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Gollakota V V Hemasundar

A NOTE ON WEIERSTRASS POINTS OF HYPERELLIPTIC RIEMANN SURFACES

Abstract: It is known that there exists a non-constant meromorphic function on a compact Riemann surface X of genus g which has a pole of order $\leq g$ at p and is holomorphic in $X \setminus \{p\}$ if and only if p is a Weierstrass point. In this note we survey some results related to Weierstrass points on a hyperelliptic Riemann surfaces and analyze the gap sequence.

Keywords: Hyperelliptic Riemann Surfaces, Weierstrass Gap Theorem, Weier-strass Points, Compact Riemann Surfaces.

Mathematic Subject Classification No.: 30F10.

1. Introduction

Let X be a compact Riemann surface of genus g and $p \in X$. Then the following conditions on X are equivalent:

1. There exists a non-constant meromorphic function on X which has a pole of order $\leq g$ at p and is holomorphic in $X \setminus \{p\}$.

2. p is a Weierstrass point.

This is a consequence of Riemann-Roch and Seree Duality theorems See [2].

One of the important questions is the existence of meromorphic functions having pole at a single point p on X and is holomorphic in $X \setminus \{p\}$.

One of the basic results in this topic is Weierstrass gap theorem, which is stated below:

Theorem 1: *For a surface of genus $g \geq 1$ there are precisely g integers*

$$1 = \lambda_1 < \lambda_2 < \cdots < \lambda g < 2g \tag{1}$$

such that there does not exist a meromorphic function on X with a pole of order λ_k at p.

The numbers λ_j, for $j = 1, \ldots, g$ are called "gaps" at p and their complement in are called "non-gaps". Further, the sequence is uniquely determined by the point p. For proof [1], [5] and [3].

Definition 2: *A compact Riemann surface X of genus $g > 1$ is said to be hyperelliptic if it is a two sheeted covering of the sphere $\mathbb{P}^1$.*

In this note we derive the following result for hyperelliptic Riemann surfaces:

Theorem 3: *Suppose X is a hyperelliptic Riemann surface of genus g. There are exactly $2g + 2$ points $p_1, \ldots, p_{2g+2}$ with the following conditions*:

1. *For $p \in X \setminus \{p_1, \ldots, p_{2g+2}\}$ there exists a non-constant meromorphic function $f \in \mathcal{M}(X))$ which has a pole of order $g + 1$ at p and is holo-morphic in $X \setminus \{p\}$.*

2. *For $p \in X \setminus \{p_1, \ldots, p_{2g+2}\}$, every meromorphic function with a single pole at p, must have a pole of order $\geq g + 1$.*

2. Some Consequences of Riemann-Roch Theorem

We define a Weierstrass point by using gap sequence as follows:

Definition 4: *Suppose* $p \in X$ *and*

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_g < 2_g$$

be the gap sequence at p. In terms of the gap sequence we define the weight of the point p, denoted by $\omega(p)$ *by*

$$\sum_{i=1}^{g} (\lambda_i - i).$$

Note that $\omega(p) \geq 0$ for all $p \in X$.

Definition 5: *A point* $p \in X$ *is called a Weierstrass point if* $\omega(p) > 0$.

One can compute the number of Weierstrass points counted according to their weights on a compact Riemann surface X of genus g. It is equal to $g^3 - g = (g-1)g(g+1)$.

It follows that there are no Weierstrass points on the surfaces of genus $g = 0$ and genus $g = 1$. Also from the Theorem 1, it follows that there is no non-constant meromorphic function on torus ($g = 1$ surface) with a single simple (= order 1) pole.

The following theorem gives the bounds for Weierstrass points on a compact Riemann surface of genus $g \geq 2$.

Theorem 6: *Suppose X is a compact Riemann surface of genus* $g \geq 2$. *Let* $W(X)$ *denotes the number of Weierstrass points on X. Then*

$$2g + 2 \leq W(X) \leq g3 - g \tag{2}$$

A point p is called a hyperelliptic Weierstrass point if the non-gap sequence starts with 2 and the hyperelliptic Riemann surfaces are characterized by the gap sequence at the Weierstrass points:

$$P = \{1, 3, \ldots, 2g-1\} \tag{3}$$

hence the non-gaps are $Q = \{2, 4, \ldots, 2g\}$

Let X be a hyperelliptic Riemann surface and p be a Weierstrass point on X.

Then we can find

$$\omega(p) = [1 + 3 + \ldots + (2g-1)] - [1 + 2 + \ldots + g] = \frac{g(g-1)}{2} \tag{4}$$

Therefore, from the Equation 4 we can see that there are precisely $2g+2$ Weierstrass points on a hyperelliptic Riemann surface.

Remark 7: *In terms of Weierstrass points, the hyperelliptic Riemann surfaces may be characterized as the surfaces that attain the lower bound on the number of Weierstrass points.*

We restate the result which we mentioned in the beginning. For Proof See [2].

Theorem 8: *Let X be a compact Riemann surface of genus g and $p \in X$. Then the following conditions on X are equivalent*:

1. *There exists a non-constant meromorphic function on X which has a pole of order $\leq g$ at p and is holomorphic in $X \setminus \{p\}$.*

2. *p is a Weierstrass point.*

Remark 9: *If p is not a Weierstrass point then there is no meromorphic function on X, with a single pole of order $\leq g$.*

The following theorem is a consequence of the Riemann-Roch theorem:

Theorem 10: *Suppose X is a compact Riemann surface of genus g and p is a point of X. Then there is a non-constant meromorphic function f on X which has a pole of order $\leq g+1$ at p and is holomorphic in $X \setminus \{p\}$.*

3. Proof of Theorem 3

Proof: Let X be a hyperelliptic Riemann surface of genus g. Therefore, there are precisely $2g+2$ Weierstrass points points on X. Let $p_1, \dots, p_{2g+2}$ be the Weierstrass points on X.

Let $p \in X \setminus \{p_1, \dots, p_{2g+2}\}$. Therefore, p is not a Weierstrass point on X. From Theorem 8, there does not exist a meromorphic function $f \in \mathcal{M}(X)$ such that f has a pole of order $\leq g$ at p and is holomorphic in $X \setminus \{p\}$.

But by Theorem 10, there must exist a meromorphic function with a pole of order $\leq g+1$ at p and is holomorphic in $X \setminus \{p\}$.

Therefore, it follows that there exists a meromorphic function f with a pole of order $g+1$ and is holomorphic in $X \setminus \{p\}$.

The second statement is nothing but rephrasing the first one.

We state another result which is a simple consequence of Weierstrass gap sequence for hyperelliptic Riemann surfaces and Theorem 8.

Remark 11: *Suppose p is a Weierstrass point on a hyperelliptic Riemann surface X of genus g. If g is even there is no meromorphic function with a pole of order $g+1$ at p and is holomorphic in $X \setminus \{p\}$.*

4. Concluding remarks for the Hyperelliptic Riemann Surface of Genus $g = 4$

As an example, we sum up everything for the case of hyperelliptic Riemann surfaces of genus $g = 4$.

1. The gap sequence at the Weierstrass points is $1 < 3 < 5 < 7 < 8 = 2g$.

2. The non-gaps are $2, 4, 6, 8 = 2g$.

3. The number of Weierstrass points are $2.g + 2 = 10$.

4. If p is a Weierstrass point there exists a meromorphic function having pole at p of order 2 and is holomorphic in $X \setminus \{p\}$.

5. If p is a Weierstrass point there exists a meromorphic function having pole at p of order 4 and is holomorphic in $X \setminus \{p\}$.

6. If p is a Weierstrass point there is no meromorphic function with a single pole of order 3 or 5.

7. If p is not a Weierstrass point, a meromorphic function with a single pole must have order minimum ≥ 5 and there exists a meromorphic function f on X with a pole of order 5 and holomorphic in $X \setminus \{p\}$. But if p is a Weierstrass point then there does not exist a meromorphic function f with a single pole at p with order 5. This follows due to gap sequence of hyperelliptic Riemann surfaces at Weierstrass points.

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ON DIFFERENT REPRESENTATION OF GROWTH INDICATORS OF ENTIRE AND MEROMORPHIC FUNCTIONS

Abstract: In this paper we introduce the idea of generalized (p, q)-th relative (α, β) order and (p, q)-th generalized relative (α, β)-type as well as (p, q)-th generalized relative (α, β)-upper weak type of an entire function with respect to another entire function. Here, we study integral representation of (p, q)-th generalized relative (α, β)-type as well as (p, q)-th generalized relative (α, β)-upper weak type of an entire function with respect to another entire function. We also establish their equivalence relation under some certain condition, where α, β are non negative continuous functions defined on $(-\infty, +\infty)$ and p, q are all positive integers. Some examples are provided to justify the results.

Keywords: Entire Functions, Growth, Generalized Relative (α, β)-Order, Generalized Relative (α, β)-Type.

Mathematic Subject Classification (2020) No.: 32A15, 30D20, 30D35.

1. Introduction

1.1 Introduction, Definitions and Notations: Let us consider that the reader is familiar with the fundamental results and standard notations of the Nevanlinna theory of meromorphic functions which are available in [6, 9, 13]. We denote by $\mathbb{C}$, the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. The maximum modulus function $M_f(r)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined as

$$M_f(r) = \max\{|f(z)|:|z| = r\}.$$

Moreover f is non constant entire then $M_f(r)$ is also strictly increasing and continuous function of r. Therefore its inverse

$$M_f^{-1} : (M_f(0), \infty) \to (0, \infty)$$

exists and is such that

$$\lim_{s\to\infty} M_f^{-1}(s) = \infty.$$

We use the standard notations and definitions of the theory of entire functions which are available in [12] and therefore we do not explain those in details.

For $x \in [0, \infty)$, we define iteration of logarithmic and exponential functions as

$$\log^{[k]} x = \log(\log^{[k-1]} x) \text{ for } k = 1, 2, 3, ...$$

$$\log^{[0]} x = x,\ \log^{[-1]} x = \exp x$$

and

$$\exp^{[k]} x = \exp(\exp^{[k-1]} x) \text{ for } k = 1, 2, 3, ...$$

$$\exp^{[0]} x = x, \exp^{[-1]} x = \log x.$$

However let K be a class of continuous non negative function α on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$. For any $\alpha \in K$, we say that $\alpha \in K_1^0$ if

$$\alpha((1 + O(1))\, x) = (1 + O(1))\, \alpha\, (x) \text{ as } x \to +\infty$$

and $\alpha \in K_2^0$, if

$$\alpha(\exp(1 + O(1))\, x) = (1 + O(1))\, \alpha\, (\exp(x)) \text{ as } x \to +\infty.$$

Finally for any $\alpha \in K$, we also say that $\alpha \in K_1$ if

$$\alpha(cx) = (1 + O(1))\alpha(x) \text{ as } x_0 \leq x \to +\infty \text{ for each } c \in (0, +\infty).$$

and $\alpha \in K_2$ if

$$\alpha(\exp(cx) = (1 + O(1))\alpha(\exp(x)) \text{ as } x_0 \leq x \to +\infty \text{ for each } c \in (0, +\infty).$$

Clearly,

$$K_1 \subset K_1^0, K_2 \subset K_2^0 \text{ and } K_2 \subset K_1.$$

Considering this, the value

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to\infty} \frac{\alpha(\log M_f(r))}{\beta(\log r)}, (\alpha \in K, \beta \in K)$$

and

$$\lambda(f) = \liminf_{r\to\infty} \frac{\alpha(\log M_f(r))}{\beta(\log r)}, (\alpha \in K, \beta \in K)$$

are respectively called generalized (α, β)-order and generalized (α, β)-lower order of an entire function f [11]. For details about generalized (α, β)-order one may see [11]. During the past decades, several authors made closed investigations on the properties of entire functions related to generalized order and in some different directions and we get many important results from [4, 5, 6, 7, 8, 10]. For the purpose of future applications, several authors rewrite the definition of generalized (α, β) order of entire and meromorphic function in the following way after giving a minor modification to the original definition [11].

Definition 1.1 [6]: *The order and lower order of a meromorphic function f are defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r}$$

and
$$\lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r}.$$

If f is an entire function, then

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r}$$

Using the inequality

$$T_f(r) \le \log M_f(r) \le 3T_f(2r) \ \{cf.[6]\}$$

one may easily verify the above definition for an entire function.

Juneja. et.al. {cf. [8]} *defined* (p,q)*-th order* $\rho^{(p,q)}(f)$ *and* (p,q)*-th lower order* $\lambda^{(p,q)}(f)$ *of an entire function f are as*

$$\rho^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}$$

and

$$\lambda^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}$$

where $p \ge q$*. The function f is said to be of regular* (p,q) *growth when* (p,q)*-th order and* (p,q)*-th lower order of f are the same. Functions which are not of regular* (p,q) *growth are said to be of irregular* (p,q) *growth.*

However the above definition is very useful for measuring the growth of entire functions. If $p = l$ *and* $q = 1$ *then* $\rho^{(l,1)}(f) = \rho^{(l)}(f)$ *and*

$\lambda^{(l,1)}(f) = \lambda^{(l)}(f)$ *where* $\rho^{(l)}(f)$ *and* $\lambda^{(l)}(f)$ *are respectively known as generalized order and generalized lower order of function f* $\{cf.[10]\}$. *Also for* $p = 2$ *and* $q = 1$ *we respectively denote* $\rho^{(2,1)}(f)$ *and* $\lambda^{(2,1)}(f)$ *by* $\rho(f)$ *and* $\lambda(f)$ *which are classical growth indicators such as order and lower order of entire function f.*

Definition 1.2: *Let* $\alpha, \beta \in K$. *Then we define generalized* $(p, q)^{th}$ (α, β)*-order denoted by* $\rho^{(p,q)}_{(\alpha,\beta)}[f]$ *and generalized* $(p, q)^{th}$ (α, β)*-lower order denoted by* $\lambda^{(p,q)}_{(\alpha,\beta)}[f]$ *of an entire function f as*

$$\rho^{(p,q)}_{(\alpha,\beta)}[f] = \limsup_{r\to\infty} \frac{\alpha^{[p]}(M_f(r))}{\beta^{[q]}(r)}$$

and

$$\lambda^{(p,q)}_{(\alpha,\beta)}[f] = \liminf_{r\to\infty} \frac{\alpha^{[p]}(M_f(r))}{\beta^{[q]}(r)},$$

where p, q *are any two positive integers with* $p \geq q$. *Further, an entire function* f *is said to be of regular* (p, q)*-growth if its* (p, q)*-th order coincides with its* (p, q)*-th lower order, otherwise* f *is said to be of irregular* (p, q)*-growth.*

Definition 1.3: *Let f and g be any two entire functions. Bernal* [1, 2] *initiated the definition of relative order* $\rho_g(f)$ *of f with respect to g which keep away from comparing growth just with* $\exp z$ *to find out order of entire functions as follows*:

$$\rho_g(f) = \inf\{\mu > 0 : M_f(r) < M_g(r^{\mu}) \text{ for all } r > r_0(\mu) > 0\}$$

$$= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

Analogously, one may define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ *as*

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}$$

However, an entire function, for which order and lower order are the same, is said to be of regular growth. The function $\exp z$ *is an example of regular growth of entire functions. Further the functions which are not of regular growth are said to be of irregular growth.*

Definition 1.4 [3]: *Let* $\alpha, \beta \in K$. *The generalized relative* (α, β) *order denoted by* $\rho_{(\alpha,\beta)}[f]_g$ *and generalized relative* (α, β)*-lower order denoted by* $\lambda_{(\alpha,\beta)}[f]_g$ *of an entire function f with respect to another entire function g are defined as*:

$$\rho_{(\alpha,\beta)}[f]_g = \limsup_{r\to\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)}$$

and

$$\lambda_{(\alpha,\beta)}[f]_g = \liminf_{r\to\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)},$$

where $p \geq 1$.

