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Gollakota V. V. Hemasundar

COMPUTING THE DIMENSION OF QUADRATIC DIFFERENTIALS OVER COMPACT RIEMANN SURFACES OF GENUS $g \geq 2$

Abstract: In this note we show that how to compute the dimension of holomorphic quadratic differentials on a compact Riemann surface of genus $g \geq 2$.

Keywords: Compact Riemann Surfaces, Holomorphic Quadratic Differentials.

Mathematical Subject Classification No.: 30G10, 32G15.

1. Introduction

A Riemann surface is a complex manifold with complex dimension 1. For a more detailed definition Refer [1]. In general on a given topological surface, there are many inequivalent complex structures. The set of inequivalent complex structres on a given topological surface is known as the moduli space. The moduli space is a complicated one and is a topic for advanced research. The interplay between the complex structure of a Riemann surface and the geometry induced by a quadratic differential was given in the paper of Hubbard and Masur [3].

Let X be a compact Riemann surface of genus g. A holomorphic quadratic differential φ is an assignment of a holomorphic function $\varphi_i(z_i)$ to each local coordinate chart z_i such that if there is another chart z_j, then

$$\varphi_j(z_j)dz_j^2 = \varphi_i(z_i)\,dz_i^2$$

or
$$\varphi_i(z_i) = \varphi_j(z_j)\left(\frac{dz_j}{dz_i}\right)^2$$

and $z_i = h(z_j)$.

In 1857, Riemann published his work on the deformation of complex structures. He called the number of independent parameters the deformation depends as moduli. It is known that the moduli for compact Riemann surfaces of genus $g \geq 2$ is equal to $3g-3$ (complex parameters) which is also the same as the dimension of holomorphic quadratic differentials on a compact Riemann surface of genus $g \geq 2$.

In this note we show how to compute the dimension of holomorphic quadratic differentials. The reader may refer to [1] for Cohomology groups, Riemann-Roch and Serre Duality theorems. The role of quadratic differentials in the moduli of Riemann surfaces may be found in [2], [4].

2. The Dimension of Holomorphic Quadratic Differentials

Theorem 1: *Let X be a compact Riemann surface of genus $g \geq 2$. The dimension of holomorphic quadratic differentials on X is $3g-3$.*

Proof: Let X be a compact Riemann surface of genus g. Let K be the canonical bundle (whose holomorphic cross sections are precisely the abelian differentials of first kind) on X.

By the Riemann Roch theorem, for any holomorphic bundle $\xi \in H^1(X, \mathcal{O}^*)$, we have,

$$\dim H^0(X, \mathcal{O}(\xi)) - \dim H^1(X, \mathcal{O}(\xi)) = 1 - g + \deg \xi \tag{1}$$

and from the Serre duality theorem

$$\dim H^1(X, \mathcal{O}(\xi)) = \dim H^0(X, \mathcal{O}(K\xi^{-1})) \tag{2}$$

Therefore, by writing

$$\xi = K^2 = K \otimes K$$

and using the Eq. 1 and Eq. 2 we can deduce that

$$\dim H^0(X,\mathcal{O}(\xi)) - \dim H^0(X,\mathcal{O}(K\xi^{-1})) = 1 - g + \deg \xi$$

which implies,

$$\dim H^0(X,\mathcal{O}(K \otimes K)) - \dim H^0(X,\mathcal{O}(K^{-1})) = 1 - g + \deg K^2$$

but,

$$\deg K = 2g - 2$$

$$\Rightarrow \deg K^{-1} = 2 - 2g < 0$$

$$\Rightarrow \dim H^0(X,\mathcal{O}(K^{-1})) = 0 \ .$$

Therefore we have,

$$\dim H^0(X,\mathcal{O}(K^2)) = \dim H^0(X,\mathcal{O}(K \otimes K)) \tag{3}$$

$$= 1 - g + 4g - 4 \tag{4}$$

$$= 3g - 3 \tag{5}$$

where $H^0(X,\mathcal{O}(K^2))$ denotes the holomorphic quadratic differentials on X. ■

Note: The dimension of the space of holomorphic quadratic differentials is independent of the complex structure of a Riemann surface of genus g.

Summary 2: *According to Riemann well known formula the complex structure of a compact Riemann surface X of genus $g \geq 2$ (with no punctures) depends on $3g - 3$ parameters.*

= the number of linearly independent quadratic differentials on X

= the maximal number of simple closed curves on X whose homotopy classes represented by non-intersecting curves which are not homotopic to each other. See [4].

