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Thomas Koshy

SUMS INVOLVING EXTENDED GIBONACCI POLYNOMIALS REVISITED

Abstract: We explore the Jacobsthal versions of four sums involving gibonacci polynomial squares.

Keywords: Extended Gibonacci Polynomials, Fibonacci Polynomial, Lucas Polynomial. Binet-Like Formulas, Jacobsthal, and Jacobsthal-Lucas Polynomials

Mathematical Subject Classification (2020) No.: Primary 11B37, 11B39, 11C08.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $_{zn}(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 3].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$ [2, 3]. Gibonacci and Jacobsthal polynomials are linked by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [2, 3].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $c_n = J_n$ or j_n, $\Delta = \sqrt{x^2 + 4}$, $2\alpha = x + \Delta$, $E = \sqrt{x^2 + 1}$, $\gamma = x + E$ and $D = \sqrt{4x + 1}$, where $c_n = c_n(x)$.

2. Gibonacci Sums

We established the following four results in [4]:

Theorem 1: *Let k be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k} x}{l_{2n+2k+1} + (-1)^{n+k} x} = \frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}. \tag{1}$$

Theorem 2: *Let k be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k+1} x}{l_{2n+2k+1} - (-1)^{n+k} x} = \frac{1}{\Delta^2}\left(\frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}\right). \tag{2}$$

Theorem 3: *Let k be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k} x f_{2n+2k+2} + x^2}{\left[l_{2n+2k+1} + (-1)^{n+k} x\right]^2} = \frac{\alpha^2}{\Delta^2} - \frac{f_{k+2}^2}{l_{k+1}^2}. \tag{3}$$

Theorem 4: *Let k be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k+1} x f_{2n+2k+2} + x^2}{\left[l_{2n+2k+1} - (-1)^{n+k} x\right]^2} = \frac{1}{\Delta^4}\left(\Delta^2\alpha^2 - \frac{l_{k+2}^2}{f_{k+1}^2}\right). \tag{4}$$

Next we explore the Jacobsthal implications of these theorems.

3. Jacobsthal Consequences

Using the Jacobsthal-gibonacci relationships in Section 1, we will now find the Jacobsthal versions of equations (1) – (4). In the interest of brevity and clarity, we let A denote the fractional expression on left-hand side of the given equation and B its right-hand side, and LHS and RHS those of the desired Jacobsthal equation, respectively.

3.1 Jacobsthal Version of Equation (1)**: Proof:** Let $A = \dfrac{(-1)^{n+k} x}{l_{2n+k+1} + (-1)^{n+k} x}$.

Replacing x with $1/\sqrt{x}$, and multiplying the numerator and denominator of the resulting expression with x^{n+k}, we get

$$A = \frac{(-1)^{n+k}}{\sqrt{x}\, l_{2n+k+1} + (-1)^{n+k}}$$

$$= \frac{(-x)^{n+k}}{x^{(2n+2k+1)/2} l_{2n+2k+1} + (-x)^{n+k}}$$

$$= \frac{(-x)^{n+k}}{j_{2n+k+1} + (-x)^{n+k}};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{(-x)^{n+k}}{j_{2n+2k+1} + (-x)^{n+k}}, \tag{5}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next, we let $B = \frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}$. Replacing x with $1/\sqrt{x}$, then multiply each numerator and denominator of the resulting expression with $x^{(k+1)/2}$. This yield

$$B = \frac{D+1}{2D} - \frac{x^{(k+1)/2} f_{k+2}}{x^{(k+1)/2} l_{k+1}}:$$

$$\text{RHS} = \frac{D+1}{2D} - \frac{J_{k+2}}{j_{k+1}}.$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

This, combined with equation (5), yields the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{(-x)^{n+k}}{j_{2n+2k+1} + (-x)^{n+k}} = \frac{D+1}{2D} - \frac{J_{k+2}}{j_{k+1}}. \tag{6}$$

where $c_n = c_n(x)$. □

It then follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k}}{L_{2n+2k+1} + (-1)^{n+k}} = \frac{5+\sqrt{5}}{10} - \frac{F_{k+2}}{L_{k+1}} [4];$$

$$\sum_{n=1}^{\infty} \frac{(-2)^{n+k}}{j_{2n+2k+1} + (-2)^{n+k}} = \frac{2}{3} - \frac{J_{k+2}}{j_{k+1}}.$$

Next we find the Jacobsthal consequence of equation (2).

3.2 Jacobsthal Version of Equation (2)**: Proof:** We have $A = \frac{(-1)^{n+k+1} x}{l_{2n+k+1} + (-1)^{n+k} x}$. Replace x with $1/\sqrt{x}$, and then multiply the numerator and denominator of the resulting expression with x^{n+k}.

We then get

$$A = \frac{(-1)^{n+k+1}}{\sqrt{x}\, l_{2n+k+1} - (-1)^{n+k}}$$

$$= \frac{(-x)^{n+k}}{x^{(2n+2k+1)/2} l_{2n+2k+1} - (-x)^{n+k}}$$

$$= \frac{-(-x)^{n+k}}{j_{2n+k+1} - (-x)^{n+k}};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{(-x)^{n+k}}{j_{2n+2k+1} - (-x)^{n+k}}, \tag{7}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next we let $B = \Delta\alpha - \frac{l_{k+2}}{f_{k+1}}$. Replacing x with $1/\sqrt{x}$, and then multiplying each numerator and denominator of the resulting expression with $x^{(n+k)/2}$, yields

$$B = \frac{x}{D^2}\left[\frac{(D+1)D}{2x} - \frac{l_{k+2}}{f_{k+1}}\right]$$

$$= \frac{1}{D^2}\left[\frac{(D+1)D}{2} - \frac{x^{(k+2)/2} l_{k+2}}{x^{k/2} f_{k+1}}\right];$$

$$\text{RHS} = \frac{1}{D^2}\left[\frac{(D+1)D}{2} - \frac{j_{k+2}}{J_{k+1}}\right],$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Combined with equation (7), this yields the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{-(-x)^{n+k}}{j_{2n+2k+1} - (-x)^{n+k}} = \frac{1}{D^2}\left[\frac{(D+1)D}{2} - \frac{j_{k+2}}{J_{k+1}}\right], \tag{8}$$

where $c_n = c_n(x)$. □

In particular, this yields

$$\sum_{n=1}^{\infty} \frac{-(-1)^{n+k}}{L_{2n+2k+1} - (-1)^{n+k}} = \frac{5+\sqrt{5}}{10} - \frac{L_{k+2}}{5F_{k+1}} \text{ [4]};$$

$$\sum_{n=1}^{\infty} \frac{-(-2)^{n+k}}{j_{2n+2k+1} - (-2)^{n+k}} = \frac{2}{3} - \frac{j_{k+2}}{9J_{k+1}}.$$

3.3 Jacobsthal Version of Equation (3)**: Proof:** Let

$A = \dfrac{2(-1)^{n+k} x f_{2n+k+2} + x^2}{\left[l_{2n+k+1} + (-1)^{n+k} x\right]^2}$. Replacing x with $1/\sqrt{x}$, and multiplying the numerator and denominator of the resulting expression with $x^{2n+2k+1}$, we get

$$A = \frac{2(-1)^{n+k}\frac{1}{\sqrt{x}} f_{2n+k+2} + \frac{1}{x}}{\left[l_{2n+k+1} + (-1)^{n+k}\frac{1}{\sqrt{x}}\right]^2}$$

$$= \frac{2(-x)^{n+k}\left[x^{(2n+2k+1)/2} f_{2n+k+2}\right] + x^{2n+2k}}{\left[x^{(2n+2k+1)/2} l_{2n+2k+1} + (-x)^{n+k}\right]^2}$$

$$= \frac{2(-x)^{n+k} J_{2n+2k+2} + x^{2n+2k}}{\left[j_{2n+2k+1} + (-x)^{n+k}\right]^2};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{2(-x)^{n+k} J_{2n+k+2} + x^{2n+2k}}{\left[j_{2n+2k+1} + (-x)^{n+k} \right]^2}, \tag{9}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Now let $B = \frac{\alpha^2}{\Delta^2} - \frac{f_{k+2}^2}{l_{k+1}^2}$. Replace x with $1/\sqrt{x}$, and multiply each numerator and denominator of the resulting expression with x^{k+1}. This yields

$$B = \frac{(D+1)^2}{4D^2} - \frac{\left[x^{(k+1)/2} f_{k+2} \right]^2}{\left[x^{(k+1)/2} l_{k+1} \right]^2};$$

$$\text{RHS} = \frac{(D+1)^2}{4D^2} - \frac{J_{k+2}^2}{J_{k+1}^2},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

This, coupled with equation (9), yields the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{2(-x)^{n+k} J_{2n+k+2} + x^{2n+2k}}{\left[j_{2n+2k+1} + (-1)^{n+k} \right]^2} = \frac{(D+1)^2}{4D^2} - \frac{J_{k+2}^2}{j_{k+1}^2}. \tag{10}$$

where $c_n = c_n(x)$. □

In particular, this yields

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k} F_{2n+k+2} + 1}{\left[L_{2n+2k+1} + (-1)^{n+k} \right]^2} = \frac{3+\sqrt{5}}{10} - \frac{F_{k+2}^2}{L_{k+1}^2} [4];$$

$$\sum_{n=1}^{\infty} \frac{2(-2)^{n+k} J_{2n+k+2} + 4^{n+k}}{\left[j_{2n+2k+1} + (-2)^{n+k} \right]^2} = \frac{4}{9} - \frac{J_{k+2}^2}{j_{k+1}^2}.$$

Next we find the Jacobsthal consequence of Theorem 4.

