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(*Invited Article*)

H. M. Srivastava

SOME GENERAL FRACTIONAL-ORDER KINETIC EQUATIONS ASSOCIATED WITH THE RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE

Abstract: In the current literature, various operators of fractional calculus (that is, fractional-order integrals and fractional-order derivatives) have been and continue to be successfully applied in the modeling and analysis of a remarkably large spectrum of applied scientific and real-world problems in the mathematical, physical, biological, engineering and statistical sciences, and indeed also in other scientific disciplines. In this article, we investigate a general family of fractional-order kinetic equations involving the Riemann-Liouville fractional derivative, which also includes a remarkably general class of functions as a part of the non-homogeneous term. The main results, which we have derived in this article, are capable of yielding solutions of a significantly large number of simpler fractional-order kinetic equations.

Keywords and Phrases: Riemann-Liouville and Related Fractional Derivative Operators, Riemann-Liouville Fractional Derivative Operator, Hyper-geometric Functions, Special (or Higher Transcendental) Functions, Fox-Wright Hypergeometric Function, Mittag-Leffler Type Functions, General Fox-Wright Function, Zeta and Related Functions, Lerch Transcendent (or Hurwitz-Lerch Zeta function).

Mathematical Subject Classification (2020) No.: Primary 26A33, 33C20, 33E12, Secondary 47B38, 47G10.

1. Introduction and Motivation

In recent years, various operators of fractional calculus (that is, operators of integrals and derivatives of any real or complex order) has received considerable attention because mainly of their demonstrated applications in the modeling and analysis of applied problems and real-world situations occurring in numerous seemingly diverse and widespread fields of science and engineering. These operators do indeed provide several potentially useful tools and techniques for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables (see, for details, [8], [9], [11] and [12]; see also [4], [6] and [26]).

Traditionally (and by far the most commonly used), the operators of fractional-order integration and fractional-order differentiation are defined by means of the right-sided Riemann-Liouville fractional integral operator ${}^{\mathrm{RL}}I^{\mu}_{a+}$ and the left-sided Riemann-Liouville fractional integral operator ${}^{\mathrm{RL}}I^{\mu}_{a-}$, and the corresponding Riemann-Liouville fractional derivative operators ${}^{\mathrm{RL}}D^{\mu}_{a+}$ and ${}^{\mathrm{RL}}D^{\mu}_{a-}$, as follows (see, for example, [3, Chapter 13], [8, pp. 69-70] and [13]):

$$\left({}^{\mathrm{RL}}I^{\mu}_{a+}f\right)(x)=\frac{1}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1} f(t)dt \qquad (x>a;\,\Re(\mu)>0), \tag{1}$$

$$\left({}^{\mathrm{RL}}I^{\mu}_{a-}f\right)(x)=\frac{1}{\Gamma(\mu)}\int_x^a (t-x)^{\mu-1} f(t)dt \qquad (x<a;\,\Re(\mu)>0) \tag{2}$$

and

$$\left({}^{RL}D^{\mu}_{a\pm}f\right)(x)=\left(\pm\frac{d}{dx}\right)^{n}\left(I^{n-\mu}_{a\pm}f\right)(x) \qquad \left(\Re(\mu)\geqq 0;\, n=[\Re(\mu)]+1\right). \tag{3}$$

Here, and in what follows, the function f is locally integrable, $\Re(\mu)$ denotes the real part of the complex number $\mu\in\mathbb{C}$ and $[\Re(\mu)]$ means the greatest integer in $\Re(\mu)$, and $\Gamma(z)$ denotes the classical (Euler's) Gamma function defined by

$$\Gamma(z) := \begin{cases} \int_0^\infty e^{-t} t^{z-1} dt & (\Re(z) > 0) \\ \dfrac{\Gamma(z+n)}{\prod_{j=0}^{n-1} (z+j)} & (z \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ n \in \mathbb{N}), \end{cases} \tag{4}$$

which happens to be one of the most fundamental and the most useful special functions of mathematical analysis, $\mathbb{N}$ and $\mathbb{Z}_0^-$ being the sets of *positive* and *non-positive* integers, respectively.

Our main object in this article is investigate some general families of fractional-order kinetic equations involving the Riemann-Liouville right-sided fractional derivative operator ${}^{\mathrm{RL}}(D_{0+}^{\mu} f)(x)$, which is given (for convenience) by (3) for $a = 0$, as well as including a remarkably general class of functions as a part of the non-homogeneous term. Our main results (Theorem 1, Theorem 2 and Theorem 3 in this article) are capable of yielding solutions of a significantly large number of simpler fractional-order kinetic equations.

2. Definitions and Preliminaries

First of all, it is easily observed that most (if not all) of the various claimed one-variable and multi-parameter (or multi-index) "generalizations" of the familiar Mittag-Leffler function $E_\alpha(z)$ and its two-parameter extension $E_{\alpha,\beta}(z)$, which are defined as follows:

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \qquad \text{and} \qquad E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{5}$$

are no more than fairly obvious special or limit cases of the substantially much more general Fox-Wright function ${}_p\Psi_q\ (p, q \in \mathbb{N}_0)$ or ${}_p\Psi_q^*\ (p, q \in \mathbb{N}_0)$, which happens to be the Fox-Wright generalization of the relatively more familiar hypergeometric function ${}_pF_q\ (p, q \in N_0)$, with p numerator parameters $a_1, \cdots, a_p$ and q denominator parameters $b_1, \cdots, b_q$ such that

$$a_j \in \mathbb{C}\ (j = 1, \cdots, p) \qquad \text{and} \qquad b_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad (j = 1, \cdots, q).$$

These general Fox-Wright functions ${}_p\Psi_q$ $(p, q \in \mathbb{N}_0)$ and ${}_p\Psi_q^*$ $(p, q \in \mathbb{N}_0)$ are indeed defined by (see, for details, [2, p. 183] and [25, p. 21]; see also [7, p. 65], [8, p. 56] and [14])

$$
{}_p\Psi_q^*\left[\begin{matrix}(a_1, A_1), \cdots, (a_p, A_p); \\ \\ (b_1, B_1), \cdots, (b_q, B_q);\end{matrix}\; z\right]
$$

$$
:= \sum_{n=0}^{\infty} \frac{(a_1)_{A_1 n} \cdots (a_p)_{A_p n}}{(b_1)_{B_1 n} \cdots (b_p)_{B_q n}} \frac{z^n}{n!}
$$

$$
= \frac{\Gamma(b_1) \cdots \Gamma(b_q)}{\Gamma(a_1) \cdots \Gamma(a_p)}\, {}_p\Psi_q\left[\begin{matrix}(a_1, A_1), \cdots, (a_p, A_p); \\ \\ (b_1, B_1), \cdots, (b_q, B_q);\end{matrix}\; z\right] \tag{6}
$$

$$
\left(\Re(A_j) > 0 \;\; (j = 1, \cdots, p); \Re(B_j) > 0 \;\; (j = 1, \cdots, q); 1 + \Re\left(\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j\right) \geqq 0\right),
$$

where, and in what follows, $(\lambda)_\nu$ denotes the general Pochhammer symbol or the *shifted* factorial, since

$$
(1)_n = n! \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\; \mathbb{N} := \{1, 2, 3, \cdots\}),
$$

which is defined (for $\lambda, \nu \in \mathbb{C}$ and in terms of the above-defined familiar Gamma function in the equation (4)) by

$$
(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases} \tag{7}
$$

it being assumed *conventionally* that $(0)_0 := 1$ and understood *tacitly* that the Γ-quotient exists. Here we suppose, in general, that

$$a_j, A_j \in \mathbb{C} \ (j = 1, \cdots, p) \ \text{ and } \ b_j, B_j \in \mathbb{C} \ (j = 1, \cdots, q)$$

and that the equality in the convergence condition in the definition (6) holds true only for suitably bounded values of $|z|$ given by

$$|z| < \nabla := \left(\prod_{j=1}^{p} A_j^{-A_j} \right) \cdot \left(\prod_{j=1}^{q} B_j^{B_j} \right).$$

We remark in passing that the above-mentioned generalized hypergeoemtric function ${}_pF_q \, (p, q \in \mathbb{N}_0)$, with p numerator parameters $a_1, \cdots, a_p$ and q denominator parameters $b_1, \cdots, b_q$, is a widely- and extensively-investigated and potentially useful special case of the general Fox-Wright function ${}_p\Psi_q \, (p, q \in \mathbb{N}_0)$ when

$$A_j = 1 \ (j = 1, \cdots, p) \ \text{ and } \ B_j = 1 \ (j = 1, \cdots, q).$$

We now turn to a series of monumental works (see, for example, [28], [29] and [30]) by Sir Edward Maitland Wright (1906-2005), with whom I had the privilege to meet and discuss researches emerging from his publications on hypergeometric and related functions during my visit to the University of Aberdeen in the year 1976, introduced and systematically studied the asymptotic expansion of the following Taylor-Maclaurin series (see [28, p. 424]):

$$\mathfrak{E}_{\alpha,\beta}(\phi; z) := \sum_{n=0}^{\infty} \frac{\phi(n)}{\Gamma(\alpha n + \beta)} z^n \qquad (\alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0), \tag{8}$$

where $\phi(t)$ is a function satisfying suitable conditions. In fact, it was my proud privilege to have also met many times and discussed mathematical researches, especially on various families of higher transcendental functions and related topics, with my Canadian colleague, Charles Fox (1897-1977) of birth and education in England, both at McGill University and Sir George Williams University (*now* Concordia University) in Montréal, mainly during the 1970s (see, for details, [14]).

The above-cited contributions by Wright were motivated essentially by the earlier developments reported for simpler cases by Magnus Gustaf (Gösta) Mittag-Leffler (1846-1927) in 1905, Anders Wiman (1865-1959) in 1905, Ernest William Barnes (1874-1953) in 1906, Godfrey Harold Hardy (1877-1947) in 1905, George Neville Watson (1886-1965) in 1913, Charles Fox (1897-1977) in 1928, and other authors. In particular, the aforementioned work [1] by *Bishop* Ernest William Barnes (1874-1953) of the Church of England in Birmingham considered the asymptotic expansions of functions in the class defined below:

$$E_{\alpha,\beta}^{(\kappa)}(s;z) := \sum_{n=0}^{\infty} \frac{z^n}{(n+\kappa)^s \, \Gamma(\alpha n+\beta)} \qquad (\alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0) \tag{9}$$

for suitably-restricted parameters κ and s. Clearly, we have the following relationship:

$$\lim_{\alpha \to \infty} \left\{ E_{\alpha,\beta}^{(\kappa)}(s;z) \right\} = \frac{1}{\Gamma(\beta)} \Phi(z, s, \kappa)$$

with the classical Lerch transcendent (or the Hurwitz-Lerch zeta function) $\Phi(z, s, \kappa)$ defined by (see, for example, [2, p. 27, Eq. 1.11 (1)]; see also [23] and [24])

$$\Phi(z, s, \kappa) := \sum_{n=0}^{\infty} \frac{z^n}{(n+\kappa)^s} \tag{10}$$

$$(k \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1).$$

The Hurwitz-Lerch zeta function $\Phi(z, s, \kappa)$ defined by (10) contains, as its *special* cases, not only the Riemann zeta function $\zeta(s)$ and the Hurwitz (or generalized) zeta function $\zeta(s, \kappa)$:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \Phi(1, s, 1) \quad \text{and} \quad \zeta(s, \kappa) := \sum_{n=0}^{\infty} \frac{1}{(n+\kappa)^s} = \Phi(1, s, \kappa) \tag{11}$$

and the Lerch zeta function $\ell_s(\xi)$ defined by (see, for details, [2, Chapter I] and [23, Chapter 2])

$$\ell_s(\xi) := \sum_{n=1}^{\infty} \frac{e^{2n\pi i\xi}}{n^s} = e^{2\pi i\xi} \Phi(e^{2\pi i\xi}, s, 1) \tag{12}$$

$$(\mathrm{i} = \sqrt{-1}; \xi \in \mathbb{R}; \Re(s) > 1),$$

but also such other important functions of *Analytic Number Theory* as the Polylogarithmic function (or *de Jonquière's function*) $\mathrm{Li}_s(z)$:

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\,\Phi(z, s, 1) \tag{13}$$

$$(s \in \mathbb{C} \quad \text{when} \quad |z| < 1;\ \Re(s) > 1 \quad \text{when} \quad |z| = 1)$$

and the Lipschitz-Lerch zeta function (see [23, p. 122, Eq. 2.5 (11)]):

$$\phi(\xi, \kappa, s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi \mathrm{i}\xi}}{(n+\kappa)^s} = \Phi\left(e^{2\pi \mathrm{i}\xi}, s, \kappa\right) =: L(\xi, s, \kappa) \tag{14}$$

$$\left(\kappa \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \Re(s) > 0 \quad \text{when} \quad \xi \in \mathbb{R} \setminus \mathbb{Z};\ \Re(s) > 1 \quad \text{when} \quad \xi \in \mathbb{Z}\right),$$

which was first studied by Rudolf Lipschitz (1832-1903) and Matyáš Lerch (1860-1922) in connection with Dirichlet's famous theorem on primes in arithmetic progressions (see, for details, [17] and [18]).

A natural unification and generalization of the Fox-Wright function ${}_p\Psi_q^*$ defined by (6) as well as the Hurwitz-Lerch zeta function $\Phi(z, s, \kappa)$ defined by (10) was indeed accomplished by introducing essentially arbitrary numbers of numerator and denominator parameters in the definition (10). For this purpose, in addition to the symbol ∇^* defined by

$$\nabla^* := \left(\prod_{j=1}^{p} \rho_j^{-\rho_j}\right) \cdot \left(\prod_{j=1}^{q} \sigma_j^{-\sigma_j}\right), \tag{15}$$

the following notations will be employed:

$$\Delta := \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j \quad \text{and} \quad \Xi := s + \sum_{j=1}^{q} \mu_j - \sum_{j=1}^{p} \lambda_j + \frac{p-q}{2}. \tag{16}$$

Then the extended Hurwitz-Lerch zeta function

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}(z, s, \kappa)$$

is defined by [27, p. 503, Equation (6.2)] (see also [15] and [24])

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}(z, s, \kappa) := \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\lambda_j)_{n\rho_j}}{n\,!\,.\prod_{j=1}^{q} (\mu_j)_{n\sigma_j}} \frac{z^n}{(n+\kappa)^s} \tag{17}$$

$$\Big(p, q \in \mathbb{N}_0; \lambda_j \in \mathbb{C}\ (j = 1, \cdots, p); \kappa, \mu_j \in \mathbb{C}\setminus\mathbb{Z}_0^-(j = 1, \cdots, q);$$

$$\rho_j, \sigma_k \in \mathbb{R}^+(j = 1, \cdots, p; k = 1, \cdots, q); \Delta > -1 \quad \text{when} \quad s, z \in \mathbb{C};$$

$$\Delta = -1 \quad \text{and} \quad s \in \mathbb{C} \quad \text{when} \quad |z| < \nabla^*;$$

$$\Delta = -1 \ \text{ and } \ \Re(\Xi) > \frac{1}{2} \ \text{ when } \ |z| = \nabla^*\Big).$$

For an interesting and potentially useful family of λ-generalized Hurwitz-Lerch zeta functions, which *further* extend the multi-parameter Hurwitz-Lerch zeta function

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}(z, s, \kappa)$$

defined by (17), was introduced and investigated systematically in a recent paper by Srivastava [16], who also discussed their potential application in Number Theory by appropriately constructing a presumably new continuous analogue of Lippert's Hurwitz measure and also considered some other statistical applications of these families of the λ-generalized Hurwitz-Lerch zeta functions in probability distribution theory (see also the references to several related earlier works cited by Srivastava [16]).