Definition 1.5 [3]: *Let* $\alpha, \beta \in K$ *where* K *is defined earlier. The generalized relative* (α, β)*-type denoted by* $\sigma_{(\alpha,\beta)}[f]_g$ *and generalized relative* (α, β)*-lower type denoted by* $\bar{\sigma}_{(\alpha,\beta)}[f]_g$ *of an entire function f with respect to another entire function g having non-zero finite generalized relative order* (α, β) *are defined as*

$$\sigma_{(\alpha,\beta)}[f]_g = \limsup_{r\to\infty} \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}}$$

and

$$\overline{\sigma}_{(\alpha,\beta)}[f]_g = \liminf_{r\to\infty} \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)^{\rho_{(\alpha,\beta)}[f]_g}}.$$

Again the generalized relative (α,β)*-upper weak type denoted by* $\tau_{(\alpha,\beta)}[f]_g$ *and generalized relative* (α,β)*-lower weak type denoted* $\overline{\tau}_{(\alpha,\beta)}[f]_g$ *of an entire function f with respect to another entire function g having non-zero finite generalized relative lower order* (α,β) *are defined as*

$$\tau_{(\alpha,\beta)}[f]_g = \limsup_{r\to\infty} \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]_g}}$$

and

$$\overline{\tau}_{(\alpha,\beta)}[f]_g = \liminf_{r\to\infty} \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)^{\lambda_{(\alpha,\beta)}[f]_g}}.$$

Definition 1.6: *Let f and g be any two entire functions with maximum modulus functions* $M_f(r)$ *and* $M_g(r)$ *respectively, then the generalized relative* (α,β)*-order and generalized relative* (α,β)*-lower order) of f with respect to another entire function g, denoted by* $\rho^{(p,q)}_{(\alpha,\beta)}[f]_g$ *(respectively* $\lambda^{[p]}_{(\alpha,\beta)}[f]_g$*) is defined as*

$$\rho^{(p,q)}_{(\alpha,\beta)}[f]_g = \limsup_{r\to\infty} \frac{\alpha^{[p]}(M_g^{-1}(M_f(r))}{\beta^{[q]}(r)}$$

and

$$\lambda^{(p,q)}_{(\alpha,\beta)}[f]_g = \liminf_{r\to\infty} \frac{\alpha^{[p]}(M_g^{-1}(M_f(r))}{\beta^{[q]}(r)}.$$

where p, q are any two positive integers with $p \geq q$*. In particular if we consider* $q = 1$*, then the Definition* 1.6 *is reduced to Definition* 1.4*. These* *definitions extend the generalized relative order and generalized relative lower order of an entire function f with respect to another entire function g. Further an entire function f is said to be of regular relative* (p, q)*-th growth if its* (p, q)*-th relative order coincides with its* (p, q)*-th relative lower order, otherwise f is said to be of irregular relative* (p, q)*-th growth.*

Definition 1.7: *Let* $\alpha, \beta \in K$ *where K is defined earlier. Let f and g be any two entire functions with maximum modulus functions* $M_f(r)$ *and* $M_g(r)$ *respectively, then we define* (p, q)*-th generalized relative* (α, β)*-type denoted by* $\sigma_{(\alpha,\beta)}^{(p,q)}[f]_g$ *of an generalized relative* (α, β) *order* $\left(0 < \rho_{(\alpha,\beta)}^{(p,q)}[f]_g < \infty\right)$ *as*

$$\sigma_{(\alpha,\beta)}^{(p,q)}[f]_g = \limsup_{r\to\infty} \frac{\exp(\alpha^{[p]}(M_g^{-1}M_f(r)))}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}},$$

where p, q are positive integer such that $p \geq q$.

Definition 1.8: *Let* $\alpha, \beta \in K$ *and f, g be any two entire functions having finite positive generalized relative* (α, β) *order* $\left(0 < \rho_{(\alpha,\beta)}^{(p,q)}[f]_g < \infty\right)$*, where p, q are positive integer. Then* (p, q)*-th generalized relative* (α, β)*-type denoted by* $\sigma_{(\alpha,\beta)}^{(p,q)}[f]_g$ *of an entire function f with respect to another entire function g is define as:*

the integral

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{k+1}}$$

converges for $k > \sigma_{(\alpha,\beta)}^{(p,q)}[f]_g$ *and diverges for* $k < \sigma_{(\alpha,\beta)}^{(p,q)}[f]_g$.

Definition 1.9: *Let* $\alpha, \beta \in K$ *and* f *and* g *be any two entire functions with maximum modulus functions* $M_f(r)$ *and* $M_g(r)$ *respectively, then we define* (p, q)*-th generalized relative* (α, β)*-lower weak type denoted by* $\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g$ *of an entire function f with respect to another entire function g having finite positive generalized relative* (α, β)*-lower order* $\left(0 < \lambda_{(\alpha,\beta)}^{(p,q)}[f]_g < \infty\right)$ *as*

$$\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g = \liminf_{r\to\infty} \frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}},$$

where (p, q) *are positive integers such that* $p \geq q$.

Definition 1.10: *Let* $\alpha, \beta \in K$ *and f, g be any two entire functions having finite positive generalized relative* (α, β)*-lower order* $\left(0 < \lambda_{(\alpha,\beta)}^{(p,q)}[f]_g < \infty\right)$, *where p, q are positive integer. Then* (p, q)*-th generalized relative* (α, β)*-lower weak type denoted by* $\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g$ *of an entire function f with respect to another entire function g is define as:*

the integral

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{k+1}}$$

converges for $k > \overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g$ *and diverges for* $k < \overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g$.

Definition 1.11: *Let* $\alpha, \beta \in K$ *and* f *and* g *be any two entire function with maximum modulus functions* $M_f(r)$ *and* $M_g(r)$ *respectively, then we define* (p, q)*-th generalized relative* (α, β)*-lower type denoted by* $\overline{\sigma}_{(\alpha,\beta)}^{(p,q)}[f]_g$ *of* *an entire function f with respect to another entire function g having finite positive generalized relative* (α, β) *order* $\left(0 < \rho^{(p,q)}_{(\alpha,\beta)}[f]_g < \infty\right)$ *as*

$$\bar{\sigma}^{(p,q)}_{(\alpha,\beta)}[f]_g = \liminf_{r\to\infty} \frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho^{(p,q)}_{(\alpha,\beta)}[f]_g}},$$

where (p, q) *are positive integers such that* $p \geq q$.

Definition 1.12: *Let* $\alpha, \beta \in K$ *and f and g be any two entire functions having finite positive generalized relative* (α, β) *order* $\left(0 < \rho^{(p,q)}_{(\alpha,\beta)}[f]_g < \infty\right)$ *where p, q are positive integer. Then* (p, q)*-th generalized relative* (α, β)*-lower type denoted by* $\bar{\sigma}^{(p,q)}_{(\alpha,\beta)}[f]_g$ *of an entire function f with respect to another entire function g is define as:*

the integral

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho^{(p,q)}_{(\alpha,\beta)}[f]_g}\right]^{k+1}}$$

converges for $k > \bar{\sigma}^{(p,q)}_{(\alpha,\beta)}[f]_g$ *and diverges for* $k < \bar{\sigma}^{(p,q)}_{(\alpha,\beta)}[f]_g$.

Definition 1.13: *Let* $\alpha, \beta \in K$ *and* $f(z)$ *and* $g(z)$ *be any two entire functions having finite positive generalized relative* (α, β) *lower order* $\left(0 < \lambda^{(p,q)}_{(\alpha,\beta)}[f]_g < \infty\right)$, *where p, q are positive integer. Then* (p, q)*-th generalized relative upper weak type* (α, β) *denoted by* $\tau^{(p,q)}_{(\alpha,\beta)}[f]_g$ *of an entire function f with respect to another entire function g is defined as*

$$\tau_{(\alpha,\beta)}^{(p,q)}[f]_g = \limsup_{r\to\infty} \frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}}.$$

Definition 1.14: *Let* $\alpha, \beta \in K$ *and f and g be any two entire functions having finite positive generalized relative* (α, β)*-lower order* $\left(0 < \lambda_{(\alpha,\beta)}^{(p,q)}[f]_g < \infty\right)$, *where p, q are positive integer. Then (p, q)-th generalized relative* (α, β)*-upper weak type denoted by* $\tau_{(\alpha,\beta)}^{(p,q)}[f]_g$ *of an entire function f with respect to another entire function g is define as:*

the integral

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{k+1}}$$

converges for $k > \tau_{(\alpha,\beta)}^{(p,q)}[f]_g$ *and diverges for* $k < \tau_{(\alpha,\beta)}^{(p,q)}[f]_g$.

In this paper, we wish to establish the equivalence of definitions of (p, q)-th generalized relative type and (p, q)-th generalized relative lower weak type with their integral representations.

2. Lemma

Lemma 2.1: *Let* $\alpha, \beta \in K$ *and f, g be any two entire functions and let the integral*

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{A}\right]^{k+1}} dr, (r_0 > 0)$$

converges where $0 < A < \infty$. *Then*

$$\lim_{r\to\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^A\right]^k} = 0.$$

Proof: Since the integral

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^A\right]^{k+1}} dr, \ (r_0 > 0)$$

converges then

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^A\right]^{k+1}} dr < \varepsilon \quad \text{if} \quad r_0 > R(\varepsilon).$$

Therefore,

$$\int_{r_0}^{\exp\left(\exp\left(\beta^{[q]}(r_0)\right)\right)^A + r_0} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^A\right]^{k+1}} dr < \varepsilon.$$

Since, $\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)$ increases with r, so

$$\int_{r_0}^{\exp\left(\exp\left(\beta^{[q]}(r_0)\right)\right)^A + r_0} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^A\right]^{k+1}} dr$$

$$\geq \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r_0)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r_0)\right)\right)^A\right]^{k+1}} \cdot \left[\exp\left(\exp\left(\beta^{[q]}(r_0)\right)\right)^A\right]$$

i.e., for all large values of r,

$$\int_{r_0}^{\exp\left(\exp\left(\beta^{[q]}(r_0)\right)\right)^A+r_0} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^A\right]^{k+1}}\,dr$$

$$\geq \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r_0)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r_0)\right)\right)^A\right]^{k}}$$

so that

$$\frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r_0)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r_0)\right)\right)^A\right]^{k}} < \varepsilon \text{ if } r_0 > R(\varepsilon)$$

Therefore,

$$\lim_{r\to\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r_0)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^A\right]^{k}} = 0.$$

This proves the Lemma. ■

3. Main Results

In this section we state the main results of the paper.

Theorem 3.1: *Let* $\alpha, \beta \in K$ *and* f, g *be any two entire function having finite positive* (p, q)*-th generalized relative* (α, β)*-order* $\rho^{(p,q)}_{(\alpha,\beta)}[f]_g$, $\left(0 < \rho^{(p,q)}_{(\alpha,\beta)}[f]_g < \infty\right)$ *and* (p, q)*-th generalized relative* (α, β)*-type* $\sigma^{(p,q)}_{(\alpha,\beta)}[f]_g$ *where* p, q *are any two positive integers. Then Definition* 1.7 *and Definition* 1.8 *are equivalent.*

Proof: Let $f(z)$ and $g(z)$ be any two entire functions such that $\rho_{(\alpha,\beta)}^{(p,q)}[f]_g$, $\left(0 < \rho_{(\alpha,\beta)}^{(p,q)}[f]_g < \infty\right)$ exists , where p, q are any two positive integers.

Case I: Let

$$\sigma_{(\alpha,\beta)}^{(p,q)}[f]_g = \infty.$$

Definition 1.7 $\Rightarrow$ Definition 1.8.

As $\sigma_{(\alpha,\beta)}^{(p,q)}[f]_g = \infty$, from Definition 1.7 we have for an arbitrary $G > 0$ and a sequence of values of r tending to infinity,

$$\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r_0)\right)\right) > G \cdot \left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g},$$

$$\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) > \left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^G. \tag{1}$$

If possible let the integral

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{G+1}} dr, \, (r_0 > 0)$$

be converge. Then by Lemma 2.1

$$\limsup_{r\to\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{G}} = 0.$$

So for all sufficiently large values of r,

$$\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) < \left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^G.$$

Therefore, by Equation (1) and Equation (2) we arrive at a contadriction.

Hence,

$$\int_{r_0}^{\infty}\frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{G+1}}dr, (r_0 > 0)$$

diverges where $G > 0$ is finite , which is Definition 1.8.

Now we show Definition 1.8 $\Rightarrow$ Definition 1.7.

Let G be any positive number. Since

$$\sigma_{(\alpha,\beta)}^{(p,q)}[f]_g = \infty,$$

from Definition 1.8 the divergence of the integral

$$\int_{r_0}^{\infty}\frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{G+1}}dr, (r_0 > 0)$$

gives an arbitrary positive ε and for a sequence of values of r tending to infinity

$$\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) > \left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{G-\varepsilon},$$

$$\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) > (G-\varepsilon)\cdot\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}$$

which implies that

$$\frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}}G > \varepsilon.$$

Since $G > 0$ is arbitrary, it follows that

$$\limsup_{r\to\infty}\frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}} = \infty$$

i.e.,

$$\sigma_{(\alpha,\beta)}^{(p,q)}[f]_g = \infty.$$

Thus, Definition 1.7 follows.

Case II: Let

$$0 \le \sigma_{(\alpha,\beta)}^{(p,q)}[f]_g = \infty.$$

First we show that Definition 1.7 $\Rightarrow$ Definition 1.8.

Sub case (A):

$$0 \le \sigma_{(\alpha,\beta)}^{(p,q)}[f]_g = \infty.$$

Let *f*, g be any two entire functions such that

$$0 \le \sigma_{(\alpha,\beta)}^{(p,q)}[f]_g = \infty$$

exists for positive integers p, q. Then according to Definition 1.7, for any arbitrary positive ε and for large values of r we obtain that

$$\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) < \left(\sigma_{(\alpha,\beta)}^{(p,q)}[f]_g + \varepsilon\right)\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}$$

i.e.,

$$\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) < \left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho^{(p,q)}_{(\alpha,\beta)}[f]_g}\right]^{\left(\sigma^{(p,q)}_{(\alpha,\beta)}[f]_g+\varepsilon\right)},$$

i.e.,

$$\frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho^{(p,q)}_{(\alpha,\beta)}[f]_g}\right]^k} < \frac{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho^{(p,q)}_{(\alpha,\beta)}[f]_g}\right]^{\left(\sigma^{(p,q)}_{(\alpha,\beta)}[f]_g+\varepsilon\right)}}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho^{(p,q)}_{(\alpha,\beta)}[f]_g}\right]^k}$$

i.e.,

$$\frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho^{(p,q)}_{(\alpha,\beta)}[f]_g}\right]^k} < \frac{1}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho^{(p,q)}_{(\alpha,\beta)}[f]_g}\right]^{k-\left(\sigma^{(p,q)}_{(\alpha,\beta)}[f]_g+\varepsilon\right)}}.$$

Therefore,

$$\int_{r_0}^{\infty}\frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho^{(p,q)}_{(\alpha,\beta)}[f]_g}\right]^{k+1}}dr, (r_0 > 0)$$

converges for $k > \sigma^{(p,q)}_{(\alpha,\beta)}[f]_g$.

Again using Definition 1.7 we obtain for a sequence of values of r tending to infinity that

$$\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) > \left(\sigma^{(p,q)}_{(\alpha,\beta)}[f]_g-\varepsilon\right)\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho^{(p,q)}_{(\alpha,\beta)}[f]_g}$$

i.e.,

$$\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) > \left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{\left(\sigma_{(\alpha,\beta)}^{(p,q)}[f]_g-\varepsilon\right)}, \quad (3)$$

so for $k < \sigma_{(\alpha,\beta)}^{(p,q)}[f]_g$, we get from Equation (3) that

$$\frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^k} > \frac{1}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{k-\left(\sigma_{(\alpha,\beta)}^{(p,q)}[f]_g-\varepsilon\right)}}.$$

Therefore,

$$\int_{r_0}^{\infty}\frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{k+1}}dr, (r_0 > 0)$$

diverges for $k < \sigma_{(\alpha,\beta)}^{(p,q)}[f]_g$. Hence,

$$\int_{r_0}^{\infty}\frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{k+1}}dr, (r_0 > 0)$$

converges for $k > \sigma_{(\alpha,\beta)}^{(p,q)}[f]_g$ and diverges for $k < \sigma_{(\alpha,\beta)}^{(p,q)}[f]_g$.

Sub case (B):

$$\sigma_{(\alpha,\beta)}^{(p,q)}[f]_g = 0.$$

When $\sigma_{(\alpha,\beta)}^{(p,q)}[f]_g = 0$ for positive integers p, q Definition (1.7) gives for all sufficiently large values of r that

$$\frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho^{(p,q)}_{(\alpha,\beta)}[f]_g}} < \varepsilon.$$

Then similarly as before we get that

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho^{(p,q)}_{(\alpha,\beta)}[f]_g}\right]^{k+1}} dr, (r_0 > 0)$$

converges for $k > 0$ and diverges for $k < 0$. Thus, combining Subcase (A) and Subcase (B) Definition 1.8 follows.