3.4. Jacobsthal Version of Equation (4)**: Proof:** We have $A = \frac{2(-1)^{n+k+1} x f_{2n+k+2} + x^2}{\left[l_{2n+k+1}^2 - (-1)^{n+k} x \right]^2}$. Replace x with $1/\sqrt{x}$, and multiply the numerator and denominator of the resulting expression with $x^{2n+2k+1}$. We then get

$$\begin{aligned} A &= \frac{2(-1)^{n+k+1} \frac{1}{\sqrt{x}} f_{2n+k+2} + \frac{1}{x}}{\left[l_{2n+2k+1} - (-1)^{n+k} \frac{1}{\sqrt{x}} \right]^2} \\ &= \frac{-2(-x)^{n+k} \left[x^{(2n+2k+1)/2} f_{2n+k+2} \right] + x^{2n+2k}}{\left[x^{(2n+2k+1)/2} l_{2n+2k+1} - (-x)^{n+k} \right]^2} \\ &= \frac{-2(-x)^{n+k} J_{2n+k+2} + x^{2n+2k}}{\left[j_{2n+2k+1} - (-x)^{n+k} \right]^2}; \end{aligned}$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{-2(-x)^{n+k} J_{2n+k+2} + x^{2n+2k}}{\left[j_{2n+2k+1} - (-x)^{n+k} \right]^2}, \tag{11}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next we let $B = \frac{1}{\Delta^4} \left(\Delta^2 \alpha^2 - \frac{l_{k+2}^2}{f_{k+1}^2} \right)$. Replacing x with $1/\sqrt{x}$, and then multiplying each numerator and denominator of the resulting expression with x^{k+2} yields

$$B = \frac{x^2}{D^4}\left\{\frac{D^2(D+1)^2}{4x^2} - \frac{[x^{(k+2)/2} l_{k+2}]^2}{x^2[x^{k/2} f_{k+1}]^2}\right\};$$

$$\text{RHS} = \frac{(D+1)^2}{4D^2} - \frac{j_{k+2}^2}{D^4 J_{k+1}^2},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Combining this with equation (11) yields the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{-2(-x)^{n+k} J_{2n+k+2} + x^{2n+2k}}{\left[j_{2n+2k+1} - (-1)^{n+k} \right]^2} = \frac{(D+1)^2}{4D^2} - \frac{j_{k+2}^2}{D^4 J_{k+1}^2}, \tag{12}$$

where $c_n = c_n(x)$. □

It follows from this equation that

$$\sum_{n=1}^{\infty} \frac{-2(-1)^{n+k} F_{2n+k+2} + 1}{\left[L_{2n+2k+1} - (-1)^{n+k} \right]^2} = \frac{3+\sqrt{5}}{10} - \frac{L_{k+2}^2}{25F_{k+1}^2} [4]:$$

$$\sum_{n=1}^{\infty} \frac{(-2)^{n+k+1} J_{2n+k+2} + 4^{n+k}}{\left[j_{2n+2k+1} - (-2)^{n+k} \right]^2} = \frac{4}{9} - \frac{J_{k+2}^2}{j_{k+1}^2}.$$

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SOME COMMON FIXED POINT THEOREMS IN COMPLEX VALUED METRIC SPACES FOR THREE SELF-MAPPINGS UNDER RATIONAL TYPE CONTRACTION

Abstract: In this paper we prove the common fixed point theorems in complex valued metric space for three self mappings. Banach's fixed point theorem plays a major role in fixed point theory. Because of its usefulness we have used Banach's contraction principle for the improvement and generalization of our result. Our result generalizes some recent results in the literature due to Azam *et al*. (2011) and Sintunavarat and Kumam (2012) by using the idea of two weakly compatible mappings. Azam *et al*. made a generalization by introducing a complex valued metric space using some contractive type conditions whereas Sintunavarat and Kumam generalized their result by replacing the constants of contraction by some control functions. Some concepts have been taken from the results obtained by Choi *et al*. (2017) and Jebril *et al*. (2019) to improve our results. The results of Choi *et al*. and Jebril *et al*. in bicoplex valued metric spaces are very effective tools to improve our result. Also an example is given to illustrate our obtained results.

Keywords and phrases: Complex Valued Metric Space, Common Fixed Point, Point of Coincidence, Weakly Compatible Mapping.

Mathematics Subject Classification (2020) No.: 47H10, 54H25.

1. Introduction, Definitions and Notations

The concept of fixed point theorem was first introduced by Poincare and Miranda [15] in 1883. After that Brouwer [4] published his famous fixed point theorem in 1912. The theorem states that **"**If B is a closed unit ball in R^n and if $T : B \to B$ is continuous then T has a fixed point in B". In 1992 Banach [5] proved his famous fixed point theorem in which contraction principle is the main tools. Banach's fixed point theorem plays a major role in fixed point theory. It has applications in many branches of mathematics. Because of its usefulness, a lot of articles have been dedicated to the improvement and generalization of that result. Most of these generalizations have been made by considering different contractive type conditions in different spaces $\{cf.$[6] [19]$\}$. In 2011, Azam *et al*. [3] made a generalization by introducing a complex valued metric space using some contractive type conditions. Very recently, Sintunavarat *et al*. [20] generalized this result by replacing the constants of contraction by some control functions. The purpose of this work is to obtain a common fixed point result for three self mappings in complex valued metric spaces which generalizes the results of [2] and [20].

We write regular complex number as $z = x + iy$ where x and y are real numbers and $i^2 = -1$. Let $\mathbb{C}_1$ be the set of complex numbers and z_1 and $z_2 \in \mathbb{C}_1$. Define a partial order relation $\precsim$ on $\mathbb{C}_1$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$.

Thus, $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(iii) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,

(iv) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e. one of (ii), (iii) and (iv) is satisfied and $(z_1) \prec (z_2)$ if only (iv) is satisfied.

Taking this into account some fundamental properties of the partial order $\precsim$ on $\mathbb{C}_1$ as follows:

(1) If $0 \precsim z_1 \precsim z_2$ then $|z_1| < |z_2|$;

(2) If $z_1 \precsim z_2$, $z_2 \precsim z_3$ then $z_1 \precsim z_3$ and

(3) If $z_1 \precsim z_2$ and $0 < \lambda < 1$ is a real number then $\lambda z_1 \precsim z_2$.

Azam *et al*. defined the complex valued metric space in the following way:

Definition 1.1 [3]: *Let X be a nonempty set where as $\mathbb{C}_1$ be the set of complex numbers. Suppose that the mapping $d : X \times X \to \mathbb{C}_1$ satisfies the following conditions*:

$(d_1) : 0 \precsim d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

$(d_2) : d(x, y) = d(y, x)$ *for all* $x, y \in X$;

$(d_3) : d(x, y) \precsim d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Definition 1.2 [3]: *Let (X, d) be a complex valued metric space, let $\{x_n\}$ be a sequence in X and $x \in X$.*

(i) *If for every $c \in \mathbb{C}_1$ with $0 \prec c$, there exists an $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$ then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n \to \infty} x_n = x$ as $n \to \infty$.*

(ii) *If for every $c \in \mathbb{C}_1$ with $0 \prec c$, there exists an $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_m) \prec c$, where $m \in \mathbb{N}$ then $\{x_n\}$ is said to be Cauchy sequence.*

(iii) *If every Cauchy sequence in* X *is convergent then* (X, d) *is said to be a complete complex valued metric space.*

Definition 1.3 [1]: *Let* T *and* S *be self mappings of a set* X. *If* $w = Tx = Sx$ *for some* x *in* X *then* x *is called a coincidence point of* T *and* S *and* w *is called a point of coincidence of* T *and* S.

Definition 1.4 [11]: *Let* S *and* T *be self mappings of a nonempty set* X. *The mappitngs* S *and* T *are weakly compatible if* $STx = TSx$ *whenever* $Sx = Tx$.

The following theorem is established by Sintunavarat and Kumam in 2012.

Theorem 1.1 [20]: *Let* (X, d) *be a complete complex valued metric space and* $S, T : X \to X$.

If there exist mappings $\Lambda, \Xi : X \to [0, 1)$ *such that for all* $x, y \in X$:

(i) $\Lambda(Sx) \leq \Lambda(x)$ *and* $\Xi(Sx) \leq \Xi(x)$;

(ii) $\Lambda(Tx) \leq \Lambda(x)$ *and* $\Xi(Tx) \leq \Xi(x)$;

(iii) $(\Lambda + \Xi)(x) < 1$;

(iv) $d(Sx, Ty) \precsim \Lambda(x)d(x, y) + \dfrac{\Xi(x)d(x, Sx)d(y, Ty)}{1 + d(x, y)}$.

Then S *and* T *have a unique common fixed point.*

2. Lemma

In this section we introduce some lemmas which are the main tools for our results.

Lemma 2.1 [3]: *Let* (X, d) *be a complex valued metric space and let* $\{x_n\}$ *be a sequence in X. Then* $\{x_n\}$ *converges to* x *if and only if* $|\, d\left(x_n, x\right)| \to 0$ *as* $n \to \infty$.

Lemma 2.2 [3]: *Let* (X, d) *be a complex valued metric space and let* $\{x_n\}$ *be a sequence in X. Then* $\{x_n\}$ *is a Cauchy sequence if and only if* $|\, d\left(x_n, x_m\right)| \to 0$ *as* $n \to \infty$, *where* $m \in \mathbb{N}$.

Lemma 2.3 [2]: *Let* X *be a nonempty set and the mappings* $S, T, f : X \to X$ *have a unique point of coincidence* v *in X. If* (S, f) *and* (T, f) *are weakly compatible then* S, T *and* f *have a unique common fixed point.*

3. Main Results

In this section we prove a common fixed point theorem and give an example to justify our obtained results.

