $\precsim$ on $\mathfrak{C}$ as follows:*

(Pi) $\zeta_1 \precsim \zeta_2 \Leftrightarrow Re(\zeta_1) \leq Re(\zeta_2)$ *and* $Im(\zeta_1) \leq Im(\zeta_2)$;

(Pii) $\zeta_1 \prec \zeta_2 \Leftrightarrow Re(\zeta_1) < Re(\zeta_2)$ *and* $Im(\zeta_1) < Im(\zeta_2)$;

(Piii) $\zeta_1 \precnsim \zeta_2 \Leftrightarrow Re(\zeta_1) = Re(\zeta_2)$ *and* $Im(\zeta_1) \leq Im(\zeta_2)$;

(Piv) $\zeta_1 = \zeta_2 \Leftrightarrow Re(\zeta_1) = Re(\zeta_2)$ *and* $Im(\zeta 1) = Im(\zeta_2)$.

Clearly if $a \leq b \Rightarrow az \precsim bz$, for all $z \in \mathfrak{C}$ and for all $a, b \in \mathfrak{R}$. Note that if $\zeta_1 \neq \zeta_2$ and one of (Pi), (Pii) and (Piii) is satisfied then $\zeta_1 \precnsim \zeta_2$ and write $\zeta_1 = \zeta_2$ if only (Piv) is satisfied. Note that

(i) $0 \precsim \zeta_1 \precnsim \zeta_2 \Rightarrow |\zeta_1| < |\zeta_2|, \forall \zeta_1, \zeta_2 \in \mathfrak{C}$;

(ii) $\zeta_1 \precnsim \zeta_2$ and $\zeta_2 \prec \zeta_3 \Rightarrow \zeta_1 \prec \zeta_3, \forall \zeta_1, \zeta_2, \zeta_3 \in \mathfrak{C}$.

Definition 2.2 [3]: *Let $\mathfrak{X}$ be a nonempty set and $\rho : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{C}$ be a mapping which satisfies the following conditions:*

(i) $0 \precsim \rho(z, w)$, *for all* $z,\ w \in \mathfrak{X}$ *and* $\rho(z, w) = 0$ *if and only if* $z = w$;

(ii) $\rho(z, w) = \rho(w, z)$ *for all* $z,\ w \in \mathfrak{X}$;

(iii) $\rho(z, w) \precsim \rho(z, z_1) + \rho(z_1, w)$, *for all* $z,\ z_1,\ w \in \mathfrak{X}$.

Then $(\mathfrak{X}, \rho)$ is called a complex-valued metric space.

Example 1: *Let* $\mathfrak{X} = [0,1]$, *consider a metric* $\rho : \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$ *defined as* $\rho(x, y) = |x - y| + i\,|x - y|, \forall x,\ y \in \mathfrak{X}$.

To verify that $(\mathfrak{X}, \rho)$ *is a complete complex valued metric space, it is enough to verify the triangular inequality condition:*

$$\begin{aligned}\rho(x, y) &= |x - y| + i\,|x - y| \\ &= |x - z + z - y| + i\,|x - z + z - y| \\ &\preceq \{|\,x - z\,| + |\,z - y\,|\} + i\{|\,x - z\,| + |\,z - y\,|\} \\ &\preceq \{|\,x - z\,| + i\,|\,x - z\,|\} + \{|\,z - y\,| + i\,|\,z - y\,|\}\end{aligned}$$

$\therefore\ \rho(x, y) \preceq \rho(x, z) + \rho(z, y)$.

Definition 2.3 [3]: *A point* $z \in \mathfrak{X}$ *is known as an interior point of a set* $Z \subseteq \mathfrak{X}$, *if we find* $0 \prec \epsilon \in \mathfrak{C}$ *such that*

$$\mathfrak{B}(z, \epsilon) = \{w \in \mathfrak{X} : \rho(z, w) \prec \epsilon\} \subseteq Z\,. \tag{1}$$

A point $z \in Z$ is known as the limit point of Z, if there exists an open ball $\mathfrak{B}(z, \epsilon)$ such that

$$\mathfrak{B}(z, \epsilon) \cap (Z \backslash \{z\}) \neq \phi \tag{2}$$

where $0 \prec \epsilon \in \mathfrak{C}$. A subset Z of $\mathfrak{X}$ is said to be open if each point of Z is an interior point of Z. Furthermore, Z is said to be closed if it contains all its limit points.

The family

$$\mathscr{B} = \{\mathfrak{B}(z,\epsilon) : z \in \mathfrak{X}, 0 \prec \epsilon\} \tag{3}$$

is a sub basis for a Hausdorff topology τ on $\mathfrak{X}$.

Definition 2.4 [4]: *Let $(\mathfrak{X}, \rho)$ be a complex valued metric space and $\{z_n\}$ be a sequence in $\mathfrak{X}$. Then,*

- *$\{z_n\}$ is called a convergent sequence if and only if there exists $z \in \mathfrak{X}$, such that for all $\epsilon \succ 0$, $\exists\, n(\epsilon) \in N$ such that for all $n \geq n(\epsilon)$, we have $\rho(z_n, z) \prec \epsilon$. So, we write $\lim_{n\to\infty} z_n = z$.*

- *$\{z_n\}$ is called a Cauchy sequence if and only if for all $\epsilon \succ 0$, $\exists n(\epsilon) \in \mathbb{N}$ such that for each $m,\ n \geq n(\epsilon)$, we have $\rho(z_n, z_m) \prec \epsilon$.*

- *$(\mathfrak{X}, \rho)$ is called complete if every Cauchy sequence in $\mathfrak{X}$ converges to a point $z \in \mathfrak{X}$.*

Definition 2.5 [8]: *Let $(\mathfrak{X}, \rho)$ be a metric space. We define the Hausdorff metric on $\mathscr{C}\mathscr{B}(\mathfrak{X})$ induced by ρ*

$$H(A, B) = \max\{\sup_{x\in A} \rho(x, B), \sup_{y\in B} \rho(A, y)\}, \tag{4}$$

for all $A, B \in \mathscr{C}\mathscr{B}(\mathfrak{X})$, where $\mathscr{C}\mathscr{B}(\mathfrak{X})$ denotes the closed and bounded subsets of $\mathfrak{X}$ and

$$\rho(x, B) = \inf\{\rho(x, \beta) : \beta \in B\}, \tag{5}$$

for all $x \in X$.

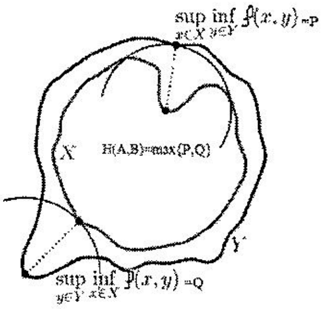

Figure 1: Hausdorff distance between the green line X and the blue line Y.

Definition 2.6 [1]: *Let* $(\mathfrak{X}, \rho)$ *be a complex valued metric space. A nonempty subset* A *of* $\mathfrak{X}$ *is called bounded from below if there exists some* $z \in \mathfrak{C}$, *such that* $z \preceq a$ *for all* $a \in A$.

Definition 2.7: *Let* $(\mathfrak{X}, \rho)$ *be a complex valued metric space. A nonempty subset* A *which is bounded from below is said to have the greatest lower bound property (g.l.b property) on* $(\mathfrak{X}, \rho)$ *if there exists a lower bound* $\omega \in \mathfrak{C}$ *such that* $z \preceq \omega$ *where* z *is any arbitrary lower bound of* A *in* $\mathfrak{X}$ *and we write it by* $\inf A$.

[6] Let $(\mathfrak{X}, \rho)$ be a complex-valued metric space. We denote the family of all nonempty closed and bounded subsets of complex-valued metric space $\mathfrak{X}$ by $\mathscr{CB}(\mathfrak{X})$. For $\nu \in \mathfrak{C}$ we represent

$$s(\nu) = \{z \in \mathfrak{C} : \nu \preceq z\} \tag{6}$$

and for $\omega \in \mathfrak{X}$ and $B \in \mathscr{CB}(\mathfrak{X})$.

$$s(\omega, B) = \bigcup_{\beta \in B} s(\rho(\omega, \beta)) = \bigcup_{\beta \in B} \{z \in \mathfrak{C} : \rho(\omega, B) \preceq z\}. \tag{7}$$

For $A, B \in \mathscr{CB}(\mathfrak{X})$, we denote

$$s(A, B) = \left(\bigcap_{\alpha \in A} s(\alpha, \beta)\right) \cap \left(\bigcap_{\beta \in B} s(\beta, A)\right). \tag{8}$$

Remark 2.1 [1]: *Let* $(\mathfrak{X}, \rho)$ *be a complex valued metric space. If* $\mathfrak{C} = \mathfrak{R}$, *then* (X, ρ) *is a metric space. Moreover, for* $A, B \in \mathscr{C}\mathscr{B}(\mathfrak{X})$, $H(A, B) = \inf s\,(A, B)$ *is the Hausdorff distance induced by* ρ.

A fuzzy set in $\mathfrak{X}$ is a function with domain $\mathfrak{X}$ and values in [0,1], $F(\mathfrak{X})$ is the collection of all fuzzy sets in $\mathfrak{X}$. If A is a fuzzy set and $x \in \mathfrak{X}$, then the function value $A(x)$ is called the grade of membership of x in A. The α-level set of fuzzy set A, is denoted by $[A]_\alpha$, and defined as:

$$[A]_\alpha = \{x : A(x) \geq \alpha\} \text{ where } \alpha \in (0,1],$$

$$[A]_0 = \overline{\{x : A(x) > 0\}}$$

Let $\mathfrak{X}$ be any nonempty set and $\mathscr{Y}$ be a complex valued metric space. A mapping T is called a fuzzy mapping, if T is a mapping from $\mathfrak{X}$ into $F(\mathscr{Y})$. A fuzzy mapping T is a fuzzy subset on $X \times \mathscr{Y}$ with membership function $T(x)(y)$. The function $T(x)(y)$ is the grade of membership of y in $T(x)$. For convenience, we denote the α-level set of $T(x)$ by $[Tx]_\alpha$ instead of $[T(x)]_\alpha$.

Definition 2.8 [8]: *A point* $x \in \mathfrak{X}$ *is called a fuzzy fixed point of a fuzzy mapping* $T : \mathfrak{X} \to F(\mathfrak{X})$ *if there exists* $\alpha \in (0,1]$ *such that* $x \in [Tx]_\alpha$.

Lemma 2.1 [8]: *Let* A *and* B *be nonempty closed and bounded subsets of a metric space* $(\mathfrak{X}, \rho)$. *If* $a \in A$, *then*

$$\rho(a, B) \leq H(A, B).$$

Lemma 2.2: *Let* A *and* B *be nonempty closed and bounded subsets of a complex valued metric space* $(\mathfrak{X}, \rho)$. *If* $a \in A$, *then*

$$\rho(a, B) \precsim \inf s\,(A, B).$$

Proof: The Lemma follows directly as a consequence of Lemma 2.1. □

Lemma 2.3 [8]: *Let A and B be nonempty closed and bounded subsets of a metric space $(\mathfrak{X}, \rho)$ and $0 < \alpha \in \mathfrak{R}$. Then, for $a \in A$, there exists $b \in B$ such that*

$$\rho(a, b) \leq H(A, B) + \alpha .$$

Lemma 2.4: *Let A and B be nonempty closed and bounded subsets of a complex valued metric space $(\mathfrak{X}, \rho)$ and $0 \prec \alpha \in \mathfrak{C}$. Then, for $a \in A$, there exists $b \in B$ such that*

$$\rho(a, b) \preceq \inf s\,(A, B) + \alpha .$$

Proof: The Lemma follows directly as a consequence of Lemma 2.3. □

3. Main Results

Theorem 3.1: *Let $(\mathfrak{X}, \rho)$ be a complete complex valued metric space. Let $T : \mathfrak{X} \to F(\mathfrak{X})$ be a fuzzy mapping and for $x \in \mathfrak{X}$, there exist $\alpha(x) \in (0, 1]$ satisfying the following condition:*

$$\inf s\,([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \preceq a_1\rho(x, [Tx]_{\alpha(x)}) + a_2\rho(y, [Ty]_{\alpha(y)} + a_3\rho(x, [Ty]_{\alpha(y)})$$

$$+ a_4\rho(y, [Ty]_{\alpha(x)}) + a_5\rho(x, y) + \omega \cdot \frac{\rho(x, [Tx]_{\alpha(x)})(1 + |\rho(x, [Tx]_{\alpha(x)})|)}{1 + |\rho(x; y)|} \tag{9}$$

for all $x, y \in X$. Also $0 \leq a_i \in \mathfrak{R}$, where $i = 1, 2, 3, \ldots, 5$ and $0 \prec \omega \in \mathfrak{C}$ with $|\tau| < 1$, where $\tau = \frac{a_1, a_3, a_5 + \omega}{1-(a_2+a_3)}$ and $(a_2 + a_3) \neq 1$. Then, T has a fixed point.