Now we show Definition 1.8 $\Rightarrow$ Definition 1.7.

From Definition 1.8 and arbitrary positive ε, the integral

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho^{(p,q)}_{(\alpha,\beta)}[f]_g}\right]^{\rho^{(p,q)}_{(\alpha,\beta)}[f]_g+\varepsilon+1}} dr, (r_0 > 0)$$

converges. Then by Lemma 2.1 we get

$$\frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho^{(p,q)}_{(\alpha,\beta)}[f]_g}\right]^{\sigma^{(p,q)}_{(\alpha,\beta)}[f]_g+\varepsilon}} = 0.$$

So, we obtain for all sufficiently large values of r that

$$\frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{\sigma_{(\alpha,\beta)}^{(p,q)}[f]_g+\varepsilon}} < \varepsilon$$

i.e.,

$$\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) < \varepsilon.\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{\sigma_{(\alpha,\beta)}^{(p,q)}[f]_g+\varepsilon},$$

$$\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) < \log\varepsilon + \left(\sigma_{(\alpha,\beta)}^{(p,q)}[f]_g + \varepsilon\right)\left\{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right\},$$

i.e.,

$$\limsup_{r\to\infty}\frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}} \le \sigma_{(\alpha,\beta)}^{(p,q)}[f]_g + \varepsilon.$$

Since $\varepsilon = 0$ is arbitrary, it follows that

$$\limsup_{r\to\infty}\frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}} \le \sigma_{(\alpha,\beta)}^{(p,q)}[f]_g. \quad (4)$$

On the other hand the divergence of the integral

$$\int_{r_0}^{\infty}\frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{\sigma_{(\alpha,\beta)}^{(p,q)}[f]_g-\varepsilon+1}}dr, (r_0 > 0)$$

implies that there exists a sequence values of *r* tending to infinity such that

$$\frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g-\varepsilon+1}} > \frac{1}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{1+\varepsilon}},$$

i.e.,

$$\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) > \left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{\sigma_{(\alpha,\beta)}^{(p,q)}[f]_g-2\varepsilon},$$

i.e.,

$$\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) > \left(\sigma_{(\alpha,\beta)}^{(p,q)}[f]_g - 2\varepsilon\right)\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g},$$

i.e.,

$$\frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}} > \left(\sigma_{(\alpha,\beta)}^{(p,q)}[f]_g - 2\varepsilon\right).$$

Since $\varepsilon > 0$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}} \geq \sigma_{(\alpha,\beta)}^{(p,q)}[f]_g. \tag{5}$$

So from Equation (4) and Equation (5) we obtain that

$$\limsup_{r\to\infty} \frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}} = \sigma_{(\alpha,\beta)}^{(p,q)}[f]_g.$$

This proves the theorem. ■

Remark 3.1: *We give an example below which validates Theorem* 3.1.

Example 1: Let $f(z) = \exp(z^2)$, $g(z) = \log z$, $(z > 0)$, $p = 4$ and $q = 2. So\ g^{-1}(z) = \exp(z)$

$$\rho^{(p,q)}_{(\alpha,\beta)}[f]_g = \limsup_{r\to\infty} \frac{\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)}{\beta^{[q]}(r)}$$

$$= \limsup_{r\to\infty} \frac{\log^{[4]}\left(\exp\left(\exp\left(r^2\right)\right)\right)}{\log^{[2]} r}$$

$$= \limsup_{r\to\infty} \frac{\log^{[2]} r^2}{\log^{[2]} r}$$

$$= \limsup_{r\to\infty} \frac{\log 2 + \log^{[2]} r}{\log^{[2]} r}$$

$$= 1.$$

$Again$

$$\sigma^{(p,q)}_{(\alpha,\beta)}[f]_g = \limsup_{r\to\infty} \frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho^{(p,q)}_{(\alpha,\beta)}[f]_g}}$$

$$= \limsup_{r\to\infty} \frac{\exp\left(\log^{[4]}\left(\exp\left(\exp(r)\right)\right)\right)}{\left(\exp\left(\log^{[2]} r\right)\right)^1}$$

$$= \limsup_{r\to\infty} \frac{\log r^2}{\log r}$$

$$= 1.$$

Next if we take $k = 3$, *that is* $k > \sigma_{(\alpha,\beta)}^{(p,q)}[f]_g$ *we see that the value of the integral for* $r_0 = 0$,

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{k+1}}dr,$$

$$= \int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\log^{[4]}\left(\exp(\exp r^2)\right)\right)}{\left[\exp\left(\exp\left(\log^{[2]}(r)\right)\right)^1\right]^4}dr$$

$$= \int_{r_0}^{\infty} \frac{r^2}{r^4}dr$$

$$= \int_{r_0}^{\infty} \frac{1}{r^2}dr$$

$$= \frac{1}{r_0},$$

which coverges. Next if we take $k = 3$, *that is* $k > \sigma_{(\alpha,\beta)}^{(p,q)}[f]_g$ *we see that the value of the integral for* $r_0 = 0$,

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\rho_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{k+1}}dr,$$

$$= \int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\log^{[4]}\left(\exp(\exp r^2)\right)\right)}{\left[\exp\left(\exp\left(\log^{[2]}(r)\right)\right)^1\right]^2}dr$$

$$= \int_{r_0}^{\infty} \frac{r^2}{r^2} dr$$

$$= [r]_{r_0}^{\infty}$$

$$= \infty,$$

which diverges.

Theorem 3.2: *Let $f,\ g \in A(K)$ be any two entire functions having finite positive (p, q)-th generalized relative (α, β)-lower order $\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g$, $\left(0 < \lambda_{(\alpha,\beta)}^{(p,q)}[f]_g < \infty\right)$ and (p, q)-th generalized relative (α, β)-lower weak type, $\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g$, where $p,\ q$ are any two positive integers. Then Definition 1.9 and Definition 1.10 are equivalent.*

Proof: Case I: Let

$$\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g = \infty.$$

Definition 1.9 $\Rightarrow$ Definition 1.10.

As $\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g = \infty,$ from Definition 1.9 we get for an arbitrary positive G and for all sufficiently large values of r that

$$\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) > G \cdot \left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g},$$

i.e.,

$$\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) > \left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^G. \tag{6}$$

If possible, let the integral

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{G+1}} dr, (r_0 > 0),$$

be convergent. Then by Lemma 2.1

$$\liminf_{r\to\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{G}} = 0.$$

So, for a sequence of values of r tending to infinity, we get that

$$\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) < \left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{G}. \qquad (7)$$

Therefore, from Equation (6) and Equation (7) we arrive at a contradiction. Hence,

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{G+1}} dr, (r_0 > 0)$$

diverges, whenever G is finite which is Definition 1.10.

Now we show Definition 1.10 $\Rightarrow$ Definition 1.9.

Let G be any positive number. Since,

$$\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g = \infty,$$

from Definition 1.10 the divergence of the integral

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{G+1}} dr, (r_0 > 0)$$

gives an arbitrary positive ε and for all sufficiently large values of r that

$$\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) > \left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{G-\varepsilon}$$

$$\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) > (G-\varepsilon)\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g},$$

which implies that

$$\frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}} > G-\varepsilon.$$

i.e.,

$$\liminf_{r\to\infty} \frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}} \geq G-\varepsilon.$$

Since, $G > 0$ is arbitrary, it follows that

$$\liminf_{r\to\infty} \frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}} = \infty.$$

i.e.,

$$\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g = \infty.$$

Thus, Definition 1.9 follows.

Case II: Let

$$0 \leq \overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g < \infty.$$

First we show that Definition 1.9 $\Rightarrow$ Definition 1.10.

Sub case (A_1)**:**

$$0 < \overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g < \infty.$$

Let *f*, g be any two entire functions such that

$$0 < \overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g < \infty$$

exists for positive integers p, q. Then according to Definition 1.9 for any arbitrary positive ε and for large value of r we obtain that

$$\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) < \left(\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g + \varepsilon\right)\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}$$

i.e.,

$$\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) < \left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{\left(\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g+\varepsilon\right)},$$

i.e.,

$$\frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^k} < \frac{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{\left(\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g+\varepsilon\right)}}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^k},$$

i.e.,

$$\frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^k} < \frac{1}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{k-\left(\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g+\varepsilon\right)}}.$$

Therefore,

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{k+1}} dr, (r_0 > 0)$$

converges for $k > \overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g$.

Again by Definition 1.9 we obtain for all sufficiently large values of r that

$$\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) > \left(\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g - \varepsilon\right)\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}$$

i.e.,

$$\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) > \left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{\left(\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g + \varepsilon\right)}. \qquad (8)$$

So, for $k < \overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g$ we get from Equation (8) that

$$\frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{k}} > \frac{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{\left(\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g + \varepsilon\right)}}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{k}}$$

i.e.,

$$\frac{\log^{[p-2]} |\hat{g}| \left(|f|\,(r)\right)}{\left[\exp\left\{\left(\log^{[q-1]} r\right)^{\lambda_g^{p,q}[f]_g}\right\}\right]^k} > \frac{1}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{k-\left(\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g + \varepsilon\right)}}.$$

Therefore,

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{k+1}} dr, (r_0 > 0)$$

diverges for $k < \overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g$. Hence,

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{k+1}} dr, (r_0 > 0)$$

converges for $k > \overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g$ and diverges for $k < \overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g$.

Subcase (B_1)**:**

$$\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g = 0$$

when $\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g = 0$, for any positive integer p, q Definition 1.9 gives for a sequence of values of r tending to infinity that

$$\frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}} < \varepsilon.$$

Then similarly as before we get that

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{k+1}} dr, (r_0 > 0)$$

converges for $k > 0$ and diverges for $k < 0$. Thus, combining Subcase (A_1) and Subcase (B_1) Definition 1.10 follows.

Now we show that Definition 1.10 $\Rightarrow$ Definition 1.9.

From Definition 1.10 and arbitrary positive ε, the integral

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g+\varepsilon+1}} dr, \; (r_0 > 0)$$

converges. Then by Lemma 2.1, we get that

$$\liminf_{r\to\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g+\varepsilon}} < \varepsilon,$$

i.e.,

$$\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) < \varepsilon \cdot \left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g+\varepsilon},$$

i.e.,

$$\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) < \log\varepsilon + \left(\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g + \varepsilon\right)\left\{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right\},$$

i.e.,

$$\liminf_{r\to\infty} \frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}} \leq \overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\liminf_{r\to\infty} \frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}} \leq \overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g. \tag{9}$$

On the other hand, the divergence of the integral

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g-\varepsilon+1}} dr, (r_0 > 0)$$

implies for all sufficiently large values of r that

$$\frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g-\varepsilon+1}} > \frac{1}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{1+\varepsilon}},$$

$$\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) > \left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g-2\varepsilon},$$

i.e.,

$$\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right) > \left(\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g - 2\varepsilon\right)\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g},$$

i.e.,

$$\frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}} > \left(\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g - 2\varepsilon\right).$$

Since $\varepsilon > 0$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}} \geq \overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g. \tag{10}$$

So, from Equation (9) and Equation (10) we obtain that

$$\liminf_{r\to\infty} \frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}} = \overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g.$$

This proves the theorem. ■

Remark 3.2: *We give an example below which validates the Theorem* 3.4

Example 2: *Let* $f(z) = \exp^{[2]} z$, $g(z) = \exp z$, $p = 3$ *and* $q = 2.So\hat{g}(z) = \log(z)$

$$\lambda_g^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)}{(\beta^{[q]}r)}$$

$$= \liminf_{r\to\infty} \frac{\log^{[3]}\left(\log \exp^{[2]}(r)\right)}{\log^{[2]} r}$$

$$= \liminf_{r\to\infty} \frac{\log^{[2]} r}{\log^{[2]} r}$$

$$= 1.$$

Again

$$\overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g = \liminf_{r\to\infty} \frac{\exp\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}}$$

$$= \liminf_{r\to\infty} \frac{\exp\left(\log^{[3]}\left(\log \exp^{[2]} r\right)\right)}{\left(\exp\left(\log^{[2]} r\right)\right)^1}$$

$$= \liminf_{r\to\infty} \frac{\log r}{\log r}$$

$$= 1.$$

Next if we take $k = 2$*, that is* $k > \overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g$ *we see that the value of the integral for* $r_0 > 0$,

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{k+1}}dr,$$

$$= \int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\log^{[3]}\left(\exp\exp^{[2]} r\right)\right)}{\left[\exp\left(\exp\left(\log^{[2]}(r)\right)\right)^{1}\right]^{3}}dr$$

$$= \int_{r_0}^{\infty} \frac{r}{r^3}dr$$

$$= \int_{r_0}^{\infty} r^{-2}dr$$

$$= \frac{1}{r_0},$$

which converges. Next if we take $k = 0$*, that is* $k < \overline{\tau}_{(\alpha,\beta)}^{(p,q)}[f]_g$ *we see that the value of the integral for* $r_0 = 0$,

$$\int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\alpha^{[p]}\left(M_g^{-1}M_f(r)\right)\right)}{\left[\exp\left(\exp\left(\beta^{[q]}(r)\right)\right)^{\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g}\right]^{k+1}}dr,$$

$$= \int_{r_0}^{\infty} \frac{\exp^{[2]}\left(\log^{[3]}\left(\log\exp^{[2]} r\right)\right)}{\left[\exp\left(\exp\left(\log^{[2]}(r)\right)\right)^{1}\right]^{1}}dr$$

$$= \int_{r_0}^{\infty} \frac{r}{r} dr$$

$$= [r]_{r_0}^{\infty}$$

$$= \infty,$$

which diverges.

Corollary 3.1: *Let* $\alpha, \beta \in K$ *and* $f(z)$ *and* $g(z)$ *be any two entire functions having finite positive* (p, q)*-th generalized relative* (α, β) *order* $\rho_{(\alpha,\beta)}^{(p,q)}[f]_g$, $\left(0 < \rho_{(\alpha,\beta)}^{(p,q)}[f]_g < \infty\right)$ *and* (p, q)*-th generalized relative* (α, β)*-lower type,* $\bar{\sigma}_{(\alpha,\beta)}^{(p,q)}[f]_g$ *where p, q are any two positive integers. Then Definition* 1.11 *and Definition* 1.12 *are equivalent.*

Corollary 3.2: *Let* $\alpha, \beta \in K$ *and f, g be any two entire functions having finite positive* (p, q)*-th generalized relative* (α, β)*-lower order* $\lambda_{(\alpha,\beta)}^{(p,q)}[f]_g$, $\left(0 < \lambda_{(\alpha,\beta)}^{(p,q)}[f]_g < \infty\right)$ *and* (p, q)*-th generalized relative* (α, β)*-upper weak type ,* $\tau_{(\alpha,\beta)}^{(p,q)}[f]_g$*, where p, q are any two positive integers. Then Definition* 1.13 *and Definition* 1.14 *are equivalent.*

Conclusion and Future Prospect

After introducing the idea of generalized relative (α, β)-order (lower order) and generalized relative (α, β)-type (lower type) of an entire function of complex variable with respect to another entire function, where α, β are non negative continuous functions defined on $(-\infty, +\infty)$, here in this paper we study different representation of type (lower type) and upper weak type (lower type) of entire functions with respect to another entire function. This assumption is also used to modify the idea of generalized relative (α, β)- type (lower type) and generalized relative (α, β)-upper weak type (lower weak type) of an entire function as well as meromorphic function by using non-decreasing unbounded function ψ, where $\psi : [0, \infty) \to (0, \infty)$ satisfying the following two conditions:

$$\text{(i)} \qquad = \lim_{r\to\infty} \frac{\log^{[p]} r}{\log^{[q]} \psi(r)} = 0$$

and

$$\text{(ii)} \qquad = \lim_{r\to\infty} \frac{\log^{[q]} \psi(\alpha r)}{\log^{[q]} \psi(r)} = 1.$$

Taking this modification we derive some results which will no doubt inspire the future researcher to derive some growth properties of entire and meromorphic functions of n complex variables.