Theorem 3.1 *Let* (X, d) *be a complex valued metric space and* $f, S, T : X \to X$. *Suppose there exist mappings* $\alpha_1, \alpha_2, \alpha_3 : X \to \left[0, 1\right)$ *such that for all* $x, y \in X$:

(i) $\alpha_i(Sx) \leq \alpha_i(fx)$, $\alpha_i(Tx) \leq \alpha_i(fx)$, $i = 1, 2, 3$;

(ii) $\alpha_1(fx) + \alpha_2(fx) + 2\alpha_3(fx) < 1$ *and*

(iii) $$d(Sx, Ty) \precsim \alpha_1(fx)d(fx, fy) + \alpha_2(fx)\frac{d(fx, Sx)d(fx, Ty)}{1 + d(fx, fy)}$$

$$+ \alpha 3(fx)\frac{d(fy, Sx) + d(fx, Ty)}{1 + d(fx, fy)} .$$

If $S(X) \cup T(X) \subseteq f(X)$ and $f(X)$ is complete then f, S and T have a unique point of coincidence. Moreover, if (S, f) and (T, f) are weakly compatible then f, S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be arbitrary. Choose a point $x_1 \in X$ such that $fx_1 = Sx_0$ which is possible as $S(X) \subseteq f(X)$. Also we may choose a point $x_2 \in X$ such that $fx_2 = Tx_1$ as $T(X) \subseteq f(X)$. Continuing this process we get

$$fx_n = \begin{cases} Sx_{n-1} & \text{if } n \text{ is odd} \\ Tx_{n-1} & \text{if } n \text{ is even.} \end{cases}$$

If $n \in \mathbb{N}$ is odd then by using the hypothesis we obtain that

$$d(fx_n, fx_{n+1}) = d(Sx_{n-1}, Tx_n)$$

$$\precsim \alpha_1(fx_{n-1})d(fx_{n-1}, fx_n) + \alpha_2(fx_{n-1}) \frac{d(fx_{n-1}, Sx_{n-1})d(fx_n, Tx_n)}{1 + d(fx_{n-1}, fx_n)}$$

$$+ \alpha_3(fx_{n-1}) \frac{d(fx_n, Sx_{n-1}) + d(fx_{n-1}, Tx_n)}{1 + d(fx_{n-1}, fx_n)}$$

$$= \alpha_1(fx_{n-1})d(fx_{n-1}, fx_n) + \alpha_2(fx_{n-1}) \frac{d(fx_{n-1}, fx_n)d(fx_n, fx_{n+1})}{1 + d(fx_{n-1}, fx_n)}$$

$$+ \alpha_3(fx_{n-1}) \frac{d(fx_n, fx_n) + d(fx_{n-1}, fx_{n+1})}{1 + d(fx_{n-1}, fx_n)}$$

Therefore,

$$|d(fx_n, fx_{n+1})| \leq \alpha_1(fx_{n-1}) \, |d(fx_{n-1}, fx_n)|$$

$$+ \alpha_2(fx_{n-1}) \left| \frac{d(fx_{n-1}, fx_n)d(fx_n, fx_{n+1})}{1 + d(fx_{n-1}, fx_n)} \right|$$

$$+ \alpha_3(fx_{n-1}) \left| \frac{d(fx_n, fx_n) + d(fx_{n-1}, fx_{n+1})}{1 + d(fx_{n-1}, fx_n)} \right|$$

$$\leq \alpha_1(Tx_{n-2}) \mid d(fx_{n-1}, fx_n) \mid$$

$$+ \alpha_2(Tx_{n-1}) \left| \frac{d(fx_{n-1}, fx_n)}{1 + d(fx_{n-1}, fx_n)} \right| \mid d(fx_n, fx_{n+1}) \mid$$

$$+ \alpha_3(Tx_{n-2}) \mid d(fx_{n-1}, fx_{n+1}) \mid$$

$$\leq \alpha_1(fx_{n-2}) \mid d(fx_{n-1}, fxn) \mid + \alpha_2(fx_{n-2}) \mid d(fx_n, fx_{n+1}) \mid$$

$$+ \alpha_3(fx_{n-2}) \mid d(fx_{n-1}, fx_n) \mid + \alpha_3(fx_{n-2}) \mid d(fx_n, fx_{n+1}) \mid$$

$$= \alpha_1(Sx_{n-3}) \mid d(fx_{n-1}, fx_n) \mid + \alpha_2(Sx_{n-3}) \mid d(fx_n, fx_{n+1}) \mid$$

$$+ \alpha_3(Sx_{n-3}) \mid d(fx_{n-1}, fx_n) \mid + \alpha_3(Sx_{n-3}) \mid d(fx_n, fx_{n+1}) \mid$$

$$\leq \alpha_1(fx_{n-3}) \mid d(fx_{n-1}, fx_n) \mid + \alpha_2(fx_{n-3}) \mid d(fx_n, fx_{n+1}) \mid$$

$$+ \alpha_3(fx_{n-3}) \mid d(fx_{n-1}, fx_n) \mid + \alpha_3(fx_{n-3}) \mid d(fx_n, fx_{n+1}) \mid$$

$$\vdots$$

$$\leq \alpha_1(fx_0) \mid d(fx_{n-1}, fx_n) \mid + \alpha_2(fx_0) \mid d(fx_n, fx_{n+1}) \mid$$

$$+ \alpha_3(fx_0) \mid d(fx_{n-1}, fx_n) \mid + \alpha_3(fx_0) \mid d(fx_n, fx_{n+1}) \mid,$$

which implies that

$$\mid d(fx_n, fx_{n+1}) \mid \leq \frac{(\alpha_2(fx_0) + \alpha_3(fx_0))}{(1 - \alpha_1(fx_0) - \alpha_3(fx_0))} \mid d(fx_{n-1}, fx_n) \mid \tag{1}$$

Again if n is even then we have

$$d(fx_n, fx_{n+1}) = d(Tx_{n-1}, Sx_n) = d(Sx_n, Tx_{n-1})$$

$$\precsim \alpha_1(fx_n)d(fx_n, fx_{n-1})$$

$$+ \alpha_2(fx_n)\frac{d(fx_n, Sx_n)d(fx_{n-1}, Tx_{n-1})}{1 + d(fx_n, fx_{n-1})}$$

$$+ \alpha_3(fx_n)\frac{d(fx_{n-1}, Sx_n) + d(fx_n, Tx_{n-1})}{1 + d(fx_n, fx_{n-1})}$$

$$= \alpha_1(fx_n)d(fx_n, fx_{n-1})$$

$$+ \alpha_2(fx_n)\frac{d(fx_n, fx_{n+1})d(fx_{n-1}, fx_n)}{1 + d(fx_n, fx_{n-1})}$$

$$+ \alpha_3(fx_n)\frac{d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n)}{1 + d(fx_n, fx_{n-1})}.$$

Therefore,

$$|d(fx_n, fx_{n+1})| \leq \alpha_1(fx_n)\,|d(fx_n, fx_{n-1})|$$

$$+ \alpha_2(fx_n)\left|\frac{d(fx_n, fx_{n+1})d(fx_{n-1}, fx_n)}{1 + d(fx_n, fx_{n-1})}\right|$$

$$+ \alpha_3(fx_n)\frac{d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n)}{1 + d(fx_n, fx_{n-1})}$$

$$\leq \alpha_1(fx_n)\,|d(fx_n, fx_{n-1})|$$

$$+ \alpha_2(fx_n)\left|\frac{d(fx_{n-1}, fx_n)}{1 + d(fx_n, fx_{n-1})}\right|\,|d(fx_n, fx_{n+1})|$$

$$+\ \alpha_3(fx_n)\,|\,d(fx_{n-1}, fx_{n+1})\,|$$

$$\leq\ \alpha_1(fx_n)\,|\,d(fx_n, fx_{n-1})\,| + \alpha_2(fx_n)\,|\,d(fx_n, fx_{n+1})\,|$$

$$+\ \alpha_3(fx_n)\,|\,d(fx_{n-1}, fx_n)\,| + \alpha_3(fx_n)\,|\,d(fx_n, fx_{n+1})\,|$$

$$=\ \alpha_1(Tx_{n-1})\,|\,d(fx_{n-1}, fx_n)\,| + \alpha_2(Tx_{n-1})\,|\,d(fx_n, fx_{n+1})\,|$$

$$+\ \alpha_3(Tx_{n-1})\,|\,d(fx_{n-1}, fx_n)\,| + \alpha_3(Tx_{n-1})\,|\,d(fx_n, fx_{n+1})\,|$$

$$\leq\ \alpha_1(fx_{n-1})\,|\,d(fx_{n-1}, fx_n)\,| + \alpha_2(fx_{n-1})\,|\,d(fx_n, fx_{n+1})\,|$$

$$+\ \alpha_3(fx_{n-1})\,|\,d(fx_{n-1}, fx_n)\,| + \alpha_3(fx_{n-1})\,|\,d(fx_n, fx_{n+1})\,|$$

$$=\ \alpha_1(Sx_{n-2})\,|\,d(fx_{n-1}, fx_n)\,| + \alpha_2(Sx_{n-2})\,|\,d(fx_n, fx_{n+1})\,|$$

$$+\ \alpha_3(Sx_{n-2})\,|\,d(fx_{n-1}, fx_n)\,| + \alpha_3(Sx_{n-2})\,|\,d(fx_n, fx_{n+1})\,|$$

$$\leq\ \alpha_1(fx_{n-2})\,|\,d(fx_{n-1}, fx_n)\,| + \alpha_2(fx_{n-2})\,|\,d(fx_n, fx_{n+1})\,|$$

$$+\ \alpha_3(fx_{n-2})\,|\,d(fx_{n-1}, fx_n)\,| + \alpha_3(fx_{n-2})\,|\,d(fx_n, fx_{n+1})\,|$$

$$\vdots$$

$$\leq\ \alpha_1(fx_0)\,|\,d(fx_{n-1}, fx_n)\,| + \alpha_2(fx_0)\,|\,d(fx_n, fx_{n+1})\,|$$

$$+\ \alpha_3(fx_0)\,|\,d(fx_{n-1}, fx_n)\,| + \alpha_3(fx_0)\,|\,d(fx_n, fx_{n+1})\,|,$$

which implies that

$$|\,d(fx_n, fx_{n+1})\,| \leq \frac{(\alpha_2(fx_0) + \alpha_3(fx_0))}{(1 - \alpha_1(fx_0) - \alpha_3(fx_0))}\,|\,d(fx_{n-1}, fx_n)\,|. \tag{2}$$