We now introduce some general families of the Riemann-Liouville type fractional integrals and fractional derivatives by making use of the following interesting unification of the definitions in (8) and (17) for a suitably-restricted function $\varphi(\tau)$ given by

$$\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa) := \sum_{n=0}^{\infty} \frac{\varphi(n)}{(n+\kappa)^s \, \Gamma(\alpha n + \beta)} z^n \qquad (\alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0), \tag{18}$$

where the parameters $\alpha,\ \beta,\ s$ and κ are appropriately constrained as above. The resulting general right-sided fractional integral operator $\mathcal{I}_{a+}^{\mu}(\varphi; z, s, \kappa, \nu)$ and the general left-sided fractional integral operator $\mathcal{I}_{a-}^{\mu}(\varphi; z, s, \kappa, \nu)$, and the corresponding fractional derivative operators $\mathcal{D}_{a+}^{\mu}(\varphi; z, s, \kappa, \nu)$ and $\mathcal{D}_{a-}^{\mu}(\varphi; z, s, \kappa, \nu)$, each of the Riemann-Liouville type, are defined by (see, for details, [20], [21] and [22])

$$\left(\mathcal{I}_{a+}^{\mu}(\varphi; z, s, \kappa, \nu) f\right)(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} \mathcal{E}_{\alpha,\beta}(\varphi; z(x-t)^{\nu}, s, \kappa)\, f(t)\, dt \tag{19}$$
$$(x > a; \Re(\mu) > 0),$$

$$\left(\mathcal{I}_{a-}^{\mu}(\varphi; z, s, \kappa, \nu) f\right)(x) = \frac{1}{\Gamma(\mu)} \int_x^a (t-x)^{\mu-1} \mathcal{E}_{\alpha,\beta}(\varphi; z(t-x)^{\nu}, s, \kappa)\, f(t)\, dt \tag{20}$$
$$(x < a; \Re(\mu) > 0)$$

and

$$\left(\mathcal{D}_{a\pm}^{\mu}(\varphi; z, s, \kappa, \nu) f\right)(x) = \left(\pm \frac{d}{dx}\right)^n \left(\mathcal{I}_{a\pm}^{n-\mu}(\varphi; z, s, \kappa, \nu) f\right)(x) \tag{21}$$
$$(\Re(\mu) \geqq 0;\ n = [\Re(\mu)] + 1),$$

where the function f is in the space $L(\mathfrak{a}, \mathfrak{b})$ of Lebesgue integrable functions on a finite closed interval $[\mathfrak{a}, \mathfrak{b}]\ (\mathfrak{b} > \mathfrak{a})$ of the real line $\mathbb{R}$ given by

$$L(\mathfrak{a}, \mathfrak{b}) = \left\{ f : \|f\|_1 = \int_{\mathfrak{a}}^{\mathfrak{b}} |f(x)|\, dx < \infty \right\}, \tag{22}$$

it being *tacitly* assumed that, in situations such as those occurring in conjunction with the usages of the definitions in (19), (20) and (21), the point $\mathfrak{a}$ in all such function spaces as (for example) the function space $L(\mathfrak{a}, \mathfrak{b})$ coincides precisely with the *lower* terminal a in the integrals involved in the definitions (19), (20) and (21).

Next, in terms of the operator $\mathcal{L}$ of the Laplace transform given by

$$\mathcal{L}\{f(\tau) : \mathfrak{s}\} := \int_0^\infty e^{-st} f(\tau)\, d\tau =: F(\mathfrak{s}) \qquad (\Re(\mathfrak{s}) > 0), \tag{23}$$

where the function $f(\tau)$ is so constrained that the integral exists, it is easily seen for the function $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$, defined above by (18), that

$$\mathcal{L}\{\tau^{\mu-1}\mathcal{E}_{\alpha,\beta}(\varphi; z\tau^{\nu}, s, \kappa) : \mathfrak{s}\} = \frac{1}{\mathfrak{s}^{\mu}} \sum_{k=0}^{\infty} \frac{\varphi(k)\Gamma(\nu k + \mu)}{(k+\kappa)^{s}\Gamma(\alpha k + \beta)} \left(\frac{z}{\mathfrak{s}^{\nu}}\right)^{k} \tag{24}$$

$$\left(\Re(\mathfrak{s}) > 0; \Re(\mu) > 0; \Re(\nu) > 0; \Re(\alpha) > 0\right),$$

provided that each member of (24) exists. Obviously, upon setting $\mu = \beta$ and $\nu = \alpha$, the Laplace transform formula (24) simplifies to the following form:

$$\mathcal{L}\{\tau^{\beta-1}\mathcal{E}_{\alpha,\beta}(\varphi; z\tau^{\alpha}, s, \kappa) : \mathfrak{s}\} = \frac{1}{\mathfrak{s}^{\mu}} \sum_{k=0}^{\infty} \frac{\varphi(k)}{(k+\kappa)^{s}} \left(\frac{z}{\mathfrak{s}^{\alpha}}\right)^{k} \tag{25}$$

$$\left(\Re(\mathfrak{s}) > 0; \Re(\alpha) > 0; \Re(\beta) > 0\right).$$

In case we apply the following limit formula:

$$\mathfrak{E}_{\alpha,\beta}(\phi; z) = \lim_{s\to 0}\{\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)\}\,|_{\varphi\equiv\phi}\ . \tag{26}$$

or, alternatively, if we make use of the definitions in (8) and (23), we find for Wright's function $\mathfrak{E}_{\alpha,\beta}(\phi; z)$ that

$$\mathcal{L}\{\tau^{\mu-1}\mathfrak{E}_{\alpha,\beta}(\phi; z\tau^{\nu}) : \mathfrak{s}\} = \frac{1}{\mathfrak{s}^{\mu}} \sum_{k=0}^{\infty} \frac{\phi(k)\Gamma(\nu k + \mu)}{\Gamma(\alpha k + \beta)} \left(\frac{z}{\mathfrak{s}^{\nu}}\right)^{k} \tag{27}$$

$$\left(\Re(\mathfrak{s}) > 0; \Re(\mu) > 0; \Re(\nu) > 0; \Re(\alpha) > 0\right),$$

which, in the special case when $\nu = \alpha$ and $\mu = \beta$, yields

$$\mathcal{L}\{\tau^{\beta-1}\mathfrak{E}_{\alpha,\beta}(\phi; z\tau^{\alpha}) : \mathfrak{s}\} = \frac{1}{\mathfrak{s}^{\beta}} \sum_{k=0}^{\infty} \phi(k) \left(\frac{z}{\mathfrak{s}^{\alpha}}\right)^{k} \tag{28}$$

$$\big(\Re(\mathfrak{s}) > 0; \Re(\alpha) > 0; \Re(\beta) > 0\big).$$

Moreover, in the case when the sequence $\{\varphi(n)\}_{n=0}^{\infty}$ is given by

$$\varphi(n) = \frac{\Gamma(\alpha n + \beta) \prod_{j=1}^{p} (\lambda_j)_{np_j}}{n!\,.\prod_{j=1}^{q} (\mu_j)_{n\sigma_j}} \quad (n \in \mathbb{N}_0), \tag{29}$$

then the Laplace transformation formula (25) would yield the following result:

$$\mathcal{L}\left\{\tau^{\mu-1}\, \Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q}(z\tau^{\nu}, s, k) : \mathfrak{s}\right\}$$

$$= \frac{\Gamma(\mu)}{\mathfrak{s}^{\mu}}\, \Phi_{\mu,\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\nu,\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}\left(\frac{z}{\mathfrak{s}^{\nu}}, s, k\right) \tag{30}$$

$$\big(\Re(\mathfrak{s}) > 0; \Re(\mu) > 0; \Re(\nu) > 0; \Re(\alpha) > 0\big)$$

for the extended Hurwitz-Lerch zeta function

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}(z, s, k)$$

defined by (17).

Finally, for the right-sided Riemann-Liouville fractional derivative operator $\mathcal{D}_{0+}^{\mu}$ of order μ in the definition (3), it is easily observed that (see, for example, [12, p. 105, Eq. (2.248)])

$$\mathcal{L}\{({}^{\mathrm{RL}}\mathcal{D}_{0+}^{\mu} f)(t) : \mathfrak{s}\} = \mathfrak{s}^{\mu} F(\mathfrak{s}) - \sum_{k=0}^{n-1} \mathfrak{s}^{k} ({}^{RL}\mathcal{D}_{0+}^{\mu-k-1} f)(0+) \tag{31}$$

$$(n - 1 < \Re(\mu) < n; n \in \mathbb{N})$$

or, equivalently, that (see, for example, [8, p. 84, Eq. (2.2.37)])

$$\mathcal{L}\{({}^{\mathrm{RL}}\mathcal{D}_{0+}^{\mu}f)(t) : \mathfrak{s}\} = \mathfrak{s}^{\mu}F(\mathfrak{s}) - \sum_{k=0}^{n-1}\mathfrak{s}^{n-k-1}\frac{d^k}{dt^k}\left\{({}^{\mathrm{RL}}I_{0+}^{n-\mu}f)(t)\right\}\Big|_{t=0}$$

$$= \mathfrak{s}^{\mu}F(\mathfrak{s}) - \sum_{k=0}^{n-1}\mathfrak{s}^{k}\frac{d^{n-k-1}}{dt^{n-k-1}}\left\{({}^{\mathrm{RL}}I_{0+}^{n-\mu}f)(t)\right\}\Big|_{t=0} \tag{32}$$

$$(n-1 < \Re(\mu) < n;\ n \in \mathbb{N})$$

where, for convenience

$$\left({}^{\mathrm{RL}}\mathcal{D}_{0+}^{\mu-k-1}f\right)(0+) := \lim_{t\to 0+}\left\{\left({}^{\mathrm{RL}}\mathcal{D}_{0+}^{\mu-k-1}f\right)(t)\right\} =: \left({}^{\mathrm{RL}}\mathcal{D}_{0+}^{\mu-k-1}f\right)(t)\Big|_{t=0}$$

and

$$\frac{d^k}{dt^k}\left\{\left({}^{\mathrm{RL}}I_{0+}^{n-\mu}f\right)(t)\right\}\Big|_{t=0} := \lim_{t\to 0+}\frac{d^k}{dt^k}\left\{\left({}^{\mathrm{RL}}I_{0+}^{n-\mu}f\right)(t)\right\}$$

$$=: \frac{d^k}{dt^k}\left\{\left({}^{\mathrm{RL}}I_{0+}^{n-\mu}f\right)(0+)\right\} \qquad (k \in \{0, 1, 2, \cdots, n-1\}).$$

Indeed, for the *ordinary* derivative $f^{(n)}(t)$ of order $n \in \mathbb{N}_0$, it is known that

$$\mathcal{L}\{f^{(n)}(t) : \mathfrak{s}\} = \mathfrak{s}^{n}F(\mathfrak{s}) - \sum_{k=0}^{n-1}\mathfrak{s}^{k}f^{(n-k-1)}(t)\Big|_{t=0} \qquad (n \in \mathbb{N}_0) \tag{33}$$

or, equivalently, that

$$\mathcal{L}\{f^{(n)}(t) : \mathfrak{s}\} = \mathfrak{s}^{n}F(\mathfrak{s}) - \sum_{k=0}^{n-1}\mathfrak{s}^{n-k-1}f^{(k)}(0+) \quad (n \in \mathbb{N}_0), \tag{34}$$

where, as well as in all of such situations in this paper, an *empty* sum is to be interpreted as 0.

3. A General Family of Fractional-Order Kinetic Equations

For an arbitrary reaction, which is characterized by a time-dependent quantity $N = N(t)$, it is possible to calculate the rate of change $\frac{dN}{dt}$ to be a balance between the destruction rate $\mathfrak{d}$ and the production rate $\mathfrak{p}$ of N, that is,

$$\frac{dN}{dt} = -\mathfrak{d} + \mathfrak{p}.$$

By means of feedback or other interaction mechanism, the destruction and the production depend on the quantity N itself, that is,

$$\mathfrak{d} = \mathfrak{d}(N) \qquad \text{and} \qquad \mathfrak{p} = \mathfrak{p}(N).$$

Since the destruction or the production at a time t depends not only on $N(t)$, but also on the past history $N(\eta) \quad (\eta < t)$ of the variable N, such dependence is, in general, complicated. This may be formally represented by the following equation (see [5]):

$$\frac{dN}{dt} = -\,\mathfrak{d}\,(N_t) + \mathfrak{p}\,(N_t)\,, \tag{35}$$

where N_t denotes the function defined by

$$N_t(t^*) = N(t - t^*) \qquad (t^* > 0)\ .$$

Haubold and Mathai [5] studied a special case of the equation (35) in the following form:

$$\frac{dN_j}{dt} = -c_j N_j(t), \tag{36}$$

that is,

$$\frac{dN_j(t)}{N_j(t)} = -c_j\, dt, \tag{37}$$

with the initial condition that

$$N_j\,(t)\Big|_{t=0} = N_0,$$

is the number density of species j at time $t = 0$ and the constant $c_j > 0$. This is known as a standard kinetic equation. The solution of the equation (36) (*without* the subscript j) is readily seen to be given by

$$N_j(t) = N_0 e^{-c_j t}, \tag{38}$$

which, upon integration, yields the following alternative form of the solution of the equation (36) (*without* the subscript j):

$$N(t) - N_0 = c \cdot {}_0D_t^{-1}\{N(t)\} , \tag{39}$$

where ${}_0D_t^{-1}$ is the standard (ordinary) integral operator and c is a constant of integration.

The fractional-order generalization of the equation (39) is given as in the following form (see [5]):

$$N(t) - N_0 = c^{\nu}\left({}^{\mathrm{RL}}I_{0+}^{\nu}N\right)(t) \tag{40}$$

in terms of the familiar right-sided Riemann-Liouville fractional integral operator ${}^{\mathrm{RL}}I_{0+}^{\nu}$ of order ν defined, as in (1), by (see, for example, [8])

$$\left({}^{\mathrm{RL}}I_{0+}^{\nu}f\right)(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-u)^{\nu-1} f\, u\,(du) \qquad \left(t > 0; \Re(\nu) > 0\right). \tag{41}$$

For a considerably large number of extensions and further generalizations of the fractional-order kinetic equation (40), the interested reader should refer (for example) to [10], [19] and [20] as well as the other relevant references which are cited in each of these earlier publications. We propose here to investigate the solution of a general family of fractional-order kinetic equations which are associated with the function $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$ defined by (18), which we have introduced in this article, as well as the Riemann-Liouville fractional derivative operator ${}^{\mathrm{RL}}D_{0+}^{\sigma}$ defined by (3). The results presented here are sufficiently general in character and are indeed capable of being specialized appropriately to include solutions of the corresponding (known or new) fractional-order kinetic equations associated with simpler functions.

Theorem 1: *Let* $c, \mu, \nu, \rho \in \mathbb{R}^+$ *and* $0 < \sigma < 1$. *Suppose also that the general function-order* $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$, *defined by* (18), *exists. If we set*

$$\chi_0(\sigma) := \left({}^{\mathrm{RL}}I_{0+}^{1-\sigma} f \right)(0+), \tag{42}$$

then the solution of the following generalized fractional-order kinetic equation:

$$N(t) - N_0\, t^{\mu-1}\, \mathcal{E}_{\alpha,\beta}(\varphi; zt^{\nu}, s, \kappa) = -c^{\rho}\left({}^{\mathrm{RL}}D_{0+}^{\sigma} N\right)(t) \tag{43}$$

is given by

$$N(t) = N_0\, t^{\mu-1} \sum_{r=0}^{\infty} (-1)^r \left(\frac{t^{\sigma}}{c^{\rho}} \right)^{r+1}$$

$$\sum_{k=0}^{\infty} \frac{\varphi(k)\Gamma(\nu k + \mu)}{(k+\kappa)^s\, \Gamma(\alpha k + \beta)\Gamma(\nu k + (r+1)\sigma + \mu)} (zt^{\nu})^k$$

$$+ \chi_0(\sigma) \sum_{r=0}^{\infty} (-1)^r \frac{t^{\sigma(r+1)-1}}{c^{\rho r}\Gamma\big(\sigma(r+1)\big)} \quad (t > 0), \tag{44}$$

provided that the right-hand side of the solution asserted by (44) *exists.*

Proof: Since, by hypothesis, $0 < \sigma < 1$, we can make use of the Laplace transform formula (32) in the following form:

$$\mathcal{L}\{\left({}^{\mathrm{RL}}D_{0+}^{\sigma} N\right)(t) : \mathfrak{s}\} = \mathfrak{s}^{\sigma}\mathcal{N}(\mathfrak{s}) - \chi_0(\sigma) \qquad (0 < \sigma < 1), \tag{45}$$

where

$$\mathcal{N}(\mathfrak{s}) := \mathcal{L}\{N(t) : \mathfrak{s}\} = \int_0^{\infty} e^{-\mathfrak{s}t} N(t)\, dt \tag{46}$$

and $\chi_0(\sigma)$ is defined by (42).