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4-TOTALMEAN CORDIAL LABELLING OF st(m_1, m_2, m_3, m_4, m_5)

Abstract: Let G be a graph. Let $f: V(G) \rightarrow \{0, 1,2, \dots\dots k \quad 1\}$ be a function where $k \in$ and $k > 1$. For each edge uv, assign the label $f(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. $f$is called a k – total mean cordial labeling of G if $\left|t_{mf}(\text{i}) - t_{mf}(\text{j})\right| \leq 1$, for all $i, j \in \{0, 1, 2, \dots\dots, k \quad 1\}$, where $t_{mf}(x)$ denotes the total number of vertices and edges labelled with x, $x \in \{0, 1, 2, \dots\dots, k \quad 1\}$. A graph with admit a k – total mean cordial labeling is called k – total mean cordial graph. In this paper we investigate the 4-total mean cordial labeling of $st(n,n), st(1,1,n), st(1,2,n)$, $st(2,2,n), st(2,3,n), st(n,n,n), st(n,n,n,n), st(n,n,n,n,n)$.

Keywords: Star, Path, Complete Bipartite Graph, Union of Graph.

Mathematics Subject Classification: 05C78.

1. Introduction

In this paper we consider simple, finite and undirected graphs only. Cordial labeling was introduced by Cahit [1]. The notion of k-total mean cordial labeling has been introduced in [5]. The 4–total mean cordial labeling behaviour of several graphs like cycle, complete graph, star, bistar, comb and crown have been studied in [5, 6, 7, 8, 9, 10, 11, 12, 13]. Super edge-magic labeling behaviour of $st(m,n), st(1,1,n), st(1,2,n), st(2,2,n), st(2,3,n)$ was studied in [4]. In this paper we investigate the 4-total mean cordial labeling of $st(n,n), st(1,1,n), st(1,2,n), st(2,2,n), st(2,3,n), st(n,n,n), st(n,n,n,n), st(n,n,n,n,n)$.

Let x be any real number. Then x stands for the smallest integer greater than or equal to x. Terms are not defined here follow from Harary [3] and Gallian [2].

2. k–Total Mean Cordial Graph

Definition 2.1: Let G be a graph. Let $f: V(G) \rightarrow \{0,1,2,\ldots\ldots,k \quad 1\}$ be a function where $k \in$ and $k > 1$. For each edge uv, assign the label $f(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called ak–total mean cordial labeling of G if $\left|t_{mf}(\mathrm{i}) - t_{mf}(\mathrm{j})\right| \leq 1$, for all $i,j \in \{0,1,2,\ldots\ldots,k \quad 1\}$, where $t_{mf}(x)$ denotes the total number of vertices and edges labelled with x, $x \in \{0,1,2,\ldots.,k \quad 1\}$. A graph with admit a k–total mean cordial labeling is called k–total mean cordial graph.

3. PRELIMINARIES

Definition 3.1 [3]: The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Definition 3.2 [3]: The complete bipartite graph $K_{1,n}$ is called the Star.

Definition 3.3 [4]: The graph$st(a_1, a_2, \ldots a_n)$ denote the disjoint union of the n stars K_{1,a_1}, K_{1,a_2},……..,K_{1,a_n}.

4. Main Results

Theorem 4.1: The graph $st(n,n)$ is a 4-total mean cordial for all values of n.

Proof: Let $V(st(n,n)) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $E(st(n,n)) = \{uu_i, vv_i : 1 \leq i \leq n\}$. Obviously $|V(st(n,n))| + |E(st(n,n))| = 4n + 2$.

Assign the labels 1,3 to the vertices u, v respectively.

Now we assign the label 0 to the n vertices $u_1, u_2, \ldots, u_n$. Next we assign the label 2 to the n vertices $v_1, v_2, \ldots, v_n$.

Clearly $t_{mf}(0) = t_{mf}(2) = n$; $t_{mf}(1) = t_{mf}(3) = n + 1$.

Theorem 4.2: The graph $st(1,1,n)$ is 4-total mean cordial for all values of n.

Proof: Let $V(st(1,1,n)) = \{u, v, w, u_1, v_1, w_i : 1 \leq i \leq n\}$ and $E(st(1,1,n)) = \{uu_1, vv_1, ww_i : 1 \leq i \leq n\}$.

Note that $|V(st(1,1,n))| + |E(st(1,1,n))| = 2n + 7$.

Assign the labels 0,0,1,2,3 to the vertices u, v, w, u_1, v_1 respectively.

Case 1: $n \equiv 0 \ (mod\ 2)$.

Let $n = 2r$, $r \in$. Assign the label 0 to the r vertices $w_1, w_2, \ldots, w_r$. Now we assign the label 3 to the r vertices $w_{r+1}, w_{r+2}, \ldots, w_{2r}$.

Case 2: $n \equiv 1 \ (mod\ 2)$.

Let $n = 2r + 1$, $r \geq 0$. Label the vertices $w_i (1 \leq i \leq 2r)$ as in Case 1. Next we assign the label 3 to the vertex w_{2r+1}.

Thus, this vertex labeling f is a 4-total mean cordial labeling follows from the Table 1.

Order of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 2r$	$r + 2$	$r + 2$	$r + 2$	$r + 1$
$n = 2r + 1$	$r + 2$	$r + 2$	$r + 3$	$r + 2$

Table 1

Theorem 4.3: The graph $st(1,2,n)$ is 4-total mean cordial for all values of n.

Proof: Let $V(st(1,2,n)) = \{u, v, w, u_1, v_1, v_2, w_i : 1 \leq i \leq n\}$ and $E(st(1,2,n)) = \{uu_1, vv_1, vv_2, ww_i : 1 \leq i \leq n\}$.

Clearly $|V(st(1,2,n))| + |E(st(1,2,n))| = 2n + 9$.

Assign the labels 0,0,1,1,3,3 to the vertices u, v, w, u_1, v_1, v_2 respectively.

Case 1: $n \equiv 0 \ (mod\ 2)$.

Let $n = 2r$, $r \in$. Now we assign the label 0 to the r vertices $w_1, w_2, \ldots, w_r$. Next we assign the label 3 to the r vertices $w_{r+1}, w_{r+2}, \ldots, w_{2r}$.

Case 2: $n \equiv 1\ (mod\ 2)$.

Let $n = 2r+1$, $r \geq 0$. As in Case 1 assign the label to the vertices $w_i (1 \leq i \leq 2r)$. Now we assign the label 3 to the vertex w_{2r+1}.

Note that this vertex labeling f is a 4-total mean cordial labeling follows from the Table 2.

Nature of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 2r$	$r+2$	$r+3$	$r+2$	$r+2$
$n = 2r+1$	$r+2$	$r+3$	$r+3$	$r+3$

Table 2

Theorem 4.4: The graph $st(2,2,n)$ is 4-total mean cordial for all values of n.

Proof: Let $V(st(2,2,n)) = \{u, v, w, u_1, u_2, v_1, v_2, w_i : 1 \leq i \leq n\}$ and $E(st(2,2,n)) = \{uu_1, uu_2, vv_1, vv_2, ww_i : 1 \leq i \leq n\}$.

Note that $|V(st(2,2,n))| + |E(st(2,2,n))| = 2n + 11$.

Assign the labels $0, 1, 1, 0, 1, 3, 3$ to the vertices $u, v, w, u_1, u_2, v_1, v_2$ respectively.

Case 1: $n \equiv 0\ (mod\ 2)$.

Let $n = 2r$, $r \in$. Assign the label 0 to the r 1 vertices $w_1, w_2, \ldots, w_{r-1}$.

Now we assign the label 3 to the $r+1$ vertices $w_r, w_{r+1}, \ldots, w_{2r}$.

Case 2: $n \equiv 1\ (mod\ 2)$.

Let $n = 2r+1$, $r \geq 0$. In this case assign the label for the vertices $w_i (1 \leq i \leq 2r)$ as in Case 1. Finally we assign the label 0 to the vertex w_{2r+1}.

Thus, this vertex labeling f is a 4-total mean cordial labeling follows from the Table 3.

n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 2r$	$r+2$	$r+3$	$r+3$	$r+3$
$n = 2r+1$	$r+3$	$r+4$	$r+3$	$r+3$

Table 3

Theorem 4.5: The graph $st(2,3,n)$ is 4-total mean cordial for all values of n.

Proof: Let $V(st(2,3,n)) = \{u, v, w, u_i, v_j, w_k : 1 \leq i \leq 2, 1 \leq j \leq 3, 1 \leq k \leq n\}$ and $E(st(2,3,n)) = \{uu_i, vv_j, ww_k : 1 \leq i \leq 2, 1 \leq j \leq 3, 1 \leq k \leq n\}$.

Obviously $|V(st(2,3,n))| + |E(st(2,3,n))| = 2n + 13$.

Assign the labels $0, 1, 1, 0, 3, 1, 3, 3$ to the vertices $u, v, w, u_1, u_2, v_1, v_2, v_3$ respectively.

Case 1: $n \equiv 0 \ (mod\ 2)$.

Let $n = 2r$, $r \in$. Assign the label 0 to the r vertices $w_1, w_2, \dots, w_r$. Next we assign the label 3 to the r vertices $w_{r+1}, w_{r+2}, \dots, w_{2r}$.

Case 2: $n \equiv 1 \ (mod\ 2)$.

Let $n = 2r + 1$, $r \geq 0$. Label the vertices $w_i (1 \leq i \leq 2r)$as in Case 1. Now we assign the label 3 to the vertex w_{2r+1}.

Note that this vertex labeling f is a 4-total mean cordial labeling follows from the Table 4.

Order of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 2r$	$r+3$	$r+4$	$r+3$	$r+3$
$n = 2r+1$	$r+3$	$r+4$	$r+4$	$r+4$

Table 4

Theorem 4.6: The graph $st(n,n,n)$ is 4-total mean cordial for all values of n.

Proof: Let $V(st(n,n,n)) = \{x,y,z,x_i,y_i,z_i : 1 \le i \le n\}$ and $E(st(n,n,n)) = \{xx_i, yy_i, zz_i : 1 \le i \le n\}$.

Clearly $|V(st(n,n,n))| + |E(st(n,n,n))| = 6n + 3$.

Assign the labels 2,0,1 to the vertices x, y, z respectively.

Case 1: $n \equiv 0 \ (mod\ 2)$.

Let $n = 2r$, $r \in$. Assign the label 0 to the $2r$ vertices $x_1, x_2, \dots, x_{2r}$. Now we assign the label 3 to the $2r$ vertices $y_1, y_2, \dots, y_{2r}$. Next we assign the label 0 to the r vertices $z_1, z_2, \dots, z_r$. Finally we assign the label 3 to the r vertices $z_{r+1}, z_{r+2}, \dots, z_{2r}$.

Case 2: $n \equiv 1 \ (mod\ 2)$.

Let $n = 2r + 1$, $r \ge 0$. As in case 1 assign the label to the vertices $x_i, y_i, z_i (1 \le i \le 2r)$. Next we assign the labels 0, 3, 3 to the vertices $x_{2r+1}, y_{2r+1}, z_{2r+1}$.

Thus, this vertex labeling f is a 4-total mean cordial labeling follows from the Table 5.

Nature of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 2r$	$3r + 1$	$3r + 1$	$3r + 1$	$3r$
$n = 2r + 1$	$3r + 2$	$3r + 2$	$3r + 3$	$3r + 2$

Table 5

Theorem 4.7: The graph $st(n,n,n,n)$ is a 4-total mean cordial for all values of n.

Proof: Let $V(st(n,n,n,n)) = \{x,y,z,w,x_i,y_i,z_i,w_i : 1 \le i \le n\}$ and $E(st(n,n.n,n)) = \{xx_i, yy_i, zz_i, ww_i : 1 \le i \le n\}$.

Note that $|V(st(n,n,n,n))| + |E(st(n,n,n,n))| = 8n + 4$.

Assign the labels $0, 1, 2, 3$ to the vertices x, y, z, w respectively.

Now we assign the label 0 to the n vertices $x_1, x_2, \ldots, x_n$. Next we assign the label 1 to the $n$vertices $y_1, y_2, \ldots, y_n$. We now assign the label 2 to the n vertices $z_1, z_2, \ldots, z_n$. Finally we assign the label 3 to the n vertices $w_1, w_2, \ldots, w_n$.

Clearly $t_{mf}(0) = t_{mf}(1) = t_{mf}(2) = t_{mf}(3) = 2n + 1$.

Theorem 4.8: The graph $st(n, n, n, n, n)$ is a 4-total mean cordial for all values of n.

Proof: Let $V(st(n, n, n, n, n)) = \{u, v, x, y, z, u_i, v_i, x_i, y_i, z_i : 1 \leq i \leq n\}$ and $E(st(n, n, n. n, n)) = \{uu_i, vv_i, xx_i, yy_i, zz_i : 1 \leq i \leq n\}$.

Clearly $|V(st(n, n, n, n, n))| + |E(st(n, n, n, n, n))| = 10n + 5$.

Assign the labels 0, 1, 2, 3, 1 to the vertices u, v, x, y, z respectively.

Next we assign the label 0 to the n vertices $u_1, u_2, \ldots, u_n$. We now assign the label 1 to the n vertices $v_1, v_2, \ldots, v_n$. Next we assign the label 2 to the n vertices $x_1, x_2, \ldots, x_n$. Now we assign the label 3 to the n vertices $y_1, y_2, \ldots, y_n$.

Case 1: $n \equiv 0 \ (mod\ 2)$.

Let $n = 2r$, $r \in$. Assign the label 0 to the r vertices $z_1, z_2, \ldots, z_r$. Next we assign the label 3 to the r vertices $z_{r+1}, z_{r+2}, \ldots, z_{2r}$.

Case 2: $n \equiv 1 \ (mod\ 2)$.

Let $n = 2r + 1$, $r \geq 0$. In this case assign the label for the vertices $z_i (1 \leq i \leq 2r)$. Now we assign the labels 3 to the vertex z_{2r+1}.

Thus, this vertex labeling f is a 4-total mean cordial labeling follows from the Table 6.

n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 2r$	$5r + 1$	$5r + 2$	$5r + 1$	$5r + 1$
$n = 2r + 1$	$5r + 3$	$5r + 4$	$5r + 4$	$5r + 4$

Table 6

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TWO PEBBLING PROPERTY OF THORN GRAPHS OF EVEN CYCLE

Abstract: Chung defined a pebbling move on a graph G, to be the removal of two pebbles from one vertex and the addition of one pebble to an adjacent vertex. The pebbling number of a connected graph is the smallest numberf(G) such that any distribution of f(G) pebbles on G allows one pebble to be moved to any specified, but arbitrary vertex by a sequence of pebbling moves. Let $p_1, p_2, \dots, p_n$ be positive integers and G be a graph such that $|V(G)| = n$. The thorn graph of the graph G with parameters $p_1, p_2, \dots, p_n$ is obtained by attaching p_i new vertices of degree 1 to the vertex v_i of the graph G, $i = 1,2, \dots, n$. In this paper, we discuss about the pebbling number of the thorn graph of cycle with n vertices also called as thorn cycle and we show that it satisfies the two-pebbling property.

Keywords: Graphs, Pebbling Number, Thorn Cycle, Two-pebbling Property.

Mathematical Subject Classification (2010) No.: 05C12, 05C25, 05C38, 05C76.

1. Introduction

Pebbling in graphs was first studied by Chung [1]. A pebbling move consists of taking two pebbles off one vertex and placing one pebble on an adjacent vertex. The pebbling number of a vertex v in G is the smallest number $f(G, v)$ such that from every placement of $f(G, v)$ pebbles, it is possible to move a pebble to v by a sequence of pebbling moves. Then the pebbling number of a graph G, denoted by f(G) is the maximum $f(G, v)$ over all the vertices v in G. Given a configuration of pebbles placed on G, let p(G) be the number of pebbles placed on the graph G, q be the number of vertices with at least one pebble and let r be the number of vertices with odd number of pebbles. We say that G satisfies the two-pebbling property (respectively, weak or odd two-pebbling property), if it is possible to move two pebbles to any specified target vertex, when the total starting number of pebbles is 2f(G)-q+1 (respectively, 2f(G)-r+1). Note that any graph which satisfies the two-pebbling property also satisfies the weak or odd two-pebbling property.

Theorem 1.1 [5]: The pebbling number of a path graph P_n of length n is 2^n.

Theorem 1.2 [5]: The pebbling number of star graph $K_{1,n}$ is $f(K_{1,n}) = \text{n} + 2 \text{ ifn} > 1$.

Definition 1.1 [3]: Let $p_1, p_2, \dots, p_n$ be positive integers and G be a graph with |V(G)| = n. The thorn graph of the graph G with $p_1, p_2, \dots, p_n$is obtained by attaching p_i new vertices of degree 1 to the vertex v_i of the graph G, $\text{i} = 1,2, \dots, \text{n}$.