Proof: Let x_0 be any arbitrary point in $\mathfrak{X}$, such that $x_1 \in [Tx_0]_{\alpha(x_0)}$. Then by Lemma 2.4 there exists $x_2 \in [Tx_1]_{\alpha(x_1)}$, such that

$$\rho(x_1, x_2) \preceq \inf s\left([Tx_0]_{\alpha(x_0)}, [Tx_1]_{\alpha(x_1)}\right) + (a_1 + a_3 + a_5 + \omega)$$

$$\preceq a_1\rho(x_0, [Tx_0]_{\alpha(x_0)}) + a_2\rho(x_1, [Tx_1]_{\alpha(x_1)}) + a_3\rho(x_0, [Tx_1]_{\alpha(x_1)})$$

$$+ a_4\rho(x_1, [Tx_0]_{\alpha(x_0)}) + a_5\rho(x_0, x_1)$$

$$+ \omega \cdot \frac{\rho(x_0, [Tx_0]_{\alpha(x_0)})(1 + |\rho(x_0, [Tx_0]_{\alpha(x_0)})|)}{(1 + |\rho(x_0, x_1)|} + (a_1 + a_3 + a_5 + \omega)$$

$$\preceq a_1\rho(x_0, x_1) + a_2\rho(x_1, x_2) + a_3[\rho(x_0, x_1) + \rho(x_1, x_2)]$$

$$+ a_5\rho(x_0, x_1) + \omega\rho(x_0, x_1) + (a_1 + a_3 + a_5 + \omega).$$

$$\rho(x_1, x_2) \preceq \frac{a_1 + a_3 + a_5 + \omega}{1 - (a_2 + a_3)}\rho(x_0, x_1) + \frac{a_1 + a_3 + a_5 + \omega}{1 - (a_2 + a_3)}. \tag{10}$$

Let $\tau = \dfrac{a_1 + a_3 + a_5 + \omega}{1 - (a_2 + a_3)}$, with condition $|\tau| = \left|\dfrac{a_1 + a_3 + a_5 + \omega}{1 - (a_2 + a_3)}\right| < 1$ and $(a_2 + a_3) \neq 1$.

Therefore the Equation (10), we have

$$\rho(x_1, x_2) \preceq \tau\rho(x_0, x_1) + \tau.$$

Again by Lemma 2.4, $\exists\ x_3 \in [Tx_2]_{\alpha(x_2)}$ such that

$$\rho(x_2, x_3) \preceq \inf s\left([Tx_1]_{\alpha(x_1)}, [Tx_2]_{\alpha(x_2)}\right) + \frac{(a_1 + a_3 + a_5 + \omega)^2}{1 - (a_2 + a_3)}$$

$$\preceq a_1\rho(x_1, [Tx_1]_{\alpha(x_1)}) + a_2\rho(x_2, [Tx_2]_{\alpha(x_2)}) + a_3\rho(x_1, [Tx_2]_{\alpha(x_2)})$$

$$+ a_4\rho(x_2, [Tx_1]_{\alpha(x_1)}) + a_5\rho(x_1, x_2)$$

$$+\omega\cdot\frac{\rho(x_1,[Tx_1]_{\alpha(x_1)})(1+|\rho(x_1,[Tx_1]_{\alpha(x_1)})|)}{(1+|\rho(x_1,x_2)|}+\frac{(a_1+a_3+a_5+\omega)^2}{1-(a_2+a_3)}$$

$$\rho(x_2,x_3)\preceq a_1\rho(x_1,x_2)+a_2\rho(x_2,x_3)+a_3[\rho(x_1,x_2)+\rho(x_2,x_3)]$$

$$+a_5\rho(x_1,x_2)+\omega\rho(x_1,x_2)+\frac{(a_1+a_3+a_5+\omega)}{1-(a_2+a_3)}$$

$$\preceq\frac{a_1+a_3+a_5+\omega}{1-(a_2+a_3)}\rho(x_1,x_2)+\frac{(a_1+a_3+a_5+\omega)^2}{(1-(a_2+a_3))^2}$$

$$\rho(x_2,x_3)\preceq\left(\frac{a_1+a_3+a_5+\omega}{1-(a_2+a_3)}\right)^2\rho(x_0,x_1)+2\left(\frac{(a_1+a_3+a_5+\omega)}{(1-(a_2+a_3))}\right)^2\quad\text{by (10)}$$

$$\therefore\ \rho(x_2,x_3)\preceq\tau^2\rho(x_0,x_1)+2\tau^2.$$

Continuing as in a preceding way, we can obtain a sequence $\{x_n\}$ such that $x_n\in[T_{x_{n+1}}]_{\alpha(x_{n+1})}$, we have

$$\rho(x_n,x_{n+1})\preceq\tau^n\rho(x_0,x_1)+n\tau^n. \tag{11}$$

Now, for any positive integers $m, n(n>m)$, we have

$$\rho(x_m,x_n)\preceq[\rho(x_m,x_{m+1})+\rho(x_{m+1},x_n)]$$

$$\preceq[\rho(x_m,x_{m+1})]+[\rho(x_{m+1},x_{m+2})+\rho(x_{m+2},x_n)]$$

$$\preceq[\rho(x_m,x_{m+1})]+\{\rho(x_{m+1},x_{m+2})\}+\ldots+\{\rho(x_{n-1},x_n)\}]$$

$$\preceq\tau^m\rho(x_0,x_1)+m\tau^m+\tau^{m+1}\rho(x_0,x_1)+(m+1)\tau^{m+1}+\cdots$$

$$+\tau^{n-1}\rho(x_0,x_1)+(n-1)\tau^{n-1}\quad\text{by (11)}$$

$$\preceq \tau^m(1+\tau+\cdots+\tau^{n-m-1})\rho(x_0, x_1)+\sum_{i=m}^{n-1} i\tau^i .$$

It follows from Cauchy root test that $\sum i\tau^i$ is convergent and hence x_n is a Cauchy sequence. Since $(\mathfrak{X}, \rho)$ is complete then there exists $z \in \mathfrak{X}$ such that $x_n \to z$ as $n \to \infty$.

Now,

$$\rho(z, [Tz]_{\alpha(z)}) \preceq [\rho(z, x_{n+1}) + \rho(x_{n+1}, [Tz]_{\alpha(z)})]$$

$$\preceq [\rho(z, x_{n+1}) + \inf s([Tx_n]_{\alpha(x_n)}, [Tz]_{\alpha(z)})]$$

Using (9) with $n \to \infty$ we get that

$$(1-(a_2+a_3))\rho(z, [Tz]_{\alpha(z)}) \preceq 0$$

So, we obtain that

$$z \in [Tz]_{\alpha(z)} .$$

Hence, $z \in \mathfrak{X}$ is a fixed point for the fuzzy mapping T. □

Example 2: *Let* $\mathfrak{X} = [0, 1]$ *and* $\rho(x, y) = \{|x-y| + i\,|\,x-y\,|\}$, *whenever* $x, y \in \mathfrak{X}$, *then* $(\mathfrak{X}, \rho)$ *is a complete metric space. Define a fuzzy mapping* $T : \mathfrak{X} \to F(\mathfrak{X})$ *by*

$$T(x)(t)\begin{cases}1, & 0 \le t \le \frac{x}{4} \\ \frac{1}{2}, & \frac{x}{4} \le t \le \frac{x}{3} \\ \frac{1}{4}, & \frac{x}{3} \le t \le \frac{x}{2} \\ 0, & \frac{x}{2} \le t \le 1\end{cases}$$

For all $x \in \mathfrak{X}$, *there exists* $\alpha(x) = 1$ *such that* $[Tx]_{\alpha(x)} = [0, \frac{x}{4}]$. *Then*

$$\inf s([Tx]_{\alpha(x)},[Ty]_{\alpha(x)}) \preceq \left\{\frac{1}{5}\,|x-\frac{x}{4}|+i\frac{1}{4}\,|x-\frac{x}{5}|+\frac{1}{10}\,|\,y-\frac{y}{4}\,|+i\frac{1}{10}\,|y-\frac{y}{4}|\right.$$

$$+\frac{1}{15}\,|x-\frac{y}{4}|+i\frac{1}{15}\,|x-\frac{y}{4}|+\frac{1}{20}\,|\,y-\frac{x}{4}\,|+i\frac{1}{20}\,|y-\frac{x}{4}|$$

$$\left.\frac{1}{20}\{|x-y|+i\,|x-y|\}-\right\}$$

$$+\frac{1}{30}\left(\frac{\{|x-\frac{x}{4}|+i|x-\frac{x}{4}|\}+(1+\{|x-\frac{x}{4}|+i|x-\frac{x}{4}|\})}{1+||x-y|+i|x-y||}\right)$$

Since, all the conditions of Theorem 3.1 *are satisfied. Therefore,* $0 \in \mathfrak{X}$ *is the fixed point of T.*

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(*Received*, September 30, 2023)

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Journal of Indian Acad. Math.
Vol. 46, No. 1 (2024) pp. 25-31
ISSN: 0970-5120

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KKM-MAPS IN NORMED SPACES

Abstract: In 1928, Sperner established a combinatorial lemma. In 1929, from this Sperner's lemma, Knaster, Kuratowski and Mazurkiewicz established the well-known KKM Theorem on the closed cover of a simplex. In this paper we extend and generalize the results of KKM-map related to normed space and a few results are derived as corollary.

Key words and phrases: KKM-Map, Strongly Continuous, Almost Affine Map.

Mathematical Subject Classification (2000) No.: 47H10, 54H25, 55M20.

1. Introduction

In 1961 Ky Fan generalized the KKM Theorem to a subset of any topological vector space. There are many generalizations and applications of this theorem in fixed point theory, approximation theory, minimax theory and variational inequalities. We can mention authors like Prolla [9], Li [8], Carbone [1] extended and improve KKM theorem in normed linear spaces'. Before we state the theorem we recall some well known definitions and other relevant results.

Let X be a subset of a vector space E. A map $F: X \to 2^E$ is called a KKM-map if $co\{x_1,...,x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i)$ for each finite subset $\{x_1,\ldots,x_n\}$ of X.

For details on KKM maps we refer Granas [2].

A function f is said to be *strongly continuous* if $x_n \to x$ weakly implies that $fx_n \to fx$ strongly. A function $g : X \to E$, where X is non-empty convex set, is said to be *almost affine* if $\| g(t) - y \| \le \| g(t_1) - y \| + (1-\lambda) \| g(t_2) - y \|$ for all $t_1, t_2 \in X$, $0 < \lambda < 1$, $t = \lambda t_1 + (1-\lambda)t_2$ and $y \in E$.

Theorem 1.1[8]: Let C be a nonempty convex set in a normed vector space X and let $f : C \to X$ be a mapping such that

(i) for every $y \in C$, $\{x \in C : \|f(x) - y\| \ge \|f(x) - x\|\}$ is a closed subset of C;

(ii) there exists a compact convex subset D of C, such that the set $\bigcap_{y \in D} \{x \in C : \|f(x) - y\| \ge \|f(x) - x\|\}$ is contained in a compact subset of C.

Then there exists a point $x_0 \in C$ such that $\|f(x_0) - x_0\| = d(f(x_0), C)$.

Theorem 1.2 [5]: Let Y be a convex set in a topological vector space E and X a nonempty subset of Y. For each x ∈ X, let F(x) be a relatively closed subset of Y such that the convex hull of every finite subset $\{x_1, \ldots, x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^{n} F(x_i)$. If there is a nonempty subset X_0 of X such that the intersection $\bigcap_{x \in X_0} F(x)$ is compact and X_0 is contained in a compact convex subset of Y then $\bigcap_{x \in X} F(x) \ne \varphi$.