Now, by applying the formulas (24) and (45), if we take the Laplace transforms of both sides of the fractional-order kinetic equation (43), we find that

$$\mathcal{N}(\mathfrak{s}) - \frac{N_0}{\mathfrak{s}^{\mu}} \sum_{k=0}^{\infty} \frac{\varphi(k)\Gamma(\nu k + \mu)}{(k+\kappa)^s \Gamma(\alpha k + \beta)} \left(\frac{z}{\mathfrak{s}^{\nu}}\right)^k$$

$$= -c^{\rho}\left[\mathfrak{s}^{\sigma}\mathcal{N}(\mathfrak{s}) - \left({}^{\mathrm{RL}}I_{0+}^{1-\sigma}N\right)(0+)\right]$$

$$= -c^{\rho}\mathfrak{s}^{\sigma}\mathcal{N}(\mathfrak{s}) + c^{\rho}\chi_0(\sigma), \tag{47}$$

which readily yields

$$\mathcal{N}(\mathfrak{s}) = \frac{N_0}{1 + c^{\rho}\mathfrak{s}^{\sigma}} \sum_{k=0}^{\infty} \frac{\varphi(k)\Gamma(\nu k + \mu)}{(k+\kappa)^s \Gamma(\alpha k + \beta)} \frac{z^k}{\mathfrak{s}^{\nu k + \mu}} + \frac{c^{\rho}\chi_0(\sigma)}{1 + c^{\rho}\mathfrak{s}^{\sigma}}. \tag{48}$$

In view of the following series expansion:

$$\frac{1}{1 + c^{\rho}\mathfrak{s}^{\sigma}} = \sum_{r=0}^{\infty} \frac{(-1)^r}{(c^{\rho}\mathfrak{s}^{\sigma})^{r+1}} \qquad \left(|c^{\rho}\mathfrak{s}^{\sigma}| > 1\right),$$

this last equation (48) can be rewritten as follows:

$$\mathcal{N}(\mathfrak{s}) = N_0 \sum_{r=0}^{\infty} \frac{(-1)^r}{c^{\rho(r+1)}} \sum_{k=0}^{\infty} \frac{\varphi(k)\Gamma(\nu k + \mu)}{(k+\kappa)^s \Gamma(\alpha k + \beta)} \frac{z^k}{\mathfrak{s}^{\nu k + \mu + \sigma(r+1)}}$$

$$+ \chi_0(\sigma) \sum_{r=0}^{\infty} \frac{(-1)^r}{c^{\rho r}\mathfrak{s}^{\sigma(r+1)}}. \tag{49}$$

Finally, we invert the Laplace transforms occurring in (49) by using the following well-known identity:

$$\mathcal{L}\{t^{\lambda} : \mathfrak{s}\} = \frac{\Gamma(\lambda + 1)}{\mathfrak{s}^{\lambda+1}}$$

$$\mathcal{L}^{-1}\left(\frac{1}{\mathfrak{s}^{\lambda+1}}\right) = \frac{t^{\lambda}}{\Gamma(\lambda+1)} \qquad \left(\Re(\lambda) > -1; \Re(\mathfrak{s}) > 0\right). \tag{50}$$

We are thus led to the solution (44) asserted by Theorem 1. This evidently completes the proof of Theorem 1. □

The distinct advantage of using the general function $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$, defined by (18), in the non-homogeneous term of the fractional-order kinetic equation (43) lies in its generality so that solutions of other kinetic equations involving relatively simpler non-homogeneous terms can be derived by appropriately specializing the solution (44) asserted by Theorem 1. We find it to be worthwhile to record the following relatively simpler versions of Theorem 1.

Theorem 2: *Let* $c, \mu, \nu, \rho \in \mathbb{R}^+$ *and* $0 < \sigma < 1$. *Suppose also that the general function* $\mathfrak{E}_{\alpha,\beta}(\phi; z)$, *defined by* (8), *exists. If* $\chi_0(\sigma)$ *is given by* (42), *then the solution of the following generalized fractional kinetic equation*:

$$N(t) - N_0\, t^{\mu-1}\, \mathfrak{E}_{\alpha,\beta}(\phi; zt^{\nu}) = -c^{\rho}\left({}^{\mathrm{RL}}D_{0+}^{\sigma}N\right)(t) \tag{51}$$

is given by

$$N(t) = N_0\, t^{\mu-1} \sum_{r=0}^{\infty} (-1)^r \left(\frac{t^{\sigma}}{c^{\rho}}\right)^{r+1} \cdot \sum_{k=0}^{\infty} \frac{\phi(k)\Gamma(\nu k + \mu)}{\Gamma(\alpha k + \beta)\Gamma(\nu k + (r+1)\sigma + \mu)} (zt^{\nu})^k$$

$$+ \chi_0(\sigma) \sum_{r=0}^{\infty} (-1)^r \frac{t^{\sigma(r+1)-1}}{c^{\rho r}\Gamma\big(\sigma(r+1)\big)} \quad (t > 0), \tag{52}$$

provided that the right-hand side of the solution asserted by (52) *exists*.

Proof: Our demonstration of Theorem 2 would run parallel to that of Theorem 1. Use is made; in this case, of the definition (8) and the Laplace transform formula (27). The details are being omitted here. □

Theorem 3: *For* $c, \mu, \nu, \rho \in \mathbb{R}^+$ *and* $0 < \sigma < 1$, *let the extended Hurwitz-Lerch zeta function*:

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}(z, s, \kappa),$$

defined by (17)*, exist. If* $\chi_0(\sigma)$ *is given by* (42)*, then the solution of the following generalized fractional kinetic equation*:

$$N(t) - N_0\, t^{\mu-1}\, \Phi^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(zt^{\nu}, s, \kappa) = -c^{\rho}\left({}^{\mathrm{RL}}D^{\sigma}_{0+}N\right)(t) \tag{53}$$

is given by

$$N(t) = N_0\, t^{\mu-1}\sum_{r=0}^{\infty}(-1)^r\left(\frac{t^{\sigma}}{c^{\rho}}\right)^{r+1}\frac{\Gamma(\mu)}{\Gamma(\sigma(r+1)+\mu)}$$

$$\cdot\, \Phi^{(\nu,\rho_1,\cdots,\rho_p;\nu,\sigma_1,\cdots,\sigma_q)}_{\mu,\lambda_1,\cdots,\lambda_p;\sigma(r+1)+\mu,\mu_1,\cdots,\mu_q}(zt^{\nu}, s, \kappa)$$

$$+\, \chi_0(\sigma)\sum_{r=0}^{\infty}(-1)^r\, \frac{t^{\sigma(r+1)-1}}{c^{\rho r}\Gamma\left(\sigma(r+1)\right)}\,(t > 0),$$

provided that the right-hand side of the solution asserted by (54) *exists.*

Proof: Theorem 3 can be proven, along the lines analogous to those of our demonstrations of Theorem 1 and Theorem 3, by applying the definition (17) and the Laplace transform formula (30). We choose to skip the details involved. □

4. Concluding Remarks and Observations

In our present investigation, we have established the explicit solution of some significantly general families of fractional-order kinetic equations involving the Riemann-Liouville right-sided fractional derivative operator $\left({}^{\mathrm{RL}}D^{\mu}_{0+}f\right)(x)$, which is given (for convenience) by (3) for $a = 0$, as well as a remarkably general class of functions as a part of the non-homogeneous term. Our main results (Theorem 1, Theorem 2 and Theorem 3 in this article) include, as a part of the non-homogeneous term, such general functions as $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$, $\mathfrak{E}_{\alpha,\beta}(\phi; z)$ and

$$\Phi^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z, s, \kappa),$$

which are defined by (18), (8) and (17), respectively. Each of these main results is indeed capable of yielding solutions of a significantly large number of (known or new) simpler fractional-order kinetic equations.

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SUMS INVOLVING EXTENDED GIBONACCI POLYNOMIALS REVISITED

Abstract: We explore the Jacobsthal versions of four sums involving gibonacci polynomial squares.

Keywords: Extended Gibonacci Polynomials, Fibonacci Polynomial, Lucas Polynomial. Binet-Like Formulas, Jacobsthal, and Jacobsthal-Lucas Polynomials

Mathematical Subject Classification (2020) No.: Primary 11B37, 11B39, 11C08.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial* ; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial.* They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 3].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$ [2, 3].

Gibonacci and Jacobsthal polynomials are linked by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [2, 3].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $c_n = J_n$ or j_n, $\Delta = \sqrt{x^2 + 4}$, $2\alpha = x + \Delta$, $E = \sqrt{x^2 + 1}$, $\gamma = x + E$ and $D = \sqrt{4x + 1}$, where $c_n = c_n(x)$.

2. Gibonacci Sums

We established the following four results in [4]:

Theorem 1: *Let k be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k} x}{l_{2n+2k+1} + (-1)^{n+k} x} = \frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}. \tag{1}$$

Theorem 2: *Let k be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k+1} x}{l_{2n+2k+1} - (-1)^{n+k} x} = \frac{1}{\Delta^2}\left(\frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}\right). \tag{2}$$

Theorem 3: *Let k be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k} x f_{2n+2k+2} + x^2}{\left[l_{2n+2k+1} + (-1)^{n+k} x\right]^2} = \frac{\alpha^2}{\Delta^2} - \frac{f_{k+2}^2}{l_{k+1}^2}. \tag{3}$$

Theorem 4: *Let k be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k+1} x f_{2n+2k+2} + x^2}{\left[l_{2n+2k+1} - (-1)^{n+k} x\right]^2} = \frac{1}{\Delta^4}\left(\Delta^2\alpha^2 - \frac{l_{k+2}^2}{f_{k+1}^2}\right). \tag{4}$$

Next we explore the Jacobsthal implications of these theorems.

3. Jacobsthal Consequences

Using the Jacobsthal-gibonacci relationships in Section 1, we will now find the Jacobsthal versions of equations (1) – (4). In the interest of brevity and clarity, we let A denote the fractional expression on left-hand side of the given equation and B its right-hand side, and LHS and RHS those of the desired Jacobsthal equation, respectively.

3.1 Jacobsthal Version of Equation (1)**: Proof:** Let $A = \dfrac{(-1)^{n+k} x}{l_{2n+k+1} + (-1)^{n+k} x}$.

Replacing x with $1/\sqrt{x}$, and multiplying the numerator and denominator of the resulting expression with x^{n+k}, we get

$$A = \frac{(-1)^{n+k}}{\sqrt{x}\, l_{2n+2k+1} + (-1)^{n+k}}$$

$$= \frac{(-x)^{n+k}}{x^{(2n+2k+1)/2} l_{2n+2k+1} + (-x)^{n+k}}$$

$$= \frac{(-x)^{n+k}}{j_{2n+2k+1} + (-x)^{n+k}};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{(-x)^{n+k}}{j_{2n+2k+1} + (-x)^{n+k}}, \tag{5}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next, we let $B = \frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}$. Replacing x with $1/\sqrt{x}$, then multiply each numerator and denominator of the resulting expression with $x^{(k+1)/2}$. This yield

$$B = \frac{D+1}{2D} - \frac{x^{(k+1)/2} f_{k+2}}{x^{(k+1)/2} l_{k+1}}:$$

$$\text{RHS} = \frac{D+1}{2D} - \frac{J_{k+2}}{j_{k+1}}.$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

This, combined with equation (5), yields the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{(-x)^{n+k}}{j_{2n+2k+1} + (-x)^{n+k}} = \frac{D+1}{2D} - \frac{J_{k+2}}{j_{k+1}}. \tag{6}$$

where $c_n = c_n(x)$. □

It then follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k}}{L_{2n+2k+1} + (-1)^{n+k}} = \frac{5+\sqrt{5}}{10} - \frac{F_{k+2}}{L_{k+1}} \text{ [4]};$$

$$\sum_{n=1}^{\infty} \frac{(-2)^{n+k}}{j_{2n+2k+1} + (-2)^{n+k}} = \frac{2}{3} - \frac{J_{k+2}}{j_{k+1}}.$$

Next we find the Jacobsthal consequence of equation (2).

3.2 Jacobsthal Version of Equation (2)**: Proof:** We have $A = \frac{(-1)^{n+k+1} x}{l_{2n+2k+1} - (-1)^{n+k} x}$. Replace x with $1/\sqrt{x}$, and then multiply the numerator and denominator of the resulting expression with x^{n+k}.

We then get

$$A = \frac{(-1)^{n+k+1}}{\sqrt{x}\, l_{2n+2k+1} - (-1)^{n+k}}$$

$$= \frac{-(-x)^{n+k}}{x^{(2n+2k+1)/2} l_{2n+2k+1} - (-x)^{n+k}}$$

$$= \frac{-(-x)^{n+k}}{j_{2n+2k+1} - (-x)^{n+k}};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{(-x)^{n+k}}{j_{2n+2k+1} - (-x)^{n+k}}, \qquad (7)$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next we let $B = \Delta\alpha - \frac{l_{k+2}}{f_{k+1}}$. Replacing x with $1/\sqrt{x}$, and then multiplying each numerator and denominator of the resulting expression with $x^{(n+k)/2}$, yields

$$B = \frac{x}{D^2}\left[\frac{(D+1)D}{2x} - \frac{l_{k+2}}{f_{k+1}}\right]$$

$$= \frac{1}{D^2}\left[\frac{(D+1)D}{2} - \frac{x^{(k+2)/2} l_{k+2}}{x^{k/2} f_{k+1}}\right];$$

$$\text{RHS} = \frac{1}{D^2}\left[\frac{(D+1)D}{2} - \frac{j_{k+2}}{J_{k+1}}\right],$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Combined with equation (7), this yields the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{-(-x)^{n+k}}{j_{2n+2k+1} - (-x)^{n+k}} = \frac{1}{D^2}\left[\frac{(D+1)D}{2} - \frac{j_{k+2}}{J_{k+1}}\right], \tag{8}$$

where $c_n = c_n(x)$. □

In particular, this yields

$$\sum_{n=1}^{\infty} \frac{-(-1)^{n+k}}{L_{2n+2k+1} - (-1)^{n+k}} = \frac{5+\sqrt{5}}{10} - \frac{L_{k+2}}{5F_{k+1}} \text{ [4]};$$

$$\sum_{n=1}^{\infty} \frac{-(-2)^{n+k}}{j_{2n+2k+1} - (-2)^{n+k}} = \frac{2}{3} - \frac{j_{k+2}}{9J_{k+1}}.$$

3.3 Jacobsthal Version of Equation (3)**: Proof:** Let

$A = \dfrac{2(-1)^{n+k} x f_{2n+k+2} + x^2}{\left[l_{2n+k+1} + (-1)^{n+k} x\right]^2}$. Replacing x with $1/\sqrt{x}$, and multiplying the numerator and denominator of the resulting expression with $x^{2n+2k+1}$, we get

$$A = \frac{2(-1)^{n+k}\frac{1}{\sqrt{x}} f_{2n+k+2} + \frac{1}{x}}{\left[l_{2n+2k+1} + (-1)^{n+k}\frac{1}{\sqrt{x}}\right]^2}$$

$$= \frac{2(-x)^{n+k}\left[x^{(2n+2k+1)/2} f_{2n+k+2}\right] + x^{2n+2k}}{\left[x^{(2n+2k+1)/2} l_{2n+2k+1} + (-x)^{n+k}\right]^2}$$

$$= \frac{2(-x)^{n+k} J_{2n+k+2} + x^{2n+2k}}{\left[j_{2n+2k+1} + (-x)^{n+k}\right]^2};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{2(-x)^{n+k} J_{2n+k+2} + x^{2n+2k}}{\left[j_{2n+2k+1} + (-x)^{n+k} \right]^2}, \tag{9}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Now let $B = \frac{\alpha^2}{\Delta^2} - \frac{f_{k+2}^2}{l_{k+1}^2}$. Replace x with $1/\sqrt{x}$, and multiply each numerator and denominator of the resulting expression with x^{k+1}. This yields

$$B = \frac{(D+1)^2}{4D^2} - \frac{\left[x^{(k+1)/2} f_{k+2} \right]^2}{\left[x^{(k+1)/2} l_{k+1} \right]^2};$$

$$\text{RHS} = \frac{(D+1)^2}{4D^2} - \frac{J_{k+2}^2}{J_{k+1}^2},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

This, coupled with equation (9), yields the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{2(-x)^{n+k} J_{2n+k+2} + x^{2n+2k}}{\left[j_{2n+2k+1} + (-1)^{n+k} \right]^2} = \frac{(D+1)^2}{4D^2} - \frac{J_{k+2}^2}{j_{k+1}^2}, \tag{10}$$

where $c_n = c_n(x)$. □

In particular, this yields

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k} F_{2n+k+2} + 1}{\left[L_{2n+2k+1} + (-1)^{n+k} \right]^2} = \frac{3+\sqrt{5}}{10} - \frac{F_{k+2}^2}{L_{k+1}^2} \; [4];$$

$$\sum_{n=1}^{\infty} \frac{2(-2)^{n+k} J_{2n+k+2} + 4^{n+k}}{\left[j_{2n+2k+1} + (-2)^{n+k} \right]^2} = \frac{4}{9} - \frac{J_{k+2}^2}{j_{k+1}^2}.$$

Next we find the Jacobsthal consequence of Theorem 4.