From (1) and (2) we can conclude that for any integer

$$|d(fx_n, fx_{n+1})| \leq \frac{(\alpha_2(fx_0)+\alpha_3(fx_0))}{(1-\alpha_1(fx_0)-\alpha_3(fx_0))}|d(fx_{n-1}, fx_n)|. \qquad (3)$$

We set $\gamma = \frac{(\alpha_2(fx_0)+\alpha_3(fx_0))}{(1-\alpha_1(fx_0)-\alpha_3(fx_0))}$. Then by condition (ii), $\gamma < 1$. So by repeated application of (3) we obtain that

$$\begin{aligned} |d(fx_n, fx_{n+1})| &\leq \gamma\, |d(fx_{n-1}, fx_n)| \\ &\leq \gamma^2\, |d(fx_{n-2}, fx_{n-1})| \\ &\;\;\vdots \\ &\leq \gamma^n\, |d(fx_0, fx_1)|\, . \end{aligned}$$

Now for all $m, n \in \mathbb{N}$ and $m > n$ we have

$$d(fx_n, fx_m) \precsim d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2}) + \ldots d(fx_{m-1}, fx_m).$$

Therefore,

$$\begin{aligned} |d(fx_n, fx_m)| &\leq |d(fx_n, fx_{n+1})| + |d(fx_{n+1}, fx_{n+2})| + \ldots + |d(fx_{m-1}, fx_m)| \\ &\leq (\gamma^n + \gamma^{n+1} + \ldots + \gamma^{m-1})\, |d(fx_0, fx_1)| \\ &\leq \frac{\gamma^n}{1-\gamma}|d(fx_0, fx_1)|. \end{aligned}$$

Since $\gamma < 1$, taking limit as $n, m \to \infty$ we have $|d(fx_n, fx_m)| \to 0$, which implies that $\{fx_n\}$ is a Cauchy sequence in $f(X)$. By completeness of $f(X)$, there exist $u, v \in X$ such that $fx_n \to v = fu$.

Now

$$
\begin{aligned}
d(fu, Tu) &\precsim d(fu, f_{2n+1}) + d(fx_{2n+1}, Tu) \\
&= d(fu, f_{2n+1}) + d(Sx_{2n}, Tu) \\
&\precsim d(fu, f_{2n+1}) + \alpha_1(fx_{2n}) d(fx_{2n}, fu) \\
&\quad + \alpha_2(fx_{2n}) \frac{d(fx_{2n}, Sx_{2n}) d(fu, Tu)}{1 + d(fx_{2n}, fu)} \\
&\quad + \alpha_3(fx_{2n}) \frac{d(fu, Sx_{2n}) + d(fx_{2n}, Tu)}{1 + d(fx_{2n}, fu)}.
\end{aligned}
$$

Which implies that

$$
\begin{aligned}
|d(fu, Tu)| &\leq |d(fu, f_{2n+1})| + \alpha_1(fx_{2n}) |d(fx_{2n}, fu)| \\
&\quad + \alpha_2(fx_{2n}) |d(fx_{2n}, Sx_{2n})| |d(fu, Tu)| \\
&\quad + \alpha_3(fx_{2n}) \{|d(fu, Sx_{2n})| + |d(fx_{2n} Tu)|\}, \left[\text{as } \frac{1}{1 + d(fx_{2n}, fu)} < 1\right] \\
&\leq |d(fu, f_{2n+1})| + \alpha_1(fx_0) |d(fx_{2n}, fu)| \\
&\quad + \alpha_2(fx_0) |d(fx_{2n}, Sx_{2n})| |d(fu, Tu)| \\
&\quad + \alpha_3(fx_0) \{|d(fu, Sx_{2n})| + |d(fx_{2n}, Tu)|\}.
\end{aligned}
$$

Taking $n \to \infty$, it follows that $|d(fu, Tu)| = 0$ and hence, $d(fu, Tu) = 0$.

Therefore, $fu = Tu = v$.

Similarly, we can show that $fu = Su = v$.

Thus, $fu = Su = Tu = v$ and so v becomes a common point of coincidence of f, S and T.

Uniqueness:

For uniqueness suppose there exists another point $w(f \neq v) \in X$ such that $fx = Sx = Tx = w$ for some $x \in X$.

Thus,

$$d(v, w) = d(Su, Tx)$$

$$\precsim \alpha_1(fu)d(fu, fx) + \alpha_2(fu)\frac{d(fu, Su)d(fx, Tx)}{1 + d(fu, fx)}$$

$$+ \alpha_3(fu)\frac{d(fx, Su) + d(fu, Tx)}{1 + d(fu, fx)}$$

$$= \alpha_1(v)d(v, w) + \alpha_2(v)\frac{d(v, v)d(w, w)}{1 + d(v, w)}$$

$$+ \alpha_3(v)\frac{d(w, v) + d(v, w)}{1 + d(v, w)}$$

$$= \alpha_1(v)d(v, w) + 2\alpha_3(v)d(v, w).$$

Which implies that

$$|d(v, w)| \leq \alpha_1(v)|d(v, w)| + 2\alpha_3(v)|d(v, w)|.$$

Since, $0 \leq \alpha_1(v) + 2\alpha_3(v) < 1$, it follows that $|d(v, w)| = 0$ and so $v = w$. If (S, f) and (T, f) are weakly compatible then by Lemma 2.3 f, S and T have a unique common fixed point in X. ■

Corollary 3.1: *Let* (X, d) *be a complex valued metric space and* $S, T : X \to X$. *Suppose there exist mappings* $\alpha_1, \alpha_2, \alpha_3 : X \to [0, 1)$ *such that for all* $x, y \in X$

(i) $\alpha_i(Sx) \leq \alpha_i(x)$, $\alpha_i(Tx) \leq \alpha_i(x)$ *for all* $i = 1, 2, 3$;

(ii) $\alpha_1(x) + \alpha_2(x) + 2\alpha_3(x) < 1$;

(iii) $$d(Sx, Ty) \precsim \alpha_1(x)d(x, y) + \alpha_2(x)\frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + \alpha_3(x)\frac{d(y, Sx) + d(x, Ty)}{1 + d(x, y)}.$$

Then S *and* T *have a unique common fixed point in* X.

Proof: The result follows from Theorem 3.1 by taking $f = I$, the identity mapping. ■

Corollary 3.2: *Let* (X,d) *be a complex valued metric space and* $S, T : X \to X$. *If* S *and* T *satisfy*

$$d(Sx, Ty) \precsim \lambda d(x, y) + \mu\frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + \eta\frac{d(y, Sx) + d(x, Ty)}{1 + d(x, y)}$$

for all $x, y \in X$, *where* λ, μ, η *are nonnegative reals with* $\lambda + \mu + 2\eta < 1$ *then* S *and* T *have a unique common fixed point.*

Proof: The desired result can be obtained from Theorem 3.1 by setting $\alpha_1(x) = \lambda$, $\alpha_2(x) = \mu$, $\alpha_3(x) = \eta$ and $f = I$. ■

Corollary 3.3: *Let* (X, d) *be a complex valued metric space and* $f, T : X \to X$ *be such that* $T(X) \subseteq f(X)$ *and* $f(X)$ *is complete. Suppose there exist mappings* $\alpha_1, \alpha_2, \alpha_3 : X \to [0, 1)$ *such that for all* $x, y \in X$:

(i) $\alpha_i(Tx) \leq \alpha_i(fx)$ *for all* $i = 1, 2, 3$;

(ii) $\alpha_1(fx) + \alpha_2(fx) + 2\alpha_3(fx) < 1$;

(iii) $$d(Tx, Ty) \precsim \alpha_1(fx)\, d(fx, fy) + \alpha_2(fx) \frac{d(fx, Tx)d(fy, Ty)}{1 + d(fx, fy)} + \alpha_3(fx) \frac{d(fy, Tx) + d(fx, Ty)}{1 + d(fx, fy)}.$$

Then f and T have a unique point of coincidence. Moreover, if f and T are weakly compatible then f and T have a a unique common fixed point in X.

Proof: The proof of the corollary follows from Theorem 3.1 by considering $S = T$. ■

Corollary 3.4: *Let* (X, d) *be a complete complex valued metric space and* $T : X \to X$. *Suppose there exist mappings* $\alpha_1, \alpha_2, \alpha_3 : X \to [0, 1)$ *such that for all* $x, y \in X$:

(i) $\alpha_i(Tx) \leq \alpha_i(x)$ *for all* $i = 1, 2, 3$;

(ii) $\alpha_1(x) + \alpha_2(x) + 2\alpha_3(x) < 1$;

(iii) $$d(Tx, Ty) \precsim \alpha_1(x)\, d(x, y) + \alpha_2(x) \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} + \alpha_3(x) \frac{d(y, Tx) + d(x, Ty)}{1 + d(x, y)}.$$

Then T has a unique fixed point in X.