Theorem 1.3 [9]: Let C be a nonempty compact convex subset of a normed space X and $g : C \to C$ a continuous, almost affine, onto mapping. Then, for each continuous mapping $f : C \to X$ there exists $x \in C$ satisfying $\|g(x) - f(x)\| = d(f(x), C)$.

2. Main Results

In [8], Li studied some applications of KKM theorem to approximation theory and fixed point there. Here, the author used one map satisfying some conditions. The aim of this paper is to generalize some results of [8] by using one more almost affine self map. This work also extends and improve KKM theorem in normed linear space.

Theorem 2.1: Let C be a nonempty convex subset of a normed linear space X and g be almost affine self map of C onto C. Let $f: C \to X$ be a map satisfying the following conditions

(i) for every $x \in C$, $\{y \in C : \|g(y) - f(y)\| \leq \|g(x) - f(y)\|\}$ is closed in C;

(ii) C has a nonempty compact convex subset C_o such that the set $\{y \in C : \|g(y) - f(y)\| \leq \|g(x) - f(y)\| \text{ for all } x \in C_0\}$ is compact.

Then there will be a point $y_0 \in C$ such that $\|g(y_0) - f(y_0)\| = d(f(y_0), C)$.

Proof: Let $F: C \to 2^X$ be defined by

$$F(x) = \{y \in C : \|g(y) - f(y)\| \leq \|g(x) - f(y)\|\}, \text{ for every } x \in C.$$

Then by condition (i), F(x) is closed in C. It is obvious that $x \in F(x)$ for all $x \in C$. Next, we have to show that $F: C \to 2^X$ is a KKM map that is, $c_0\{x_1, ..., x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$ for any finite subset $\{x_1, ..., x_n\}$ of X.

Let $z = \sum_{i=1}^{n} \lambda_i x_i$, where $0 \leq \lambda_i \leq 1$, $\sum_{i=1}^{n} \lambda_i = 1$, $i = 1, 2, ..., n$.

If $z \notin \bigcup_{i=1}^{n} F(x_i)$, then $\|g(z) - f(z)\| > \|g(x_i) - f(z)\|\}$ for $i = 1, 2, ..., n$.

$$\Rightarrow \lambda_1\|g(z) - f(z)\| + ... + \lambda_n\|g(z) - f(z)\| > \lambda_1\|g(x_1) - f(z)\| + ... + \lambda_n\|g(x_n) - f(z)\|$$

$$\Rightarrow \sum_{i=1}^{n} \lambda_i\|g(z) - f(z)\| > \|g(\lambda_1 x_1 + ... + \lambda_n x_n) - f(z)\| \; [\because \text{ g is almost affine map}]$$

$\Rightarrow \|g(z) - f(z)\| > \|g(z) - f(z)\|$ [since $z = \sum_{i=1}^{n} \lambda_i x_i, \sum_{i=1}^{n} \lambda_i = 1$] which is meaningless.

Hence, $z \in \bigcup_{i=1}^{n} F(x_i)$. Thus, $F: C \to 2^X$ is a KKM-map.

By (ii), the set

$$\bigcap_{x \in C_0} F(x) = \{y \in C : \|g(y) - f(y)\| \leq \|g(x) - f(y)\| \text{ for all } x \in C_0\} \text{ is compact.}$$

Then by Theorem 1.2, $\bigcap_{x \in C} F(x) \neq \varphi$.

Therefore, there exists a point $y_0 \in C$ such that $y_0 \in \bigcap_{x \in C} F(x)$ which implies $\|g(y_0) - f(y_0)\| \leq \|g(x) - f(y_0)\| = \min_{x \in C} \| g(x) - f(y_0)\| = d(f(y_0), C)$ since g is onto. Hence proved.

Remark: If $g = I$, identity map then we get Theorem 1.1 of Li [8] as a corollary of this Theorem 2.1.

Corollary 2.2: Let C be a nonempty convex subset of a normed linear space X and g be continuous almost affine self map of C onto C. Let $f : C \to X$ be a continuous map satisfying the condition

(i) C has a nonempty compact convex subset C_o such that the set $\{y \in C : \|g(y) - f(y)\| \leq \|g(x) - f(y)\| \text{ for all } x \in C_0\}$ is compact.

Then there will be a point $y_0 \in C$ such that $\|g(y_0) - f(y_0)\| = d(f(y_0), C)$.

Proof: For every $x \in C$, let $F : C \to 2^X$ be defined by

$$F(x) = \{y \in C : \|g(y) - f(y)\| \leq \|g(x) - f(y)\|\}$$

Since f and g are continuous, F(x) is closed and hence (i) of above Theorem 2.1 is satisfied. Therefore, the result follows.

Remark: In case when C itself is also a compact convex set the above result reduces to Theorem1.3 (Theorem 1of Prolla [9]). In addition to this, if $g = I$, identity function condition, we have the Theorem of Ky Fan [4] (Theorem 2.1, p.74 [12])

Corollary 2.3: Let C be a nonempty convex subset of a normed linear space X and *g* be strongly continuous, almost affine self map of C onto C. Let $f : C \to X$ be a strongly continuous map satisfying the following condition

C has a nonempty weakly compact convex subset C_o such that the set $\{y \in C : \|g(y) - f(y)\| \leq \|g(x) - f(y)\| \text{ for all } x \in C_0\}$ is weakly compact.

Then there will be a point $y_0 \in C$ such that $\|g(y_0) - f(y_0)\| = d(f(y_0), C)$.

Proof: Suppose $F : C \to 2^X$ be defined by

$$F(x) = \{y \in C : \|g(y) - f(y)\| \leq \|g(x) - f(y)\|\}$$

Since f and g are strongly continuous functions, F(x) is weakly closed in C and $x \in F(x)$ for all $x \in C$. As g is almost affine map, so F is a KKM map.

Also $\bigcap_{x \in C_0} F(x) = \{y \in C : \|g(y) - f(y)\| \leq \|g(x) - f(y)\| \text{ for all } x \in C_0\}$ is weakly compact. Satisfying the above corollary 2.2 for a weak topology of X, the result follows.

Remark: In case when C itself is a nonempty weakly compact set corollary 2.3 reduces to Theorem 3 of Carbone [1]. Further, when $g = I$, we have the result of Kapoor (Theorem 3.25, p.133[12]).

Corollary 2.4: Let C be a nonempty convex subset of a normed linear space X and g be almost affine self map of C onto C. Let $f : C \to C$ be a one to one map satisfying the following condition

(i) for every $x \in C$, $\{y \in C : \|g(y) - f(y)\| \leq \|g(x) - f(y)\|\}$ is closed in C;

(ii) C has a nonempty compact convex subset C_0 such that the set $\{y \in C : \|g(y) - f(y)\| \leq \|g(x) - f(y)\| \text{ for all } x \in C_0\}$ is compact.

Then there will be a coincidence point $y_0 \in C$ i.e., $g(y_0) = f(y_0)$ for some $y_0 \in X$.

Proof: Let us define $F : C \to 2^C$ by

$$F(x) = \{y \in C : \|g(y) - f(y)\| \leq \|g(x) - f(y)\|\} \text{ for all } x \in C.$$

By condition (i), F(x) is closed in C.

Hence, by Theorem 2.1, $\|g(y_0) - f(y)\| = d(f(y_0, C)\|$

Since f is self map and one to one, for each $y_0 \in C$ implies $f(y_0) \in C$.

Hence, we get $f(y_0) = g(y_0)$, that is y_0 is a coincidence point.

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(*Received*, November 5, 2023)

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Journal of Indian Acad. Math. ISSN: 0970-5120
Vol. 46, No. 1 (2024) pp. 33-43

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KKM-MAPS VIA SET VALUED MAPS

Abstract: In this paper we extend some theorems relating to KKM maps in three set valued maps and a few results are derived as corollary.

Key words and phrases: Set Valued Map, KKM-Map.

Mathematical Subject Classification (2000) No.: 47H10, 54H25, 55M20.

1. Introduction and Preliminaries

In some of the research works of different researchers in KKM theory we see that they proved theorems of one set valued map and two set valued map by using KKM map. In 1968, Browder [2] proved the existence of fixed point for a set valued map by using the partition of unity. We can mention authors like Brosowski [1], Granas [5], Yannelis and Prabhakar [8], Lin [6] extended and improved KKM theorem on set valued maps.

Before giving our main results we quote some related definition, theorems and corollaries as follows.

By a *set valued map F* from a set *X* to a set *Y*, denoted by $F : X \to 2^Y$, we mean a correspondence which with each $x \in X$ associates a set $F(x) \in 2^Y$ (the space of subsets of *Y*).

Let X be a subset of a vector space E. A map $\mathrm{F} : \mathrm{X} \to 2^{\mathrm{E}}$ is called a

KKM-map if $co\{x_1,...,x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i)$ for each finite subset $\{x_1,\ldots,x_n\}$ of X.

For details on KKM maps we refer Granas [5].

In 1961, Ky Fan [3] gave the following theorem.

Theorem 1.1: (Fan-KKM or KKMF). Let X be an arbitrary subset of a topological vector space E. To each $x \in X$, let F(x) be a closed in E and let the convex hull of every finite subset $\{x_1,..., x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^{n} F(x_i)$. If F(x) is compact for at least one $x \in X$ then $\bigcap_{x \in X} F(x) \neq \varphi$.

It is to be noted that the above theorem has applications in various fields including fixed point theory, variational inequality and game theory.

In 1968, Browder [2] proves the following theorem.

Theorem 1.2: Let X be a nonempty compact convex subset of Hausdorff topological vector spaces E. Let $T : X \rightarrow 2^X$ be a satisfy

(i) T(x) is convex and nonempty for each $x \in X$;

(ii) $T^{-1}(y)$ is open in X for each $y \in X$.

Then there exist an $y_0 \in X$ such that $y_0 \in T(y_0)$.

In 1984, Ky Fan [4] gave the following theorem as an extension of Theorem 1.1, where compactness is relaxed. This important result was used by several researchers in recent years.

Theorem 1.3: Let Y be a convex set in a topological vector space E and X a nonempty subset of Y. For each x ∈ X, let F(x) be a relatively closed subset of Y such that the convex hull of every finite subset $\{x_1,..., x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^{n} F(x_i)$. If there is a nonempty subset X_0 of X such that the intersection $\bigcap_{x \in X_0} F(x)$ is compact and X_0 is contained in a compact convex subset of Y then $\bigcap_{x \in X} F(x) \neq \varphi$.

In 1985, Yannelis and Prabhakar [8] has given the following result.

Theorem 1.4: Let X be a nonempty compact convex subset of Hausdorff topological vector spaces E. Let $T : X \to 2^X$ satisfy

(i) $x \notin co(T(x))$ for each $x \in X$;

(ii) $T^{-1}(y)$ is open in X for each $y \in X$.

Then there exist an $x_0 \in X$ such that $T(x_0) = \varphi$.

In 1986, Lin [6] gave the following theorem.

Theorem 1.5: Let X be a nonempty convex subset of Hausdorff topological vector spaces E. Let $T : X \to 2^X$ satisfy

(i) $x \notin co(T(x))$ for each $x \in X$;

(ii) $T^{-1}(y) = \{x \in X \mid y \in T(x)\}$ is open in X for each $y \in X$;

(iii) X has a nonempty compact convex subset X_0 such that $B = \{x \in X : y \notin T(x) \text{ for all } y \in X_0\}$ is compact.

Then there exist an $x_0 \in X$ such that $T(x_0) = \varphi$.

The above result remains true if the following condition is given in place of (iii).

(iii)′ Let X_0 be a nonempty compact convex subset of X, and K a nonempty compact set of X. For every $x \in X$ K there exists a $y \in X_0$ such that $y \in T(x)$.

Corollary 1.6 [7]: Let X be a nonempty convex subset of Hausdorff topological vector spaces E. Let $T : X \to 2^X$ satisfy

(i) $x \notin T(x)$ for each $x \in X$;

(ii) T(x) is convex or empty, for each $x \in X$;

(iii) $T^{-1}(y)$ is open in X for each $y \in X$;

(iv) X has a nonempty compact convex subset X_0 such that $B = \{x \in X: y \notin T(x) \text{ for all } y \in X_0\}$ is compact.

Then there exists an $x_0 \in X$ such that $T(x_0) = \varphi$.

The following is an extension of Theorem 1.2.