3.4 Jacobsthal Version of Equation (4)**: Proof:** We have. $A = \dfrac{2(-1)^{n+k+1} x f_{2(n+k)+2} + x^2}{\left[l_{2(n+k)+1}^2 - (-1)^{n+k} x \right]^2}$ Replace x with $1/\sqrt{x}$, and multiply the numerator and denominator of the resulting expression with $x^{2n+2k+1}$. We then get

$$A = \frac{2(-1)^{n+k+1} \frac{1}{\sqrt{x}} f_{2n+k+2} + \frac{1}{x}}{\left[l_{2n+2k+1} - (-1)^{n+k} \frac{1}{\sqrt{x}} \right]^2}$$

$$= \frac{-2(-x)^{n+k} \left[x^{(2n+2k+1)/2} f_{2n+k+2} \right] + x^{2n+2k}}{\left[x^{(2n+2k+1)/2} l_{2n+2k+1} - (-x)^{n+k} \right]^2}$$

$$= \frac{-2(-x)^{n+k} J_{2n+k+2} + x^{2n+2k}}{\left[j_{2n+2k+1} - (-x)^{n+k} \right]^2};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{-2(-x)^{n+k} J_{2n+k+2} + x^{2n+2k}}{\left[j_{2n+2k+1} - (-x)^{n+k} \right]^2}, \tag{11}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next we let $B = \dfrac{1}{\Delta^4}\left(\Delta^2 \alpha^2 - \dfrac{l_{k+2}^2}{f_{k+1}^2} \right)$. Replacing x with $1/\sqrt{x}$, and then multiplying each numerator and denominator of the resulting expression with x^{k+2} yields

$$B = \frac{x^2}{D^4}\left\{\frac{D^2(D+1)^2}{4x^2} - \frac{[x^{(k+2)/2}l_{k+2}]^2}{x^2[x^{k/2}f_{k+1}]^2}\right\};$$

$$\text{RHS} = \frac{(D+1)^2}{4D^2} - \frac{j_{k+2}^2}{D^4 J_{k+1}^2},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Combining this with equation (11) yields the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{-2(-x)^{n+k} J_{2n+k+2} + x^{2n+2k}}{\left[j_{2n+2k+1} - (-1)^{n+k}\right]^2} = \frac{(D+1)^2}{4D^2} - \frac{j_{k+2}^2}{D^4 J_{k+1}^2}, \qquad (12)$$

where $c_n = c_n(x)$. □

It follows from this equation that

$$\sum_{n=1}^{\infty} \frac{-2(-1)^{n+k} F_{2n+k+2} + 1}{\left[L_{2n+2k+1} - (-1)^{n+k}\right]^2} = \frac{3+\sqrt{5}}{10} - \frac{L_{k+2}^2}{25F_{k+1}^2} [4];$$

$$\sum_{n=1}^{\infty} \frac{(-2)^{n+k+1} J_{2n+k+2} + 4^{n+k}}{\left[j_{2n+2k+1} - (-2)^{n+k}\right]^2} = \frac{4}{9} - \frac{J_{k+2}^2}{j_{k+1}^2}.$$

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GENERALIZED NEVANLINNA ORDER (α, β) BASED SOME GROWTH PROPERTIES OF COMPOSITE ANALYTIC FUNCTIONS

Abstract: In this paper our main aim is to introduce some idea about generalized Nevanlinna order (α, β) and generalized Nevanlinna lower order (α, β) of an analytic function, where α and β are continuous non negative function in extended complex plane (∞, $+\infty$). Here we also discuss about some growth properties relating to the composition of two analytic functions on the basis of generalized Nevanlinna order (α, β) and generalized Nevanlinna lower order (α, β) as compared to the growth of their corresponding left and right factors.

Keywords and Phrases: Analytic Function, Growth, Generalized Nevanlinna Order (α, β), Generalized Nevanlinna Lower Order (α, β).

Mathematical Subject Classification (2010) No.: 30D30, 30D35.

1. Introduction, Definitions and Notations

In this paper, we assume that readers are familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions in the complex plane. Throughout this paper, by a meromorphic function f(x), we mean a meromorphic function in the complex plane. We use $T_f(r)$ and $M_f(r)$ to denote the characteristic function of a meromorphic function and the maximum modulus of an entire function. In the following, we will recall some notations about meromorphic functions and entire functions.

Let f be a meromorphic function defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ let $n_f(t, a)$ $(\bar{n}_f(t, a))$ the number of a-points (distinct a-points) of f in $|z| \leq t$, where an ∞-point is a pole of f. Also

$$N_f(r, a) = \int_0^r \frac{n_f(t, a) - n_f(0, a)}{t} \, dt + n_f(0, a) \log r$$

and

$$\bar{N}_f(r, a) = \int_0^r \frac{\bar{n}_f(t, a; f) - \bar{n}_f(0, a)}{t} \, dt + \bar{n}_f(0, a) \log r.$$

The function $N_f(r, a)$ $(\bar{N}_f(r, a))$ are called the counting function of a-points (distinct a-points) of f. In many occasions $N_f(r, \infty)$ and $\bar{N}_f(r, \infty)$ are denoted by $N_f(r)$ and $\bar{N}_f(r)$ respectively.

The function $m_f(r)$, which is called the proximity function of f is defined by

$$m_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta,$$

where

$$\begin{aligned} \log^+ x &= \log x, \text{ if } x \geq 1 \\ &= 0, \text{ if } 0 \leq x < 1. \end{aligned}$$

For $a \in \mathbb{C}$ we denote by $m(r, \frac{1}{f-a})$ the function $m_f(r, a)$ and we mean by $m_f(r, \infty)$ the function $m_f(r)$.

The function $T_f(r) = m_f(r) + N_f(r)$ is called the Nevanlinna's characteristic function of f.

If f is entire, the function $T_f(r) = m_f(r)$ is called the Nevanlinna's characteristic function of f.

Now let L be a class of continuous non negative on $(-\infty, +\infty)$ function α such that $\alpha(r) = \alpha(r_0) \geq 0$ for $x \leq x_0$ with $\alpha(r) \to +\infty$ as $x \to +\infty$. Further we assume that throughout the present paper $\alpha, \alpha_1, \alpha_2, \alpha_3, \beta \in L$.

Considering the above, Sheremeta introduced the concept of generalized order (α, β) of an entire function. For details about generalized order (α, β) one may see [6] . During the past decades, several authors made close investigations on the propertics of entire functions related to generalized order (α, β) in some different directions. For the purpose of further applications, in this paper we write the definition of the generalized Nevanlinna order (α, β) and generalized Nevanlinna lower order (α, β) of an analytic function in the following way:

Definition 1.1: (*Generalized Nevanlinna order* (α, β) *and generalized Nevanlinna lower order* (α, β)).

The generalized Nevanlinna order (α, β) *and generalized Nevanlinna lower order* (α, β) *of an analytic function f denoted by* $\rho_{(\alpha,\beta)}[f]$ *and* $\lambda_{(\alpha,\beta)}[f]$ *respectively are defined as*:

$$\begin{matrix} \rho_{(\alpha,\beta)}[f] \\ \lambda_{(\alpha,\beta)}[f] \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\exp(T_f(r)))}{\beta(r)}.$$

Now one may give the definitions of generalized Nevanlinna hyper order (α, β) and generalized Nevanlinna logarithmic order (α, β) of an analytic function f as:

Definition 1.2: (*Generalized Nevanlinna hyper order* (α, β) *and generalized Nevanlinna hyper lower order* (α, β)).

The generalized Nevanlinna hyper order (α, β) *and generalized Nevanlinna hyper lower order* (α, β) *of an analytic function f denoted by* $\bar{\rho}_{(\alpha,\beta)}[f]$ *and* $\bar{\lambda}_{(\alpha,\beta)}[f]$ *respectively are defined as*:

$$\begin{matrix} \bar{\rho}_{(\alpha,\beta)}[f] \\ \bar{\lambda}_{(\alpha,\beta)}[f] \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(T_f(r))}{\beta(r)}.$$

Definition 1.3: (*Generalized logarithmic order* (α, β) *and generalized Nevanlinna logarithmic lower order* (α, β)).

The generalized Nevanlinna logarithmic order (α, β) *and generalized Nevanlinna logarithmic lower order* (α, β) *of an analytic function* f *denoted by* $\rho_{(\alpha,\beta)}[f]$ *and* $\lambda_{(\alpha,\beta)}[f]$ *respectively are defined as*:

$$\begin{matrix} \rho^{\log}_{(\alpha,\beta)}[f] \\ \lambda^{\log}_{(\alpha,\beta)}[f] \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\exp(T_f(r)))}{\beta(\log r)}.$$

However the main aim of this paper is to investigate some growth properties of Nevanlinna's Characteristic function relating to the composition of two analytic functions on the basis of generalized Nevanlinna order (α, β), generalized Nevanlinna hyper order (α, β) and generalized Nevanlinna logarithmic order (α, β) as compared to the growth of their corresponding left and right factors.

2. Main Results

In this section we present the main results of the paper.

Theorem 2.1: *Let* f *and* g *be any two non-constant analytic functions such that* $0 < \lambda_{(\alpha_1,\beta)}[f \circ g] \le \rho_{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \lambda_{(\alpha_2,\beta)}[f] \le \rho_{(\alpha_2,\beta)}[f] < \infty$. *Then*

$$\frac{\lambda_{(\alpha_1,\beta)}[f\circ g]}{\rho_{(\alpha_2,\beta,)}[f]} \le \liminf_{x\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \le \frac{\lambda_{(\alpha_1,\beta)}[f\circ g]}{\lambda_{(\alpha_2,\beta)}[f]},$$

$$\le \limsup_{x\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \le \frac{\rho_{(\alpha_1,\beta)}[f\circ g]}{\lambda_{(\alpha_2,\beta)}[f]}.$$

Proof: *From the definitions of* $\lambda_{(\alpha_1,\beta)}[f \circ g]$ *and* $\rho_{(\alpha_2,\beta)}[f]$ *for arbitrary positive* ϵ *and for all sufficiently large values of* r *we have*

$$\alpha_1(\exp(T_{f\circ g}(r))) \ge (\lambda_{(\alpha_1,\beta)}[f\circ g] - \epsilon)(\beta(r)) \tag{1}$$

and

$$\alpha_2(\exp(T_f(r))) \leq (\rho_{(\alpha_2,\beta)}[f] + \epsilon)(\beta(r)) \tag{2}$$

Now from equation (1) *and* (2) *it follows for all sufficiently large values of r that*

$$\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \geq \frac{(\lambda_{(\alpha_1,\beta)}[f \circ g] - \epsilon)}{(\rho_{(\alpha_2,\beta,)}[f] + \epsilon)}.$$

As $\epsilon(> 0)$ *is arbitrary, we obtain that*

$$\liminf_{r\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \geq \frac{\lambda_{(\alpha_1,\beta)}[f \circ g]}{\rho_{(\alpha_2,\beta)}[f]} \tag{3}$$

which is the first part of the theorem.

Again for a sequence of values of r tending to infinity, we get that

$$\alpha_1(\exp(T_{f\circ g}(r))) \leq (\lambda_{(\alpha_1,\beta)}[f \circ g] + \epsilon)(\beta(r)) \tag{4}$$

and for all sufficiently large values of r

$$\alpha_2(\exp(T_f(r))) \geq (\lambda_{(\alpha_2,\beta)}[f] - \epsilon)(\beta(r)) \tag{5}$$

Combining equation (4) *and* (5) *we have for a sequence of values of r tending to infinity that*

$$\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \leq \frac{(\lambda_{(\alpha_1,\beta)}[f \circ g] + \epsilon)}{(\lambda_{(\alpha_2,\beta)}[f] - \epsilon)}.$$

As $\epsilon(> 0)$ *is arbitrary, we obtain that*

$$\liminf_{r\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \leq \frac{\lambda_{(\alpha_1,\beta)}[f \circ g]}{\lambda_{(\alpha_2,\beta)}[f]} \tag{6}$$

Also for a sequence of values of r tending to infinity that

$$\alpha_2(\exp(T_f(r))) \leq (\lambda_{(\alpha_2,\beta)}[f] + \epsilon)(\beta(r)) \tag{7}$$

Again from equation (1) *and* (7), *we have for a sequence of values of r tending to infinity that*

$$\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \geq \frac{(\lambda_{(\alpha_1,\beta)}[f\circ g] - \epsilon)}{(\lambda_{(\alpha_2,\beta,)}[f] + \epsilon)}\,.$$

As $\epsilon(>0)$ *is arbitrary, we obtain that*

$$\limsup_{r\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \geq \frac{\lambda_{(\alpha_1,\beta)}[f\circ g]}{\lambda_{(\alpha_2,\beta)}[f]}\,. \tag{8}$$

Again for all sufficiently large values of r, we get that

$$\alpha_2(\exp(T_{f\circ g}(r))) \leq (\rho_{(\alpha_1,\beta)}[f\circ g] + \epsilon)(\beta(r)) \tag{9}$$

Now from equation (5) *and* (9), *it follows for all sufficiently large values of r that*

$$\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \leq \frac{(\rho_{(\alpha_1,\beta)}[f\circ g] + \epsilon)}{(\lambda_{(\alpha_2,\beta)}[f] - \epsilon)}\,.$$

As $\epsilon(>0)$ *is arbitrary, we obtain that*

$$\limsup_{r\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \leq \frac{\rho_{(\alpha_1,\beta)}[f\circ g]}{\lambda_{(\alpha_2,\beta)}[f]}\,. \tag{10}$$

Thus, the theorem follows from (3), (6), (8) *and* (10).

The following theorem can be proved in the line of Theorem 2.1 and so the proof is omitted.

Theorem 2.2: *Let f and g be any two non-constant analytic functions such that* $0 < \lambda_{(\alpha_1,\beta)}[f \circ g] \le \rho_{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \lambda_{(\alpha_3,\beta)}[g] \le \rho_{(\alpha_3,\beta)}[g] < \infty$. *Then*

$$\frac{\lambda_{(\alpha_1,\beta)}[f \circ g]}{\rho_{(\alpha 3,\beta)}[g]} \le \liminf_{r\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_3(\exp(T_g(r)))} \le \frac{\lambda_{(\alpha_1,\beta)}[f \circ g]}{\lambda_{(\alpha 3,\beta)}[g]},$$

$$\le \limsup_{r\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_3(\exp(T_g(r)))} \le \frac{\rho_{(\alpha_1,\beta)}[f \circ g]}{\lambda_{(\alpha_3,\beta)}[g]}.$$

Theorem 2.3: *Let f and g be any two non-constant analytic functions such that* $0 < \rho_{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \rho_{(\alpha_2,\beta)}[f] < \infty$. *Then*

$$\liminf_{r\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \le \frac{\rho_{(\alpha_1,\beta)}[f \circ g]}{\rho_{(\alpha_2,\beta)}[f]} \le \limsup_{r\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))}.$$

Proof: *From the definitions of* $\rho_{(\alpha_2,\beta)}[f]$, *for arbitrary positive* ϵ *and for a sequence of values of* r *tending to infinity we have*

$$\alpha_2(\exp(T_f(r))) \ge (\rho_{(\alpha_2,\beta)}[f] - \epsilon)(\beta(r)) \tag{11}$$

Now from equation (9) *and* (11) *it follows for a sequence of values of* r *tending to infinity that*

$$\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \le \frac{(\rho_{(\alpha_1,\beta)}[f \circ g] + \epsilon)}{(\rho_{(\alpha_2,\beta,)}[f] - \epsilon)}.$$

As $\epsilon(> 0)$ *is arbitrary, we obtain that*

$$\liminf_{r\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \le \frac{\rho_{(\alpha_1,\beta)}[f \circ g]}{\rho_{(\alpha_2,\beta)}[f]} \tag{12}$$

Again for a sequence of values of r *tending to infinity, we get that*

$$\alpha_1(\exp(T_{f\circ g}(r))) \ge (\rho_{(\alpha_1,\beta)}[f \circ g] - \epsilon)(\beta(r)) \tag{13}$$

Combining equation (2) *and* (13), *we have for a sequence of values of* r *tending to infinity that*

$$\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \geq \frac{(\rho_{(\alpha_1,\beta)}[f\circ g]-\epsilon)}{(\rho_{(\alpha_2,\beta,)}[f]+\epsilon)}.$$

As $\epsilon(>0)$ *is arbitrary, we obtain that*

$$\limsup_{r\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \geq \frac{\rho_{(\alpha_1,\beta)}[f\circ g]}{\rho_{(\alpha_2,\beta)}[f]}. \tag{14}$$

Thus, the theorem follows from (12) *and* (14).

The following theorem can be proved in the line of Theorem 2.3 and so the proof is omitted.

Theorem 2.4: *Let* f *and* g *be any two non-constant analytic functions such that* $0<\rho_{(\alpha_1,\beta)}[f\circ g]<\infty$, $0<\rho_{(\alpha_3,\beta)}[g]<\infty$. *Then*

$$\liminf_{r\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_3(\exp(T_g(r)))} \leq \frac{\rho_{(\alpha_1,\beta)}[f\circ g]}{\rho_{(\alpha_3,\beta)}[g]} \leq \limsup_{r\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_3(\exp(T_g(r)))}.$$

The following theorem is a consequence of Theorem 2.1 and Theorem 2.3 and so the proof is omitted.