Proof: The proof of the corollary follows from Theorem 3.1 by considering $S = T$ and $f = I$. ■

Corollary 3.5: *Let (X, d) be a complex valued metric space and $T : X \to X$. Suppose T satisfies*

$$d(Tx, Ty) \precsim \lambda d(fx, fy) + \mu \frac{d(fx, Sx)d(fy, Ty)}{1 + d(fx, fy)}$$

$$+ \eta \frac{d(fy, Sx) + d(fx, Ty)}{1 + d(fx, fy)}.$$

for all $x, y \in X$, where λ, μ, η are nonnegative reals with $\lambda + \mu + 2\eta < 1$. If $T(X) \subseteq f(X)$ and $f(X)$ is complete then f and T have a unique point of coincidence. Moreover, if f and T are weakly compatible then f and T have a unique common fixed point in X.

Proof: Putting $S = T$, $\alpha_1(x) = \lambda$, $\alpha_2(x) = \mu$, $\alpha_3(x) = \eta$ in Theorem 3.1 we can prove this result. ■

Example 3.1: *Let $X = [1, \infty)$. Define $T, f : X \to X$ b y $Tx = \frac{3x-1}{2}$ and $fx = \frac{5x-2}{3}$ in X. If d_u is the usual metric on X then T and f are not the contraction mappings on X as for all $x, y \in X$, $d_u(Tx, Ty) = \frac{3}{2} | x - y |$ and $d_u(fx, fy) = \frac{5}{3} | x - y |$.*

So we can not apply Banach contraction theorem to find the unique fixed point of T and f.

Now we consider a complex valued metric $d : X \times X \to \mathbb{C}_1$ by

$$d(x, y) = | x - y | + i | x - y |$$

Then (X, d) *is a complete complex valued metric space.*

Now,

$$d(Tx, Ty) = \frac{3}{2}[| x - y | + i_1 | x - y |]$$

$$= \frac{9}{10} d(fx, fy) \leq hd(fx, fy), \quad \text{where } 0 < h = \frac{9}{10} < 1 .$$

Since, $T(X) = f(X) = X$ *, we have all the conditions of* Corollary 3.5 *with* $\lambda = h, \ \mu = 0 = \eta$.

So applying Corollary 3.5 *we can obtain a unique fixed point* 1 *of* T *and* f *in* X.

4. Future Prospect

In the line of the works as carried out in the paper one may think of the deduction of fixed point theorems using fuzzy metric, quasi metric, partial metric and other different types of metrics under the flavour of bicomplex analysis. This may be an active area of research to the future workers in this branch.

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SPLIT DOMATIC NUMBER OF A GRAPH

Abstract: A dominating set $D \subseteq V(G)$ is a split dominating set if the induced subgraph $<V - D>$ is disconnected. The minimum cardinality of a split dominating set is called the split domination number of G, denoted by $\gamma_s(G)$. The maximum order of a partition of $V(G)$ into split dominating sets of G is called the split domatic number of G and is denoted by $d_s(G)$. In this paper, we study several aspects of these two parameters and find certain classes of graphs that are domatically full.

Key words and phrases: Domination, Domination Number, Split Domination, Split Domination Number, Domatic Number, Split Domatic Number.

Mathematical Subject Classification No.: 05C69.

1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively.

For graph theoretical terms we refer to Harary [2] and for terms related to domination we refer Haynes *et al*. [3] and [4]. A subset D of V is said to be a dominating set in G if every vertex in $V - D$ is adjacent to at least one vertex in D. The maximum order of a partition of V into dominating sets of G is called the *domatic number* of G and is denoted by $d(G)$. G is *domatically full* if $d(G) = \delta(G) + 1$. A dominating set D of a graph $G = (V, E)$ is a split dominating set if the induced subgraph $< V - D >$ is disconnected. The split domination number $\gamma_s(G)$ of G is the minimum cardinality of a split dominating set. The maximum order of a partition of $V(G)$ into split dominating sets of G is called the split domatic number of G and is denoted by $d_s(G)$. The *corona* of two graphs G_1 and G_2 is the graph $G = G_1 \ o \ G_2$ formed from one copy of G_1 and $| V(G_1) |$ copies of G_2, where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2. In this paper, we study several aspects of these parameters and find certain classes of graphs that are domatically full.

Kulli and Janakiram introduced the concept of split domination in graphs [5]. The following results are very useful in the subsequent sections.

Theorem 1.1 [1]: *(If $\delta(G) \geq 1$ then $\gamma(G) + d(G) \leq \left\lfloor \frac{p}{2} \right\rfloor + 2$ and equality requires that $\{\gamma(G), d(G)\} = \{\left\lfloor \frac{p}{2} \right\rfloor, 2\}$.*

Theorem 1.2 [2]: *For any graph G, $\chi(G) \leq 1 + \Delta(G)$.*

Theorem 1.3 [6]: *Let G be any unicyclic graph with cycle C_p. Then $\gamma(G) = \gamma_{ns}(G)$ if and only if $G \cong G_1, G_2, G_3$ or G_4 where $G_1 \cong H \circ K_1$ where H is any unicyclic graph, $G_2 \cong$ any unicyclic graph with cycle $C_p (p \geq 4)$ in which every support is adjacent to exactly one pendent vertex, every vertex not on the cycle is a pendent vertex and exactly three consecutive vertices on the cycle have degree 2 and G_3, G_4 are as given in* Figure 1.

Figure 1

Example 1.4: *For* $G \cong G_1$, *where* G_1 *is given in* Figure 2, $\gamma_s(G) = 3$.

Figure 2

It is interesting to observe that the property 'split' is one among very few properties which are neither hereditary nor super hereditary. Motivated by this feature, we study several aspects of this parameter.

2. Split Domination Number of a Graph

The following are immediate from the definition:

Proposition 2.1: (1) $\gamma_s(G)$ *does not exist if and only if* $G \cong K_p$.

(2) *If* G *is any graph with* $\Delta(G) = p - 1$ *and* $\delta(G) = 1$ *then* $\gamma_s(G) = 1$. *Converse is not true. If* G *is the graph given in* Figure 3, $\gamma_s(G) = 1$, $\Delta(G) = p - 1$ *but* $\delta(G) \neq 1$.

Figure 3

(3) *For any connected graph* G *which is not isomorphic to complete graph,* $\gamma_s(G) \leq p - 2$.

(4) *If* H *is a connected spanning subgraph of* G, *then* $\gamma_s(H) \leq \gamma_s(G)$.

(5) *If* $G \cong K_{m,n}$, *then* $\gamma(\bar{G}) = \gamma_s(\bar{G}) = 2$.

(6) *Let* G *be any connected graph* $\ncong P_3$ *such that either* $\Delta(G) = \delta(G) = p - 2$ *or* $\Delta(G) = p - 2 =$ *number of pendent vertices. Then* $\gamma_s(G) \leq 2$.

(7) *If* G *is any disconnected graph without isolated vertices, then* $\gamma_s(G) = p - 2$ *if and only if* $G \cong 2K_2$.

(8) *If* G *is a galaxy then* $\gamma(G) = \gamma_s(G) =$ *number of components of* G. *Converse is not true. If* G *is the graph given in* Figure 4, $\gamma(G) = \gamma_s(G) = 2 =$ *number of components of* G*, but* G *is not a galaxy.*

Figure 4

Theorem 2.2: *Let* T *be any tree.* $\gamma_s(T) = 2$ *if and only if* T *has exactly two supports and* $diam(T) \leq 5$.

Proof: Suppose $\gamma_s(T) = 2$. If T has exactly one support then $T \cong K_{1,p}$. But $\gamma_s(K_{1,p}) = 1$ and so T has at least two supports. If T has 3 or more supports then clearly $\gamma_s(T) \geq 3$ and so T has exactly two supports. If $diam(T) > 6$, then again $\gamma_s(T) > 2$ and so $diam(T) \leq 5$. Converse is obvious. □

The following is immediate.

Corollary 2.3: *Let T be any tree with exactly two supports. Then $\gamma_s(T) = 3$ if and only if $diam(T) = 6, 7$ or 8.*

Theorem 2.4: *Let T be a tree. Every minimum dominating set is a split dominating set if and only if $T \ncong P_p \circ K_1$.*

Proof: Assume that every minimum dominating set is a split dominating set. If $T \ncong P_p \circ K_1$, the set of all pendent vertices is a minimum dominating set which is not a split dominating set and so $T \ncong P_p \circ K_1$. If $T \ncong P_p \circ K_1$, every minimum dominating set contains a non-pendent vertex and so is a split dominating set. □

Theorem 2.5: *Let T be any tree such that $T \ncong K_{1,p}$. Then*

$$\gamma_s(\bar{T}) = \begin{cases} p - \Delta(T) & if \;\; diam(T) = 3 \\ p - \Delta(T) - 1 & if \;\; diam(T) \geq 4. \end{cases}$$

Proof: Let $v \in V(T)$ with $deg \; v = \Delta(T)$.

Case (i): $diam(T) = 3$.

Since, $diam(T) = 3$, there exists a pendent vertex u adjacent to v and a pendent vertex w non-adjacent to v. Let $S = N[v] - \{u\}$.

Claim: $V - S$ is a minimum split dominating set of $\bar{T}$.

$w \in V - S$ dominates v and every vertex of $N(v) - \{u\}$ is dominated by u. v is an isolated vertex of $< S >$ in $\bar{T}$ and so $V - S$ is a split dominating set of $\bar{T}$. The vertices of $V(T) - N[v]$ are all adjacent to a single vertex of $N(v)$ since otherwise T has a cycle. Hence, $u \in N[v]$ is essential to dominate that vertex in $\bar{T}$. If any other vertex of $V(T) - N[v]$ lies in S, then that vertex is adjacent to v in $\bar{T}$ so that S is not a split dominating set. Hence, $V - S$ is a minimum split dominating set of $\bar{T}$ so that $\gamma_s(\bar{T}) = p - \Delta(T)$.