Theorem 1.7[7]: Let X be a nonempty convex subset of Hausdorff topological vector spaces E. Let $S, T : X \to 2^X$ satisfy

(i) $S(x) \subset T(x)$ for each $x \in X$;

(ii) $x \notin T(x)$ for each $x \in X$;

(iii) T(x) is convex or empty for each $x \in X$;

(iv) $S^{-1}(y) = \{x \in X \mid y \in S(x)\}$ is open in X for each $y \in X$;

(v) X has a nonempty compact convex subset X_0 such that $C = \{x \in X : y \notin S(x) \text{ for all } y \in X_0\}$ is compact.

Then there exist an $x_0 \in X$ such that $S(x_0) = \varphi$.

Theorem 1.8[7]: Let X be a nonempty convex subset of Hausdorff topological vector spaces E. Let $S, T : X \to 2^X$ satisfy the following conditions.

(i) $S(x) \subset T(x)$ for each $x \in X$;

(ii) $S(x) \neq \varphi$ for each $x \in X$,

(iii) T(x) is convex for each $x \in X$,

(iv) $S^{-1}(y)$ is open in X for each $x \in X$,

(v) X has a nonempty compact convex subset X_0 such that $C = \{x \in X : y \notin S(x) \text{ for all } y \in X_0\}$ is compact.

Then T has a fixed point in X.

2. Main Results

We develop KKM-maps using set valued maps. Further, we give theorems, corollaries and give also results on existence of fixed points under certain condition.

Theorem 2.1: Let X be nonempty convex subset of topological vector spaces E. Let $S, T, R : X \to 2^X$ be set-valued maps satisfying the following conditions:

(i) $S(x) \subset T(x) \subset R(x)$ for every $x \in X$;

(ii) For every $x \in X$, $x \in S(x)$;

(iii) For every $y \in X$, $\{x \in X \mid y \in R(x)\}$ is closed in X;

(iv) For every $x \in X$, the set $\{y \in X \mid y \notin T(x)\}$ is convex or empty;

(v) X has a nonempty compact convex subset X_0 such that $\bigcap_{y \in X_0} \{x \in X : y \in R(x)\}$ is compact.

Then there exist an $x_0 \in X$ such that $y \in R(x_0)$ for every $y \in X$.

Proof: Let $F : X \to 2^X$ be defined by $F(y) = \{x \in X : y \in R(x)\}$, for all $y \in X$. Then F(y) is closed in X by condition (iii). Also $y \in F(y)$ and $F(y) \neq \varphi$ by conditions (i) and (ii). Next we have to prove that $F : X \to 2^X$ is a KKM-map. Suppose $y_1,...,y_n \in X$ and $0 \le \lambda_i \le 1$, $i = 1,...., n$. Let $z = \sum_{i=1}^{n} \lambda_i y_i$. We need to show that $z \in \bigcup_{i=1}^{n} F(y_i)$.

On the contrary, let $z \notin \bigcup_{i=1}^{n} F(y_i)$. It implies that $z \notin F(y_i) = \{x \in X : y_i \in R(x)\}$ for each $y_i, i = 1,...,n$. Then, $y_i \notin R(z)$ i.e., $y_i \notin T(z)$ for $i = 1,..., n$ by condition (i). By condition (iv) applied to this z, the set $\{y \in X : y \notin T(z)\}$ is convex or empty for all $z \in X$. If the set $\{y \in X \mid y \notin T(z)\}$ is empty, the result follows. If not, the set $\{y \in X \mid y \notin T(z)\}$ is convex.

Then, $z = \sum_{i=1}^{n} \lambda_i y_i \in \{y \in X : y \notin T(z)\}$ which implies $z \notin T(z)$. So, $z \notin S(z)$, a contradiction to (ii). Hence, $z \in \bigcup_{i=1}^{n} F(y_i)$ i.e., F is KKM map. By condition (v), $\bigcap_{y \in X_0} F(y)$ is compact.

Hence the conditions of theorem 1.3 are satisfied for the topology of E.

Therefore, $\bigcap_{y \in X} F(y) \neq \varphi$ i.e., $\exists\ x_0 \in X$ s.t. $x_0 \in \bigcap_{y \in X} F(y)$ i.e., $x_0 \in F(y) \Rightarrow y \in R(x_0)$, $\forall\ y \in X$.

Remark: If X be a non empty compact convex subset of a topological vector spaces E, the condition (v) is redundant and accordingly may be dropped.

Corollary 2.2: Let X be non-empty convex subset of topological vector space E. Let $S, T : X \to 2^X$ be set -valued maps satisfying

(i) $S(x) \subset T(x)$ for each $x \in X$;

(ii) For each $x \in X$, $x \in S(x)$;

(iii) For each $y \in X$, the set $\{x \in X \mid y \in T(x)\}$ is closed in X;

(iv) For each $x \in X$, the set $\{y \in X \mid y \notin S(x)\}$ is convex or empty;

(v) X has a nonempty compact convex subset X_0 such that $\bigcap_{y \in X_0} \{x \in X : y \in T(x)\}$ is compact.

Then there exist an $x_0 \in X$ such that $y \in T(x_0)$ for every $y \in X$.

Proof: One can prove very easily by letting S = T and T = R in above theorem.

Corollary 2.3: Let X be non empty convex subsets of topological vector spaces E. Let $S : X \to 2^X$ be set-valued map satisfying the following conditions:

(i) For every $x \in X$, $x \in S(x)$;

(ii) For every $y \in X$, the set $\{x \in X \mid y \in S(x)\}$ is closed in X;

(iii) For every $x \in X$, the set $\{y \in X \mid y \notin S(x)\}$ is convex or empty;

(iv) X has a nonempty compact convex subset X_0 such that $\bigcap_{y \in X_0} \{x \in X : y \in S(x)\}$ is compact.

Then there exists an $x_0 \in X$ such that $y \in S(x_0)$ for every $y \in X$.

Proof: By letting S = T = R in above theorem, the proof follows.

Corollary 2.4: Let X be nonempty convex subsets of vector spaces E. Let $S, T, R : X \to 2^X$ be set valued maps satisfying the following conditions.

(i) $S(x) \subset T(x) \subset R(x)$ for each $x \in X$;

(ii) For every $x \in X$, $x \in S(x)$;

(iii) For every $y \in X$, $\{x \in X : y \in R(x)\}$ is weakly closed in X;

(iv) For each $x \in X$, the set $\{y \in X : y \notin T(x)\}$ is convex or empty;

(v) X has a nonempty compact convex subset X_0 such that $\bigcap_{y \in X_0} \{x \in X: y \in R(x)\}$ is weakly compact.

Then there exists an $x_0 \in X$ such that $y \in R(x_0)$ for every $y \in X$.

Proof: Let $F : X \to 2^X$ be defined by $F(y) = \{x \in X : y \in R(x)\}$ for all $y \in X$. Then F(y) is weakly closed in X by condition (iii). Further by condition (v), $\bigcap_{y \in X_0} F(y)$ is weakly compact.

Hence the conditions of theorem 2.1 are satisfied for the weak topology of E.

Therefore, there exist $x_0 \in X$ such that $x_0 \in \bigcap_{y \in X} F(y)$. That is $x_0 \in F(y)$ implies $y \in R(x_0)$ for every $y \in X$.

Corollary 2.5: Let X be nonempty compact convex subset of topological vector spaces E. Let $S, T, R : X \to 2^X$ be set-valued maps satisfying the following conditions.

(i) $S(x) \subset T(x) \subset R(x)$ for each $x \in X$;

(ii) For each $x \in X$, $x \in S(x)$;

(iii) For each $x \in X$, $\overline{\{y \in X : y \in R(x)\}}$ is a subset of X;

(iv) For each $y \in X$, the set $\{x \in X : y \notin T(x)\}$ is convex or empty.

Then $\bigcap_{y \in X} \overline{\{x \in X : y \in R(x)\}} \neq \varphi$.

Proof: Its proof follows from that of Theorem 2.1.

Theorem 2.6: Let X be a nonempty convex subset of a topological vector space E. Let $S, T, R : X \to 2^X$ be set valued maps satisfying the following conditions.

(i) $S(x) \subset T(x) \subset R(x)$ for each $x \in X$;

(ii) For each $y \in X$, the set $\{x \in X : y \in S(x)\}$ is open in X;

(iii) For each $x \in X$, the set $\{y \in X : y \in T(x)\}$ is convex and $\{y \in X : y \in S(x)\}$ is nonempty;

(iv) X has a nonempty compact convex subset X_0 such that the set $\{x \in X : y \notin S(x) \text{ for all } y \in X_0\}$ is compact.

Then R has a fixed point i.e., $x_0 \in R(x_0)$ for all $x_0 \in X$.

Proof: Let $G(x) = X - S(x)$, $H(x) = X - T(x)$ and $I(x) = X - R(x)$.

Then $I(x) \subset H(x) \subset G(x)$. Let $F : X \to 2^X$ be a map defined by

$$F(y) = \{x \in X : y \in G(x)\} = \{x \in X : y \notin S(x)\} \text{ for all } y \in X.$$

Then F(y) is closed in X by condition (ii). By condition (iii), the set $\{y \in X : y \notin H(x)\} = \{y \in X : y \in T(x)\}$ is convex for each $x \in X$

By condition (iv), $\bigcap_{y \in X_0} F(y) = \{x \in X : y \in G(x) \text{ for all } y \in X_0\}$

$= \{x \in X : y \notin S(x) \text{ for all } y \in X_0\}$ is compact.

If we assume that R has no fixed point i.e., $x \notin R(x)$ for all $x \in X$. Then $x \in I(x)$ for all $x \in X$. Theorem 2.1 implies that there exist an $x_0 \in X$ such that $y \in G(x_0)$ i.e., $y \notin S(x_0)$ for all $y \in X$.

Therefore, $\{y \in X : y \in S(x_0)\} = \varphi$, a contradiction to (iii).

Thus, there exists a $x_0 \in X$ such that $x_0 \in R(x_0)$. Hence, R has a fixed point.

Theorem 2.7: Let X be nonempty convex subset of topological vector spaces E. Let $S, T : X \to 2^X$ be set-valued maps satisfying the following conditions.

(i) For each $x \in X, S(x) \subset T(x)$;

(ii) $x \notin co(T(x))$ for each $x \in X$;

(iii) $S^{-1}(y) = \{x \in X : y \in S(x)\}$ is open in X for each $y \in X$;

(iv) X has a nonempty compact convex subset X_0 such that the set $\bigcap_{y \in X_0} \{x \in X: y \notin S(x)\}$ is compact.

Then there exists an $x_0 \in X$ such $S(x_0) = \varphi$.

Proof: Let $F(y) = X - S^{-1}(y) = \{x \in X : y \notin S(x)\}$ for each $y \in X$. Then F(y) is closed in X for each $y \in X$ by condition (iii).

We have to show that F is KKM map. Let $z \in co\{y_1, ..., y_n\}$. If $z \notin \bigcup_{i=1}^{n} F(y_i)$, then $z \notin F(y_i)$ which implies $z \in S^{-1}(y_i) \Rightarrow y_i \in S(z) \Rightarrow y_i \in T(z)$ by condition (i), for all $i = 1, ..., n$. $\Rightarrow$ $co\{y_1, ..., y_n\} \in coT(z)$ which implies $z \in coT(z)$ a contradiction to (ii)

Hence, F is KKM-map. Further, by condition (iv), the set $\bigcap_{y \in X_0} F(y)$ is compact. Therefore, by theorem 1.3, there exists $x_0 \in \bigcap_{y \in X_0} F(y)$ which implies $x_0 \notin S^{-1}y$ for all $y \in X$. Then, $y \notin S(x_0)$ for all $y \in X$. Hence, $S(x_0) = \varphi$.

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(*Received*, December 12, 2023)

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Journal of Indian Acad. Math.
Vol. 46, No. 1 (2024) pp. 45-54
ISSN: 0970-5120

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A FAMILY OF GIBONACCI SUMS: ALTERNATE GENERALIZATIONS

Abstract: We explore an alternate generalization of a sum involving a family of gibonacci polynomial squares and its consequences.

Keywords: Extended Gibonacci Polynomials, Fibonacci Polynomial, Lucas Polynomial, Binet-like Formulas.

Mathematical Subject Classification (MSC2020) No.: 11B37, 11B39, 11C08.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*.