The thorn graph of the graph G will be denoted by G or by G $(p_1, p_2, \dots, p_n)$ if the respective parameters need to be specified. In this paper, we will consider the thorn graph with every $p_i \geq 2(i = 1, 2, \dots, n)$.

Definition 1.2 [2]: Given a configuration of pebbles placed on G, a transmitting subgraph of G is a path $v_1, v_2, \dots, v_n$ such that there are atleast two pebbles on v_i and atleast one pebble on each of the other vertices in the path, possibly except v_n. Thus, we can transmit a pebble fromv_1 to v_n.

Throughout this paper, G will denote a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The graph C_n denotes the cycle graph with n vertices.

2. Pebbling number of thorn cycle C_n

Definition 2.1: Let C_n be a cycle with n vertices where $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and $E(C_n) = \{e_1, e_2, \dots, e_n\}$. Let $X_i = \{x_{i1}, x_{i2}, \dots, x_{in}\}$ when $p_i \geq 2$ and $i = 1, 2, \dots, n$. Consider the graph C_n obtained from C_n such that $V(C_n) = \{v_i \cup x_i / \ i = 1, 2, \dots, n\}$ and $E(C_n) = E(C_n) \cup \{v_i x_{ij} / \ i = 1, 2, \dots, n \ \&$ $j = 1, 2, \dots, p_i\}$. Then C_n is called the thorn cycle with n vertices.

In this paper, we consider the cycle C_n with even vertices.

Lemma 2.1: Let C_n be the thorn graph of the cycle C_n with n vertices $\{v_1, v_2, \dots, v_n\}$. Let $X_i = \{x_{ij} / \ j = 1, 2, \dots, p_i$and each x_{ij} is adjacent to $v_i\}$ where

$i = 1, 2, \ldots, n$. Let $X = \ {}_{i=1}^{n} X_i$. Let r_k be the number of vertices with odd number of pebbles in $X \quad X_k$. If we distribute $d = 2^{\frac{n}{2}+2} + \sum_{i=1}^{n} p_i \quad 2$ pebbles are placed on the vertices of C_n such that $p(x_{k1}) = 0, p(x_{kj}) < 2 \forall j = 2, 3, \ldots, p_k$, $1 \leq k \leq n$ an $\,d p(C_n) = 0$ where p is the number of pebbles placed. Then

(i) $p(X \quad X_k) = 2^{\frac{n}{2}+2} + \sum \ \ p_i \quad p_k \quad 1$ and

(ii) $r_k < \sum p_i \quad p_k$.

Proof: (i) Distributing $d = 2^{\frac{n}{2}+2} + \sum_{i=1}^{n} p_i \quad 2$ pebbles on the vertices of C_n such that $p(x_{k1}) = 0, p(C_n) = 0, p(x_{kj}) < 2 \ \forall j = 2,3, \ldots, p_k, 1 \leq k \leq n$. Then $X \quad X_k$ receives $p(X \quad X_k) = d \quad (p_k \quad 1)$. Hence, $p(X \quad X_k) = 2^{\frac{n}{2}+2} + \sum \ \ p_i \quad p_k \quad 1$.

(ii) Now, let us prove that $r_k < \sum p_i \quad p_k$. Clearly r_k cannot be greater than $\sum p_i \quad p_k$. It is enough to prove that $r_k \neq \sum p_i \quad p_k$.

Suppose $r_k = \sum p_i \quad p_k$.

Case 1: If r_k is even, then both $\sum p_i$ and p_k are even or both $\sum p_i$ and p_k are odd.

Subcase 1.1: Let us assume that both $\sum p_i$ and p_k are even. Hence, $p_k \quad 1$ is odd. So $d \quad (p_k \quad 1)$ is odd. Distributing $d \quad (p_k \quad 1)$ pebbles on r_k vertices, there exists odd vertices with even number of pebbles. Hence, $r_k \quad 1$ vertices have odd number of pebbles.

That is, number of vertices with odd number of pebbles is odd which is a contradiction to the number of vertices with odd number of pebbles, r_k is even.

Subcase 1.2: Now, let us assume that both $\sum p_i$ and p_k are odd. If p_k is odd, then $p_k \quad 1$ is even and $d \quad (p_k \quad 1)$ is odd. Discussing as in the subcase 1.1, we get a contradiction to r_k is even. Therefore, we get $r_k < \sum p_i \quad p_k$.

Case 2: If r_k is odd then either $\sum p_i$ is even and p_k is odd or $\sum p_i$ is odd and p_k is even.

Subcase 2.1: Let us assume that $\sum p_i$ is even and p_k is odd. Then, $p_k \quad 1$ is even and $d \quad (p_k \quad 1)$ is even. Now distributing these even pebbles on r_k vertices, there exists even vertices with odd number of pebbles. That is, number of vertices with odd number of pebbles is even which is a contradiction to number of vertices with odd number of pebbles, r_k is odd.

Subcase 2.2: Now, let us assume that $\sum p_i$ is odd and p_k is even. If p_k is even, then $p_k \quad 1$ is odd and $d \quad (p_k \quad 1)$ is even. Discussing as in the subcase 2.1, we get a contradiction to r_k is odd. Therefore, $r_k < \sum p_i \quad p_k$.

Theorem 2.1: Let C_n be the thorn graph of cycle with n vertices where n is even. Then the pebbling number of the thorn cycle is $f(C_n) = 2^{\frac{n}{2}+2} + \sum p_i \quad 2$ where $p_i \geq 2 \, , i = 1,2, \ldots , n \, , n \geq 4$.

Proof: Let $V(C_n) = \{v_1, v_2, \ldots , v_n\}$. Let $X = {}_{i=1}^{n} X_i$ where Let $X_i = \{x_{ij} / \ j = 1,2, \ldots , p_i$ and each x_{ij} is adjacent to $v_i\}$.

Assume that $2^{\frac{n}{2}+2} + \sum p_i \quad 3$ pebbles are placed on the vertices of C_n as follows.

(i) $p(C_n) = 0 \; and \; p(x_{11}) = 0$.

(ii) $p(x_{ij}) = 1, i = 2,3, \ldots , \frac{n}{2} \quad 1, \frac{n}{2}, \frac{n}{2} + 2, \ldots , n$ and $j = 1,2, \ldots , p_i$.

(iii) $p(x_{1j}) = 1, j = 2,3, \ldots , p_1$.

(iv) $p\left(x_{\frac{n}{2}+1}, j\right) = 1, j = 2,3, \ldots \, , p_{\frac{n}{2}+1}$.

(v) $p\left(x_{\frac{n}{2}+1}, 1\right) = 2^{\frac{n}{2}+2} \quad 1$.

If x_{11} be our target vertex, then by the above distribution a pebble cannot be moved to x_{11} as the length of the path $(x_{\frac{n}{2}+1}, x_{11})$ is $\frac{n}{2}$.

Therefore, $f(C_n) \geq 2^{\frac{n}{2}+2} + \sum p_i \quad 2$.

Now let us show that $f(C_n) \leq 2^{\frac{n}{2}+2} + \sum p_i \quad 2$.

Case 1: Suppose that the target vertex is v_k where $1 \leq k \leq n \; and \; p(v_k) = 0$.

If $p(x_{kj}) \geq 2$ for some $j = 1,2,\ldots,p_k$ then we can move one pebble from $x_{kj}\ to\ v_k$. If $p(x_{kj}) < 2$ for all $j = 1,2,\ldots,p_k$ and let $p(C_n) = s$ for some $s \geq 0$. Then the number of pebbles on $X \quad X_k$ is at least $2^{\frac{n}{2}+2} + \sum p_i \quad 2 \quad p_k \quad s$. Then the number of pebbles that can be brought to C_n is at least $\frac{2^{\frac{n}{2}+2}+\sum p_i-2-p_k-s-r_k}{2}$. Since, $r_k < \sum p_i \quad p_k$, C_n will get at least $\frac{2^{\frac{n}{2}+2}-s-2}{2}$ pebbles. Then the total number of pebbles on C_n will be at least $\frac{2^{\frac{n}{2}+2}-s-2}{2} + s = 2^{\frac{n}{2}+1} + \frac{s-2}{2} > 2^{\frac{n}{2}}\ as\ s \geq 0$ an d $n \geq 4$. Hence, one pebble can be moved to v_k.

Case 2: Suppose that the target vertex is x_{ij} where $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,p_i$. Without loss of generality, let us assume that x_{11} be our target vertex and $p(x_{11}) = 0$. If $p(v_1) \geq 2$ then one pebble can be moved to x_{11}. If $p(v_1) = 1$, then if there exists atleast one vertex $x_{1j} (j \neq 1)$ such that $p(x_{1j}) \geq 2$ then $\{x_{ij}, v_1, x_{11}\}$ forms a transmitting subgraph. Hence, one pebble can be moved to x_{11}. If $p(x_{1j}) < 2$ for all $j = 1,2,\ldots,p_1$, then the number of pebbles on $C_n \quad X_1$ is at least $2^{\frac{n}{2}+2} + \sum p_i \quad 2 \quad (p_1 \quad 1) = 2^{\frac{n}{2}+2} + \sum p_i \quad p_1 \quad 1$, then proceeding as in case 1, one pebble can be moved to v_1 and from v_1 one pebble can be moved to x_{11}. If $p(v_1) = 0$, then the following cases arise.

Subcase 2.1: If $p(x_{ij_1}) \geq 4$ for only one $j_1 \neq 1$ and $p(x_{1r}) < 2$ for all $r \neq 1, j_1$, then two pebbles can be moved from x_{1j_1} to v_1 and hence one pebble can be moved to x_{11}.

Subcase 2.2: If there exists at least two vertices x_{1j_1}, x_{1j_2} with $p(x_{1j_1}) \geq 2$ and $p(x_{1j_2}) \geq 2$ where $j_1, j_2 \neq 1$ among the vertices $x_{11}, x_{12}, \ldots, x_{1p_1}$ then we can move one pebble from x_{1j_1} to v_1. So $\{x_{1j_2}, v_1, x_{11}\}$ forms a transmitting subgraph. Hence, one pebble can be moved to x_{11}.

Subcase 2.3: If $2 \leq p(x_{1j_1}) < 4$ for only one $j_1 \neq 1$ and $p(x_{1r}) < 2$ for all $r \neq 1, j_1$ then we can move one pebble from x_{1j_1} to v_1. Then the number of pebbles on $C_n \quad X_1$ is at least

$$2^{\frac{n}{2}+2} + \sum p_i \quad 2 \quad (3 + p_1 \quad 2) = 2^{\frac{n}{2}+2} + \sum p_i \quad p_1 \quad 3.$$

Now proceeding as in Case 1, C_n will get atleast $2^{\frac{n}{2}+1} + \frac{s-3}{2} \geq 2^{\frac{n}{2}}$ as $s \geq 0\ and\ n \geq 4$. Hence, another pebble can be moved to v_1.

Thus, v_1 gets two pebbles and one pebble can be moved to x_{11}.

Subcase 2.4: If $p(x_{1r}) < 2$ for all $r \neq 1$, then the number of pebbles on $C_n \quad X_1$ is at least $2^{\frac{n}{2}+2} + \sum p_i \quad 2 \quad (p_1 \quad 1)$. Let $p(C_n) = s, s \geq 0$. Then the number of pebbles on $X \quad X_1$ is at least $2^{\frac{n}{2}+2} + \sum p_i \quad p_1 \quad 1 \quad s$. Let r_1 be the number of vertices with odd pebbles on $X \quad X_1$. Then the number of pebbles that can be brought to C_n is atleast $\frac{2^{\frac{n}{2}+2}+\sum p_i - p_1 - 1 - s - r_1}{2} \geq \frac{2^{\frac{n}{2}+2}+\sum p_i - p_1 - 1 - s - \sum p_i + p_1 + 1}{2}$ as $r_1 < \sum p_i \quad p_1$. Then the total number of pebbles C_n have will be at least $\frac{2^{\frac{n}{2}+2}-s}{2} + s \geq 2^{\frac{n}{2}+1} + \frac{s}{2} \geq 2^{\frac{n}{2}+1}$. Thus, two pebbles can be moved to v_1 and hence one pebble can be moved to x_{11}. Hence, the result.

3. Two Pebbling Property

Definition 3.1 [4]: We say a graph G satisfies the 2-pebbling properly if two pebbles can be moved to any specified vertex, when the total starting number of pebbles is 2f(G) $\quad$ q + 1, where q is the number of vertices with at least one pebble.

Theorem 3.1: Let C_n be the thorn graph of the cycle $C_n$with n vertices. Then C_n satisfies the two pebbling properly when n is even.

Proof: Let P be the number of pebbles on the thorn cycle Cn* and q be the number of vertices with attest one pebble and $p + q = 2\left(2^{\frac{n}{2}+2} + \sum p_i \quad 2\right) + 1$. We consider the following two types of possible target vertices.

Case 1: Suppose that the target vertex is v_k where $1 \leq k \leq n$, if $p(v_k) = 1$, then the number of pebbles on all the vertices except v_k is $2\left(2^{\frac{n}{2}+2} + \sum p_i \quad 2\right) + 1 \quad q \quad 1 > 2^{\frac{n}{2}+2} + \sum p_i \quad 2$. Since, $q \leq n + \sum p_i$. Since, $f(C_n) = 2^{\frac{n}{2}+2} + \sum p_i \quad 2$, we can put one more pebble on v_k using the $2\left(2^{\frac{n}{2}+2} + \sum p_i \quad 2\right) + 1 \quad q \quad 1$ pebbles.

If $p(v_k) = 0$, then we consider the following cases.

Subcase 1.1: Suppose that $p(x_{kj}) \geq 2$ for some x_{kj} $(j = 1, 2, \ldots, p_k)$. Then we can move one pebble to v_k. Using the remaining $2\left(2^{\frac{n}{2}+2} + \sum p_i \quad 2\right) + 1 \quad q \quad 2$ pebbles, we can move another pebble to v_k.

Subcase 1.2: Suppose that $p(x_{kj}) < 2$ for some x_{kj} $(j = 1, 2, \dots, p_k)$. Since, $q < \quad n \quad 1 + \sum p_i$ as $p(v_k) = 0$ we have

$$p \geq 2\left(2^{\frac{n}{2}+2} + \sum p_i \quad 2\right) + 1 \quad (n \quad 1 + \sum p_i) = 2^{\frac{n}{2}+3} + \sum p_i \quad (n+2).$$

Now $p(C_n \quad X_k) \geq 2^{\frac{n}{2}+3} + \sum p_i \quad (n+2) \quad p_k$. Let $p(C_n) = s, s \geq 0$. Then the number of pebbles on $X \quad X_k$ is atleast $2^{\frac{n}{2}+3} + \sum p_i \quad (n+2) \quad p_k \quad s$. Let r_k be the number of vertices with odd number of pebbles in $X \quad X_k$. The number of pebbles that can be brought to C_n is atleast $\frac{2^{\frac{n}{2}+3}+\sum p_i-(n+2)-p_k-s-r_k}{2}$, where $r_k < \sum p_i \quad p_k$. Therefore, C_n will get at least $2^{\frac{n}{2}+2} \quad \frac{s+(n+1)}{2}$ pebbles. Now C_n has atleast $2^{\frac{n}{2}+2} + s \quad \frac{s+(n+1)}{2} > 2^{\frac{n}{2}+1}$ pebbles. Hence, two pebbles can be moved to v_k.

Case 2: Suppose that the target vertex is x_{kj} where $j = 1, 2, \dots, p_k$. Without loss of generality, let us assume that the target vertex is x_{k1}. If $p(x_{k1}) = 1$, then the number of pebbles on all the vertices except x_{k1} is

$$2\left(2^{\frac{n}{2}+2} + \sum p_i \quad 2\right) + 1 \quad q \quad 1 > 2^{\frac{n}{2}+2} + \sum p_i \quad 2 \text{ as } q \leq n + \sum p_i.$$

Since, $f(C_n) = 2^{\frac{n}{2}+2} + \sum p_i \quad 2$, we can put one more pebble on x_{k1}. If $p(x_{k1}) = 0$, then we consider the following cases.

Subcase 2.1: If $p(v_k) \geq 2$, then we can move one pebble from v_k to x_{k1}. Using the remaining $2\left(2^{\frac{n}{2}+2} + \sum p_i \quad 2\right) + 1 \quad q \quad 2$ pebbles, we can move another pebble to x_{k1}.