Case (ii): $diam(T) \geq 4$.

Let $S' = N[v]$. As $diam(T) \geq 4$, either there exists $x \in V(T)$ with $d(v, x) \geq 3$ or there exists $x, y \in V(T)$ with $d(v, x) = 2$ and $d(v, y) = 2$. In the former case, x is adjacent to all vertices of S' in $\overline{T}$ and in the latter case $\{x, y\}$ dominates all vertices of S' in $\overline{T}$. Also v is an isolated vertex in $< S' >$ in $\overline{T}$. So $V - S'$ is a split dominating set of $\overline{T}$. As in case (i), it is minimum. Hence, $\gamma_s(\overline{T}) = p - (\Delta(T) + 1)$. □

Corollary 2.6: *For any tree* $T \ncong K_{1,p}$, $\gamma_s(T) + \gamma_s(\overline{T}) = p - \Delta(T) + 1$ *if and only if T is obtained from* P_5 *or* P_6 *by adding zero or more number of pendents to the supports.*

Proof: If $diam(T) = 3$ then $\gamma_s(\overline{T}) = p - \Delta(T)$ and so $\gamma_s(T) + \gamma_s(\overline{T}) = p - \Delta(T) + 1 \Rightarrow \gamma_s(T) = 1$ which is impossible as $T \ncong K_{1,p}$.

Suppose $diam(T) = 4$. Then by Theorem 2.5, $\gamma_s(\overline{T}) = p - \Delta(T) - 1$ and so $\gamma_s(T) = 2$. By Theorem 2.2, T has exactly two supports and $diam(T) \leq 5$. Thus, $diam(T) = 4$ or 5 and T is obtained from P_5 or P_6 by adding zero or more number of pendents to the supports.

Converse is obvious. □

Theorem 2.7: *Let T be any tree with* $diam(T) = 3$. *Then*

$$\gamma_s(T)\gamma_s(\overline{T}) \leq 2(p - \Delta(T)).$$

Proof: By Theorem 2.5, $\gamma_s(\overline{T}) = p - \Delta(T)$ if $diam(T) = 3$. If $diam(T) = 3$ then T has exactly 2 supports and so by Theorem 2.2, $\gamma_s(T) = 2$. Hence, $\gamma_s(T)\gamma_s(\overline{T}) \leq 2(p - \Delta(T))$. □

Remark 2.8: *Converse of Theorem 2.7 is not true. Consider* P_5. $\gamma_s(P_5)\gamma_s(\bar{P}_5) = 4 < 6$. *But* $diam(P_5) = 4$.

We now relate $\gamma_s(G)$ with other graph theoretic parameters.

Theorem 2.9: *For any connected graph* $G, \gamma_s(G) + \chi(G) \leq p + \Delta(G) - 2$. *Equality holds for* $G \cong C_4$, P_4.

Proof: By Theorem 1.2, and by Proposition **??**(3) we have $\gamma_s(G) + \chi(G) \leq p + \Delta(G) - 1$. By Theorem 1.2, if $\chi(G) = \Delta(G) + 1$ then G is either an odd cycle or a complete graph. For a complete graph, $\gamma_s(G)$ is not defined. For an odd cycle C_p, $\gamma_s(C_p) < p - 2$. Hence, the above bound can be improved as $\gamma_s(G) + \chi(G) \leq p + \Delta(G) - 2$. Equality holds for $G \cong C_4$, P_4. □

Theorem 2.10: *For any connected graph* $G, \gamma_s(G) + k(G) \leq p + \Delta(G) - 2$. *Equality holds if* $G \cong C_4$.

Proof: For any connected graph G, $\gamma_s(G) \leq p - 2$ and $k(G) \leq \Delta(G)$ so that $\gamma_s(G) + k(G) \leq p + \Delta(G) - 2$. Equality holds if $\gamma_s(G) = p - 2$ and $k(G) = \Delta(G)$. If $G \cong C4$, $\gamma_s(G) = 2$, $k(G) = 2$ and so the bound is sharp.

Theorem 2.11: *For any connected graph* $G, \gamma_s(G) + diam(G) \leq 2p - 3$. *Equality holds if* $G \cong P_4$.

Proof: For any connected graph G, $\gamma_s(G) \leq p - 2$ and $diam(G) \leq p - 1$ so that $\gamma_s(G) + diam(G) \leq 2p - 3$. Clearly equality holds if $G \cong P_4$. □

Theorem 2.12: *Let* G *be any unicyclic graph with cycle* C_p. *Then* $\gamma(G) = \gamma_{ns}(G) = \gamma_s(G)$ *if and only if* $G \cong G_1$, G_2, G_3 *or* G_4 *where* $G_1 \cong H \circ K_1$ *for any unicyclic graph* H, $G_2 \cong$ *any unicyclic graph with cycle* C_p $(p \geq 4)$ *in which every support is adjacent to exactly one pendent vertex,* *every vertex not on the cycle is a pendent vertex and exactly three consecutive vertices on the cycle have degree* 2 *and* G_3, G_4 *are as given in* Figure 5.

Figure 5

Proof: Follows by Theorem 1.3 since for every such graph there exists a split dominating set with cardinality $\gamma(G)$. □

Theorem 2.13: *For any tree* T, $\gamma(T) = \gamma_s(T) = \gamma_{ns}(T) = \gamma_{sns}(T) = 2$ *if and only if* $T \cong P4$.

Proof: Since a tree cannot contain a cycle, $\gamma_{sns}(T) = 2$ which implies $p = 4$. So $T \cong K_{1,3}$ or P_4. If $T \cong K_{1,3}$ then $\gamma(T) = 1$ and so $T \cong P_4$.

Converse is obvious. □

Proposition 2.14: *Let* G *be any connected graph and* $G' = G \circ K_1$, *where* $G \circ K_1$ *is the corona of* G *and* K_1. *Then* $\gamma(G') = \gamma_s(G') = \gamma_{ns}(G') = p$, *where* $p = |V(G)|$.

Proof: Since G is connected, the set of all pendent vertices of G' forms a γ_{ns}-set of G and so $\gamma_{ns}(G') = p$. Similarly $V(G)$ forms a γ_s-set of G' which is also a γ-set of G'. Hence, $\gamma(G') = \gamma_s(G') = \gamma_{ns}(G') = p$. □

3. Split Domatic number of a Graph

Definition 3.1: Let $G = (V, E)$ be a graph. The maximum order of a partition of $V(G)$ into split dominating sets of G is called the split domatic number of G and is denoted by $d_s(G)$.

Remark 3.2: $d_s(G)$ cannot be determined for any graph with $\Delta(G) = p - 1$.

In the following proposition we summarize a number of elementary results which determine $d_s(G)$ for special classes of graphs. The proofs of these results are simple and are omitted.

Proposition 3.3: (1) *If* $G \cong P_p\,(p \geq 4)$ *then* $d_s(P_p) = 2$.

(2) *If* $G \cong K_{m,n}$ *then* $d_s(K_{m,n}) = 2$.

(3) *If* $G \cong G \circ K_1$ *then* $d_s(G \circ K_1) = 2$.

(4) *If* $G \cong \bar{C}_p$ *where* $p = 4$, 5 *or* 6 *then* $d_s(\bar{C}_p) = 2$.

(5) *If* $G \cong \bar{P}_p$ *where* $p = 4$, 5 *or* 6 *then* $d_s(\bar{P}_p) = 2$.

(6) *For any graph* G, $d_s(G) \leq \delta(G) + 1$.

Definition 3.4: A graph G is split domatically full if $d_s(G) = \delta(G) + 1$.

Proposition 3.5: *If* $G \cong C_p(p = 3k,\ k > 1)$, *then* $C_p^{'}s$ *are domatically full.*

Proof: Let $V(C_p) = \{v_1, v_2, \ldots, v_{3k}\}$. The sets $\{v_1, v_4, v_7, \ldots, v_{3k-2}\}$, $\{v_2, v_5, \ldots, v_{3k-1}\}$ and $\{v_3, v_6, \ldots, v_{3k}\}$ form a partition of $V(G)$ into split dominating sets and so $d_s(C_p) = 3 = \delta(C_p) + 1$. □

Corollary 3.6: *Let* $G \cong C_p(p \geq 4)$. *Then* G *is split domatically full if and only if* $p \equiv 0(\text{mod}3)$.

Proof: By Theorem 1.1, $d(C_p) = 2$ if $p \equiv 1,\ 2(\mathrm{mod} 3)$ and hence, the result follows. □

Using cycles, we now construct a special class of split domatically full graphs.

Example 3.7: Let $I = \{C_p : C_p$ is a cycle on p vertices, $p = 3k, k > 1\}$. An operation $\circ$ is defined on I as follows:

If $C_i, C_j, C_k \in I$, then $C_i \circ C_j$ is obtained by joining any one vertex of C_i to any one vertex of C_j by an edge. $(C_i \circ C_j) \circ C_k$ is obtained by joining any one vertex of $C_i \circ C_j$ to any one vertex of C_k by an edge. The process is repeated finite number of times and we define G^* to be the collection of all such graphs. As each C_p in I is domatically full and union of split dominating sets of the individual cycles give split dominating sets of the newly constructed graphs, every element of G^* is split domatically full.

Proposition 3.8: *If T is any tree other than a star, then T is split domatically full.*

Proof: By Theorem 1.1, $d(T) = 2$ for any tree with at least two vertices. In any tree with at least two distinct supports, it is easy to observe that there exist two disjoint dominating sets which are also split dominating sets. If T is a star then $d_s(T) = 1$.