They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $\Delta = \sqrt{x^2 + 4}$ and $2\alpha = x + \Delta$.

It follows by the Binet-like formulas that $\lim_{m\to\infty} \frac{1}{g_{m+r}} = 0$ and $\lim_{m\to\infty} \frac{g_{m+r}}{g_m} = \alpha^r$.

1.1 Fundamental Gibonacci Identities: Gibonacci polynomials satisfy the following properties [2, 3, 4, 5, 6, 7]:

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1} f_k^2, & \text{if } g_n = f_n \\ (-1)^{n+k} \Delta^2 f_k^2, & \text{otherwise}; \end{cases} \tag{1}$$

$$g_{n+k+r}g_{n-k} - g_{n+k}g_{n-k+r} = \begin{cases} (-1)^{n+k+1} f_r f_{2k}, & \text{if } g_n = f_n \\ (-1)^{n+k} \Delta^2 f_r f_{2k}, & \text{otherwise}; \end{cases} \tag{2}$$

where k and r are positive integers. These properties can be confirmed using the Binet-like formulas.

It follows from these two identities that

$$g_{(2pn+t)k+r}g_{(2pn+t-2p)k+r} - g_{(2pn+t-p)k+r}^2 = \begin{cases} (-1)^{tk+r+1} f_{pk}^2, & \text{if } g_n = f_n \\ (-1)^{tk+r} \Delta^2 f_{pk}^2, & \text{otherwise}; \end{cases} \tag{3}$$

$$g_{(2pn+t)k}g_{2pn+t-2p)k+r} - g_{(2pn+t)k+r}g_{(2pn+t-2p)k} = \begin{cases} (-1)^{tk} f_r f_{2pk}, & \text{if } g_n = f_n \\ (-1)^{tk+1} \Delta^2 f_r f_{2pk}, & \text{otherwise}; \end{cases} \tag{4}$$

where k, p, r, and t are positive integers and $t \leq 2p$ [7].

2. A Telescoping Gibonacci Sum

Using recursion, we will now investigate a telescoping gibonacci sum.

Lemma 1: *Let k, p, r, t, and λ be positive integers, where $t \le 2p$. Then*

$$\sum_{n=1}^{\infty}\left[\frac{g^{\lambda}_{(2pn+t-2p)k}}{g^{\lambda}_{(2pn+t-2p)k+r}} - \frac{g^{\lambda}_{(2pn+t)k}}{g^{\lambda}_{(2pn+t)k+r}}\right] = \frac{g^{\lambda}_{tk}}{g^{\lambda}_{tk+r}} - (-\beta)^{\lambda r}. \tag{5}$$

Proof: With recursion [2, 3], we will first confirm that

$$\sum_{n=1}^{m}\left[\frac{g^{\lambda}_{(2pn+t-2p)k}}{g^{\lambda}_{(2pn+t-2p)k+r}} - \frac{g^{\lambda}_{(2pn+t)k}}{g^{\lambda}_{(2pn+t)k+r}}\right] = \frac{g^{\lambda}_{tk}}{g^{\lambda}_{tk+r}} - \frac{g^{\lambda}_{(2pm+t)k}}{g^{\lambda}_{(2pm+t)k+r}}. \tag{6}$$

To realize this goal, in the interest of brevity, we let A_m denote the left-hand side of this equation and B_m its right-hand side. Then

$$B_m - B_{m-1} = A_m - A_{m-1}.$$

By recursion, this yields

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 \\ &= 0, \end{aligned}$$

establishing the validity of equation (6).

Since, $\lim_{m\to\infty} \frac{g_m}{g_{m+r}} = \frac{1}{\alpha^r} = (-\beta)^r$, the given result now follows, as desired. □

3. Gibonacci Sums

Coupled with identities (3) and (4), Lemma 1 with $\lambda = 1$ plays a significant role in our explorations.

To this end, in the interest of brevity, we now let

$$\mu = \begin{cases} 1, & \text{if } g_n = f_n \\ \Delta^2, & \text{otherwise}; \end{cases} \quad \text{and} \quad \nu = \begin{cases} -1, & \text{if } g_n = f_n \\ 1, & \text{otherwise.} \end{cases}$$

With these tools at our fingertips, we now embark on our discourse with the following result.

Theorem 1: *Let k, p, r, and t be positive integers, where $t \le 2p$. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk}\mu\nu f_r f_{2pk}}{g_{(2pn+t-p)k+r}^2 + (-1)^{tk+r}\mu\nu f_{pk}^2} = \frac{g_{tk}}{g_{tk+r}} - (-\beta)^r. \tag{7}$$

Proof: Suppose $g_n = f_n$. Coupled with identities (3) and (4), Lemma 1 then yields

$$\frac{(-1)^{tk+1} f_r f_{2pk}}{f_{(2pn+t-p)k+r}^2 - (-1)^{tk+r} f_{pk}^2} = \frac{f_{(2pn+t)k+r} f_{(2pn+t-2p)k} - f_{(2pn+t)k} f_{(2pn+t-2p)k+r}}{f_{(2pn+t)k+r} f_{(2pn+t-2p)k+r}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} f_r f_{2pk}}{f_{(2pn+t-p)k+r}^2 - (-1)^{tk+r} f_{pk}^2} = \sum_{n=1}^{\infty} \left[\frac{f_{(2pn+t-2p)k}}{f_{(2pn+t-2p)k+r}} - \frac{f_{(2pn+t)k}}{f_{(2pn+t)k+r}} \right]$$

$$= \frac{f_{tk}}{f_{tk+r}} - (-\beta)^r.$$

On the flip side, let $g_n = l_n$. With the same two identities and Lemma 1, we then get

$$\frac{(-1)^{tk} \Delta^2 f_r f_{2pk}}{l_{(2pn+t-p)k+r}^2 + (-1)^{tk+r} \Delta^2 f_{pk}^2} = \frac{l_{(2pn+t)k+r} l_{(2pn+t-2p)k} - l_{(2pn+t)k} l_{(2pn+t-2p)k+r}}{l_{(2pn+t)k+r} l_{(2pn+t-2p)k+r}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} \Delta^2 f_r f_{2pk}}{l_{(2pn+t-p)k+r}^2 + (-1)^{tk+r} \Delta^2 f_{pk}^2} = \sum_{n=1}^{\infty} \left[\frac{l_{(2pn+t-2p)k}}{l_{(2pn+t-2p)k+r}} - \frac{l_{(2pn+t)k}}{l_{(2pn+t)k+r}} \right]$$

$$= \frac{l_{tk}}{l_{tk+r}} - (-\beta)^r.$$

The given result now follows by combining the two cases, as desired. □

In particular, with $r = 1$ and $k \le 3$, the theorem yields [3, 4, 7]:

Case 1: Let $p = 1$. Then $t \le 2$. With $t = 1$, we get

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n+1}^2 - 1} = \frac{3}{2} - \frac{\sqrt{5}}{2}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n+1}^2 + 5} = -\frac{1}{6} + \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n+1}^2 + 1} = -\frac{1}{3} + \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n+1}^2 - 5} = \frac{1}{12} - \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{6n+1}^2 - 4} = \frac{7}{48} - \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6n+1}^2 + 20} = -\frac{3}{112} + \frac{\sqrt{5}}{80};$$

with $t = 2$, the formula yields

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n+2}^2 + 1} = -1 + \frac{\sqrt{5}}{2}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n+2}^2 - 5} = \frac{1}{4} - \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n+3}^2 + 1} = -\frac{11}{30} + \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n+3}^2 - 5} = \frac{5}{66} - \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{6n+4}^2 + 4} = -\frac{29}{208} + \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6n+4}^2 - 20} = \frac{13}{464} - \frac{\sqrt{5}}{80}.$$

Case 2: With $p = 2$, we have $t \le 4$. When $t = 1$, the theorem implies

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n}^2 - 1} = \frac{1}{2} - \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n}^2 + 5} = -\frac{1}{18} + \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{8n-1}^2 + 9} = -\frac{1}{21} + \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{8n-1}^2 - 45} = \frac{1}{84} - \frac{\sqrt{5}}{210};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n-2}^2 - 64} = \frac{7}{864} - \frac{\sqrt{5}}{228}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n-2}^2 + 320} = -\frac{1}{10,080} + \frac{\sqrt{5}}{1,440};$$

with $t = 2$, we get

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n+1}^2 + 1} = -\frac{1}{3} + \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n+1}^2 - 5} = \frac{1}{12} - \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{8n+1}^2 + 9} = -\frac{11}{210} + \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{8n+1}^2 - 45} = \frac{5}{462} - \frac{\sqrt{5}}{210};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n+1}^2 + 64} = -\frac{29}{3,744} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n+1}^2 - 320} = \frac{13}{8,352} - \frac{\sqrt{5}}{1,440};$$

using $t = 3$, the theorem yields

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n+2}^2 - 1} = \frac{7}{18} - \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n+2}^2 + 5} = -\frac{1}{14} + \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{8n+3}^2 + 9} = -\frac{29}{546} + \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{F_{8n+3}^2 - 45} = \frac{13}{1,218} - \frac{\sqrt{5}}{210};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n+4}^2 - 64} = \frac{41}{5,280} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n+4}^2 + 320} = -\frac{55}{35,424} + \frac{\sqrt{5}}{1,440};$$

and finally, letting $t = 4$, we get

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n+3}^2 + 1} = -\frac{11}{30} + \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n+3}^2 - 5} = \frac{5}{66} - \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{8n+5}^2 + 9} = -\frac{19}{357} + \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{8n+5}^2 - 45} = \frac{17}{1,596} - \frac{\sqrt{5}}{210};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n+7}^2 + 64} = -\frac{521}{67,104} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n+7}^2 - 320} = \frac{233}{150,048} - \frac{\sqrt{5}}{1,440}.$$

3.1 Gibonacci Delights: Using the above results, we can extract delightful dividends [3, 4].

$$\sum_{n=2}^{\infty} \frac{1}{F_{2n+1}^2 + 1} = \sum_{n=1}^{\infty} \frac{1}{F_{4n+1}^2 + 1} + \sum_{n=1}^{\infty} \frac{1}{F_{4n+3}^2 + 1} = -\frac{41}{30} + \frac{\sqrt{5}}{3};$$

$$\sum_{n=2}^{\infty} \frac{1}{F_{2n}^2 - 1} = \sum_{n=1}^{\infty} \frac{1}{F_{4n}^2 - 1} + \sum_{n=1}^{\infty} \frac{1}{F_{4n+2}^2 - 1} = \frac{8}{9} - \frac{\sqrt{5}}{3};$$

$$\sum_{n=2}^{\infty} \frac{1}{L_{2n}^2 + 5} = \sum_{n=1}^{\infty} \frac{1}{L_{4n}^2 + 5} + \sum_{n=1}^{\infty} \frac{1}{L_{4n+2}^2 + 5} = -\frac{8}{63} + \frac{\sqrt{5}}{15};$$

$$\sum_{n=2}^{\infty} \frac{1}{F_{2n+1}^2 - 1} = \sum_{n=1}^{\infty} \frac{1}{F_{4n+1}^2 - 1} + \sum_{n=1}^{\infty} \frac{1}{F_{4n+3}^2 - 1} = -\frac{7}{10} + \frac{\sqrt{5}}{3};$$

$$\sum_{n=2}^{\infty} \frac{1}{L_{2n+1}^2 - 5} = \sum_{n=1}^{\infty} \frac{1}{L_{4n+1}^2 - 5} + \sum_{n=1}^{\infty} \frac{1}{L_{4n+3}^2 - 5} = \frac{7}{44} - \frac{\sqrt{5}}{15};$$

$$\sum_{n=3}^{\infty} \frac{1}{F_{2n+1}^2 + 9} = \sum_{n=1}^{2} \left(\sum_{i=-1}^{\infty} \frac{1}{F_{8n+2i+1}^2 + 9} \right) = -\frac{228}{1,105} + \frac{2\sqrt{5}}{21};$$

$$\sum_{n=2}^{\infty} \frac{1}{L_{2n+1}^2 - 45} = \sum_{n=1}^{\infty} \left(\sum_{i=-1}^{2} \frac{1}{F_{8n+2i+1}^2 - 45} \right) = \frac{267}{6,061} - \frac{2\sqrt{5}}{105};$$

$$\sum_{n=2}^{\infty} \frac{1}{F_{6n+1}^2 + 64} = \sum_{n=1}^{\infty} \frac{1}{F_{12n+1}^2 + 64} + \sum_{n=1}^{\infty} \frac{1}{F_{12n+7}^2 + 64} = -\frac{2,255}{145,392} + \frac{\sqrt{5}}{144};$$

$$\sum_{n=2}^{\infty} \frac{1}{L_{6n+1}^2 - 320} = \sum_{n=1}^{\infty} \frac{1}{L_{12n+1}^2 - 320} + \sum_{n=1}^{\infty} \frac{1}{L_{12n+7}^2 - 320} = \frac{2,255}{725,232} - \frac{\sqrt{5}}{720};$$

$$\sum_{n=2}^{\infty} \frac{1}{F_{6n+4}^2 - 64} = \sum_{n=1}^{\infty} \frac{1}{F_{12n-2}^2 - 64} + \sum_{n=1}^{\infty} \frac{1}{F_{12n+4}^2 - 64} = \frac{377}{23,760} - \frac{\sqrt{5}}{144};$$

$$\sum_{n=2}^{\infty} \frac{1}{L_{6n+4}^2 + 320} = \sum_{n=1}^{\infty} \frac{1}{L_{12n-2}^2 + 320} + \sum_{n=1}^{\infty} \frac{1}{L_{12n+4}^2 + 320} = -\frac{377}{123,984} + \frac{\sqrt{5}}{720}.$$