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CONVERGENCE OF EXTENDED GIBONACCI POLYNOMIAL SERIES

Abstract: We explore the convergence of both gibonacci polynomial series, and then extract their Pell and Jacobsthal versions.

Keywords: Extended Gibonacci Polynomials Series, Fibonacci Polynomials, Lucas Polynomials, Pell Polynomials, Jacobsthal Polynomials.

Mathematical Subject Classification (2020) No.: Primary 11B37, 11B39, 11C08.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is a positive integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 5, 6].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [6].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial* [3, 6]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$ and $j_n(1) = L_n$.

Gibonacci and Jacobsthal polynomials are linked by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2}\, l_n(1/\sqrt{x})$ [6].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $c_n = J_n(x)$ or $j_n(x)$, $\Delta = \sqrt{x^2+4}$, $2\alpha(x) = x + \Delta$, and $2\beta(x) = x - \Delta$. It follows by the *Binet-like formulas* [6] that $\lim_{m\to\infty} \frac{g_{m+k}}{g_m} = \alpha^k(x)$.

1.1 Gibonacci Generating Functions: Fibonacci polynomials f_n and Lucas polynomials g_n are generated by the generating functions $f(t)$ and $l(t)$ [6]:

$$f(t) = \frac{t}{1 - xt - t^2} = \sum_{n=1}^{\infty} f_n t^n \ ; \tag{1}$$

$$l(t) = \frac{2 - xt}{1 - xt - t^2} = \sum_{n=0}^{\infty} l_n t^n \ , \tag{2}$$

respectively, where $x \geq 1$.

By the *Ratio test* [8], both series converge if $|t| < \lim_{n\to\infty} \frac{g_n}{g_{n+1}}$; that is, if $t < \frac{1}{\alpha(x)} = \frac{2}{\alpha + \Delta}$.

2. Gibonacci Polynomial Series

With the brief background above, we now explore the convergence of both gibonacci polynomial series for a special family of values of t, beginning with the Fibonacci series.

2.1 Fibonacci Polynomial Series: The following theorem identifies the value of the Fibonacci polynomial series (1) for a special family of t.

Theorem 1: *Let k and λ be arbitrary positive integers, and $u = 1 + \lambda x$. Then*

$$\sum_{n=1}^{\infty} \left(\sum_{r=1}^{k} \frac{1}{u^{rn}} \right) f_n = \sum_{r=1}^{k} \frac{u^r}{u^{2r} - xu^r - 1}. \tag{3}$$

Proof: With $x \geq 1$, we have $\sqrt{x^2 + 4} < x + 2$; so $x + \Delta < 2 + 2x$. Consequently, $0 < \frac{1}{1+x} < \frac{2}{x + \Delta}$; that is, $0 < \frac{1}{1+x} < \frac{2}{\alpha(x)}$.

With $\lambda \geq 2$, we have $1 + \lambda x > 1 + x$. So, $0 < \frac{1}{1+\lambda x} < \frac{1}{1+x} < \frac{1}{\alpha(x)}$.

Since $(1 + \lambda x)^{rn} > (1 + x)^{rn} > 1 + x$, it follows that

$$0 < \frac{1}{(1 + \lambda x)^{rn}} < \frac{1}{(1 + x)^{rn}} < \frac{1}{1+x} < \frac{1}{\alpha(x)},$$

where $r \geq 1$.

Consequently, the Fibonacci series $\sum_{n=1}^{\infty} t^n f_n$ converges when $t = \frac{1}{(1+\lambda x)^r} = \frac{1}{u^r}$ and $r \geq 1$.

This implies

$$f(1/u^r) = \frac{1/u^r}{1 - \frac{x}{u^r} - \frac{1}{u^{2r}}}$$

$$= \frac{u^r}{u^{2r} - xu^r - 1}.$$

Consequently, we have

$$\sum_{n=1}^{\infty} \left(\sum_{r=1}^{k} \frac{1}{u^{rn}} \right) f_n = \sum_{r=1}^{k} f(1/u^r)$$

$$= \sum_{r=1}^{k} \frac{u^r}{u^{2r} - xu^r - 1}$$

as desired. □

With $u = 1 + \lambda x$, it follows by the theorem that

$$\sum_{n=1}^{\infty} \frac{1}{u^n} f_n = \frac{u}{u^2 - xu - 1} ;$$

$$\sum_{n=1}^{\infty} \frac{1 + u^n}{u^{2n}} f_n = \frac{u}{u^2 - xu - 1} + \frac{u^2}{u^4 - xu^2 - 1}$$

$$\sum_{n=1}^{\infty} \frac{1 + u^n + u^{2n}}{u^{3n}} f_n = \frac{u}{u^2 - xu - 1} + \frac{u^2}{u^4 - xu^4 - 1} + \frac{u^3}{u^6 - xu^3 - 1}$$

$$\sum_{n=1}^{\infty} \frac{1 + u^n + u^{2n} + u^{3n}}{u^{4n}} f_n = \frac{u}{u^2 - xu - 1} + \frac{u^2}{u^4 - xu^4 - 1} + \frac{u^3}{u^6 - xu^3 - 1} + \frac{u^4}{u^8 - xu^4 - 1}$$