Theorem 2.5: *Let* f *and* g *be any two non-constant analytic functions such that* $0<\lambda_{(\alpha_1,\beta)}[f\circ g]\leq\rho_{(\alpha_1,\beta)}[f\circ g]<\infty$ *and* $0<\lambda_{(\alpha_2,\beta)}[f]\leq\rho_{(\alpha_2,\beta)}[f]<\infty$. *Then*

$$\liminf_{r\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \leq \min\left\{\frac{\lambda_{(\alpha_1,\beta)}[f\circ g]}{\lambda_{(\alpha_2,\beta)}[f]}, \frac{\rho_{(\alpha_1,\beta)}[f\circ g]}{\rho_{(\alpha_2,\beta)}[f]}\right\}$$

$$\leq \max\left\{\frac{\lambda_{(\alpha_1,\beta)}[f\circ g]}{\lambda_{(\alpha 2,\beta)}[f]}, \frac{\rho_{(\alpha_1,\beta)}[f\circ g]}{\rho_{(\alpha_2,\beta)}[f]}\right\} \leq \limsup_{r\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))}.$$

Analogously one may state the following theorem without its proof.

Theorem 2.6: *Let f and g be any two non-constant analytic functions such that* $0 < \lambda_{(\alpha_1,\beta)}[f \circ g] \le \rho_{(\alpha_1,\beta)}[f \circ g] < \infty$ *and* $0 < \lambda_{(\alpha_3,\beta)}[f] \le \rho_{(\alpha_3,\beta)}[f] < \infty$. *Then*

$$\liminf_{r\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_3(\exp(T_f(r)))} \le \min\left\{\frac{\lambda_{(\alpha_1,\beta)}[f\circ g]}{\lambda_{(\alpha_3,\beta)}[f]}, \frac{\rho_{(\alpha_1,\beta)}[f\circ g]}{\rho_{(\alpha_3,\beta)}[f]}\right\}$$

$$\le \max\left\{\frac{\lambda_{(\alpha_1,\beta)}[f\circ g]}{\lambda_{(\alpha_3,\beta)}[f]}, \frac{\rho_{(\alpha_1,\beta)}[f\circ g]}{\rho_{(\alpha_3,\beta)}[f]}\right\} \le \limsup_{r\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_3(\exp(T_f(r)))}.$$

We may now state the following two theorems based on Definition 1.2 and Definition 1.3 respectively.

Theorem 2.7: *Let f and g be any two non-constant analytic functions such that* $0 < \bar{\lambda}_{(\alpha_1,\beta)}[f \circ g] \le \bar{\rho}_{(\alpha_1,\beta)}[f \circ g] < \infty$ *and* $0 < \bar{\lambda}_{(\alpha_2,\beta)}[f] \le \bar{\rho}_{(\alpha_2,\beta)}[f] < \infty$. *Then*

$$\frac{\bar{\lambda}_{(\alpha_1,\beta)}[f\circ g]}{\bar{\rho}_{(\alpha_2,\beta)}[f]} \le \liminf_{r\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \le \min\left\{\frac{\bar{\lambda}_{(\alpha_1,\beta)}[f\circ g]}{\bar{\lambda}_{(\alpha_2,\beta)}[f]}, \frac{\bar{\rho}_{(\alpha_1,\beta)}[f\circ g]}{\bar{\rho}_{(\alpha_2,\beta)}[f]}\right\}$$

$$\le \max\left\{\frac{\bar{\lambda}_{(\alpha_1,\beta)}[f\circ g]}{\bar{\lambda}_{(\alpha_2,\beta)}[f]}, \frac{\bar{\rho}_{(\alpha_1,\beta)}[f\circ g]}{\bar{\rho}_{(\alpha_2,\beta)}[f]}\right\} \le \limsup_{r\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \le \frac{\bar{\rho}_{(\alpha_1,\beta)}[f\circ g]}{\bar{\lambda}_{(\alpha_2,\beta)}[f]}.$$

Theorem 2.8: *Let f and g be any two non-constant analytic functions such that* $0 < \lambda^{\log}_{(\alpha_1,\beta)}[f \circ g] \le \rho^{\log}_{(\alpha_1,\beta)}[f \circ g] < \infty$ *and* $0 < \lambda^{\log}_{(\alpha_2,\beta)}[f \circ g] \le \rho^{\log}_{(\alpha_2,\beta)}[f] < \infty$. *Then*

$$\frac{\lambda^{\log}_{(\alpha_1,\beta)}[f\circ g]}{\rho^{\log}_{(\alpha_2,\beta)}[f]} \le \liminf_{r\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \le \min\left\{\frac{\lambda^{\log}_{(\alpha_1,\beta)}[f\circ g]}{\lambda^{\log}_{(\alpha_2,\beta)}[f]}, \frac{\rho^{\log}_{(\alpha_1,\beta)}[f\circ g]}{\rho^{\log}_{(\alpha_2,\beta)}[f]}\right\}$$

$$\leq \max\left\{\frac{\lambda^{\log}_{(\alpha_1,\beta)}[f\circ g]}{\lambda^{\log}_{(\alpha_2,\beta)}[f]}, \frac{\rho^{\log}_{(\alpha_1,\beta)}[f\circ g]}{\rho^{\log}_{(\alpha_2,\beta)}[f]}\right\} \leq \limsup_{r\to\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \leq \frac{\rho^{\log}_{(\alpha_1,\beta)}[f\circ g]}{\lambda^{\log}_{(\alpha_2,\beta)}[f]}.$$

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GENERALIZED RELATIVE ORDER (α, β) AND GENERALIZED RELATIVE TYPE (α, β) BASED SOME GROWTH PROPERTIES OF MEROMORPHIC FUNCTION WITH RESPECT TO AN ENTIRE FUNCTION

Abstract: In this paper our main aim is to introduce some idea about generalized relative order (α, β) and generalized relative type (α, β) of a meromorphic function with respect to an entire function where α and β are continuous non negative function in extended complex plane (∞, $+\infty$). Here we also discuss about some growth properties relating to the composition of entire and meromorphic functions on the basis of generalized relative order (α, β) and generalized relative type (α, β) as compared to the growth of their corresponding left and right factors.

Keywords and Phrases: Meromorphic Function, Analytic Function, Growth, Generalized Relative Nevanlinna Order (α, β), Generalized Relative Nevanlinna Type (α, β).

Mathematical Subject Classification (2010) No.: 30D30, 30D35.

1. Introduction, Definitions and Notations

In this paper, we assume that readers are familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions in the complex plane. Throughout this paper, by a meromorphic function f(z), we mean a meromorphic function in the complex plane. We use $T_f(r)$ and $M_f(r)$ to denote the characteristic function of a meromorphic function and the maximum modulus of an entire function. In the following, we will recall some notations about meromorphic functions and entire functions.

Let *f* be a meromorphic function defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ let $n_f(t, a)$ $(\bar{n}_f(t, a))$ the number of a-points (distinct a-points) of f in $|z| \leq t$, where an ∞-point is a pole of *f*. Also

$$N_f(r, a) = \int_0^r \frac{n_f(t, a) - n_f(0, a)}{t} \, dt + n_f(0, a) \log r$$

and

$$\bar{N}_f(r, a) = \int_0^r \frac{\bar{n}_f(t, a; f) - \bar{n}_f(0, a)}{t} \, dt + \bar{n}_f(0, a) \log r.$$

The function $N_f(r, a)$ $(\bar{N}_f(r, a))$ are called the counting function of a-points (distinct a-points) of *f*. In many occasions $N_f(r, \infty)$ and $\bar{N}_f(r, \infty)$ are denoted by $N_f(r)$ and $\bar{N}_f(r)$ respectively.

The function $m_f(r)$, which is called the proximity function of f is defined by

$$m_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta,$$

where

$$\log+ x = \log x, \text{ if } x \geq 1$$
$$= 0, \text{ if } 0 \leq x < 1.$$

For $a \in \mathbb{C}$ we denote by $m(r, \frac{1}{f-a})$ the function $m_f(r, a)$ and we mean by $m_f(r, \infty)$ the function $m_f(r)$.

The function $T_f(r) = m_f(r) + N_f(r)$ is called the Nevanlinna's characteristic function of f.

If f is entire, the function $T_f(r) = m_f(r)$ is called the Nevanlinna's characteristic function of f.

Moreover, if f is non constant entire then $T_f(r)$ is also strictly increasing and continuous function of r. Therefore its inverse $T_f^{-1} : (T_f(0), \infty) \rightarrow (0, \infty)$ exists and is such that $\lim\limits_{s \rightarrow \infty} T_f^{-1}(s) = \infty$.

Now let L be a class of continuous non negative on $(-\infty, +\infty)$ function α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \rightarrow +\infty$ as $x \rightarrow +\infty$. For any $\alpha \in L$, we say that $\alpha \in L_1$, if $\alpha(cx) = (1 + o(1)\alpha)(x)$ as $x_0 \leq x \rightarrow +\infty$ for each $c \in (0, +\infty)$ and $\alpha \in L_2$, if $\alpha(\exp(cx)) = (1 + o(1))\, \alpha\, (\exp(x))$ as $x_0 \leq x \rightarrow +\infty$ for each $c \in (0, +\infty)$. Clearly $L_2 \subset L_1$.

Considering the above, Sheremeta introduced the concept of generalized order (α, β) of an entire function. For details about generalized order (α, β) one may see [6] . During the past decades, several authors made close investigations on the propertics of entire functions related to generalized order (α, β) in some different directions. For the purpose of further applications, in this paper we write the definition of the generalized order (α, β) of entire and meromorphic function in the following way:

Definition 1.1: *(Generalized order (α, β) and generalized lower order (α, β)).Let $\alpha \in L_2$ and $\beta \in L_1$. The Generalized order (α, β) and generalized lower order (α, β) of a **meromorphic** function f denoted by $\rho_{(\alpha,\beta)}[f]$ and $\lambda_{(\alpha,\beta)}[f]$ respectively are defined as*:

$$\begin{matrix} \rho_{(\alpha,\beta)}[f] \\ \lambda_{(\alpha,\beta)}[f] \end{matrix} = \lim_{r \rightarrow \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\exp(T_f(r)))}{\beta(r)}.$$

If f is an entire function, then

$$\begin{matrix} \rho_{(\alpha,\beta)}[f] \\ \lambda_{(\alpha,\beta)}[f] \end{matrix} = \lim_{r \rightarrow \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\exp(M_f(r)))}{\beta(r)}.$$

Using the inequality $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$, *for an entire function f, one may easily verify that*

$$\begin{matrix} \rho_{(\alpha,\beta)}[f] \\ \lambda_{(\alpha,\beta)}[f] \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(M_f(r))}{\beta(r)} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\exp(T_f(r)))}{\beta(r)}.$$

The function f is said to be of regular generalized (α, β) *growth when generalized order* (α, β) *and generalized lower order* (α, β) *of f are the same. Functions which are not of regular generalized* (α, β) *growth are said to be of irregular generalized* (α, β) *growth.*

Definition 1.2: (*Generalized type* (α, β) *and generalized lower type* (α, β)).

Let $\alpha \in L_2$ *and* $\beta \in L_1$. *The generalized type* (α, β) *and generalized lower type* (α, β) *of a meromorphic function f having finite positive generalized order* (α, β) $(0 < \rho_{(\alpha,\beta)}[f] < \infty)$, *denoted by* $\sigma_{(\alpha,\beta)}[f]$ *and* $\bar{\sigma}_{(\alpha,\beta)}[f]$ *respectively are defined as:*

$$\begin{matrix} \sigma_{(\alpha,\beta)}[f] \\ \bar{\sigma}_{(\alpha,\beta)}[f] \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\exp(\alpha(\exp(T_f(r))))}{(\exp \beta(r))^{\rho_{(\alpha,\beta)}[f]}}.$$

Analogously, to determine the relative growth of two meromorphic functions having same non zero finite generalized lower order (α, β) ,one can introduced the definition of generalized weak type (α, β) and generalized upper weak type (α, β) of a meromorphic function f having finite positive generalized lower order (α, β) in the following way:

Definition 1.3: (*Generalized upper weak type* (α, β) *and generalized weak type* (α, β)).

Let $\alpha \in L_2$ *and* $\beta \in L_1$. *The generalized upper weak type* (α, β) *and generalized weak type* (α, β) *of a meromorphic function f having finite positive* *generalized lower order* (α, β) $(0 < \lambda_{(\alpha,\beta)}[f] < \infty)$, *denoted by* $\tau_{(\alpha,\beta)}[f]$ *and* $\overline{\tau}_{(\alpha,\beta)}[f]$ *respectively are defined as*:

$$\begin{matrix} \tau_{(\alpha,\beta)}[f] \\ \overline{\tau}_{(\alpha,\beta)}[f] \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\exp(\alpha(\exp(T_f(r))))}{(\exp \beta(r))^{\lambda_{(\alpha,\beta)}[f]}}.$$

It is obvious that $0 \leq \overline{\tau}_{(\alpha,\beta)}[f] \leq \tau_{(\alpha,\beta)}[f] \leq \infty$.

Mainly the growth investigation of entire and meromorphic functions has usually been done through their maximum moduli or Nevanlinna's characteristic function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire and meromorphic function w.r.t.a new entire function, the notions of relative growth indicators will come. Now in order to make some progress in the study of relative order, one may introduce the definitions of generalized relative order (α, β) and generalized relative lower order (α, β) of a meromorphic function w.r.t. another entire function in the following way:

Definition 1.4: (*Generalized relative order* (α, β) *and generalized relative lower order* (α, β)).

Let $\alpha, \beta \in L_1$. *The Generalized relative order* (α, β) *and generalized relative lower order* (α, β) *of a meromorphic function* f *with respect to an entire function* g *denoted by* $\rho_{(\alpha,\beta)}[f]_g$ *and* $\lambda_{(\alpha,\beta)}[f]_g$ *respectively are defined as*:

$$\begin{matrix} \rho_{(\alpha,\beta)}[f]_g \\ \lambda_{(\alpha,\beta)}[f]_g \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)}.$$

The previous definitions are easily generated as particular cases, e.g. if $g = z$, *Definition* 1.4 *reduces to Definition* 1.1. *If* $\alpha(r) = \beta(r) = \log r$, *then we get the definition of relative order of meromorphic function* f *with respect to an entire function* g *introduced by Lahiri et al. and if* $g = \exp z$

and $\alpha(r) = \beta(r) = log\, r$ *then* $\rho_{(\alpha,\beta)}[f]_g = \rho(f)$. *Also if* $\alpha(r) = \log^{[p]} r$, $\beta(r) = \log^{[q]} r$ *and* $g = z$, *then Definition* 1.4 *becomes the classical one given in.*

Further if generalized relative order (α, β) *and generalized relative lower order* (α, β) *of a meromorphic function f with respect to an entire function g are the same, then f is called a function of regular generalized relative* (α, β) *growth w.r.t. g. Otherwise, f is called a irregular generalized relative* (α, β) *growth w.r.t. g.*

Now in order to refine the above growth scale, one may introduce the definitions of other growth indicators, such as generalized relative type (α, β) *and generalized relative lower type* (α, β) *of a meromorphic function f with respect to an entire function g which are as follows*:

Definition 1.5: (*Generalized relative type* (α, β) *and generalized relative lower type* (α, β)). *Let* $\alpha, \beta \in L_1$. *The Generalized relative type* (α, β) *denoted by* $\sigma_{(\alpha,\beta)}[f]_g$ *and generalized relative lower type* (α, β) *denoted by* $\bar{\sigma}_{(\alpha,\beta)}[f]_g$ *of a meromorphic function f with respect to an entire function g having non-zero finite generalized relative order* (α, β) *are defined as*:

$$\begin{matrix} \sigma_{(\alpha,\beta)}[f]_g \\ \bar{\sigma}_{(\alpha,\beta)}[f]_g \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp \beta(r))^{\rho_{(\alpha,\beta)}[f]_g}}.$$

Analogously, to determine the relative growth of a meromorphic function f having same non zero finite generalized relative lower order (α, β) *with respect to an entire function g, one can introduce generalized relative upper weak type* (α, β) *denoted by* $\tau_{(\alpha,\beta)}[f]_g$ *and generalized relative weak type* (α, β) *denoted by* $\bar{\tau}_{(\alpha,\beta)}[f]_g$ *of f with respect to g of finite positive generalized relative lower order* (α, β) *in the following way*:

Definition 1.6: (*Generalized relative upper weak type* (α, β) *and generalized relative weak type* (α, β)).