Hence, if $T \ncong K_{1,p-1}$ then $d_s(T) = 2 = \delta(G) + 1$. □

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EXTENDED GIBONACCI CONJECTURES

Abstract: We present two conjectures, one involving Fibonacci numbers and the other Jacobsthal numbers.

Keywords: Fibonacci, Lucas, Jacobsthal, and Jacobsthal-Lucas Numbers, Extended Gibonacci Numbers.

Mathematical Subject Classification (2020) No.: Primary 11B37, 11B39, 11C08.

1. Introduction

Extended gibonacci numbers G_n are defined by the recurrence $G_n = aG_{n-1} + bG_{n-2}$, where a, b, G_1, and G_2 are arbitrary integers and $n \geq 3$.

Suppose $a = 1 = b$. When $G_1 = 1 = G_2$, $G_n = F_n$, the nth *Fibonacci number*; and when $G_1 = 1$ and $G_2 = 3$, $G_n = L_n$, the nth *Lucas number*. They can also be defined by *Binet-like* formulas [2, 3].

On the other hand, let $a = 1$ and $b = 2$. When $G_1 = 1 = G_2$, $G_n = J_n$, the nth *Jacobsthal number*; and when $G_1 = 1$ and $G_2 = 5$, $G_n = j_n$, the nth *Jacobsthal-Lucas number* [3].

The following table shows the first 10 Fibonacci, Lucas, Jacobsthal, and Jacobsthal-Lucas numbers.

Table 1
First 10 Fibonacci, Lucas, Jacobsthal, and Jacobsthal-Lucas numbers

n	1	2	3	4	5	6	7	8	9	10
F_n	(1)	(1)	(2)	(3)	(5)	(8)	13	21	34	(55)
L_n	(1)	(3)	(4)	(7)	(11)	18	29	47	76	123
J_n	(1)	(1)	(3)	(5)	(11)	21	43	85	(171)	341
j_n	(1)	(5)	(7)	17	31	65	127	257	511	1,025

A quick look at the first 100 Fibonacci numbers in [2, 3] gives a fascinating observation. There are exactly seven *palindromic* Fibonacci numbers $\leq F_{100}$, and six (the smallest *perfect number* [1]) of them are single-digit integers and are all circled in Table 1.

Interestingly, the corresponding Jacobsthal table in [3] contains two added bonuses. In addition to the four single-digit integers, there are two additional palindromic numbers, namely $J_5 = 11$ and $J_9 = 171$, again a total of six Jacobsthal palindromes $\leq J_{100}$, also circled in Table 1.

With a computer program, Z. Gao established that there are *no* additional Fibonacci palin-dromes $\leq F_{170,000}$, and *no* additional Jacobsthal palindromes $\leq J_{120,000}$ [4].

2. Extended Gibonacci Conjectures

Based on Fibonacci and Jacobsthal tables in [3], and the data collected by Gao, we conjecture that:

1. There are exactly seven Fibonacci palindromes: 1, 1, 2, 3, 5, 8, and 55; and

2. There are exactly six Jacobsthal palindromes: 1, 1, 3, 5, 11, and 171.

Clearly, similar conjectures can be conceived for both Lucas and Jacobsthal-

Lucas numbers as well. For the curious-minded, we add that $L_{25} = 167,761$ is palindromic.

We encourage gibonacci enthusiasts to either confirm or disprove each conjecture.

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SUMS INVOLVING EXTENDED GIBONACCI POLYNOMIALS

Abstract: We explore four sums involving gibonacci polynomial squares and their Pell versions.

Keywords: Extended Gibonacci Polynomials, Fibonacci Polynomial, Lucas Polynomial, Binet-Like Formulas, Pell Polynomials, Pell-Lucas Polynomials.

Mathematical Subject Classification (2020) No.: Primary 11B37, 11B39, 11C08.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 4].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively [4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, and $b_n = p_n$ or q_n, $c_n = J_n$ or j_n, $\Delta = \sqrt{x^2 + 4}$, $2\alpha = x + \Delta$, $E = \sqrt{x^2 + 1}$, $\gamma = x + E$, and $D = \sqrt{4x + 1}$ where $c_n = c_n(x)$.

It follows by the Binet-like formulas that $\lim_{m\to\infty} \frac{f_{m+1}}{l_m} = \frac{\alpha}{\Delta}$ and $\lim_{m\to\infty} \frac{l_{m+1}}{f_m} = \Delta\alpha$.

1.1 Fundamental Gibonacci Identities: Gibonacci polynomials satisfy the following properties:

$$\Delta^2 f_{n+1} f_n = l_{2n+1} - (-1)^n x; \tag{1}$$

$$l_{n+1} l_n = l_{2n+1} + (-1)^n x; \tag{2}$$

$$l_n f_{n+2} - l_{n+1} f_{n+1} = (-1)^n x; \tag{3}$$

$$l_n f_{n+2} + l_{n+1} f_{n+1} = 2 f_{2n+2} + (-1)^n x; \tag{4}$$

$$f_n l_{n+2} - f_{n+1} l_{n+1} = (-1)^{n+1} x; \tag{5}$$

$$f_n l_{n+2} + f_{n+1} l_{n+1} = 2 f_{2n+2} - (-1)^n x. \tag{6}$$

These properties can be confirmed using the Binet-like formulas.

It follows by identities (3) – (6) that

$$l_n^2 f_{n+2}^2 - l_{n+1}^2 f_{n+1}^2 = 2(-1)^n x f_{2n+2} + x^2; \tag{7}$$

$$f_n^2 l_{n+2}^2 - f_{n+1}^2 l_{n+1}^2 = 2(-1)^{n+1} x f_{2n+2} + x^2. \tag{8}$$

2. Telescoping Gibonacci Sums

We now establish two telescoping gibonacci sums, where $k \geq 0$ and $\lambda \geq 1$ are integers.

Lemma 1:

$$\sum_{n=1}^{\infty}\left(\frac{f_{n+k+2}^{\lambda}}{l_{n+k+1}^{\lambda}}-\frac{f_{n+k+1}^{\lambda}}{l_{n+k}^{\lambda}}\right)=\frac{\alpha^{\lambda}}{\Delta^{\lambda}}-\frac{f_{k+2}^{\lambda}}{l_{k+1}^{\lambda}}. \tag{9}$$

Proof: Since $\sum_{n=1}^{m}\left(\frac{f_{n+k+2}^{\lambda}}{l_{n+k+1}^{\lambda}}-\frac{f_{n+k+1}^{\lambda}}{l_{n+k}^{\lambda}}\right)$ is a telescoping sum, we have

$$\sum_{n=1}^{m}\left(\frac{f_{n+k+2}^{\lambda}}{l_{n+k+1}^{\lambda}}-\frac{f_{n+k+1}^{\lambda}}{l_{n+k}^{\lambda}}\right)=\frac{f_{m+k+2}^{\lambda}}{l_{m+k+1}^{\lambda}}-\frac{f_{k+2}^{\lambda}}{l_{k+1}^{\lambda}}.$$

This yields the desired result. □

Lemma 2:

$$\sum_{n=1}^{\infty}\left(\frac{l_{n+k+2}^{\lambda}}{f_{n+k+1}^{\lambda}}-\frac{l_{n+k+1}^{\lambda}}{f_{n+k}^{\lambda}}\right)=\Delta^{\lambda}\alpha^{\lambda}-\frac{l_{k+2}^{\lambda}}{f_{k+1}^{\lambda}}. \tag{10}$$

Proof: Using the fact that $\lim_{m\to\infty}\frac{l_{m+1}}{f_m}=\Delta\alpha$, the proof follows as above. So, in the interest of brevity, we omit the details. □

These two lemmas play a pivotal role in our discourse.

3. Gibonacci Sums

With the above identities and lemmas at our disposal, we are now ready for further explorations.

The next two theorems invoke the lemmas with $\lambda = 1$.

Theorem 1: *Let k be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k} x}{l_{2n+2k+1} + (-1)^{n+k} x} = \frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}. \tag{11}$$

Proof: It follows by identities (2) and (3) that

$$l_{n+k+1} l_{n+k} = l_{2n+2k+1} + (-1)^{n+k} x;$$

$$l_{n+k} f_{n+k+2} - l_{n+k+1} f_{n+k+1} = (-1)^{n+k} x.$$

By Lemma 1, we then have

$$\frac{(-1)^{n+k} x}{l_{2n+2k+1} + (-1)^{n+k} x} = \frac{l_{n+k} f_{n+k+2} - l_{n+k+1} f_{n+k+1}}{l_{n+k+1} l_{n+k}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k} x}{l_{2n+2k+1} + (-1)^{n+k} x} = \sum_{n=1}^{\infty} \left(\frac{f_{n+k+2}}{l_{n+k+1}} - \frac{f_{n+k+1}}{l_{n+k}} \right)$$

$$= \frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}},$$

as desired. □

It then follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+1} + (-1)^n} = -\frac{1}{2} + \frac{\sqrt{5}}{10}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+3} - (-1)^n} = \frac{1}{6} - \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+5} + (-1)^n} = -\frac{1}{4} + \frac{\sqrt{5}}{10}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+7} - (-1)^n} = \frac{3}{14} - \frac{\sqrt{5}}{10}.$$

The next result invokes Lemma 2 with $\lambda = 1$.