We now conclude our exploration with showcasing the counterparts of equations (8)-(11) in [7]. They follow from the Theorem by letting $t = p$; $t = p = k$; $t = 2p$; and $t = 2p = 2k$, respectively.

$$\sum_{n=1}^{\infty} \frac{(-1)^{pk} \mu\nu f_r f_{2pk}}{g_{2pnk+r}^2 + (-1)^{pk+r} \mu\nu f_{pk}^2} = \frac{g_{pk}}{g_{pk+r}} - (-\beta)^r. \tag{8}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{p^2} \mu\nu f_r f_{2p^2}}{g_{2p^2n+r}^2 + (-1)^{p^2+r} \mu\nu f_{p^2}^2} = \frac{g_{p^2}}{g_{p^2+r}} - (-\beta)^r. \tag{9}$$

$$\sum_{n=1}^{\infty} \frac{\mu\nu f_r f_{2pk}}{g_{pk(2n+1)+r}^2 + (-1)^r \mu\nu f_{pk}^2} = \frac{g_{2pk}}{g_{2pk+r}} - (-\beta)^r. \tag{10}$$

$$\sum_{n=1}^{\infty} \frac{\mu\nu f_r f_{2p^2}}{g_{p^2(2n+1)+r}^2 + (-1)^r \mu\nu f_{p^2}^2} = \frac{g_{2p^2}}{g_{2p^2+r}} - (-\beta)^r. \tag{11}$$

With the labels

$$A = 21,890; \qquad C = 284,240; \qquad E = 21,607,408;$$

$$B = 25,840; \qquad D = 635,664; \qquad F = 48,315,632,$$

they yield

$$\sum_{n=1}^{\infty} \frac{1}{F_{10n+1}^2 - 25} = \frac{9}{440} - \frac{\sqrt{5}}{110}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{10n+1}^2 + 125} = -\frac{2}{495} + \frac{\sqrt{5}}{550};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{18n+1}^2 - 1,156} = \frac{123}{C} - \frac{\sqrt{5}}{5,168}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{18n+1}^2 + 5,780} = -\frac{55}{D} + \frac{\sqrt{5}}{B};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{10n+6}^2 - 25} = \frac{199}{9,790} + \frac{\sqrt{5}}{110}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{10n+6}^2 + 125} = \frac{89}{A} - \frac{\sqrt{5}}{550};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{18n+10}^2 - 1,156} = -\frac{9,349}{E} + \frac{\sqrt{5}}{5,168}; \quad \sum_{n=1}^{\infty} \frac{1}{L_{18n+10}^2 - 5,780} = \frac{4,181}{F} - \frac{\sqrt{5}}{B}.$$

respectively.

It then follows that

$$\sum_{n=2}^{\infty} \frac{1}{F_{5n+1}^2 - 25} = \sum_{n=1}^{\infty} \frac{1}{F_{10n+1}^2 - 25} + \sum_{n=1}^{\infty} \frac{1}{F_{10n+6}^2 - 25} \qquad = -\frac{1,597}{39,160} + \frac{\sqrt{5}}{55};$$

$$\sum_{n=2}^{\infty} \frac{1}{F_{9n+1}^2 - 1,156} = \sum_{n=1}^{\infty} \frac{1}{F_{18n+1}^2 - 1,156} + \sum_{n=1}^{\infty} \frac{1}{F_{18n+10}^2 - 1,156} \quad = \frac{1}{17,476,580}.$$

4. Acknowledgment

The author is grateful to Z. Gao for a careful reading of the article and his computational assistance.

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(*Received*, March 21, 2024)

Journal of Indian Acad. Math. ISSN: 0970-5120
Vol. 46, No. 2 (2024) pp. 55-65

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A FAMILY OF JACOBSTHAL SUMS: ALTERNATE GENERALIZATIONS

Abstract: We explore an alternate generalization of a sum involving Jacobsthal polynomial squares and its consequences.

Keywords: Jacobsthal Polynomial, Jacobsthal-Lucas Polynomial, Extended Gibonacci Polynomials, Fibonacci Polynomial, Lucas Polynomial.

Mathematical Subject Classification No.: MSC2020, 11B37, 11B39, 11C08.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 3].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$ [2, 3].

Gibonacci and Jacobsthal polynomials are linked by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [3, 4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $c_n = J_n$ or j_n, $\Delta = \sqrt{x^2 + 4}$, $2\beta = x - \Delta$, $D = \sqrt{4x + 1}$, and $2\omega = 1 - D$. Then $\beta(1/\sqrt{x}) = \dfrac{1 - D}{2\sqrt{x}} = \dfrac{\omega}{\sqrt{x}}$.

2. A Gibonacci Sum: An Alternate Generalization

Before presenting an interesting gibonacci sum, again in the interest of brevity and expediency, we now let [5, 6]

$$\mu = \begin{cases} 1, & if\ g_n = f_n \\ \Delta^2, & otherwise; \end{cases} \qquad \nu = \begin{cases} -1, & if\ g_n = f_n\ or\ c_n = J_n \\ 1, & if\ g_n = l_n\ or\ c_n = j_n; \end{cases} \quad \text{and}$$

$$D^* = \begin{cases} 1, & if\ c_n = J_n \\ D^2, & otherwise. \end{cases}$$

With these tools as building blocks, we established the following gibonacci sum in [6], the cornerstone of our discourse.

Theorem 1: *Let k, p, r, and t be positive integers, where $t \le 2p$. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} \mu\nu f_r f_{2pk}}{g^2_{(2pn+t-p)k+r} + (-1)^{tk+r} \mu\nu f^2_{pk}} = \frac{g_{tk}}{g_{tk+r}} - (-\beta)^r. \tag{1}$$

The objective of our discourse is to explore the Jacobsthal counterpart of this delightful sum.

3. A Jacobsthal Polynomial Sum

To achieve our goal, we will employ the gibonacci-Jacobsthal relationships in Section 1. Again, in the interest of conciseness and clarity, we let A denote the fractional expression on the left side of the given gibonacci equation and B that on its right side, and LHS and RHS the left-hand side and right-hand side of the corresponding Jacobsthal equation, as in [4, 5].

With this brief background, we now begin our endeavor.

Proof: Case 1: Suppose $g_n = f_n$. We have $A = \dfrac{(-1)^{tk+1} f_r f_{2pk}}{f^2_{(2pn+t-p)k+r} - (-1)^{tk+r} f^2_{pk}}$.

Replacing x with $1/\sqrt{x}$ and then multiplying the numerator and denominator with $x^{(2pn+t)k+r-1}$. We then get

$$A = \frac{(-1)^{tk+1} x^{(2pn+t-p)k+r/2} [x^{(r-1)/2} f_r][x^{(2pk-1)/2} f_{2pk}]}{\{x^{[(2pn+t-p)k+r-1]/2} f_{(2pn+t-p)k+r}\}^2 - (-1)^{tk+r} x^{(2pn+t-2p)k+r} [x^{(pk-1)/2} f_{pk}]^2};$$

$$= \frac{(-1)^{tk+1} x^{(2pn+t-2p)k+r/2} J_r J_{2pk}}{J^2_{(2pn+t-p)k+r} - (-1)^{tk+r} x^{(2pn+t-2p)k+r} J^2_{pk}};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{(-1)^{tk+1} x^{(2pn+t-2p)k+r/2} J_r J_{2pk}}{J^2_{(2pn+t-p)k+r} - (-1)^{tk+r} x^{(2pn+t-2p)k+r} J^2_{pk}}.$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Turning to the right side, we have $B = \dfrac{f_{tk}}{f_{tk+r}} - (-\beta)^r$.

Now, replace x with $1/\sqrt{x}$, and then multiply the numerator and denominator with $x^{(tk+r-1)/2}$. This yields

$$B = \frac{x^{r/2}[x^{(tk-1)/2} f_{tk}]}{[x^{(tk+r-1)/2} f_{tk+r}]} - \frac{(-\omega)^r}{x^{r/2}};$$

$$\text{RHS} = \frac{x^{r/2} J_{tk}}{J_{tk+r}} - \frac{(-\omega) r}{x^{r/2}},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Combining the two sides, we get the Jacobsthal version of equation (1):

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} x^{(2pn+t-2p)k} J_r J_{2pk}}{J^2_{(2pn+t-p)k+r} - (-1)^{tk+r} x^{(2pn+t-2p)k+r} J^2_{pk}} = \frac{J_{tk}}{J_{tk+r}} - \frac{(-\omega)^r}{x^r}, \qquad (2)$$

where $c_n = c_n(x)$.

Next, we explore the Jacobsthal-Lucas version of theorem 1.

Case 2: With $g_n = l_n$, we have $A = \dfrac{(-1)^{tk} \Delta^2 f_r f_{2pk}}{l^2_{(2pn+t-p)k+r} + (-1)^{tk+r} \Delta^2 f^2_{pk}}$. As above, x with $1/\sqrt{x}$ but then multiplying the numerator and denominator with $x^{(2pn+t-p)k+r}$, we then have

$$A = \frac{(-1)^{tk} D^2 \cdot x^{(2pn+t-2p)k+r/2} [x^{(r-1)/2} f_r][x^{(2pk-1)/2} f_{2pk}]}{\{x^{[(2pn+t-p)k+r]/2} l_{(2pn+t-p)k+r}\}^2 + (-1)^{tk+r} D^2 x^{(2pn+t-2p)k+r} [x^{(pk-1)/2} f_{pk}]^2};$$

$$= \frac{(-1)^{tk} D^2 x^{(2pn+t-2p)k+r/2} J_r J_{2pk}}{j^2_{(2pn+t-p)k+r} + (-1)^{tk+r} D^2 x^{(2pn+t-2p)k+r} J^2_{pk}};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{(-1)^{tk} D^2 x^{(2pn+t-2p)k+r/2} J_r J_{2pk}}{j^2_{(2pn+t-p)k+r} + (-1)^{tk+r} D^2 x^{(2pn+t-2p)k+r} J^2_{pk}}.$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Correspondingly, we have $B = \dfrac{l_{tk}}{l_{tk+r}} - (-\beta)^r$. Replacing x with $1/\sqrt{x}$ and then multiplying the numerator and denominator with $x^{(tk+r)/2}$, we get

$$B = \frac{x^{r/2}[x^{tk/2} l_{tk}]}{[x^{(tk+r)/2} l_{tk+r}]} - \frac{(-\omega)^r}{x^{r/2}};$$

$$\text{RHS} = \frac{x^{r/2} j_{tk}}{j_{tk+r}} - \frac{(-\omega) r}{x^{r/2}},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Equating the two sides yields the corresponding Jacobsthal-Lucas version of equation (1):

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} D^2 x^{(2pn+t-2p)k} J_r J_{2pk}}{j_{(2pn+t-p)k+r}^2 + (-1)^{tk+r} D^2 x^{(2pn+t-2p)k+r} J_{pk}^2} = \frac{j_{tk}}{j_{tk+r}} - \frac{(-\omega)^r}{x^r}, \qquad (3)$$

where $c_n = c_n(x)$.