Let $\alpha, \beta \in L_1$. *The Generalized relative upper weak type* (α, β) *and generalized relative weak type* (α, β) *of a meromorphic function f with respect to an entire function g having non-zero finite generalized relative lower order* (α, β) *denoted by* $\tau_{(\alpha,\beta)}[f]_g$ *and* $\overline{\tau}_{(\alpha,\beta)}[f]_g$ *respectively are defined as*:

$$\begin{matrix} \tau_{(\alpha,\beta)}[f]_g \\ \overline{\tau}_{(\alpha,\beta)}[f]_g \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp \beta(r))^{\lambda_{(\alpha,\beta)}[f]_g}}.$$

However the main aim of this paper is to investigate some growth properties of entire and meromorphic functions using generalized relative order (α, β) and generalized relative type (α, β) of a meromorphic function with respect to an entire function which improve and extend some earlier result (*see, e.g.,*). Throughout this paper we assume that $\alpha, \beta \in L_1$, $\gamma \in L_2$ and all the growth indicators are non zero finite.

2. Main Results

In this section we preseent the main results of the paper.

Theorem 2.1: *Let f be a meromorphic function and g,h and k be non-constant entire functions such that* $0 < \lambda_{(\alpha,\beta)}[f(h)]_g \le \rho_{(\alpha,\beta)}[f(h)]_g < \infty$ *and* $0 < \lambda_{(\alpha,\beta)}[f]_k \le \rho_{(\alpha,\beta)}[f]_k < \infty$. *Then*

$$\frac{\lambda_{(\alpha,\beta)}[f(h)]_g}{\rho_{(\alpha,\beta)}[f]_k} \le \liminf_{r\to\infty} \frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \le \min\left\{\frac{\lambda_{(\alpha,\beta)}[f(h)]_g}{\lambda_{(\alpha,\beta)}[f]_k}, \frac{\rho_{(\alpha,\beta)}[f(h)]_g}{\rho_{(\alpha,\beta)}[f]_k}\right\}$$

$$\le \max\left\{\frac{\lambda_{(\alpha,\beta)}[f(h)]_g}{\lambda_{(\alpha,\beta)}[f]_k}, \frac{\rho_{(\alpha,\beta)}[f(h)]_g}{\rho_{(\alpha,\beta)}[f]_k}\right\} \le \limsup_{r\to\infty} \frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \le \frac{\rho_{(\alpha,\beta)}[f(h)]_g}{\lambda_{(\alpha,\beta)}[f]_k}.$$

Proof: *From the definitions of* $\lambda_{(\alpha,\beta)}[f(h)]_g$, $\rho_{(\alpha,\beta)}[f(h)]_g$, $\lambda_{(\alpha,\beta)}[f]_k$, $\rho_{(\alpha,\beta)}[f]_k$ *and for ar bitrary positive* ϵ *and for all sufficiently large values of* r *we have*

$$\alpha(T_g^{-1}(T_{f(h)}(r))) \geq (\lambda_{(\alpha,\beta)}[f(h)]_g - \epsilon)\beta(r), \quad (1)$$

$$\alpha(T_g^{-1}(T_{f(h)}(r))) \leq (\rho_{(\alpha,\beta)}[f(h)]_g + \epsilon)\beta(r), \quad (2)$$

$$\alpha(T_k^{-1}(T_f(r))) \geq (\lambda_{(\alpha,\beta)}[f]_k - \epsilon)\beta(r), \quad (3)$$

and

$$\alpha(T_k^{-1}(T_f(r))) \leq (\rho_{(\alpha,\beta)}[f]_k + \epsilon)\beta(r). \quad (4)$$

Again for a sequence of values of r *tending to infinity,*

$$\alpha(T_g^{-1}(T_{f(h)}(r))) \leq (\lambda_{(\alpha,\beta)}[f(h)]_g + \epsilon)\beta(r), \quad (5)$$

$$\alpha(T_g^{-1}(T_{f(h)}(r))) \geq (\rho_{(\alpha,\beta)}[f(h)]_g - \epsilon)\beta(r), \quad (6)$$

$$\alpha(T_k^{-1}(T_f(r))) \leq (\lambda_{(\alpha,\beta)}[f]_k + \epsilon)\beta(r), \quad (7)$$

and

$$\alpha(T_k^{-1}(T_f(r))) \geq (\rho_{(\alpha,\beta)}[f]_k - \epsilon)\beta(r). \quad (8)$$

Now from equation (1) *and* (4) *it follows for all sufficiently large values of* r *that*

$$\frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \geq \frac{\lambda_{(\alpha,\beta)}[f(h)]_g - \epsilon}{\rho_{(\alpha,\beta)}[f]_k + \epsilon}.$$

As $\epsilon(> 0)$ *is arbitrary, we obtain that*

$$\liminf_{r\to\infty} \frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \geq \frac{\lambda_{(\alpha,\beta)}[f(h)]_g}{\rho_{(\alpha,\beta)}[f]_k}, \quad (9)$$

which is the first part of the theorem.

Combining equation (5) *and* (3), *we have for a sequence of values of* r *tending to infinity that*

$$\frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \leq \frac{\lambda_{(\alpha,\beta)}[f(h)]_g + \epsilon}{\lambda_{(\alpha,\beta)}[f]_k - \epsilon} .$$

As $\epsilon(> 0)$ *is arbitrary, we obtain that*

$$\liminf_{r\to\infty} \frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \leq \frac{\lambda_{(\alpha,\beta)}[f(h)]_g}{\lambda_{(\alpha,\beta)}[f]_k} , \tag{10}$$

Again from equation (1) *and* (7), *we have for a sequence of values of* r *tending to infinity that*

$$\frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \geq \frac{\lambda_{(\alpha,\beta)}[f(h)]_g - \epsilon}{\lambda_{(\alpha,\beta)}[f]_k + \epsilon} .$$

As $\epsilon(> 0)$ *is arbitrary, we obtain that*

$$\limsup_{r\to\infty} \frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \geq \frac{\lambda_{(\alpha,\beta)}[f(h)]_g}{\lambda_{(\alpha,\beta)}[f]_k} , \tag{11}$$

Now from equation (3) *and* (2), *it follows for all sufficiently large values of* r *that*

$$\frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \leq \frac{\rho_{(\alpha,\beta)}[f(h)]_g + \epsilon}{\lambda_{(\alpha,\beta)}[f]_k - \epsilon} .$$

As $\epsilon(> 0)$ *is arbitrary, we obtain that*

$$\limsup_{r\to\infty} \frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \le \frac{\rho_{(\alpha,\beta)}[f(h)]_g}{\lambda_{(\alpha,\beta)}[f]_k}, \tag{12}$$

which is the last part of the theorem.

Again from equation (2) *and* (8), *we have for a sequence of values of* r *tending to infinity that*

$$\frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \le \frac{\rho_{(\alpha,\beta)}[f(h)]_g + \epsilon}{\rho_{(\alpha,\beta)}[f]_k - \epsilon}.$$

As $\epsilon(>0)$ *is arbitrary, we obtain that*

$$\liminf_{r\to\infty} \frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \le \frac{\rho_{(\alpha,\beta)}[f(h)]_g}{\rho_{(\alpha,\beta)}[f]_k}, \tag{13}$$

Combining equation (4) *and* (6), *we have for a sequence of values of* r *tending to infinity that*

$$\frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \ge \frac{\rho_{(\alpha,\beta)}[f(h)]_g - \epsilon}{\rho_{(\alpha,\beta)}[f]_k + \epsilon}.$$

As $\epsilon(>0)$ *is arbitrary, we obtain that*

$$\limsup_{r\to\infty} \frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \ge \frac{\rho_{(\alpha,\beta)}[f(h)]_g}{\rho_{(\alpha,\beta)}[f]_k}, \tag{14}$$

So, the second part of the theorem follows from equation (10) *and* (13), *the third part is trivial and fourth part follows from* (11) *and* (14).

Thus, the theorem follows from (9), (10), (11), (12), (13) *and* (14).

Remark 2.1: *If we take* "$0 < \lambda_{(\alpha,\beta)}[h]_k \leq \rho_{(\alpha,\beta)}[h]_k < \infty$" *instead of* "$0 < \lambda_{(\alpha,\beta)}[f]_k \leq \rho_{(\alpha,\beta)}[f]_k < \infty$" *and other conditions remain same, the conclusion of Theorem* (2.1) *remains true with* "$\lambda_{(\alpha,\beta)}[f]_k$", "$\rho_{(\alpha,\beta)}[f]_k$" *and* "$\alpha(T_k^{-1}(T_f(r)))$" *replaced by* "$\lambda_{(\alpha,\beta)}[h]_k$", "$\rho_{(\alpha,\beta)}[h]_k$" *and* "$\alpha(T_k^{-1}(T_h(r)))$" *respectively in the denominator.*

Theorem 2.2: *Let f be a meromorphic function and g,h and k be non-constant entire functions such that* $0 < \lambda_{(\alpha,\beta)}[f]_k \leq \rho_{(\alpha,\beta)}[f]_k < \infty$ *and* $\lambda_{(\alpha,\beta)}[f(h)]_g = \infty$. *Then*

$$\lim_{r\to\infty} \frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} = \infty.$$

Proof: *If possible, let the conclusion of the theorem does not hold. Then we can find a constant* $\delta > 0$ *such that for a sequence of values of r tending to infinity*

$$\alpha(T_g^{-1}(T_{f(h)}(r))) \leq \delta.\, \alpha(T_k^{-1}(T_f(r))). \tag{15}$$

Again from the definition of $\rho_{(\alpha,\beta)}[f]_k$, *it follows that for all sufficiently large values of r*

$$\alpha(T_k^{-1}(T_f(r))) \leq (\rho_{(\alpha,\beta)}[f]_k + \epsilon)\beta(r). \tag{16}$$

From (15) *and* (16), *for a sequence of values of r tending to infty,*

$$\alpha(T_g^{-1}(T_{f(h)}(r))) \leq \delta\, (\rho_{(\alpha,\beta)}[f]_k + \epsilon)\beta(r).$$

$$i.e., \frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\beta(r)} \leq \delta\, (\rho_{(\alpha,\beta)}[f]_k + \epsilon)$$

$$i.e., \liminf_{r\to\infty} \frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\beta(r)} \le \lambda_{(\alpha,\beta)}[f(h)]_k < \infty.$$

This is a contradiction. Hence, the theorem holds.

Remark 2.2: *If we take* "$0 < \lambda_{(\alpha,\beta)}[h]_k \le \rho_{(\alpha,\beta)}[h]_k < \infty$" *instead of* "$0 < \lambda_{(\alpha,\beta)}[f]_k \le \rho_{(\alpha,\beta)}[f]_k < \infty$" *and other conditions remain same, the conclusion of Theorem* (2.2) *remains true with* "$\alpha(T_k^{-1}(T_f(r)))$" *replaced by* "$\alpha(T_k^{-1}(T_h(r)))$" *in the denominator.*

Remark 2.3: *Theorem* (2.2) *and Remark* (2.2) *are also valid with "limit superior" instead of "limit" if* "$\lambda_{(\alpha,\beta)}[f(h)]_g = \infty$" *is replaced by* "$\rho_{(\alpha,\beta)}[f(h)]_g = \infty$" *and the other conditions remain the same.*

Theorem 2.3: *Let f be a meromorphic function and g,h and k be non-constant entire functions such that* $0 < \bar{\sigma}_{(\alpha,\beta)}[f(h)]_g \le \sigma_{(\alpha,\beta)}[f(h)]_g < \infty$ *and* $0 < \bar{\sigma}_{(\alpha,\beta)}[f]_k \le \sigma_{(\alpha,\beta)}[f]_k < \infty$. *Then*

$$\frac{\bar{\sigma}_{(\alpha,\beta)}[f(h)]_g}{\sigma_{(\alpha,\beta)}[f]_k} \le \liminf_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \le \min\left\{\frac{\bar{\sigma}_{(\alpha,\beta)}[f(h)]_g}{\bar{\sigma}_{(\alpha,\beta)}[f]_k}, \frac{\sigma_{(\alpha,\beta)}[f(h)]_g}{\sigma_{(\alpha,\beta)}[f]_k}\right\}$$

$$\le \max\left\{\frac{\bar{\sigma}_{(\alpha,\beta)}[f(h)]_g}{\bar{\sigma}_{(\alpha,\beta)}[f]_k}, \frac{\sigma_{(\alpha,\beta)}[f(h)]_g}{\sigma_{(\alpha,\beta)}[f]_k}\right\} \le \limsup_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \le \frac{\sigma_{(\alpha,\beta)}[f(h)]_g}{\bar{\sigma}_{(\alpha,\beta)}[f]_k}.$$

Proof: *From the definitions of* $\sigma_{(\alpha,\beta)}[f(h)]_g$, $\bar{\sigma}_{(\alpha,\beta)}[f(h)]_g$, $\sigma_{(\alpha,\beta)}[f]_k$, $\bar{\sigma}_{(\alpha,\beta)}[f]_k$ *and for arbitrary positive* ϵ *and for all sufficiently large values of* r *we have*

$$\exp(\alpha(T_g^{-1}(T_{f(h)}(r)))) \ge (\bar{\sigma}_{(\alpha,\beta)}[f(h)]_g - \epsilon)(\exp \beta(r))^{\rho(\alpha,\beta)[f(h)]_g}, \quad (17)$$

$$\exp(\alpha(T_k^{-1}(T_f(r)))) \le (\sigma_{(\alpha,\beta)}[f]_k + \epsilon)(\exp \beta(r))^{\rho(\alpha,\beta)[f]_k}, \quad (18)$$

$$\exp(\alpha(T_k^{-1}(T_f(r)))) \geq (\bar{\sigma}_{(\alpha,\beta)}[f]_k - \epsilon)(\exp \beta(r))^{\rho(\alpha,\beta)[f]_k}, \quad (19)$$

and

$$\exp(\alpha(T_g^{-1}(T_{f(h)}(r)))) \leq (\sigma_{(\alpha,\beta)}[f(h)]_g + \epsilon(\exp \beta(r))^{\rho(\alpha,\beta)[f(h)]_g}. \quad (20)$$

Again for a sequence of values of r tending to infinity,

$$\exp(\alpha(T_g^{-1}(T_{f(h)}(r)))) \leq (\bar{\sigma}_{(\alpha,\beta)}[f(h)]_g + \epsilon)(\exp \beta(r))^{\rho(\alpha,\beta)[f(h)]_g}, \quad (21)$$

$$\exp(\alpha(T_k^{-1}(T_f(r)))) \geq (\sigma_{(\alpha,\beta)}[f]_k - \epsilon)(\exp \beta(r))^{\rho(\alpha,\beta)[f]_k}, \quad (22)$$

$$\exp(\alpha(T_k^{-1}(T_f(r)))) \leq (\bar{\sigma}_{(\alpha,\beta)}[f]_k + \epsilon)(\exp \beta(r))^{\rho(\alpha,\beta)[f]_k}, \quad (23)$$

and

$$\exp(\alpha(T_g^{-1}(T_{f(h)}(r)))) \geq (\sigma_{(\alpha,\beta)}[f(h)]_g - \epsilon(\exp \beta(r))^{\rho(\alpha,\beta)[f(h)]_g}. \quad (24)$$

Now from equation (17) *and* (18) *and the condition* $\rho_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[f]_k$ *it follows for all sufficiently large values of r that*

$$\frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \geq \frac{\bar{\sigma}_{(\alpha,\beta)}[f(h)]_g - \epsilon}{\sigma_{(\alpha,\beta)}[f]_k + \epsilon}.$$

As $\epsilon(> 0)$ *is arbitrary, we obtain that*

$$\liminf_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \geq \frac{\bar{\sigma}_{(\alpha,\beta)}[f(h)]_g}{\sigma_{(\alpha,\beta)}[f]_k}, \quad (25)$$

which is the first part of the theorem.

Combining equation (21) *and* (19) *and the condition* $\rho_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[f]_k$, *we have for a sequence of values of r tending to infinity that*

$$\frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \leq \frac{\bar{\sigma}_{(\alpha,\beta)}[f(h)]_g + \epsilon}{\bar{\sigma}_{(\alpha,\beta)}[f]_k - \epsilon}.$$

As $\epsilon(>0)$ *is arbitrary, we obtain that*

$$\liminf_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \leq \frac{\bar{\sigma}_{(\alpha,\beta)}[f(h)]_g}{\bar{\sigma}_{(\alpha,\beta)}[f]_k}, \tag{26}$$

Again from equation (17) *and* (23) *and the condition* $\rho_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[f]_k$, *we have for a sequence of values of* r *tending to infinity that*

$$\frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \geq \frac{\bar{\sigma}_{(\alpha,\beta)}[f(h)]_g - \epsilon}{\bar{\sigma}_{(\alpha,\beta)}[f]_k + \epsilon}.$$

As $\epsilon(>0)$ *is arbitrary, we obtain that*

$$\limsup_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \geq \frac{\bar{\sigma}_{(\alpha,\beta)}[f(h)]_g}{\bar{\sigma}_{(\alpha,\beta)}[f]_k}, \tag{27}$$

Now from equation (19) *and* (20) *and the condition* $\rho_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[f]_k$, *it follows for all sufficiently large values of* r *that*

$$\frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \leq \frac{\sigma_{(\alpha,\beta)}[f(h)]_g + \epsilon}{\bar{\sigma}_{(\alpha,\beta)}[f]_k - \epsilon}.$$

As $s(>0)$*is arbitrary, we obtain that*

$$\limsup_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \leq \frac{\sigma_{(\alpha,\beta)}[f(h)]_g}{\sigma_{(\alpha,\beta)}[f]_k}, \tag{28}$$

which is the last part of the theorem.