Subcase 2.2: Consider $p(v_k) = 1$. If there is atleast one vertex $x_{kj_1} (j_1 \neq 1)$ with $p(x_{kj_1}) \geq 2$ then $\{x_{kj_1}, v_k, x_{k1}\}$ forms a transmitting subgraph. Using the remaining $2\left(2^{\frac{n}{2}+2} + \sum p_i \quad 2\right) + 1 \quad q \quad 3$ pebbles, we can move another pebble to x_{k1}. If $p(x_{kr}) < 2$ for all $r \neq 1$, and if $p(C_n) = s, s \geq 0$ then the number of pebbles placed on $X \quad X_k$ is at least $2\left(2^{\frac{n}{2}+2} + \sum p_i \quad 2\right) + 1 \quad q \quad (p_k \quad 1) \quad s$. Let r_k be the number of vertices with odd pebbles in $X \quad X_k$, then the number of pebbles that can be brought to C_n is atleast $\frac{2\left(2^{\frac{n}{2}+2}+\sum p_i-2\right)+1-q-(p_k-1)-s-r_k}{2}$. Now as q is the number of vertices with atleast one pebble, we have $q \leq s + \sum p_i$. Hence,

C_n gets at least $\frac{2^{\frac{n}{2}+3}+\sum p_i-2-s-p_k-s-r_k}{2}$. We have $r_k \leq \sum p_i \quad p_k \quad 1$, then the number of pebbles that can be brought to C_n is atleast $\frac{2^{\frac{n}{2}+3}-2s-1}{2}$.

Now, C_n has atleast $2^{\frac{n}{2}+2} \quad 2\frac{s}{2} \quad \frac{1}{2}+s$ pebbles. That is

$$p(C_n) \geq 2^{\frac{n}{2}+2} \quad \frac{1}{2} = 2^{\frac{n}{2}+1}+2^{\frac{n}{2}}+2^{\frac{n}{2}} \quad \frac{1}{2} > 2^{\frac{n}{2}+1}+2^{\frac{n}{2}}.$$

Hence three pebbles can be moved to v_k and thus two pebbles can be moved to x_{k1}.

Subcase 2.3: If $p(v_k)=0$ and if there exists atleast two vertices $x_{kj_1}, x_{kj_2}(j_1, j_2 \neq 1)$ with $p(x_{kj_1}) \geq 2, p(x_{kj_2}) \geq 2$ then we can move two pebble each from $x_{kj_1}$and x_{kj_2} to v_k. Thus, v_k get two pebbles and one pebble can be moved to x_{k1}. Using the remaining $2\left(2^{\frac{n}{2}+2}+\sum p_i \quad 2\right)+1 \quad q \quad 4$ pebbles we can move another pebble to x_{k1} as $q \leq n \quad 2+\sum p_i \quad 1$. If there is only one vertex $x_{kj_1}(j_1 \neq 1)$ with $p(x_{kj_1}) \geq 4$ an d$p(x_{kr})<2$ for all $r \neq 1, j_1$ then we can move two pebbles from x_{kj_1} to v_k. So $\{v_k, x_{k1}\}$ forms a transmitting subgraph. Now we have atleast $2\left(2^{\frac{n}{2}+2}+\sum p_i \quad 2\right)+1 \quad q \quad 4 \quad (p_k \quad 2)$ remaining pebbles.

Let $p(C_n)=s, s \geq 0$, then we have $q \leq s+\sum p_i$. Then by proceeding as in Subcase 2.2, C_n will get at least $2^{\frac{n}{2}+2} \quad 2 > 2^{\frac{n}{2}+1}$. Hence, two pebbles can be moved to v_k and one pebble can be placed on x_{k1}. If there is only one vertex $x_{kj_1}(j_1 \neq 1)$ with $2 \leq p(x_{kj_1}) \leq 3$ and $p(x_{kr})<2$ for all $r \neq 1, j_1$ we can move one pebble from x_{kj_1} to v_k. Then we have atleast $2\left(2^{\frac{n}{2}+2}+\sum p_i \quad 2\right)+1 \quad q \quad 3 \quad (p_k \quad 2)$ remaining pebbles. Again, by proceeding as in subcase 2.2, C_n will at least get $2^{\frac{n}{2}+2} \quad \frac{3}{2} > 2^{\frac{n}{2}+1}+2^{\frac{n}{2}}$. Hence, we can move three pebbles to v_k, and two pebbles can be moved to x_{k1}.

If $p(x_{kr})<2$ for all $r(r \neq 1)$ and if $p(C_n)=s, s \geq 0$, then the number of pebbles placed on $X \quad X_k$ is atleast $2\left(2^{\frac{n}{2}+2}+\sum p_i \quad 2\right)+1 \quad q \quad (p_k \quad 1) \quad s$. Now, proceeding as in Subcase 2.2, C_n will get at least $2^{\frac{n}{2}+2} \quad s+s$. Hence, $p(C_n) \geq 2^{\frac{n}{2}+2}$. Hence, four pebbles can be moved to v_k. Thus, two pebbles can be moved to x_{k1}.

4. Conclusion and open problem

In this paper, we determined the pebbling number of the thorn even cycle and also we have proved that the thorn path satisfies the 2-pebbling property. The pebbling number of the thorn odd cycle is an open problem.

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CERTAIN CLASS OF UNIVALENT FUNCTIONS ASSOCIATED WITH MITTAG-LEFFLER FUNCTION

Abstract: In this paper we introduce a subclass of analytic univalent functions defined with a differential operator asoociated to Mittag-Leffler function. Also we have studied the coefficient estimate, growth and distortion theorem, radii of starlikeness, convexity and close to convexity for the class.

Keywords: Univalent Function, Radius of Starlikeness, Convex Function.

Mathematical Subject Classification No.: 30C45.

1. Introduction

Let A be the class of analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the unit disc $U = \{z : |z| < 1\}$ with normalization $f(0) = 0$, $f'(0) = 1$. The subclass S of class A, consisting of functions of type (1.1) that are univalent in U.

Also let T be the subclass of S consisting functions of the type

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0, \quad z \in U) \tag{1.2}$$

which was introduced and studied by Silverman[8].

Now if $g(z) \in A$ has the form

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n , \tag{1.3}$$

the convolution (i.e. Hadamard product) of f and g is denoted by $f * g$ and is defined as

$$(f * g)(z) = (g * f)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \quad (z \in U). \tag{1.4}$$

The Mittag-Leffler function $E_\nu(z)$ introduced by Mittag Leffler[4] and its generalization E_ν, τ studied by Wiman[10] given by

$$E_\nu(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\nu n + 1)}$$

and

$$E_{\nu,\tau} = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\nu n + \tau)}$$

where $\nu, \tau \in \mathbb{C}$, $Re(\nu) > 0$ and $Re(\tau) > 0$.

The function $Q_{\nu,\tau}$ defined by Srinivasulu[9]

$$Q_{\nu,\tau}(z) = z\Gamma(\tau) E_{\nu,\tau}(z) \tag{1.5}$$

and further the differential operator $D_\lambda^m(\nu,\tau)f : A \to A$ studied by him is given by

$$D_\lambda^0(\nu,\tau)f(z) = f(z) * Q_{\nu,\tau}(z)$$

$$D_\lambda^1(\nu,\tau)f(z) = (1-\lambda)f(z) * Q_{\nu,\tau}(z) + \lambda z(f(z) * Q_{\nu,\tau}(z))$$
$$\vdots$$
$$D_\lambda^m(\nu,\tau)f(z) = D_\lambda^1(D_\lambda^{m-1}(\nu,\tau)f(z)) .$$

It is easy to see that if $f(z)$ is given by equation (1.2) then the definition of the operator D_λ^m takes the form

$$D_\lambda^m(\nu,\tau)f(z) = z - \sum_{n=2}^{\infty} \phi_n^m(\lambda,\nu,\tau)a_n z^n$$

where

$$\phi_n^m(\lambda,\nu,\tau) = \frac{\Gamma(\tau)}{\Gamma(\nu(n-1)+\tau)}[\lambda(n-1)+1]^m.$$

In this paper, with the operator D_λ^m, we define the following new class.

Definition 1.1: *The function* $f(z)$ *of the form* (1.1) *is in the class* $S_\lambda^m(\nu,\tau,\alpha,\beta,\gamma)$ *if it satisfies the inequality*

$$\left|\frac{\{D_\lambda^m(\nu,\tau)f(z)\}'-1}{2\gamma[\{D_\lambda^m(\nu,\tau)f(z)\}'-\alpha]-[\{D_\lambda^m(\nu,\tau)f(z)\}'-1]}\right| > \beta \qquad (1.6)$$

where $\nu,\tau \in \mathbb{C}$ *with* $Re(\nu) > 0$ *and* $Re(\tau) > 0$, $0 \le \lambda$, β , $\gamma \le 1$ *and* $0 \le \alpha < 1$.

Further we define $T_\lambda^m(\nu, \tau, \alpha, \beta, \gamma) = S_\lambda^m(\nu, \tau, \alpha, \beta, \gamma) \cap T$.

We note that, such classes were earlier studied extensively by Aouf and Cho[1], Aouf *et.al.*[6] and others {[3], [2], [5], [7]}.

2. Main Results

Theorem 2.1: *A function* $f(z)$ *of the form* (1.1) *belongs to the class* $S_\lambda^m(\nu, \tau, \alpha, \beta, \gamma)$ *if*

$$\sum_{n=2}^{\infty} n\phi_n^m(\lambda,\nu,\tau)[1+\beta(2\gamma-1)]a_n \le 2\gamma\beta)(1-\alpha)$$

where $\nu,\tau \in \mathbb{C}$ *with* $Re(\nu)>0$ *and* $Re(\tau)>0$, $0 \le \lambda$, β , $\gamma \le 1$ *and* $0 \le \alpha < 1$.

***Proof*:** Assume that the inequality holds true and let $|z| = 1$ from (1.6) we have

$$|\{D_\lambda^m(\nu,\tau)f(z)\}' - 1| - \beta\,|2\gamma[\{D_\lambda^m(\nu,\tau)f(z)\}' - \alpha] - [\{D_\lambda^m(\nu,\tau)f(z)\}' - 1]|$$

$$= \left| -\sum_{n=2}^{\infty} \phi_n^m(\lambda,\nu,\tau) n a_n z^{n-1} \right|$$

$$-\beta \left| 2\gamma(1-\alpha) - 2\gamma\sum_{n=2}^{\infty} \phi_n^m(\lambda,\nu,\tau) n a_n z^{n-1} + \sum_{n=2}^{\infty} \phi_n^m(\lambda,\nu,\tau) n a_n z^{n-1} \right|$$

$$\le \sum_{n=2}^{\infty} \phi_n^m(\lambda,\nu,\tau) n|a_n| - 2\gamma\beta(1-\alpha) + \beta(2\gamma-1)\sum_{n=2}^{\infty} \phi_n^m(\lambda,\nu,\tau) n|a_n|$$

$$= \sum_{n=2}^{\infty} [1+\beta](2\gamma-1)]\phi_n^m(\lambda,\nu,\tau) n|a_n| - 2\gamma\beta(1-\alpha) \le 0.$$

□

Theorem 2: *A function* $f(z)$ *of the form* (1.2) *belongs to the* $T_\lambda^m(\nu, \tau, \alpha, \beta, \gamma)$ *if and only if*

$$\sum_{n=2}^{\infty} n\phi_n^m(\lambda,\nu,\tau)[1+\beta(2\gamma-1)]a_n \le 2\gamma\beta)(1-\alpha) \tag{2.1}$$

where $\nu,\tau \in \mathbb{C}$ *with* $Re(\nu)>0$ *and* $Re(\tau)>0$, $0 \le \lambda$, β , $\gamma \le 1$ *and* $0 \le \alpha < 1$.

***Proof*:** In the view of **Theorem 2.1**, we need only to prove the necessity. Assume that, $f(z)$ belongs to the class $T_\lambda^m(\nu, \tau, \alpha, \beta, \gamma)$ then we have

$$\left|\frac{\{D_\lambda^m(\nu,\tau)f(z)\}'-1}{2\gamma(1-\alpha)-(2\gamma-1)\sum_{n=2}^{\infty}\phi_n^m(\lambda,\nu,\tau)na_nz^{n-1}}\right|$$

$$=\left|\frac{-\sum_{n=2}^{\infty}\phi_n^m(\lambda,\nu,\tau)na_nz^{n-1}}{2\gamma(1-\alpha)-(2\gamma-1)\sum_{n=2}^{\infty}\phi_n^m(\lambda,\nu,\tau)na_nz^{n-1}}\right|<\beta\,.$$

Since $Re(z)\le|z|$

$$\mathrm{Re}\left\{\frac{-\sum_{n=2}^{\infty}\phi_n^m(\lambda,\nu,\tau)na_nz^{n-1}}{2\gamma(1-\alpha)-(2\gamma-1)\sum_{n=2}^{\infty}\phi_n^m(\lambda,\nu,\tau)na_nz^{n-1}}\right\}<\beta. \tag{2.2}$$

Now, choosing the values of z on the real axis so that $\{D_\lambda^m(\nu,\tau)\}'$ is real. Upon clearing the denominator in (2.2) and letting $z\to 1$ through real axis we get

$$\sum_{n=2}^{\infty}\phi_n^m(\lambda,\nu,\tau)na_n\le 2\gamma\beta(1-\alpha)-\beta(2\gamma-1)\sum_{n=2}^{\infty}\phi_n^m(\lambda,\nu,\tau)na_n$$

which implies the inequality (2.1). □

Corollary 2.2.1: *If* $f(z)\in T_\lambda^m(\nu,\,\tau,\,\alpha,\,\beta,\,\gamma)$ *if and only if*

$$a_n\le\frac{2\gamma\beta(1-\alpha)}{n[1+\beta(2\gamma-1)]\phi_n^m(\lambda,\nu,\tau)} \tag{2.3}$$

where equality holds for the function

$$f(z)=z-\frac{2\gamma\beta(1-\alpha)}{n[1+\beta(2\gamma-1)]\phi_n^m(\lambda,\nu,\tau)}z^n. \tag{2.4}$$

Theorem 2.3: *Let* $f_1(z)=z$ *and*

$$f(z) = z - \frac{2\gamma\beta(1-\alpha)}{n[1+\beta(2\gamma-1)]\phi_n^m(\lambda,\nu,\tau)} z^n \tag{2.5}$$

then $f(z) \in T_\lambda^m(\nu, \tau, \alpha, \beta, \gamma)$ *if and only if it can be expressed as*

$$f(z) = \sum_{n=1}^{\infty} w_n f_n(z), \quad w_n \geq 0, \quad \sum_{n=1}^{\infty} w_n = 1.$$

Proof: Suppose $f(z)$ can be written as in (2.5) then

$$f(z) = z - \sum_{n=2}^{\infty} w_n \frac{2\gamma\beta(1-\alpha)}{n[1+\beta(2\gamma-1)]\phi_n^m(\lambda,\nu,\tau)} z^n.$$

Now,

$$\sum_{n=2}^{\infty} w_n \frac{2\gamma\beta(1-\alpha)^n[1+\beta(2\gamma-1)]\phi_n^m(\lambda,\nu,\tau)}{2\gamma\beta(1-\alpha)^n[1+\beta(2\gamma-1)]\phi_n^m(\lambda,\nu,\tau)} = \sum_{n=2}^{\infty} w_n = 1 - w_1 \leq 1.$$

Thus, $f(z) \in T_\lambda^m(\nu, \tau, \alpha, \beta, \gamma)$.