In particular, we then get

$$\sum_{n=1}^{\infty} \frac{F_n}{2^n} = 2; \qquad \sum_{n=1}^{\infty} \frac{1 + 2^n}{2^{2n}} F_n = \frac{26}{11}; [2]$$

$$\sum_{n=1}^{\infty} \frac{1 + 2^n + 2^{n2}}{2^{3n}} F_n = \frac{138}{55}; \qquad \sum_{n=1}^{\infty} \frac{1 + 2^n + 2^{n2} + 2^{3n}}{2^{4n}} F_n = \frac{33,862}{13,145}.$$

With $\lambda = 2$, it follows by Theorem 1 that

$$\sum_{n=1}^{\infty} \frac{F_n}{3^n} = \frac{3}{5}; \qquad \sum_{n=1}^{\infty} \frac{1 + 3^n}{3^{2n}} F_n = \frac{258}{355};$$

$$\sum_{n=1}^{\infty} \frac{1 + 3^n + 3^{n2}}{3^{3n}} F_n = \frac{190,443}{248,855}; \quad \sum_{n=1}^{\infty} \frac{1 + 3^n + 3^{n2} + 3^{3n}}{3^{4n}} F_n = \frac{1,254,037,452}{1,612,331,545}.$$

***An Interesting Case*:** Let $\lambda = M_\nu = 2^\nu - 1$, a *Mersenne number* [4]. With $x = 1$, we have $u = 2^\nu$. It then follows from equation (3) that

$$\sum_{n=1}^{\infty} \left(\sum_{r=1}^{k} \frac{1}{2^{\nu r n}} \right) f_n = \sum_{r=1}^{k} \frac{2^{\nu r}}{2^{2\nu r} - 2^{\nu r} - 1}.$$

With $\nu = 5$, we then get

$$\sum_{n=1}^{\infty} \frac{F_n}{32^n} = \frac{32}{991}; \qquad \sum_{n=1}^{\infty} \frac{1 + 32^n}{32^{2n}} F_n = \frac{34,536,416}{1,038,123,041}.$$

Next we explore the Lucas version of Theorem 1. It also identifies a family of values of t for which Lucas polynomial series (2) converges.

2.2 Lucas Polynomial Series: As before, the Lucas polynomial series also converges by the *Ratio test* for $|t| < \dfrac{1}{\alpha(x)}$.

Theorem 2: *Let k and λ be arbitrary positive integers, and $u = 1 + \lambda x$. Then*

$$\sum_{n=1}^{\infty}\left(\sum_{r=1}^{k}\frac{1}{u^{rn}}\right) l_n = \sum_{r=1}^{k}\frac{2u^{2r} - ru^r}{u^{2r} - xu^r - 1}. \qquad (4)$$

Proof **:** As in the proof of Theorem 1, $u > \alpha(x) > 0$ and hence $0 < \dfrac{1}{u^r} < \dfrac{1}{\alpha(x)}$ for every positive integer r. Conseque[redacted] converges for $t = \dfrac{1}{u^r}$.

This implies

$$l(1/u^r) = \frac{2 - \dfrac{x}{ur}}{1 - \dfrac{x}{u^r} - \dfrac{1}{u^{2r}}}$$

$$= \frac{2u^{2r} - xu^r}{u^{2r} - xu^r - 1}.$$

Consequently, we have

$$\sum_{n=0}^{\infty}\left(\sum_{r=1}^{k}\frac{1}{u^{rn}}\right) l_n = \sum_{r=1}^{k} l(1/u^r)$$

$$= \sum_{r=1}^{k}\frac{2u^{2r} - ru^r}{u^{2r} - xu^r - 1}$$ [redacted]

as desired.