Again from equation (20) *and* (22) *and the condition* $\rho_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[f]_k$ *, we have for a sequence of values of* r *tending to infinity that*

$$\frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \leq \frac{\sigma_{(\alpha,\beta)}[f(h)]_g + \epsilon}{\sigma_{(\alpha,\beta)}[f]_k - \epsilon} .$$

As $\epsilon(> 0)$ *is arbitrary, we obtain that*

$$\liminf_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \leq \frac{\sigma_{(\alpha,\beta)}[f(h)]_g}{\sigma_{(\alpha,\beta)}[f]_k} , \quad (29)$$

Combining equation (18) *and* (24) *and the condition* $\rho_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[f]_k$ *, we have for a sequence of values of* r *tending to infinity that*

$$\frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \geq \frac{\sigma_{(\alpha,\beta)}[f(h)]_g - \epsilon}{\bar{\sigma}_{(\alpha,\beta)}[f]_k + \epsilon} .$$

As $\epsilon(> 0)$ *is arbitrary, we obtain that*

$$\limsup_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r)))}) \geq \frac{\sigma_{(\alpha,\beta)}[f(h)]_g}{\sigma_{(\alpha,\beta)}[f]_k} , \quad (30)$$

Thus, the theorem follows from (25), (26), (27), (28), (29) *and* (30)*.*

Remark 2.4: *If we take* "$0 < \bar{\sigma}_{(\alpha,\beta)}[h]_k \leq \sigma_{(\alpha,\beta)}[h]_k < \infty$" *and* $\rho_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[h]_k$ *, instead of* "$0 < \bar{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$" *and* $\rho_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[f]_k$ *other conditions remain same, the conclusion of Theorem* (2.3) *remains true with* "$\sigma_{(\alpha,\beta)}[f]_k$", "$\bar{\sigma}_{(\alpha,\beta)}[f]_k$"

and "$exp(\alpha(T_k^{-1}(T_f(r))))$" *replaced by* "$\sigma_{(\alpha,\beta)}[f]_k$", "$\bar{\sigma}_{(\alpha,\beta)}[h]_k$" "$\exp(\alpha(T_k^{-1}(T_h(r))))$" *respectively in the denominator.*

Remark 2.5: *If we take* "$0 < \bar{\tau}_{(\alpha,\beta)}[f]_k \leq \tau_{(\alpha,\beta)}[f]_k < \infty$" *and* $\rho_{(\alpha,\beta)}[f(h)]_g = \lambda_{(\alpha,\beta)}[h]_k$, *instead of* "$0 < \bar{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$" *and* $\rho_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[f]_k$ *other conditions remain same, the conclusion of Theorem* (2.3) *remains true with* "$\sigma_{(\alpha,\beta)}[f]_k$", "$\bar{\sigma}_{(\alpha,\beta)}[f]_k$" *and* "$\exp(\alpha(T_k^{-1}(T_f(r))))$" *replaced by* "$\tau_{(\alpha,\beta)}[f]_k$", "$\bar{\tau}_{(\alpha,\beta)}[h]_k$" "$\exp(\alpha(T_k^{-1}(T_h(r))))$" *respectively in the denominator.*

Remark 2.6: *If we take* "$0 < \bar{\tau}_{(\alpha,\beta)}[h]_k \leq \tau_{(\alpha,\beta)}[h]_k < \infty$" *and* $\rho_{(\alpha,\beta)}[f(h)]_g = \lambda_{(\alpha,\beta)}[h]_k$, *instead of* "$0 < \bar{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$" *and* $\rho_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[f]_k$ *other conditions remain same, the conclusion of Theorem* (2.3) *remains true with* "$\sigma_{(\alpha,\beta)}[f]_k$", "$\bar{\sigma}_{(\alpha,\beta)}[f]_k$" *and* "$\exp(\alpha(T_k^{-1}(T_f(r))))$" *replaced by* "$\tau_{(\alpha,\beta)}[h]_k$", "$\bar{\tau}_{(\alpha,\beta)}[h]_k$" "$\exp(\alpha(T_k^{-1}(T_h(r))))$" *respectively in the denominator.*

Now in the line of Theorem (2.3), one can easily prove the following theorem using the notion of Generalized relative upper weak type (α, β) and generalized relative weak type (α, β) and therefore the proof is omitted.

Theorem 2.4: *Let f be a meromorphic function and g, h and k be non-constant entire functions such that* $0 < \bar{\tau}_{(\alpha,\beta)}[f(h)]_g \leq \tau_{(\alpha,\beta)}[f(h)]_g < \infty$ *and* $0 < \bar{\tau}_{(\alpha,\beta)}[f]_k \leq \tau_{(\alpha,\beta)}[f]_k < \infty$ *and* $\lambda_{(\alpha,\beta)}[f(h)]_g = \lambda_{(\alpha,\beta)}[f]_k$. *Then*

$$\frac{\bar{\tau}_{(\alpha,\beta)}[f(h)]_g}{\tau_{(\alpha,\beta)}[f]_k} \leq \liminf_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \leq \min\left\{\frac{\bar{\tau}_{(\alpha,\beta)}[f(h)]_g}{\bar{\tau}_{(\alpha,\beta)}[f]_k}, \frac{\tau_{(\alpha,\beta)}[f(h)]_g}{\tau_{(\alpha,\beta)}[f]_k}\right\}$$

$$\leq \max\left\{\frac{\overline{\tau}_{(\alpha,\beta)}[f(h)]_g}{\overline{\tau}_{(\alpha,\beta)}[f]_k}, \frac{\tau_{(\alpha,\beta)}[f(h)]_g}{\tau_{(\alpha,\beta)}[f]_k}\right\} \leq \limsup_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r)))}{\exp(\alpha(T_k^{-1}(T_f(r)))} \leq \frac{\tau_{(\alpha,\beta)}[f(h)]_g}{\overline{\tau}_{(\alpha,\beta)}[f]_k}.$$

Remark 2.7: *If we take* "$0 < \overline{\tau}_{(\alpha,\beta)}[h]_k \leq \tau_{(\alpha,\beta)}[h]_k < \infty$" *and* $\lambda_{(\alpha,\beta)}[f(h)]_g = \lambda_{(\alpha,\beta)}[h]_k$, *instead of* "$0 < \overline{\tau}_{(\alpha,\beta)}[f]_k \leq \tau_{(\alpha,\beta)}[f]_k < \infty$" *and* $\lambda_{(\alpha,\beta)}[f(h)]_g = \lambda_{(\alpha,\beta)}[f]_k$ *and other conditions remain same, the conclusion of Theorem* (2.4) *remains true with* "$\tau_{(\alpha,\beta)}[f]_k$", "$\overline{\tau}_{(\alpha,\beta)}[f]_k$" *and* "$\exp(\alpha(T_k^{-1}(T_f(r))))$" *replaced by* "$\tau_{(\alpha,\beta)}[f]_k$", "$\tau_{(\alpha,\beta)}[h]_k$" "$\exp(\alpha(T_k^{-1}(T_h(r))))$" *respectively* in the denominator.

Remark 2.8: *If we take* "$0 < \overline{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$" *and* $\lambda_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[f]_k$, *instead of* "$0 < \overline{\tau}_{(\alpha,\beta)}[f]_k \leq \tau_{(\alpha,\beta)}[f]_k < \infty$" *and* $\lambda_{(\alpha,\beta)}[f(h)]_g = \lambda_{(\alpha,\beta)}[f]_k$ *and other conditions remain same, the conclusion of Theorem* (2.4) *remains true with* "$\tau_{(\alpha,\beta)}[f]_k$", "$\overline{\tau}_{(\alpha,\beta)}[f]_k$" *replaced by* "$\sigma_{(\alpha,\beta)}[f]_k$", "$\overline{\sigma}_{(\alpha,\beta)}[h]_k$" *respectively in the denominator.*

Remark 2.9: *If we take* "$0 < \overline{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$" *and* $\lambda_{(\alpha,\beta)}[f(h)]_g = \lambda_{(\alpha,\beta)}[h]_k$, *instead of* "$0 < \overline{\tau}_{(\alpha,\beta)}[f]_k \leq \tau_{(\alpha,\beta)}[f]_k < \infty$" *and* $\lambda_{(\alpha,\beta)}[f(h)]_g = \lambda_{(\alpha,\beta)}[f]_k$ *and other conditions remain same, the conclusion of Theorem* (2.4) *remains true with* "$\tau_{(\alpha,\beta)}[f]_k$", "$\overline{\tau}_{(\alpha,\beta)}[f]_k$" *and* "$\exp(\alpha(T_k^{-1}(T_f(r))))$" *replaced by* "$\sigma_{(\alpha,\beta)}[h]_k$", "$\overline{\sigma}_{(\alpha,\beta)}[h]_k$" "$\exp(\alpha(T_k^{-1}(T_h(r))))$" *respectively in the denominator.*

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MULTIPLE WEAK SOLUTIONS FOR A CLASS OF SIXTH ORDER BOUNDARY VALUE PROBLEM: NEW FINDINGS AND APPLICATIONS

Abstract: In this paper, we study the existence of two and infinitely many weak solutions for a class from sixth-order differential equation, in which modelling for describing the behaviour of phase fronts in materials that are undergoing a transition between the liquid and solid. The results are proved by using some critical point theorems.

Key words and phrases: Multiple solutions, Sixth-Order Equations, Variational Methods, Critical Point.

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1. Introduction

In this paper, we study the following problem:

$$\begin{cases} -u^{(vi)}(x) + Au^{(iv)}(x) - Bu''(x) + Cu(x) = \lambda f(x, u(x)), \; x \in [0,1] \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0 \end{cases} \tag{1.1}$$

where $A, B, C \in \mathbb{R}$ and parameter $\lambda > 0$, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Study sixth-order differential equations was first started by studying the following problem:

$$\frac{\partial u}{\partial x} = \frac{\partial^6 u}{\partial^6 x} + A\,\frac{\partial^4 u}{\partial^4 x} + B\,\frac{\partial^2 u}{\partial^2 x} + f(x, u). \tag{1.2}$$

One of the most important applications problem (1.2) is the model that describes the phase fronts behavior in the materials.

In recent years, BVPs for sixth-order ordinary differential equations have been studied extensively, see [1, 2, 3, 5, 7, 10, 11] and the references therein in [5], Gyulov *et al*. obtained the existence and multiplicity the solutions for the following boundary value problem

$$\begin{cases} -u^{(vi)}(x) + Au^{(iv)}(x) - Bu''(x) + Cu(x) = \lambda f(x, u(x)), 0 < x < L, \\ u(0) = u(L) = u''(0) = u''(L) = u^{(iv)}(0) = u^{(iv)}(L) = 0 \end{cases} \tag{1.3}$$

where $A, B, C \in \mathbb{R}$ and $f : [0, L] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In [7], Li obtained the existence and multiplicity of positive solutions for the following problem

$$\begin{cases} -u^{(vi)}(x) + A(x)u^{(iv)}(x) + B(x)u''(x) + C(x)u(x) + f(x, u(x)) = 0,\ x \in [0, 1] \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0 \end{cases} \tag{1.4}$$

where $A(x), B(x), C(x) \in C([0, 1])$ and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous. Bonanno *et al*. in [1], applied critical point theory and variational methods to prove the existence and multiplicity of solutions for the following problem

$$-u^{(vi)}(x) + Au^{(iv)}(x) - Bu''(x) + Cu(x) = \lambda f(x, u(x)), x \in [a, b] \tag{1.5}$$

where $\lambda > 0$, A, B and C are given real constants, $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a function. Recently, Bonanno and Livrea in [2] obtained infinitely many solutions for the nonlinear sixth-order problem (1.1). They used the variational methods and an oscillating behavior on the nonlinear term to demonstrate the existence of these solutions.

In this article, we discuss the existence of two and infinitely many weak solutions for the problem (1.1), under suitable conditions on the nonlinear term. We also present examples to illustrate the results.

2. Preliminaries and Basic Notation

In this section, we first introduce some notations and some necessary definitions. Set

$$X = \{u \in H^3(0,1) \cap H_0^1(0,1) \mid u''(0) = u''(1) = 0\}. \tag{2.1}$$

X is the Sobolev space, consider the inner product

$$< u, v > := \int_0^1 (u'''(x)v'''(x) + u''(x)v''(x) + u'(x)v'(x) + u(x)v(x))dx,$$

which induces the norm

$$\|u\| := (|u'''|_2^2 + |u''|_2^2 + |u'|_2^2 + |u|_2^2)^{\frac{1}{2}} \tag{2.2}$$

Proposition 2.1: (see [2]) *If* $k = \frac{1}{\pi^2}$, *for every* $u \in X$, *we have*

$$\|u^{(i)}\|_2^2 \le k^{j-i} \|u^{(j)}\|_2^2 \; i = 0, 1, 2 \;\; j = 1, 2, 3 \text{ with } i < j, \tag{2.3}$$

where $\|u\|_2 := (\int_0^1 |u(x)|^2 \, dx)^{\frac{1}{2}}$ *is norm in* $L^2(0,1)$.

We introduce the function $N : X \to \mathbb{R}$ as follows,

$$N(u) := \|u'''\|_2^2 + A \|u''\|_2^2 + B \|u'\|_2^2 + C \|u\|_2^2, \quad \forall \; u \in X,$$

where A, B and C are real constants and satisfied in the following condition:

$$(H) \max\{-Ak, -Ak - Bk^2, -Ak - Bk^2 - Ck^3\} < 1.$$

Lemma 2.2: (see [2]) *Put*

$$\|u\|_X = \sqrt{N(u)}. \; u \in X,$$

and assume that the condition (H) *holds. Then,* $\|u\|_X$ *is a norm equivalent to the norm defined in* (2.2) *and* $(X, \|.\|_X)$ *with following inner product*

$$< u, v >:= \int_0^1 (u'''(x)v'''(x) + Au''(x)v''(x) + Bu'(x)v'(x) + Cu(x)v(x))\, dx \ ,$$

is a Hilbert space.

Clearly $(X, \|.\|_X) \to (C^0(0,1).\ \|.\|_\infty)$ and the embedding is compact.

Lemma 2.3: (*see* [2]) *Assume that* (H) *holds, one has*

$$\|u\|_\infty \leq \frac{k}{2\sqrt{\delta}} \|u\|_X, \ \ \forall \ u \in X \, .$$

for every $u \in X$, *and* $\delta > 0$ *is given in* [2].

We say that a function $u \in X$ is called a weak solution of the problem (1.1) if

$$\int_0^1 (u'''(x)v'''(x) + Au''(x)v''(x) + Bu'(x)v'(x) + Cu(x)v(x))dx$$

$$-\lambda \int_0^1 f(x, u(x))v(x)dx = 0, \ \forall \ v \in X \, .$$

Consider $I_\lambda : X \to R$ defined by

$$I_\lambda(u) = \frac{1}{2} \|u\|_X^2 - \lambda \int_0^1 F(x, u(x))dx \, , \qquad (2.4)$$

where

$$F(x,t) = \int_0^t f(x\xi)d\xi \quad \text{for all} \ \ (x,t) \in [0,1] \times \mathbb{R} \, .$$

We observe that $I_\lambda \in C^1(X, \mathbb{R})$ for any $v \in X$,

$$I_{\lambda}'(u)v = \int_0^1 (u'''(x)v'''(x) + Au''(x)v''(x) + Bu'(x)v'(x) + Cu(x)v(x))dx \qquad (2.5)$$

$$-\lambda\int_0^1 f(x, u(x))v(x)dx = 0,\ \forall\ v \in X\ . \qquad (2.6)$$

Thus, the solutions of Problem (1.1) are the critical point of I_{λ}.

Definition 2.4: *Assume X be a real reflexive Banach space. We say J satisfies Palais-Smale condition (denotes by PS condition for short), if any sequence $\{u_k\} \subset X$ for which $\{J(u_k)\}$ is bounded and $J'(u_k) \to 0$ as $k \to 0$ possesses a convergent subsequence.*

The proofs of our results are based the following theorems.