Conversely, let us assume that $f(z) \in T_\lambda^m(\nu, \tau, \alpha, \beta, \gamma)$ then by using (2.3) we get

$$w_n = \frac{n[1+\beta(2\gamma-1)]\phi_n^m(\lambda,\nu,\tau)}{2\gamma\beta(1-\alpha)} a_n, \quad n \geq 2$$

and

$$w_1 = 1 - \sum_{n=2}^{\infty} w_n$$

Then we get

$$f(z) = \sum_{n=1}^{\infty} w_n f_n(z).$$

□

Theorem 2.4: *The class* $T_\lambda^m(\nu, \tau, \alpha, \beta, \gamma)$ *is a convex set.*

Proof: Consider,

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \quad a_{n,j} \geq 0, \quad j = 1, 2$$

belongs to the class $T_\lambda^m(\nu, \tau, \alpha, \beta, \gamma)$. It is sufficient to show that the function $h(z)$ given by

$$h(z) = \zeta f_1(z) + (1-\zeta) f_2(z), \quad 0 \leq \zeta \leq 1$$

is in the class $T_\lambda^m(\nu, \tau, \alpha, \beta, \gamma)$. We get,

$$h(z) = z - \sum_{n=2}^{\infty} [\zeta a_{n,1} + (1-\zeta) a_{n,2}] z^n.$$

Now, from **Theorem (2.2)** and with easy calculation we get,

$$\sum_{n=2}^{\infty} \phi_n^m(\lambda, \nu, \tau) n[1 + \beta(2\gamma - 1)]\zeta a_{n,1} + \sum_{n=2}^{\infty} \phi_n^m(\lambda, \nu, \tau) n[1 + \beta(2\gamma - 1)](1-\zeta) a_{n,2}$$

$$\leq \zeta 2\gamma\beta(1-\alpha) + (1-\zeta) 2\gamma\beta(1-\alpha)$$

$$\leq 2\gamma\beta(1-\alpha)$$

which gives us $h(z) \in T_\lambda^m(\nu, \tau, \alpha, \beta, \gamma)$. Hence, $T_\lambda^m(\nu, \tau, \alpha, \beta, \gamma)$ is convex set. □

Now, we will obtain the radii of close to convexity and starlikeness for the class $T_\lambda^m(\nu, \tau, \alpha, \beta, \gamma)$.

Theorem 2.5: *Let the function* $f(z)$ *be defined by* (1.2) *belongs to the class* $T_\lambda^m(\nu, \tau, \alpha, \beta, \gamma)$ *then* $f(z)$ *is close to convex of order* $\delta (0 \leq \delta < 1)$ *in the disc* $|z| < r_1$. *Where,*

$$r_1 = \inf_{n\geq 2}\left\{\frac{(1-\delta)[1+\beta(2\gamma-1)]\phi_n^m(\lambda,\nu,\tau)}{\gamma\beta(1-\gamma)}\right\}^{\frac{1}{n-1}}.$$

The result is sharp with extremal function $f(z)$ *is given by* (2.4).

Proof: Given $f \in T_\lambda^m(\nu, \tau, \alpha, \beta, \gamma)$ and f is close to convex of order δ. We have,

$$|f'(z)-1| < 1-\delta. \tag{2.6}$$

Consider, the left hand side of (2.6)

$$|f'(z)-1| = \sum_{n=2}^{\infty} na_n |z|^{n-1}.$$

The last equation is bounded above by $1-\delta$

$$\sum_{n=2}^{\infty}\frac{n}{1-\delta} a_n |z|^{n-1} \leq 1$$

but, $f(z) \in T_\lambda^m(\nu, \tau, \alpha, \beta, \gamma)$ if and only if

$$\sum_{n=2}^{\infty}\frac{n[1+\beta(2\gamma-1)]\phi_n^m(\lambda,\nu,\tau)}{2\gamma\beta(1-\gamma)} a_n \leq 1.$$

Thus, equation (2.6) is true if

$$\frac{n}{1-\delta}|z|^{n-1} \leq \frac{n[1+\beta(2\gamma-1)]\phi_n^m(\lambda,\nu,\tau)}{2\gamma\beta(1-\gamma)}.$$

Or equivalently

$$|z| \leq \left\{\frac{(1-\delta)[1+\beta(2\gamma-1)]\phi_n^m(\lambda,\nu,\tau)}{2\gamma\beta(1-\gamma)}\right\}^{\frac{1}{n-1}}.$$

which completes the proof of Theorem (2.5). □

Theorem 2.6: *Let the function* $f(z)$ *be defined by* (1.2) *belongs to the class* $T_\lambda^m(\nu, \tau, \alpha, \beta, \gamma)$ *then* $f(z)$ *is starlike of order* $\delta(0 \le \delta < 1)$ *in the disc* $|z| < r_2$.

Where,

$$r_2 = \inf_{n \ge 2} \left\{ \frac{(1-\delta)n[1+\beta(2\gamma-1)]\phi_n^m(\lambda, \nu, \tau)}{(n-\delta)\gamma\beta(1-\alpha)} \right\}^{\frac{1}{n-1}}.$$

The result is sharp with the extremal function $f(z)$ *given by* (2.4).

***Proof*:** Given $f \in T_\lambda^m(\nu, \tau, \alpha, \beta, \gamma)$ and f is starlike of order δ. We have,

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \delta. \tag{2.7}$$

Now, for the left hand side of equation (2.7) we have,

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \le \frac{\sum_{n=2}^{\infty} (n-1) a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}.$$

The last equation is less than $1 - \delta$ if

$$\sum_{n=2}^{\infty} \frac{n-\delta}{1-\delta} a_n |z|^{n-1} < 1$$

using the fact that $f(z) \in T_\lambda^m(\nu, \tau, \alpha, \beta, \gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{n[1+\beta(2\gamma-1)]\phi_n^m(\lambda, \nu, \tau)}{2\gamma\beta(1-\alpha)} a_n \le 1.$$

Thus, equation (2.7) is true if

$$\frac{n-\delta}{1-\delta}|z|^{n-1} \leq \frac{n[1+\beta(2\gamma-1)\phi_n^m(\lambda, \nu, \tau)}{2\gamma\beta(1-\alpha)}.$$

Or equivalently

$$|z|^{n-1} \leq \frac{(1-\delta)n[1+\beta(2\gamma-1)]\phi_n^m(\lambda, \nu, \tau)}{(n-\delta)2\gamma\beta(1-\alpha)}$$

which gives the condition for starlikeness. □

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SUMS INVOLVING EXTENDED GIBONACCI POLYNOMIALS

Abstract: We explore four sums involving gibonacci polynomial squares, and their Pell and Jacobsthal versions.

Keywords: Fibonacci Polynomials, Extended Gibonacci Polynomials, Lucas Pell Polynomials, Jacobsthal Polynomial, Jacobsthal Lucas Polynomials.

Mathematical Subject Classification (2020) No.: Primary 11B37, 11B39, 11C08.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 5].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively [5].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$ [2, 5].

Gibonaeci and Jacobsthal polynomials are linked by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [3, 4, 5].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $c_n = J_n$ or j_n, $\Delta = \sqrt{x^2 + 4}$, $2\alpha = x + \Delta$, $E = \sqrt{x^2 + 1}$, $\gamma = x + E$, and $D = \sqrt{4x + 1}$, where $c_n = c_n(x)$.

It follows by the Binet-like formulas that $\lim_{m \to \infty} \frac{f_{m+1}}{l_m} = \frac{\alpha}{\Delta}$ and $\lim_{m \to \infty} \frac{l_{m+1}}{f_m} = \Delta\alpha$.

1.1 Fundamental Gibonacci Identities: Gibonacci polynomials satisfy the following properties:

$$\Delta^2 f_{n+1} f_n = l_{2n+1} - (-1)^n x; \tag{1}$$

$$l_{n+1} l_n = l_{2n+1} + (-1)^n x; \tag{2}$$

$$l_n f_{n+2} - l_{n+1} f_{n+1} = (-1)^n x; \tag{3}$$

$$l_n f_{n+2} + l_{n+1} f_{n+1} = 2 f_{2n+2} + (-1)^n x; \tag{4}$$

$$f_n l_{n+2} - f_{n+1} l_{n+1} = (-1)^{n+1} x; \tag{5}$$

$$f_n l_{n+2} + f_{n+1} l_{n+1} = 2 f_{2n+2} - (-1)^n x. \tag{6}$$

These properties can be confirmed using the Binet-like formulas. It follows by identities (3) – (6) that

$$l_n^2 f_{n+2}^2 - l_{n+1}^2 f_{n+1}^2 = 2(-1)^n x f_{2n+2} + x^2; \tag{7}$$

$$f_n^2 l_{n+2}^2 - f_{n+1}^2 l_{n+1}^2 = 2(-1)^{n+1} x f_{2n+2} + x^2. \tag{8}$$

2. Telescoping Gibonacci Sums

We now establish two telescoping gibonacci sums, where $k \geq 0$ and $\lambda \geq 1$ are integers.

Lemma 1:

$$\sum_{n=1}^{\infty} \left(\frac{f_{n+k+2}^{\lambda}}{l_{n+k+1}^{\lambda}} - \frac{f_{n+k+1}^{\lambda}}{l_{n+k}^{\lambda}} \right) = \frac{\alpha^{\lambda}}{\Delta^{\lambda}} - \frac{f_{k+2}^{\lambda}}{l_{k+1}^{\lambda}}. \tag{9}$$

***Proof*:** Since $\sum_{n=1}^{m} \left(\frac{f_{n+k+2}^{\lambda}}{l_{n+k+1}^{\lambda}} - \frac{f_{n+k+1}^{\lambda}}{l_{n+k}^{\lambda}} \right)$ is a telescoping sum, we have

$$\sum_{n=1}^{m} \left(\frac{f_{n+k+2}^{\lambda}}{l_{n+k+1}^{\lambda}} - \frac{f_{n+k+1}^{\lambda}}{l_{n+k}^{\lambda}} \right) = \frac{f_{m+k+2}^{\lambda}}{l_{m+k+1}^{\lambda}} - \frac{f_{k+2}^{\lambda}}{l_{k+1}^{\lambda}}.$$

This yields the desired result. □

Lemma 2:

$$\sum_{n=1}^{\infty} \left(\frac{l_{n+k+2}^{\lambda}}{f_{n+k+1}^{\lambda}} - \frac{l_{n+k+1}^{\lambda}}{f_{n+k}^{\lambda}} \right) = \Delta^{\lambda} \alpha^{\lambda} - \frac{l_{k+2}^{\lambda}}{f_{k+1}^{\lambda}}. \tag{10}$$

***Proof*:** Using the fact that $\lim_{m \to \infty} \frac{l_{m+1}}{f_m} = \Delta\alpha$, the proof follows as above. So, in the interest of brevity, we omit the details. □

These two lemmas play a pivotal role in our discourse.

3. Additional Gibonacci Polynomial Sums

With the above identities and lemmas at our disposal, we are now ready for further explorations.

The next two theorems invoke the lemmas with $\lambda = 1$.

Theorem 1: *Let k be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k} x}{l_{2n+2k+1} + (-1)^{n+k} x} = \frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}. \tag{11}$$

***Proof*:** It follows by identities (2) and (3) that

$$l_{n+k+1} l_{n+k} = l_{2n+2k+1} + (-1)^{n+k} x;$$

$$l_{n+k} f_{n+k+2} - l_{n+k+1} f_{n+k+1} = (-1)^{n+k} x.$$

By Lemma 1, we then have

$$\frac{(-1)^{n+k} x}{l_{2n+2k+1} + (-1)^{n+k} x} = \frac{l_{n+k} f_{n+k+2} - l_{n+k+1} f_{n+k+1}}{l_{n+k+1} l_{n+k}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k} x}{l_{2n+2k+1} + (-1)^{n+k} x} = \sum_{n=1}^{\infty} \left(\frac{f_{n+k+2}}{l_{n+k+1}} - \frac{f_{n+k+1}}{l_{n+k}} \right)$$

$$= \frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}},$$

as desired. □

This implies,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k}}{L_{2n+2k+1} + (-1)^{n+k}} = \frac{5+\sqrt{5}}{10} - \frac{F_{k+2}}{L_{k+1}}.$$

It then follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+1} + (-1)^n} = \frac{-5+\sqrt{5}}{10}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+3} - (-1)^n} = \frac{5-3\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+5} + (-1)^n} = \frac{-5+2\sqrt{5}}{20}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+7} - (-1)^n} = \frac{15-7\sqrt{5}}{70}.$$

The next result invokes Lemma 2 with $\lambda = 1$.

Theorem 2: *Let k be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k+1}x}{l_{2n+2k+1} - (-1)^{n+k}x} = \frac{1}{\Delta^2}\left(\frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}\right). \tag{12}$$

***Proof*:** Using identities (1) and (4), we get

$$\Delta^2 f_{n+k+1} f_{n+k} = l_{2n+2k+1} - (-1)^{n+k}x;$$

$$f_{n+k}l_{n+k+2} - f_{n+k+1}l_{n+k+1} = (-1)^{n+k+1}x.$$

By Lemma 2, we then have

$$\frac{(-1)^{n+k+1}x}{l_{2n+2k+1} - (-1)^{n+k}x} = \frac{f_{n+k}l_{n+k+2} - f_{n+k+1}l_{n+k+1}}{\Delta^2 f_{n+k+1}f_{n+k}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k+1}x}{l_{2n+2k+1} - (-1)^{n+k}x} = \frac{1}{\Delta^2}\sum_{n=1}^{\infty}\left(\frac{l_{n+k+2}}{f_{n+k+1}} - \frac{l_{n+k+1}}{f_{n+k}}\right)$$

$$= \frac{1}{\Delta^2}\left(\Delta\alpha - \frac{l_{k+2}}{f_{k+1}}\right),$$

as desired. □

This yields

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k+1}}{L_{2n+2k+1} - (-1)^{n+k}} = \frac{5+\sqrt{5}}{10} - \frac{L_{k+2}}{5F_{k+1}}.$$

Consequently, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+1} - (-1)^n} = \frac{1-\sqrt{5}}{10}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+3} + (-1)^n} = \frac{-3+\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+5} - (-1)^n} = \frac{2-\sqrt{5}}{10}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{2n+7} + (-1)^n} = \frac{-7+3\sqrt{5}}{30}.$$

Gibonacci Delights : By combining these two theorems, we can extract interesting dividends: Adding equations (11) and (12), we get

$$\sum_{n=1}^{\infty} \frac{2x^2}{l_{2n+2k+1}^2 - x^2} = \frac{\Delta^2+1}{\Delta^2}\left(\frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}\right).$$

This implies

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n+2k+1}^2 - 1} = \frac{15+3\sqrt{5}}{50} + \frac{3F_{k+2}}{5L_{k+1}}$$

In particular, this yields

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n+1}^2 - 1} = \frac{15-3\sqrt{5}}{50}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n+3}^2 - 1} = \frac{5-3\sqrt{5}}{50};$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n+5}^2 - 1} = \frac{15-6\sqrt{5}}{100}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n+7}^2 - 1} = \frac{45-21\sqrt{5}}{350}.$$

Likewise, subtraction of the two equations yields

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k} x l_{2n+2k+1}}{l_{2n+2k+1}^2 - x^2} = \frac{\Delta^2 - 1}{\Delta^2}\left(\frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}\right).$$

This implies,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k} L_{2n+2k+1}}{L_{2n+2k+1}^2 - 1} = \frac{10 + 2\sqrt{5}}{25} - \frac{2F_{k+2}}{5L_{k+1}}.$$

It then follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^n L_{2n+1}}{L_{2n+1}^2 - 1} = \frac{2\sqrt{5}}{25}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n L_{2n+3}}{L_{2n+3}^2 - 1} = \frac{10 + 6\sqrt{5}}{75};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n L_{2n+5}}{L_{2n+5}^2 - 1} = \frac{5 + 4\sqrt{5}}{50}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n L_{2n+7}}{L_{2n+7}^2 - 1} = \frac{20 + 14\sqrt{5}}{175}.$$

The next two theorems employ the lemmas with $\lambda = 2$.

Theorem 3: *Let k be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k} x f_{2n+2k+2} + x^2}{[l_{2n+2k+1} + (-1)^{n+k} x]^2} = \frac{\alpha^2}{\Delta^2} - \frac{f_{k+2}^2}{l_{k+1}^2}. \tag{13}$$

***Proof*:** Lemma 1, coupled with identities (2) and (7), yields

$$\frac{2(-1)^{n+k} x f_{2n+2k+2} + x^2}{[l_{2n+2k+1} + (-1)^{n+k} x]^2} = \frac{l_{n+k}^2 f_{n+k+2}^2 - l_{n+k+1}^2 f_{n+k+1}^2}{l_{n+k+1}^2 l_{n+k}^2}$$

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k} x f_{2n+2k+2} + x^2}{[l_{2n+2k+1} + (-1)^{n+k} x]^2} = \sum_{n=1}^{\infty}\left(\frac{f_{n+k+2}^2}{l_{n+k+1}^2} - \frac{f_{n+k+1}^2}{l_{n+k}^2}\right)$$

$$= \frac{\alpha^2}{\Delta^2} - \frac{f_{k+2}^2}{l_{k+1}^2},$$

as desired. □

Consequently, we have

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k} F_{2n+2k+2} + 1}{[L_{2n+2k+1} + (-1)^{n+k}]^2} = \frac{\alpha^2}{5} - \frac{F_{k+2}^2}{L_{k+1}^2}.$$

In particular, we then get

$$\sum_{n=1}^{\infty} \frac{2(-1)^n F_{2n+2} + 1}{[L_{2n+1} + (-1)^n]^2} = \frac{-7+\sqrt{5}}{10}; \qquad \sum_{n=1}^{\infty} \frac{2(-1)^{n+1} F_{2n+4} + 1}{[L_{2n+3} - (-1)^n]^2} = \frac{-23+9\sqrt{5}}{90};$$

$$\sum_{n=1}^{\infty} \frac{2(-1)^n F_{2n+6} + 1}{[L_{2n+5} + (-1)^n]^2} = \frac{-21+8\sqrt{5}}{80}; \quad \sum_{n=1}^{\infty} \frac{2(-1)^{n+1} F_{2n+8} + 1}{[L_{2n+7} - (-1)^n]^2} = \frac{-103+49\sqrt{5}}{490}.$$

The next result invokes Lemma 2.