This equation, coupled with formula (2), yields the desired Jacobsthal counterpart of Theorem 1, as the following theorem showcases.

Theorem 2: *Let k, p, r, and t be positive integers, where $t \le 2p$. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} D^* \nu x^{(2pn+t-2p)k} J_r J_{2pk}}{c_{(2pn+t-p)k+r}^2 + (-1)^{tk+r} D^* \nu x^{(2pn+t-2p)k+r} J_{pk}^2} = \frac{c_{tk}}{c_{tk+r}} - \frac{(-\omega)^r}{x^r}. \qquad (4) \ \square$$

Employing the gibonacci-Jacobsthal relationships in a compact way, we now present an alternate proof of this theorem.

3.1 A Delightful Alternate Method: To begin, first, we let

$$d^* = \frac{1-\nu}{4} = \begin{cases} 1/2, & \text{if } g_n = f_n \\ 0, & \text{otherwise.} \end{cases}$$

It follows from the gibonacci-Jacobsthal links in Section 1 that

$$f_n(1/\sqrt{x}) = \frac{J_n(x)}{x^{(n-1)/2}}; \quad l_n(1/\sqrt{x}) = \frac{j_n(x)}{x^{n/2}}; \quad g_n(1/\sqrt{x}) = \frac{c_n(x)}{x^{n/2-d^*}}.$$

With these tools at our fingertips, we are now ready for a sophisticated proof of Theorem 2.

Proof: Replacing x with $1/\sqrt{x}$ in the rational expression on the left side of equation (1) and using the above substitutions, we get

$$A = \frac{(-1)^{tk}\mu\nu[J_r/x^{(r-1)/2}][J_{2pk}/x^{(2pk-1)/2}]}{\left[c_{(2pn+t-p)k+r}/x^{\frac{(2pn+t-p)k+r}{2}-d^*}\right]^2 + (-1)^{tk+r}\mu\nu\left[\frac{J_{pk}}{x^{(pk-1)/2}}\right]^2};$$

$$= \frac{(-1)^{tk}\mu\nu x^{(2pn+t-p)k+r-2d^*-\frac{2pk+r-2}{2}} J_r J_{2pk}}{c^2_{(2pn+t-p)k+r} + (-1)^{tk+r}\mu\nu x^{[(2pn+t-p)k-2d^*-(pk-1)+r]} J^2_{pk}};$$

$$= \frac{(-1)^{tk} D^*\nu x^{(2pn+t-2p)k+\frac{r}{2}} J_r J_{2pk}}{c^2_{(2pn+t-p)k+r} + (-1)^{tk+r} D^*\nu x^{(2pn+t-2p)k+r} J^2_{pk}};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{(-1)^{tk} D^*\nu x^{(2pn+t-2p)k+\frac{r}{2}} J_r J_{2pk}}{c^2_{(2pn+t-p)k} - (-1)^{tk} D^*\nu x^{(2pn+t-2p)k+r} J^2_{pk}}.$$

where $c_n = c_n(x)$.

The right side of equation (1) yields

$$B = \frac{g_{tk}}{g_{tk+r}} - (-\beta)^r;$$

$$= \frac{c_{tk} \,/\, x^{tk/2-d^*}}{c_{tk+r} \,/\, x^{(tk+r)/2-d^*}} - \frac{(-\omega)^r}{x^{r/2}};$$

$$\text{RHS} = \frac{x^{r/2} c_{tk}}{c_{tk+r}} - \frac{(-\omega^*)^r}{x^{r/2}},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Combing the two sides yields the same Jacobsthal version, as expected:

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} D^* \nu x^{(2pn+t-2p)k} J_r J_{2pk}}{c^2_{(2pn+t-p)k} - (-1)^{tk} D^* \nu x^{(2pn+t-2p)k+r} J^2_{pk}} = \frac{c_{tk}}{c_{tk+r}} - \frac{(-\omega^*)^r}{x^r},$$

where $c_n = c_n(x)$. □

Finally, we now explore a host of gibonacci and Jacobsthal implications of Theorem 2. To this end, we define the following labels:

A = 10,080;	F = 25,840;	K = 117,390;	Q = 1,040,130;
B = 13,650;	G = 29,241;	L = 149,872;	R = 1,392,300;
C = 15,504;	H = 30,030;	M = 253,890;	S = 7,514,766;
D = 15,810;	I = 57,330;	N = 263,169;	T = 8,912,862;
E = 15,840;	J = 67,184;	P = 873,810;	V = 66,584,322.

3.2 Gibonacci and Jacobsthal Consequences: With $J_n(1) = F_n$, $j_n(1) = L_n$, $J_n(2) = J_n$, $j_n(2) = j_n$, Theorem 2 yields

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} F_r F_{2pk}}{F^2_{(2pn+t-p)k+r} - (-1)^{tk+r} F^2_{pk}} = \frac{F_{tk}}{F_{tk+r}} - (-\beta)^r; \tag{5}$$

$$\sum_{n=1}^{\infty} L \frac{(-1)^{tk} 5 F_r F_{2pk}}{F^2_{(2pn+t-p)k+r} + (-1)^{tk+r} 5 F^2_{pk}} = \frac{L_{tk}}{L_{tk+r}} - (-\beta)^r; \tag{6}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} 2^{(2pn+t-2p)k} J_r J_{2pk}}{J_{(2pn+t-p)k+r}^2 - (-1)^{tk+r} 2^{(2pn+t-2p)k+r} J_{pk}^2} = \frac{J_{tk}}{J_{tk+r}} - \frac{1}{2^r}; \tag{7}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} 9 \cdot 2^{(2pn+t-2p)k} J_r J_{2pk}}{j_{(2pn+t-p)k+r}^2 + (-1)^{tk+r} 9 \cdot 2^{(2pn+t-2p)k+r} J_{pk}^2} = \frac{j_{tk}}{j_{tk+r}} - \frac{1}{2^r}. \tag{8}$$

In particular, with $p \in \{2, 3\}, k \leq 3, r = 1$, and $t \leq 2$ equations (5) and (6) yield

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n}^2 - 1} = \frac{1}{2} - \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n}^2 + 5} = -\frac{1}{18} + \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{8n-1}^2 + 9} = -\frac{1}{21} + \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{8n-1}^2 - 45} = \frac{1}{84} - \frac{\sqrt{5}}{210};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n-2}^2 - 64} = \frac{7}{864} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n-2}^2 + 320} = -\frac{1}{672} + \frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n+1}^2 + 1} = -\frac{1}{3} + \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n+1}^2 - 5} = -\frac{1}{12} - \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{8n+1}^2 + 9} = -\frac{11}{210} + \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{8n+1}^2 - 45} = \frac{5}{462} - \frac{\sqrt{5}}{210};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n+1}^2 + 64} = -\frac{29}{3,744} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n+1}^2 - 320} = \frac{13}{8,352} - \frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{6n-1}^2 - 4} = \frac{3}{16} - \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6n-1}^2 + 20} = -\frac{1}{48} + \frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n-3}^2 + 64} = -\frac{1}{144} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n-3}^2 - 320} = \frac{1}{576} - \frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{18n-5}^2 - 1,156} = \frac{7}{C} - \frac{\sqrt{5}}{5,168}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{18n-5}^2 + 5,780} = -\frac{3}{3,556} + \frac{\sqrt{5}}{F};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{6n}^2 + 4} = -\frac{1}{8} + \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6n}^2 - 20} = \frac{1}{32} - \frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n-1}^2 + 64} = -\frac{1}{1,440} + \frac{\sqrt{5}}{1,440}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n-1}^2 - 320} = \frac{5}{E} - \frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{18n-2}^2 + 1,156} = -\frac{1}{J} + \frac{\sqrt{5}}{5,168}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{18n-2}^2 - 5,780} = \frac{13}{L} - \frac{\sqrt{5}}{F}.$$

Using equations (7) and (8), we now present their Jacobsthal counterparts:

$$\sum_{n=1}^{\infty} \frac{2^{4n-3}}{J_{4n}^2 - 2^{4n-2}} = \frac{1}{10}; \qquad \sum_{n=1}^{\infty} \frac{2^{4n-3}}{j_{4n}^2 + 9 \cdot 2^{4n-2}} = \frac{1}{150};$$

$$\sum_{n=1}^{\infty} \frac{2^{2(4n-3)}}{J_{8n-3}^2 + 25 \cdot 2^{8n-5}} = \frac{1}{510}; \qquad \sum_{n=1}^{\infty} \frac{2^{2(4n-3)}}{j_{8n-3}^2 - 225 \cdot 2^{8n-5}} = \frac{1}{3,570};$$

$$\sum_{n=1}^{\infty} \frac{2^{3(4n-3)}}{J_{12n-2}^2 - 441 \cdot 2^{12n-8}} = \frac{1}{B}; \qquad \sum_{n=1}^{\infty} \frac{2^{3(4n-3)}}{j_{12n-2}^2 + 3,969 \cdot 2^{12n-8}} = \frac{1}{R};$$

$$\sum_{n=1}^{\infty} \frac{2^{4n-2}}{J_{4n+1}^2 + 2^{4n-1}} = \frac{1}{30}; \qquad \sum_{n=1}^{\infty} \frac{2^{4n-2}}{j_{4n+1}^2 - 9 \cdot 2^{4n-1}} = \frac{1}{126};$$

$$\sum_{n=1}^{\infty} \frac{2^{2(4n-2)}}{J_{8n+1}^2 + 25 \cdot 2^{8n-3}} = \frac{1}{1,870}; \qquad \sum_{n=1}^{\infty} \frac{2^{2(4n-2)}}{j_{8n+1}^2 - 225 \cdot 2^{8n-3}} = \frac{1}{D};$$

$$\sum_{n=1}^{\infty} \frac{2^{3(4n-2)}}{J_{12n+1}^2 + 441 \cdot 2^{12n-5}} = \frac{1}{K}; \qquad \sum_{n=1}^{\infty} \frac{2^{3(4n-2)}}{j_{12n+1}^2 - 3,969 \cdot 2^{12n-5}} = \frac{1}{Q};$$

$$\sum_{n=1}^{\infty} \frac{2^{6n-5}}{J_{6n-1}^2 - 9 \cdot 2^{6n-4}} = \frac{1}{42}\,; \qquad \sum_{n=1}^{\infty} \frac{2^{6n-5}}{j_{6n-1}^2 + 81 \cdot 2^{6n-4}} = \frac{1}{630}\,;$$

$$\sum_{n=1}^{\infty} \frac{2^{2(6n-5)}}{J_{12n-9}^2 + 441 \cdot 2^{12n-9}} = \frac{1}{8,190}\,; \qquad \sum_{n=1}^{\infty} \frac{2^{2(6n-5)}}{j_{12n-9}^2 - 3,969 \cdot 2^{12n-9}} = \frac{1}{I}\,;$$

$$\sum_{n=1}^{\infty} \frac{2^{3(6n-5)}}{J_{18n-5}^2 - G \cdot 2^{18n-14}} = \frac{1}{P}\,; \qquad \sum_{n=1}^{\infty} \frac{2^{3(6n-5)}}{j_{18n-5}^2 + N \cdot 2^{18n-14}} = \frac{1}{T}$$

$$\sum_{n=1}^{\infty} \frac{2^{6n-4}}{J_{6n-1}^2 + 9 \cdot 2^{6n-3}} = \frac{1}{126}\,; \qquad \sum_{n=1}^{\infty} \frac{2^{6n-4}}{j_{6n-1}^2 - 81 \cdot 2^{6n-3}} = \frac{1}{882}\,;$$

$$\sum_{n=1}^{\infty} \frac{2^{2(6n-4)}}{J_{12n-1}^2 + 441 \cdot 2^{12n-7}} = \frac{1}{H}\,; \qquad \sum_{n=1}^{\infty} \frac{2^{2(6n-4)}}{j_{12n-1}^2 - 3,969 \cdot 2^{12n-7}} = \frac{1}{M}\,;$$

$$\sum_{n=1}^{\infty} \frac{2^{3(6n-4)}}{J_{18n-2}^2 + G \cdot 2^{18n-11}} = \frac{1}{S}\,; \qquad \sum_{n=1}^{\infty} \frac{2^{3(6n-4)}}{j_{18n-2}^2 + N \cdot 2^{18n-11}} = \frac{1}{V}\,,$$

respectively.