Theorem 2: *Let k be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k+1}x}{l_{2n+2k+1} - (-1)^{n+k}x} = \frac{1}{\Delta^2}\left(\frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}\right). \tag{12}$$

Proof: Using identities (1) and (4), we get

$$\Delta^2 f_{n+k+1} f_{n+k} = l_{2n+2k+1} - (-1)^{n+k}x;$$

$$f_{n+k} l_{n+k+2} - f_{n+k+1} l_{n+k+1} = (-1)^{n+k+1}x.$$

By Lemma 2, we then have

$$\frac{(-1)^{n+k+1}x}{l_{2n+2k+1} - (-1)^{n+k}x} = \frac{f_{n+k} l_{n+k+2} - f_{n+k+1} l_{n+k+1}}{\Delta^2 f_{n+k+1} f_{n+k}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k+1}x}{l_{2n+2k+1} - (-1)^{n+k}x} = \frac{1}{\Delta^2}\sum_{n=1}^{\infty}\left(\frac{l_{n+k+2}}{f_{n+k+1}} - \frac{l_{n+k+1}}{f_{n+k}}\right)$$

$$= \frac{1}{\Delta^2} - \left(\Delta\alpha - \frac{l_{k+2}}{f_{k+1}}\right),$$

as desired. □

Consequently, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+1} - (-1)^n} = \frac{1}{10} - \frac{\sqrt{5}}{10}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+3} + (-1)^n} = -\frac{3}{10} + \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+5} - (-1)^n} = \frac{1}{5} - \frac{\sqrt{5}}{10}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+7} + (-1)^n} = -\frac{7}{30} + \frac{\sqrt{5}}{10}.$$

Gibonacci Delights: By combining these two theorems, we can extract interesting dividends.

Adding equations (11) and (12), we get

$$\sum_{n=1}^{\infty} \frac{2x^2}{l_{2n+2k+1}^2 - x^2} = -\frac{\Delta^2 + 1}{\Delta^2}\left(\frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}\right).$$

In particular, this yields

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n+1}^2 - 1} = \frac{3}{10} - \frac{3\sqrt{5}}{50}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n+3}^2 - 1} = \frac{1}{10} - \frac{3\sqrt{5}}{50};$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n+5}^2 - 1} = \frac{3}{20} - \frac{3\sqrt{5}}{50}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n+7}^2 - 1} = \frac{9}{70} - \frac{21\sqrt{5}}{350}.$$

Likewise, subtraction of the two equations yield

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k} x l_{2n+2k+1}}{l_{2n+2k+1}^2 - x^2} = \frac{\Delta^2 - 1}{\Delta^2}\left(\frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}\right).$$

This implies

$$\sum_{n=1}^{\infty} \frac{(-1)^n L_{2n+1}}{L_{2n+1}^2 - 1} = \frac{2\sqrt{5}}{25}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n L_{2n+3}}{L_{2n+3}^2 - 1} = -\frac{2}{15} + \frac{2\sqrt{5}}{15};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n L_{2n+5}}{L_{2n+5}^2 - 1} = \frac{1}{10} + \frac{2\sqrt{5}}{25}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n L_{2n+7}}{L_{2n+7}^2 - 1} = -\frac{4}{35} + \frac{2\sqrt{5}}{25}.$$

The next two theorems employ the lemmas with $\lambda = 2$.

Theorem 3: *Let k be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k} x f_{2n+2k+2} + x^2}{\left[l_{2n+2k+1} + (-1)^{n+k} x\right]^2} = \frac{\alpha^2}{\Delta^2} - \frac{f_{k+2}^2}{l_{k+1}^2}. \qquad (13)$$

Proof: Lemma 1, coupled with identities (2) and (7), yields

$$\frac{2(-1)^{n+k}xf_{2n+2k+2}+x^2}{\left[l_{2n+2k+1}+(-1)^{n+k}x\right]^2}=\frac{l_{n+k}^2f_{n+k+2}^2-l_{n+k+1}^2f_{n+k+1}^2}{l_{n+k+1}^2l_{n+k}^2}$$

$$\sum_{n=1}^{\infty}\frac{2(-1)^{n+k}f_{2n+2k+2}+x^2}{\left[l_{2n+2k+1}+(-1)^{n+k}x\right]^2}=\sum_{n=1}^{\infty}\left(\frac{f_{n+k+2}^2}{l_{n+k+1}^2}-\frac{f_{n+k+1}^2}{l_{n+k}^2}\right)$$

$$=\frac{\alpha^2}{\Delta^2}-\frac{f_{k+2}^2}{l_{k+1}^2}.$$

as desired. □

In particular, we then get

$$\sum_{n=1}^{\infty}\frac{2(-1)^nF_{2n+2}+1}{\left[L_{2n+1}+(-1)^n\right]^2}=-\frac{7}{10}+\frac{\sqrt{5}}{10};\qquad \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}F_{2n+4}+1}{\left[L_{2n+3}-(-1)^n\right]^2}=-\frac{23}{90}+\frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty}\frac{2(-1)^nF_{2n+6}+1}{\left[L_{2n+5}+(-1)^n\right]^2}=-\frac{21}{80}+\frac{\sqrt{5}}{10};\qquad \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}F_{2n+8}+1}{\left[L_{2n+7}-(-1)^n\right]^2}=-\frac{103}{490}+\frac{\sqrt{5}}{10}.$$

The next result invokes Lemma 2.

Theorem 4: *Let k be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty}\frac{2(-1)^{n+k+1}xf_{2n+2k+2}+x^2}{\left[l_{2n+2k+1}-(-1)^{n+k}x\right]^2}=\frac{1}{\Delta^4}\left(\Delta^2\alpha^2-\frac{l_{k+2}^2}{f_{k+1}^2}\right). \tag{14}$$

Proof: With identities (1) and (8), Lemma 2 yields

$$\frac{2(-1)^{n+k+1}xf_{2n+2k+2}+x^2}{\left[l_{2n+2k+1}-(-1)^{n+k}x\right]^2}=\frac{f_{n+k}^2l_{n+k+2}^2-f_{n+k+1}^2l_{n+k+1}^2}{\Delta^4f_{n+k+1}^2f_{n+k}^2}$$

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k+1} x f_{2n+2k+2} + x^2}{\left[l_{2n+2k+1} - (-1)^{n+k} x\right]^2} = \frac{1}{\Delta^4} \sum_{n=1}^{\infty} \left(\frac{l_{n+k+2}^2}{f_{n+k+1}^2} - \frac{l_{n+k+1}^2}{f_{n+k}^2} \right)$$

$$= \frac{1}{\Delta^4} \left(\Delta^2 \alpha^2 \frac{l_{k+2}^2}{f_{k+1}^2} \right),$$

confirming the given result. □

It then follows that

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1} F_{2n+2} + 1}{\left[L_{2n+1} - (-1)^n\right]^2} = -\frac{3}{50} + \frac{\sqrt{5}}{10}; \quad \sum_{n=1}^{\infty} \frac{2(-1)^{n} F_{2n+4} + 1}{\left[L_{2n+3} + (-1)^n\right]^2} = -\frac{17}{50} + \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1} F_{2n+6} + 1}{\left[L_{2n+5} - (-1)^n\right]^2} = -\frac{19}{100} + \frac{\sqrt{5}}{10}; \quad \sum_{n=1}^{\infty} \frac{2(-1)^{n} F_{2n+8} + 1}{\left[L_{2n+7} + (-1)^n\right]^2} = -\frac{107}{450} + \frac{\sqrt{5}}{10}.$$

Finally, we explore the Pell versions of the theorems.

4. Pell Implications

Using the relationship $b_n(x) = g_n(2x)$, we can find the Pell versions of equations (11) – (14):

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k} x}{q_{2n+2k+1} + 2(-1)^{n+k} x} = \frac{\gamma}{4E} - \frac{p_{k+2}}{2q_{k+1}};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k} x}{q_{2n+2k+1} - 2(-1)^{n+k} x} = \frac{1}{8E^2} \left(\frac{\gamma}{2E} - \frac{p_{k+2}}{q_{k+1}} \right);$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k} x p_{2n+2k+2} + x^2}{\left[q_{2n+2k+1} + 2(-1)^{n+k} x\right]^2} = \frac{1}{4} \left(\frac{\gamma^2}{4E^2} - \frac{p_{k+2}^2}{q_{k+1}^2} \right);$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k+1} x p_{2n+2k+2} + x^2}{\left[q_{2n+2k+1} - 2(-1)^{n+k} x\right]^2} = \frac{1}{64E^4} \left(4E^2 \gamma^2 - \frac{q_{k+2}^2}{p_{k+1}^2} \right).$$

They yield

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k}}{Q_{2n+2k+1} + (-1)^{n+k}} = \frac{2+\sqrt{2}}{4} - \frac{P_{k+2}}{Q_{k+1}};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k+1}}{Q_{2n+2k+1} - (-1)^{n+k}} = \frac{2+\sqrt{2}}{32} - \frac{P_{k+2}}{16Q_{k+1}};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k} P_{2n+2k+2} + 1}{\left[Q_{2n+2k+1} + (-1)^{n+k}\right]^2} = \frac{3+2\sqrt{2}}{8} - \frac{P_{k+2}^2}{4Q_{k+1}^2};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k+1} P_{2n+2k+2} + 1}{\left[Q_{2n+2k+1} - (-1)^{n+k}\right]^2} = \frac{3+2\sqrt{2}}{8} - \frac{Q_{k+2}^2}{16P_{k+1}^2}.$$

respectively.

5. Chebyshev and Vieta Consequences

Chebyshev polynomials T_n and U_n, Vieta polynomials V_n and v_n, and gibonacci polynomials are linked by the relationships $V_n(x) = i^{n-1} f_n(-ix)$, $v_n(x) = i^n l_n(-ix)$, $V_n(x) = U_{n-1}(x/2)$, and $v_n(x) = 2T_n(x/2)$ [2, 3, 4], where $i = \sqrt{-1}$; they can be employed to find the Chebyshev and Vieta versions of the theorems. In the interest of brevity, we omit them; but we encourage gibonacci enthusiasts to explore them.

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