Theorem 4: *Let k be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k+1} x f_{2n+2k+2} + x^2}{[l_{2n+2k+1} - (-1)^{n+k} x]^2} = \frac{1}{\Delta^4}\left(\Delta^2\alpha^2 - \frac{l_{k+2}^2}{f_{k+1}^2}\right). \tag{14}$$

***Proof*:** With identities (1) and (8), Lemma 2 yields

$$\frac{2(-1)^{n+k+1} x f_{2n+2k+2} + x^2}{[l_{2n+2k+1} - (-1)^{n+k} x]^2} = \frac{f_{n+k}^2 l_{n+k+2}^2 - f_{n+k+1}^2 l_{n+k+1}^2}{\Delta^4 f_{n+k+1}^2 f_{n+k}^2}$$

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k+1} x f_{2n+2k+1} + x^2}{[l_{2n+2k+1} - (-1)^{n+k} x]^2} = \frac{1}{\Delta^4}\sum_{n=1}^{\infty}\left(\frac{l_{n+k+2}^2}{f_{n+k+1}^2} - \frac{l_{n+k+1}^2}{f_{n+k}^2}\right)$$

$$= \frac{1}{\Delta^4}\left(\Delta^2\alpha^2 - \frac{l_{k+2}^2}{f_{k+1}^2}\right),$$

confirming the given result. □

This theorem implies,

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k+1} F_{2n+2k+2} + 1}{[L_{2n+2k+1}^2 - (-1)^{n+k}]^2} = \frac{1}{25}\left(5\alpha^2 - \frac{L_{k+2}^2}{F_{k+1}^2}\right).$$

It then follows that

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1} F_{2n+2} + 1}{[L_{2n+1} - (-1)^n]^2} = \frac{-3 + 5\sqrt{5}}{50}; \qquad \sum_{n=1}^{\infty} \frac{2(-1)^n F_{2n+4} + 1}{[L_{2n+3} + (-1)^n]^2} = \frac{-17 + 5\sqrt{5}}{50};$$

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1} F_{2n+6} + 1}{[L_{2n+5} - (-1)^n]^2} = \frac{-19 + 10\sqrt{5}}{100}; \qquad \sum_{n=1}^{\infty} \frac{2(-1)^n F_{2n+8} + 1}{[L_{2n+7} + (-1)^n]^2} = \frac{-107 + 45\sqrt{5}}{450}.$$

Next we explore the Pell versions of the theorems.

4. Pell Implications

Using the relationship $b_n(x) = g_n(2x)$, we can find the Pell versions of equations (11) – (14):

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k} x}{q_{2n+2k+1} + 2(-1)^{n+k} x} = \frac{\gamma}{4E} - \frac{p_{k+2}}{2q_{k+1}};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k+1} x}{q_{2n+2k+1} - 2(-1)^{n+k} x} = \frac{1}{8E^2}\left(\frac{\gamma}{2E} - \frac{p_{k+2}}{q_{k+1}}\right);$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k} x p_{2n+2k+2} + x^2}{[q_{2n+2k+1} + 2(-1)^{n+k} x]^2} = \frac{1}{4}\left(\frac{\gamma^2}{4E^2} - \frac{p_{k+2}^2}{q_{k+1}^2}\right);$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k+1} x p_{2n+2k+2} + x^2}{[q_{2n+2k+1} - 2(-1)^{n+k} x]^2} = \frac{1}{64E^4}\left(4E^2\gamma^2 - \frac{q_{k+2}^2}{p_{k+1}^2}\right)$$

They yield

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k}}{Q_{2n+2k+1} + (-1)^{n+k}} = \frac{2+\sqrt{2}}{4} - \frac{P_{k+2}}{Q_{k+1}};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k+1}}{Q_{2n+2k+1} - (-1)^{n+k}} = \frac{2+\sqrt{2}}{32} - \frac{P_{k+2}}{16Q_{k+1}};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k} P_{2n+2k+2} + 1}{[Q_{2n+2k+1} + (-1)^{n+k}]^2} = \frac{3+\sqrt{2}}{8} - \frac{P_{k+2}^2}{4Q_{k+1}^2};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+k+1} P_{2n+2k+2} + 1}{[Q_{2n+2k+1} - (-1)^{n+k}]^2} = \frac{3+2\sqrt{2}}{8} - \frac{Q_{k+2}^2}{16P_{k+1}^2},$$

respectively.

Next we explore the Jacobsthal versions of the theorems.

5. Jacobsthal Consequences

Using the Jacobsthal-gibonacci relationships in Section 1, we will now find the Jacobsthal versions of equations (11) – (14). In the interest of brevity and clarity, we let A denote the fractional expression on left-hand side of the given equation and B its right-hand side, and LHS and RHS those of the desired Jacobsthal equation, respectively.

5.1 Jacobsthal Version of Equation (11): ***Proof*:** Let $A = \dfrac{(-1)^{n+k} x}{l_{2n+k+1} + (-1)^{n+k} x}$. Replacing x with $1/\sqrt{x}$, and multiplying the numerator and denominator of the resulting expression with x^{n+k}, we get

$$A = \frac{(-1)^{n+k}}{\sqrt{x} l_{2n+2k+1} + (-1)^{n+k}}$$

$$= \frac{(-x)^{n+k}}{x^{(2n+2k+1)/2} l_{2n+2k+1} + (-1)^{n+k}}$$

$$= \frac{(-x)^{n+k}}{j_{2n+2k+1} + (-x)^{n+k}} \; ;$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{(-x)^{n+k}}{j_{2n+2k+1} + (-x)^{n+k}} \, , \tag{15}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next we let $B = \frac{\alpha}{\Delta} - \frac{f_{k+2}}{l_{k+1}}$. Replace x with $1/\sqrt{x}$, and then multiply each numerator and denominator of the resulting expression with $x^{(k+1)/2}$. This yields

$$B = \frac{D+1}{2D} - \frac{x^{(k+1)/2} f_{k+2}}{x^{(k+1)/2} l_{k+1}} \; ;$$

$$\text{RHS} = \frac{D+1}{2D} - \frac{J_{k+2}}{j_{k+1}} \, ,$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

This, combined with equation (15), yields the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{(-x)^{n+k}}{j_{2n+2k+1} + (-x)^{n+k}} = \frac{D+1}{2D} - \frac{J_{k+2}}{j_{k+1}} \, , \tag{16}$$

where $c_n = c_n(x)$. □

It then follows that

$$\sum_{n=1}^{\infty} \frac{(-x)^{n+k}}{L_{2n+2k+1} + (-1)^{n+k}} = \frac{5+\sqrt{5}}{10} - \frac{F_{k+2}}{L_{k+1}};$$

$$\sum_{n=1}^{\infty} \frac{(-2)^{n+k}}{j_{2n+2k+1} + (-2)^{n+k}} = \frac{2}{3} - \frac{J_{k+2}}{j_{k+1}}.$$

Next we find the Jacobsthal consequence of equation (12).

5.2 Jacobsthal Version of Equation (12): ***Proof*:** We have $A = \dfrac{(-1)^{n+k+1}x}{l_{2n+k+1} - (-1)^{n+k}x}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator of the resulting expression with x^{n+k}, we get

$$A = \frac{(-1)^{n+k+1}}{\sqrt{x}l_{2n+2k+1} - (-1)^{n+k}}$$

$$= \frac{-(-x)^{n+k}}{x^{(2n+2k+1)/2}l_{2n+2k+1} - (-1)^{n+k}}$$

$$= \frac{-(-x)^{n+k}}{j_{2n+2k+1} - (-x)^{n+k}};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{-(-x)^{n+k}}{j_{2n+2k+1} - (-x)^{n+k}}, \tag{17}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next we let $B = \Delta\alpha - \dfrac{l_{k+2}}{f_{k+1}}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator of the resulting expression with $x^{(n+k)/2}$ yields

$$B = \frac{x}{D^2}\left[\frac{(D+1)D}{2x} - \frac{l_{k+2}}{f_{k+1}}\right]$$

$$= \frac{1}{D^2}\left[\frac{(D+1)D}{2} - \frac{x^{(k+2)/2} l_{k+2}}{x^{k/2} f_{k+1}}\right];$$

$$\text{RHS} = \frac{1}{D^2}\left[\frac{(D+1)D}{2} - \frac{j_{k+2}}{J_{k+1}}\right],$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Combined with equation (17), this yields the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{-(-x)^{n+k}}{j_{2n+2k+1} - (-x)^{n+k}} = \frac{1}{D^2}\left[\frac{(D+1)D}{2} - \frac{j_{k+2}}{J_{k+1}}\right], \tag{18}$$

where $c_n = c_n(x)$. □

In particular, this yields

$$\sum_{n=1}^{\infty} \frac{-(-x)^{n+k}}{L_{2n+2k+1} - (-1)^{n+k}} = \frac{5+\sqrt{5}}{10} - \frac{L_{k+2}}{5F_{k+1}};$$

$$\sum_{n=1}^{\infty} \frac{-(-2)^{n+k}}{j_{2n+2k+1} - (-2)^{n+k}} = \frac{2}{3} - \frac{j_{k+2}}{9J_{k+1}}.$$

5.3 Jacobsthal Version of Equation (13): ***Proof*:** Let $A = \dfrac{2(-1)^{n+k} x f_{2n+k+2} + x^2}{[l_{2n+k+1} + (-1)^{n+k} x]^2}$. Replacing x with $1/\sqrt{x}$, and multiplying the numerator and denominator of the resulting expression with $x^{2n+2k+1}$, we get

$$A = \frac{2(-1)^{n+k}\,\frac{1}{\sqrt{x}}\,f_{2n+k+2} + \frac{1}{x}}{\left[l_{2n+2k+1} + (-1)^{n+k}\,\frac{1}{\sqrt{x}}\right]^2}$$

$$= \frac{2(-x)^{n+k}[x^{(2n+2k+1)/2}f_{2n+2k+2}] + x^{2n+2k}}{[x^{(2n+2k+1)/2}l_{2n+2k+1} + (-1)^{n+k}]^2}$$

$$= \frac{2\,(-x)^{n+k}\,J_{2n+2k+2} + x^{2n+2k}}{[j_{2n+2k+1} + (-x)^{n+k}]^2}\;;$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{2\,(-x)^{n+k}\,J_{2n+2k+2} + x^{2n+2k}}{[j_{2n+2k+1} + (-x)^{n+k}]^2}, \tag{19}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Now let $B = \frac{\alpha^2}{\Delta^2} - \frac{f_{k+2}^2}{l_{k+1}^2}$. Replacing x with $1/\sqrt{x}$, and then multiply each numerator and denominator of the resulting expression with x^{k+1}. This yields

$$B = \frac{(D+1)^2}{4D^2} - \frac{[x^{(k+1)/2}f_{k+2}]^2}{[x^{(k+1)/2}l_{k+2}]^2};$$

$$\text{RHS} = \frac{(D+1)^2}{4D^2} - \frac{J_{k+2}^2}{j_{k+1}^2},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

This, coupled with equation (19), yields the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{2\,(-x)^{n+k}\,J_{2n+2k+2} + x^{2n+2k}}{\left[j_{2n+2k+1} + (-x)^{n+k}\right]^2} = \frac{(D+1)^2}{4D^2} - \frac{J_{k+2}^2}{j_{k+1}^2}, \tag{20}$$

where $c_n = c_n(x)$. □

In particular, this yields

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+k} F_{2n+k+2} + 1}{\left[L_{2n+2k+1} + (-x)^{n+k}\right]^2} = \frac{3+\sqrt{5}}{10} - \frac{F_{k+2}^2}{L_{k+1}^2};$$

$$\sum_{n=1}^{\infty} \frac{2(-2)^{n+k} J_{2n+k+2} + 4^{n+k}}{\left[j_{2n+2k+1} + (-2)^{n+k}\right]^2} = \frac{4}{9} - \frac{J_{k+2}^2}{j_{k+1}^2}.$$

Next we find the Jacobsthal consequence of Theorem 4.

5.4 Jacobsthal Version of Equation (14): ***Proof:*** We have $A = \dfrac{2(-1)^{n+k+1} x f_{2(n+k)+2} + x^2}{[l_{2(n+k)+1}^2 + (-1)^{n+k} x]^2}$. Replace x with and then multiply the numerator and denominator of the resulting expression with $x^{2n+2k+1}$. We then get

$$A = \frac{2(-1)^{n+k+1} \dfrac{1}{\sqrt{x}} f_{2n+k+2} + \dfrac{1}{x}}{\left[l_{2n+2k+1} - (-1)^{n+k} \dfrac{1}{\sqrt{x}}\right]^2}$$

$$= \frac{-2(-x)^{n+k}[x^{(2n+2k+1)/2} f_{2n+2k+2}] + x^{2n+2k}}{[x^{(2n+2k+1)/2} l_{2n+2k+1} - (-x)^{n+k}]^2}$$

$$= \frac{-2(-x)^{n+k} J_{2n+k+2} + x^{2n+2k}}{[j_{2n+2k+1} - (-x)^{n+k}]^2};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{-2(-x)^{n+k} J_{2n+k+2} + x^{2n+2k}}{[j_{2n+2k+1} - (-x)^{n+k}]^2}, \tag{21}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Now let $B = \frac{1}{\Delta^4}\left(\Delta^2\alpha^2 - \frac{l_{k+2}^2}{f_{k+1}^2}\right)$. Replacing x with $1/\sqrt{x}$, and then multiply each numerator and denominator of the resulting expression with x^{k+1} yields

$$B = \frac{x^2}{D^2}\left\{\frac{D^2(D+1)^2}{4x^2} - \frac{[x^{(k+2)/2}l_{k+2}]^2}{x^2[x^{(k/2}f_{k+1}]^2}\right\};$$

$$\text{RHS} = \frac{(D+1)^2}{4D^2} - \frac{j_{k+2}^2}{D^4 J_{k+1}^2},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Combining this with equation (21) yields the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{-2(-x)^{n+k} J_{2n+k+2} + x^{2n+2k}}{\left[j_{2n+2k+1} - (-x)^{n+k}\right]^2} = \frac{(D+1)^2}{4D^2} - \frac{j_{k+2}^2}{D^4 J_{k+1}^2}, \tag{22}$$

where $c_n = c_n(x)$. □

It follows from this equation that

$$\sum_{n=1}^{\infty} \frac{-2(-1)^{n+k} F_{2n+k+2} + 1}{\left[L_{2n+2k+1} - (-x)^{n+k}\right]^2} = \frac{3+\sqrt{5}}{10} - \frac{L_{k+2}^2}{25F_{k+1}^2};$$

$$\sum_{n=1}^{\infty} \frac{(-2)^{n+k+1} J_{2n+k+2} + 4^{n+k}}{\left[j_{2n+2k+1} - (-2)^{n+k}\right]^2} = \frac{4}{9} - \frac{J_{k+2}^2}{j_{k+1}^2}.$$

6. Chebyshev and Vieta Consequences

Chebyshev polynomials T_n and U_n, Vieta polynomials V_n and v_n, and gibonacci polynomials are linked by the relationships $V_n(x) = i^{n-1} f_n(-ix)$,

$v_n(x) = i^n l_n(-ix)$, $V_n(x) = U_{n-1}(x/2)$ and $v_n(x) = 2T_n(x/2)$ [3, 4, 5] where $i = \sqrt{-1}$; they can be employed to find the Chebyshev and Vieta versions of the theorems. In the interest of brevity, we omit them; but we encourage gi- bonacci enthusiasts to explore them.

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