Theorem 2.5: [9, Theorem 4.10] *Let $I_{\lambda} \in C^1(X, \mathbb{R})$, and I_{λ} satisfies the Palais-Smale condition. Assume that there exist $u_0, u_1 \in X$ and a bounded neighborhood Ω of u_0 satisfying $u_1 \notin \Omega$ and*

$$\inf_{\nu\in\partial\Omega} I_{\lambda}(\nu) > \max\{\varphi(u_0), I_{\lambda}(u_1)\},$$

then there exists a critical point u of I_{λ}, i.e., $I_{\lambda}'(u) = 0$ with

$$I_{\lambda}(u) > \max\{I_{\lambda}(u_0), I_{\lambda}(u_1)\}.$$

Theorem 2.6: [15, Theorem 38] *For the functional $I_{\lambda} : M \subseteq X \to [-\infty, +\infty]$ with $M \neq \varnothing$, $\min_{u\in M} I_{\lambda}(u) = \alpha$ has a solution in case the following conditions hold:*

(i_1) *X is a real reflexive Banach space,*

(i_2) *M is bounded and weak sequentially closed,*

(i_3) *I_λ is weak sequentially lower semi-continuous on M, i.e., by definition, for each sequence $\{u_n\}$ in M such that $u_n \rightharpoonup u$ as $n \to \infty$, we have $I_\lambda(u) \le \lim_{n\to\infty} \inf I_\lambda(u_n)$ holds.*

Theorem 2.7: *Consider X be an infinite-dimensional Banach space and $I_\lambda \in C^1(X, \mathbb{R})$ be an even functional which satisfies the (PS)-condition and $I_\lambda(0) = 0$. If $X = V \oplus E$ where V is finite dimensional and I_λ satisfies the conditions*

(j_1) *there are constants $\rho, \alpha > 0$ such that*

$$I_\lambda(u) \ge \alpha, \text{ if } \|u\| = \rho, \ u \in E,$$

(j_2) *for each finite-dimensional subspace $E_n \subseteq X$ there is D_n such that*

$$I_\lambda(u) \ge 0, \text{ if } \|u\| \ge D_n, \ u \in E_n,$$

then I_λ possesses an unbounded sequence of critical points.

We refer the reader to the paper [12, 13] in which Theorem 2.7 was successfully employed to some boundary value problems. To read more on the applications of Theorem 2.5 and 2.6, we refer to the papers [4, 6, 14].

3. Main Results

We utilize the following assumptions throughout this paper.

(f_0) there exist a constants $\nu > 2$ and $T > 0$ such that

$$0 < \nu F(x,t) \le t f(x,t), \text{ for } |t| > T \text{ and } x \in [0,1].$$

(f_1) $f : V \times \mathbb{R} \to \mathbb{R}$ continues and there exists constant $L > 0$ such that

$$\left|f(x,t)\right| \le c(1+ |t|^{q-1}), \quad \text{for } |t| \le L \text{ and } x \in [0,1]$$

where $q > 2$.

(f_2) $\lim_{t\to 0} \frac{f(x,t)}{t^2} = 0, \quad$ for $x \in [0,1]$ uniformly.

We use the following lemmas to prove our main results.

Lemma 3.1: *Assume that the condition* (f_0) *holds. Then* $I_\lambda(u)$ *satisfies the* (PS)*-condition.*

Proof: Assume that $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $\{I_\lambda(u_n)\}_{n\in\mathbb{N}}$ is bounded and $I_\lambda^{'}(u_n) \to 0$ as $n \to +\infty$. Then, there exists a positive constant c_0 such that $|I_\lambda(u_n)| \le c_0$ and $|I_\lambda^{'}(u_n)| \le c_0$ for all $n \in N$. Therefore, from the definition of $I_\lambda^{'}$ and (A_1), we have

$$\begin{aligned} c_0 + c_1 \|u_n\|_X &\ge \nu I_\lambda(u_n) - I_\lambda^{'}(u_n)(u_n) \\ &\ge \left(\frac{\nu}{2} - 1\right) \|u_n\|_X^2 + \lambda \int_0^1 (f(x, u_n(x))u_n(x) - \nu F(x, u_n(x)))\, dx \\ &\ge \left(\frac{\nu}{2} - 1\right) \|u_n\|_X^2 . \end{aligned} \tag{3.1}$$

therefore for some $c_1 > 0$, since $\nu > 2$ this implies that $\{u_n\}$ is bounded. Since X is Banach space and $\{u_n\}$ is bounded, there exist a subsequence, still denoted by $\{u_n\}$ and a function u in X such that

$$u_n \rightharpoonup u, \text{ in } X, \text{ and } u_n \to u \text{ in } C_1([0,1]). \tag{3.2}$$

By definition $I_\lambda^{'}(u)$, we get

$$< I_{\lambda}'(u_n), u_n - u > = \int_0^1 \Big(u_n'''(x)(u_n'''(x) - u'''(x)) + Au_n''(x)(u_n''(x) - u''(x))$$

$$+ Bu_n'(x)(u_n'(x) - u(x)) + Cu_n(x)(u_n(x) - u(x))\Big)\, dx$$

$$- \lambda \int_0^1 f(x, u_n(x))(u_n(x) - u(x))\, dx.$$

Therefore, we have

$$< I_{\lambda}'(u_n) - I_{\lambda}'(u_n), u_n - u > =$$

$$\int_0^1 \Big(u_n'''(x)(u_n'''(x) - u'''(x)) + Au_n''(x)(u_n''(x) - u''(x))$$

$$+ Bu_n'(x)(u_n'(x) - u(x)) + Cu_n(x)(u_n(x) - u(x))\Big)\, dx$$

$$- \lambda \int_0^1 f(x, u_n(x))(u_n(x) - u(x))\, dx$$

$$-\Big(\int_0^1 \Big(u'''(x)(u_n'''(x) - u'''(x)) + Au''(x)(u_n''(x) - u''(x))$$

$$+ Bu'(x)(u_n'(x) - u'(x)) + Cu(x)(u_n(x) - u(x))\Big)\, dx$$

$$- \lambda \int_0^1 f(x, u(x))(u_n(x) - u(x))\, dx\Big)$$

$$= \int_0^1 \Big((u_n'''(x) - u'''(x))^2) + A(u_n''(x) - u''(x))^2$$

$$+ Bu_n'(x) - u'(x))^2 + Cu_n(x) - u(x))^2 \Big)\, dx$$

$$- \lambda \int_0^1 (f(x, u_n(x)) - f(x, u(x)))(u_n(x) - u(x))\, dx$$

$$\geq \|u_n - u\|_X^2 - \lambda \int_0^1 (f(x, u_n(x)) - f(x, u(x)))(u_n(x) - u(x))\, dx.$$

From the continuity of f, we get

$$\int_0^1 \Big((u_n'''(x) - u'''(x))^2) + A(u_n''(x) - u''(x))^2 + B(u_n'(x) - u'(x))^2$$

$$+ C(u_n(x) - u(x))^2 \Big) dx \to 0, \ n \to \infty, \quad (3.3)$$

and

$$\lambda \int_0^1 (f(x, u_n(x)) - f(x, u(x)))(u_n(x) - u(x))\, dx \to 0, n \to \infty, \quad (3.4)$$

from (3.1) , (3.2), we can conclude

$$< I_\lambda'(u_n) - I_\lambda'(u_n), u_n - u > \to 0.$$

Therefore by (3.3) to (3.4), we have

$$\|u_n - u\|_X^2 \to 0.$$

Thus, the sequence u_n converges strongly to u in X. Therefore, I_λ satisfies the (PS)-condition. □

Theorem 3.2: *Assume that the assumptions* (f_0), (f_1) *and* (f_2) *hold. Then*:

if $f(x, t) \geq 0$ *for all* $(x, t) \in [0, 1] \times \mathbb{R}$, *the problem* (1.1) *has at least two weak solutions*.

Proof: Clearly, $I_\lambda(0) = 0$. From the Lemma 3.1, we can see I_λ satisfies the (PS)-condition. We will show that there exists $R > 0$ such that the functional I_λ has a local minimum $u_0 \in BR = \{u \in X; \|u\|\ X < R\}$. Assume that $\{u_n\} \subseteq \bar{B}_R$ and $u_n \rightharpoonup u$, as $n \to \infty$ by Mazur Theorem [8], there exists sequence $\{v_n\}$ of convex combinations such that

$$v_n = \sum_{j=1}^{n} a_{n_j} u_j, \sum_{j=1}^{n} a_{n_j} = 1, \qquad a_{n_j} \geq 0, j \in N$$

and $v_n \to u$ in X. Clearly, $\bar{B}_R$ is a closed convex set, therefore $\{v_n\} \subseteq \bar{B}_R$ and $u \in \bar{B}_R$. Since, I_λ is weakly sequentially lower semi-continuous on $\bar{B}_R$ and X is a reflexive Banach space, so, from Theorem 2.6 we can know that I_λ has a local minimum $u_0 \in \bar{B}_R$. Assume that $I_\lambda(u_0) = \min_{u\in\bar{B}_R} I_\lambda(u)$, we will show that $I_\lambda(u_0) < \inf_{u\in\partial\bar{B}_R} I_\lambda(u)$. By ($f_1$) and ($f_2$), there exits $\alpha > 0$ such that

$$F(x,t) \le \alpha|t|^2 + c|t|^q \;, \tag{3.5}$$

let $\alpha > 0$ be small enough such that $\alpha < \dfrac{2\delta^2}{\lambda k^2}$, therefore

$$\begin{aligned}
I_\lambda(u) &\ge \frac{1}{2}\,\|u\|_x^2 - \lambda\alpha \int_0^1 |u(x)|^2\, d\mu - \lambda c \int_0^1 |u(x)|^q\, d\mu \\
&\ge \frac{1}{2}\,\|u\|_X^2 - \lambda\alpha\,\|u\|_\infty^2 - \lambda c \int_0^1 |u(x)|^q\, d\mu \\
&\ge \frac{1}{2}\,\|u\|_X^2 - \lambda\alpha\,\frac{k^2}{4\delta^2}\,\|u\|_X^2 - \lambda c\,\|u\|_X^q \\
&\ge \left(\frac{1}{2} - \lambda\alpha\,\frac{k^2}{4\delta^2}\right)\|u\|_X^2 - \lambda c\left(\frac{k}{2\delta}\right)^q \|u\|_X^q
\end{aligned}$$

Since, $q > 2$, when $\|u\|_X < 1$ there exist $r > 0$, such that $I_\lambda(u) \ge r > 0$ for every $\|u\|_X = r$, we choosing $R = r$, thus, $I_\lambda(u) > 0 = I_\lambda(0) \ge I_\lambda(u_0)$ for $u \in \partial B_R$. Hence, $u_0 \in B_R$ and $I'_\lambda(u_0) = 0$. Since, u_0 is a minimum point of I_λ on X, there exists $R > 0$ sufficiently large such that $I_\lambda(u_0) \le 0 < \inf_{u\in\partial B_R} I_\lambda(u)$, where $B_R = \{u \in X; \|u\|_X < R\}$. Now, we will show that there exists u_1 with $\|u_1\|_X > R$ such that $I_\lambda(u_1) < \inf_{u\in\partial B_R} I_\lambda(u)$. Letting $k_1 \in X$ and $u_1 = \tau k_1$, $\tau > 0$ and $\|k_1\|_X = 1$. From (f_0) we get there exist constants a_1, $a_2 > 0$ such that $F(x,t) \ge a_1\,\|t\|^\nu - a_2$ for all $x \in [0,1]$. Thus,

$$I_\lambda(u_1) = \frac{1}{2}\,\|\tau k_1\|_X^2 - \lambda \int_0^1 F(x, \tau k_1)(x))\,d\mu$$

$$\leq \frac{1}{2}\,\tau^2\,\|k_1\|_X^2 - \lambda \tau^\nu a_1 \int_0^1 |k_1(x)|^\nu\,d\mu + \lambda a_2\,.$$

Since, $\nu > 2$, there exists sufficiently large $\tau > R > 0$ so $I_\lambda(\tau k_1) < 0$. Hence, $\max\{I_\lambda(u_0), I_\lambda(u_1)\} < \inf_{\partial B_R} I_\lambda(u)$. Then, Theorem 2.5 gives the critical point u^*. Therefore, u_0 and u^* are two critical points of I_λ, which are two weak solutions of the Problem (1.1). □

Theorem 3.3: *Assume that the assumption* (f_0) *and the following condition hold*:

(f_4) *there exists* $q > 2$ *such that*

$$\left|f(x,t)\right| \leq c\,|t|^{q-1}, \quad \text{as} \quad |t| \to 0\,.$$

Then Problem (1.1) *has infinitely many pairs of weak solutions.*

Proof: We want to apply Theorem 2.7. By lemma 3.1 the functional I_λ defined in (2.4) satisfies the (*PS*)-condition.

Now, we need to assumptions (j_1) and (j_2) of Theorem 2.7. By condition (f_4) and Lemma 2.3, we have

$$I_\lambda(u) = \frac{1}{2}\,\|u\|_X^2 - \lambda \int_0^1 F(x, u(x))\,dx$$

$$\geq \frac{1}{2}\,\|u\|_X^2 - c\int_0^1 |u|^q\,dx$$

$$\geq \frac{1}{2}\,\|u\|_X^2 - c\,\|u\|_\infty^q$$

$$\geq \frac{1}{2}\,\|u\|_X^2 - c\,\frac{k^q}{2^q \delta^{\frac{q}{2}}}\,\|u\|_X^q\,.$$

Since, $q > 2$, we have that for $\|u\| = \rho$ sufficiently small $I_\lambda(u) \geq \alpha > 0$. Let E_n be a n-dimensional subspace of X, by the equivalence of any two norms on finite-dimensional space, by integrating the condition (f_0) there exist constants $a_1, a_2 > 0$ such that

$$F(t, x) \geq a_1 |x|^\nu - a_2$$

for all $t \in [0,1]$ and $x \in \mathbb{R}$. Now, for any $u \in E_n$, we have

$$I_\lambda(u) \leq \frac{1}{2} \|u\|_X^2 - \lambda \int_0^1 F(x, u)(x))\, dx$$

$$\leq \frac{1}{2} \|u\|_X^2 - \lambda \int_0^1 a_1 |u(x)|^\nu\, dx + \lambda a_2.$$

Since, $\nu > 2$, there exists sufficiently large $D_n > 0$, such that $I_\lambda(u) \leq 0$ for $\|u\| \geq R_n$. Therefore, all the assumptions of Theorem 2.7 are established. Thus, the functional I_λ possesses an unbounded sequence of critical points on X. And it proves the result. □

Now, illustrate our results by the following examples.

Example 3.4: *Consider the following problem*

$$\begin{cases} -u^{(vi)}(x) + 2u^{(iv)}(x) + u''(x) - 3u = \lambda f(x, u(x)),\ x \in [0,1] \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases} \tag{3.6}$$

where $A = 2$, $B = -1$, $C = -3$. *Set* $f(x,t) = t^4$ *for all* $x \in [0,1]$, *thus, we have* $F(x,t) = \frac{1}{5} t^5$ *for all* $x \in [0,1]$. *Hence,* $\lim_{\xi \to +\infty} \frac{\xi f(x,\xi)}{F(x,\xi)} = 5 < \infty$, *so, by choosing* $\nu = 5 > 2$ *and* $T = 1$ *the condition* (f_0) *satisfied. Also* $f(x,t) \geq 0$ *for all* $x \in [0,1]$, *and* $\lim_{t \to 0} \frac{f(x,t)}{t^2} = 0$. *By selecting* $q = 5$ *and* $L = 1$, *we get*

$|f(x,t)| \le c(1+|t|^4)$ *for* $|t| \le 1$ *and for same* $c > 0$. *Therefore, all the assumptions in* Theorem 3.2 *are fulfilled. Hence, the* Problem (3.6) *has at least two weak solutions.*

Example 3.5: *Consider the following problem*

$$\begin{cases} -u^{(vi)}(x) + u^{(iv)}(x) - u''(x) + 3u = \lambda f(x, u(x)), \ x \in [0,1] \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases} \tag{3.7}$$

where $A = 1$, $B = 1$, $C = 3$. *Put*

$$f(x,t) = \begin{cases} 8t^5, & t \le 1 \\ 8t^7, & t > 1, \end{cases}$$

for all $x \in [0,1]$. *We have*

$$F(x,t) = \begin{cases} \frac{4}{3} t^6, & t \le 1 \\ t^8 + \frac{1}{3}, & t > 1, \end{cases}$$

for all $x \in [0,1]$. *Hence,* $\lim_{\xi \to +\infty} \frac{\xi f(t,\xi)}{F(t,\xi)} = 8 < \infty$ *and* $\lim_{\xi \to -\infty} \frac{\xi f(t,\xi)}{F(t,\xi)} = 6 < \infty$, *thus by choosing* $\nu = 8 > 2$ *and* $T = 1$ *the condition* (f_0) *satisfied. Also by choosing* $q = 6$ *and* $c = 8$, *we have* $|f(x,t)| \le 7|t|^5$ *for* $|t| \le 1$, *therefore, the condition* (f_1) *satisfied. We clearly see that all the assumptions present in* Theorem 3.2 *are established. Thus, the* Problem (3.7) *has infinitely many pairs of weak solution.*

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