Finally, we encourage gibonacci enthusiasts to explore the gibonacci and Jacobsthal sums with $p = 5, k, r \in \{1, 2\}$, and $t \leq 5$.

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(*Received*, April 6, 2024)

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Journal of Indian Acad. Math. ISSN: 0970-5120
Vol. 46, No. 2 (2024) pp. 67-74

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CLASSIFYING NUMBERS USING DIVISOR FUNCTION TO STUDY HIGHLY COMPOSITE NUMBERS AND TWIN PRIMES CONJECTURE

Abstract: In this note we classify the positive integers by using divisor function into different equivalence classes. We connect highly composite numbers with this definition. We consider pairs of consecutive odd numbers $(2n+1, 2n+3)$ with $\tau(2n+1) = \tau(2n+3)$ where τ is a divisor function in each equivalence class and call the pair as “twin odd numbers with equal number of divisors” (twin odds with ends). Using these equivalence classes we generalize the twin primes conjecture and study some special cases. We also provide computational data using Python to obtain some interesting results.

Keywords: Divisor Function, Highly Composite Numbers and Twin Primes Conjecture.

Mathematical Subject Classification (2010) No.: 11A25, 11A41.

1. Introduction

We consider the number theoretic divisor function $\tau(n)$ as follows:

Definition 1: *We define*

$$\tau(n) = \sum_{d|n} 1, d > 0$$

From the definition, $\tau(n)$ denotes the number of positive divisors of n. We consider the set of natural numbers $\mathbb{N}$ and define an equivalence relation on $\mathbb{N}$ as follows:

Two natural numbers $m \sim n$ are equivalent if and only if m and n have the same number of positive divisors. Therefore, for $m, n \in \mathbb{N}$

$$m \sim n \Leftrightarrow \tau(m) = \tau(n).$$

Clearly $\sim$ is an equivalence relation. Hence, we have got a disjoint union of equivalence classes of $\mathbb{N}$ and we denote them by $D_1, D_2, \ . \ . \ . \ D_j . \ . \ .$ where D_j contains the numbers with j positive divisors. Unless it is mentioned the divisors are meant to be positive divisors.

Clearly, D_1 contains only one element 1. D_2 denotes the set of prime numbers and D_3 contains squares of prime numbers.

Theorem 2: D_j *is infinite for* $j > 1$.

Proof: Since the number of primes are infinite D_2 is infinite. To show D_j is infinite we observe that for every prime p we consider $1, \ p, p^2, . . . , p^{j-1}$ to conclude D_j is an infinite set. ■

In this note we study the following:

1. We relate highly composite numbers with this definition.
2. We consider subsets TO_j of D_j which consists of consecutive odd numbers and analyze them.
3. We generalize the twin primes conjecture on this basis.

Interested reader may refer to [1] to see some special cases of consecutive integers.

2. Highly Composite Numbers

In 1915, Srinivasa Ramanujan published a paper in Proceedings of London Mathematical Society with title "Highly Composite number". See [2].

A number N is highly composite if $M < N$ implies $\tau(M) < \tau(N)$. We define it another way with the following notation:

Definition 3: *A number $N \in D_n$ is said to be highly composite if every number $< N$ lies in D_k for some $k < n$.*

Example 4: *The first few highly composite numbers are* $1, 2, 4, 6, 12, 24, \ldots$ *with corresponding number of divisors* $1, 2, 3, 4, 6, 8, \ldots$.

Theorem 5: *There are infinitely many highly composite numbers.*

Proof: The proof follows by a simple reasoning. Suppose there is a highly composite number $N \in D_n$ with n divisors. Then we choose $N + K \in D_{n+k}$ which has $n + k$ divisors for some $k = 1, 2, \ldots$. Therefore, the next highly composite number will be the least number with $n + k$ divisors. This process can not be terminated. Hence, there are infinitely many composite numbers. ■

The idea of the proof can be well understood from the sequence of highly composite numbers given in the example above. Once 6 is chosen with 4 divisors the next number $6 + 6 = 12$ has $4 + 2 = 6$ divisors. So there is no highly composite number with 5 divisors !

Suppose we consider the sequence (l_j) where l_j is the least number from D_j. Clearly, highly composite numbers form a sub sequence of l_j.

Remark 6: *We can define the highly composite numbers is a maximal strictly monotonic increasing subsequence of l_j.*

Here the word maximal means the strictly monotonic sequence does not contain in any other monotonic increasing sequence. In the above table the second column denotes the first number of D_j and the third column contains the sequence of highly composite numbers ! It is clear from the table that there are no highly composite numbers for, 250 with number of divisors 5, 13, 14, 15, 17, 19.

Table of equivalence classes containing equal number of divisors

	First number	Highly Composite Numbers	Divisors
D1	1	1	1
D2	2	2	2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127,131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241
D3	4	4	4, 9, 25, 49, 121, 169
D4	6	6	6, 8, 10, 14, 15, 21, 22, 26, 27, 33, 34, 35, 38, 39, 46, 51, 55, 57, 58, 62, 65, 69, 74, 77, 82, 85, 86, 87, 91, 93, 94, 95,106, 111, 115, 118, 119, 122, 123, 125, 129, 133, 134, 141, 142, 143, 145, 146, 155, 158, 159, 161, 166, 177, 178, 183, 185,187, 194, 201, 202, 203, 205, 206, 209, 213, 214, 215, 217, 218, 219, 221, 226, 235, 237, 247, 249
D5	16		16, 81
D6	12	12	12, 18, 20, 28, 32, 44, 45, 50, 52, 63, 68, 75, 76, 92, 98, 99, 116, 117, 124, 147, 148, 153, 164, 171, 172, 175, 188, 207,212, 236, 242, 243, 244, 245
D7	64		64
D8	24	24	24, 30, 40, 42, 54, 56, 66, 70, 78, 88, 102, 104, 105, 110, 114, 128, 130, 135, 136, 138, 152, 154, 165, 170, 174, 182,184, 186, 189, 190, 195, 222, 230, 231, 232, 238, 246, 248, 250
D9	36	36	36, 100, 196, 225
D10	48	48	48, 80, 112, 162, 176, 208
D11			No numbers found.

	First number	Highly Composite Numbers	Divisors
D12	60	60	60, 72, 84, 90, 96, 108, 126, 132, 140, 150, 156, 160, 198, 200, 204, 220, 224, 228, 234
D13			No numbers found.
D14	192		192
D15	144		144
D16	120	120	120, 168, 210, 216
D17			No numbers found
D18	180	180	180
D19			No numbers found
D20	240	240	240

Generated using Python

3. Generalization of Twin Primes

We consider special pairs of odd numbers which are existing in a very natural way and generalize the set of twin primes. We feel that these pairs of odd numbers are interesting and needs attention for further research in this area. Interestingly, twin primes conjecture turns out to be a special class in this classification.

Definition 7: *We define a pair of two consecutive odd numbers* $(2n+1, 2n+3)$ *is* ***twin odd numbers with ends*** *if both the numbers have equal number of positive divisors.*

In an obvious way all twin primes belong to this class of numbers with two divisors.

Let us denote the set of pairs of odd numbers belonging to the equivalence class D_j by TO_j. Therefore, TO_j is a subset of D_j which contains pairs of consecutive odd numbers contained in D_j.

Conjecture 8: (*Twin primes conjecture*) TO_2 *is an infinite set.*

See [3].

Theorem 9: $TO_3 = \varnothing$.

Proof: It is easy to see that for any such pair $(2n+1, 2n+3)$, the divisors of $2n+1$ are $1, p, p^2$ and the divisors of $2n+3$ are $1, q, q^2$ for some odd primes p and q. Even for any twin primes (p, q) it follows that $q^2 - p^2 > 2$.

Hence, there are no twin odd numbers with 3 positive divisors. ■

Theorem 10: $TO_5 = \varnothing$.

Proof: It follows from the observation that the prime factorization of such numbers can not contain more than two distinct primes. Following in the same argument as above we can conclude the result. ■

Problem 11: $TO_k = \varnothing$ *if* k *is odd*

Remark 12: *Above theorems show that* TO_k *is not always non-empty.*

Now we study the structure of the set TO_4. That is, twin odd numbers with 4 divisors. First we give an example to show that this set is non-empty.

For example, $(33, 35)$, $(55, 57)$... which are pairs of consecutive odd numbers with four divisors.

Theorem 13: *Any pair of twin odd numbers in* TO_4 *will have the divisor (positive) sets*:

$$X = \{1, p_1, q_1, p_1q_1\} \text{ and } Y = \{1, p_2, q_2, p_2q_2\}$$

where p_1, q_1, p_2, q_2 *are distinct primes or* $q_1 = p_1^2$. *In the later case only one of the divisor set is of the form* $\{1, p_1, p_1^2, p_1^3\}$

Proof: Let a, b constitute the pair of twin end odds and $\tau(a) = 4 = \tau(b)$. Let $\{1, p_1, p_2, a\}$ be the set of divisors of a such that

$$1 < p_1 < p_2 < a$$

with p_1 is the least prime divisor. Then if p_2 is a prime then a is equal to $p_1 \cdot p_2$ and if p_2 is composite then $p_2 = p_1^2$ which leads to $a = p_1^3$.

Both the divisor sets can not be of the form $\{1, p_1, p_1^2, p_1^3\}$ for a prime p in which case $|a - b| > 2$ which contradicts the hypothesis $|a - b| = 2$.

This completes the proof. ■

The following tables will illustrate the possible divisor sets for $(a, b) \in TO_4$.

Table 1: Divisor sets of Type 1

1	p_1	q_1	p_1q_1	$(,)$	p_2q_2	p_2	q_2
1	3	11	33	(33, 35)	35	5	7
1	5	11	55	(55, 57)	57	3	19

Table 2: Divisor sets of Type 2

1	p_1	p_1^2	p_1^3	$(,)$	p_2q_2	p_2	q_2
1	5	5^2	125	(123, 125)	123	3	41
1	11	11^2	1331	(1331, 1333)	1333	3	41
1	19	361	6859	(6859, 6861)	6861	3	2287

Conjecture 14: *Is TO_4 an infinite set ?*

Problem 15: *Is TO_{2k} is non-empty for every $k \in \mathbb{N}$?*

With our new notation, generalized twin primes conjecture will be stated as follows:

Conjecture 16: (*Generalized twin primes conjecture*) *Is* TO_{2k} *an infinite set for every* $k \in \mathbb{N}$ *?*

The following questions will follow naturally.

Problem 17: *Find* k *for which* TO_k *is an empty set.*

Problem 18: *Find* k *for which* TO_k *is an infinite set.*

Acknowledgement: I thank Ms. Krishnasri Gollakota, postgraduate student of Department of Physics, University of Mumbai for the computational data provided using Python.

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(*Received*, May 4, 2024)

FORM IV
(See Rule 8)

1.	Title of the Journal	Journal of Indian Academy of Mathematics
2.	Place of Publication	5, 1st floor, I. K. Girls School Campus, 14/1 Ushaganj, Near G.P.O. Indore - 452 001 India
3.	Periodicity of Publication	Bi-Annual (Twice in a Year)
4.	Language in which it is published	English
5.	Publisher's name Nationality Address	C. L. Parihar (*Editor*) Indian 5, 1st floor, I. K. Girls School Campus, 14/1 Ushaganj, Near G.P.O. Indore - 452 001 India
6.	Printer's Name Nationality Address	Piyush Gupta Indian 316, Subhash Nagar, Mumfordganj, Prayagraj - 211002
7.	Editor's Name Nationality Address	C. L. Parihar Indian Indore
8.	Name of the Printing Press, where the publication is printed	Radha Krishna Enterprises 6B/4B/9A, Beli Road, Prayagraj - 211002
9.	Name and addresses of the individuals who own the newspaper/journal and partners or shareholders holding more than one per cent of the total capital	No Individual: It is run by Indian Academy of Mathematics 5, 1st floor, I. K. Girls School Campus, 14/1 Ushaganj, Near G.P.O. Indore - 452 001 India

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Editor
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Dated: December 31, 2021

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Published by: The Indian Academy of Mathematics, Indore-452 016, India, Mobile: 7869410127
Composed & Printed by: Piyush Gupta, Prayagraj-211 002, (U.P.) Mob: 07800682251

CONTENTS