With $\lambda = 1 = x$, it follows from equation (4) that

$$\sum_{n=0}^{\infty}\left(\sum_{r=1}^{k}\frac{1}{2^{rn}}\right)L_n = \sum_{r=1}^{k}\frac{2^{2r+1}-2^{r}}{2^{2r}-2^{r}-1}.$$

This yields,

$$\sum_{n=0}^{\infty}\frac{L_n}{2^n} = 6; \qquad \sum_{n=0}^{\infty}\frac{1+2^n}{2^{2n}}L_n = \frac{94}{11};$$

$$\sum_{n=0}^{\infty}\frac{1+2^n+2^{2n}}{2^{3n}}L_n = \frac{118}{11}; \qquad \sum_{n=0}^{\infty}\frac{1+2^n+2^{2n}+2^{3n}}{2^{4n}}L_n = \frac{33,658}{2,629}$$

When $x = 1$ and $\lambda = 2$, equation (4) gives

$$\sum_{n=0}^{\infty}\frac{L_n}{3^n} = 3; \qquad \sum_{n=0}^{\infty}\frac{1+3^n}{3^{2n}}L_n = \frac{366}{71};$$

$$\sum_{n=0}^{\infty}\frac{1+3^n+3^{2n}}{3^{3n}}L_n = \frac{358,167}{49,771}; \qquad \sum_{n=0}^{\infty}\frac{1+3^n+3^{2n}+3^{3n}}{3^{4n}}L_n = \frac{2,969,627,604}{322,466,309}.$$

***An Interesting Case*:** With $\lambda = M_\nu = 2^\nu - 1$, $x = 1$, and $u = 2^\nu$, equation (4) yields

$$\sum_{n=0}^{\infty}\left(\sum_{r=1}^{k}\frac{1}{2^{\nu rn}}\right)L_n = \sum_{r=1}^{k}\frac{2.4^{\nu r}-2^{\nu r}}{4^{\nu r}-2^{\nu r}-1}.$$

This implies,

$$\sum_{n=0}^{\infty}\frac{L_n}{4^n} = \frac{28}{11}; \qquad \sum_{n=0}^{\infty}\frac{1+4^n}{4^{2n}}L_n = \frac{12,148}{2,629};$$

$$\sum_{n=0}^{\infty} \frac{L_n}{8^n} = \frac{24}{11}; \qquad \sum_{n=0}^{\infty} \frac{1+8^n}{8^{2n}} L_n = \frac{186,152}{44,341};$$

$$\sum_{n=0}^{\infty} \frac{L_n}{16^n} = \frac{496}{239}; \qquad \sum_{n=0}^{\infty} \frac{1+16^n}{16^{2n}} L_n = \frac{63,643,408}{15,601,681};$$

Next we explore the Pell implications of formulas (3) and (4).

3. PellVersions

Let k and λ be arbitrary positive integers, and $w = 1 + 2\lambda x$. With the gibonacci-Pell relationship $b_n(x) = g_n(2x)$, it follows from equations (3) and (4) that

$$\sum_{n=1}^{\infty} \left(\sum_{r=1}^{k} \frac{1}{w^{rn}} \right) p_n = \sum_{r=1}^{k} \frac{w^r}{w^{2r} - 2xw^r - 1} \tag{5}$$

$$\sum_{n=0}^{\infty} \left(\sum_{r=1}^{k} \frac{1}{w^{rn}} \right) q_n = \sum_{r=1}^{k} \frac{2w^{2r} - 2xw^r}{w^{2r} - 2xw^r - 1} \tag{6}$$

respectively.

With $W = w(1) = 1 + 2\lambda$, they yield

$$\sum_{n=1}^{\infty} \left(\sum_{r=1}^{k} \frac{1}{W^{rn}} \right) P_n = \sum_{r=1}^{k} \frac{W^r}{W^{2r} - 2W^r - 1}$$

$$\sum_{n=0}^{\infty} \left(\sum_{r=1}^{k} \frac{1}{W^{rn}} \right) Q_n = \sum_{r=1}^{k} \frac{W^{2r} - W^r}{W^{2r} - 2W^r - 1}$$

respectively.

In particular, with $\lambda = 1$, we then have

$$\sum_{n=1}^{\infty} \frac{P_n}{3^n} = \frac{3}{2}; \qquad \sum_{n=1}^{\infty} \frac{1+3^n}{3^{2n}} P_n = \frac{51}{31};$$

$$\sum_{n=1}^{\infty} \frac{1+3^n+3^{2n}}{3^{3n}} P_n = \frac{35,211}{20,894}; \qquad \sum_{n=1}^{\infty} \frac{1+3^n+3^{2n}+3^{3n}}{3^{4n}} P_n = \frac{56,743,098}{33,419,953}$$

$$\sum_{n=0}^{\infty} \frac{Q_n}{3^n} = 3; \qquad \sum_{n=0}^{\infty} \frac{1+3^n}{3^{2n}} Q_n = \frac{129}{31};$$

$$\sum_{n=0}^{\infty} \frac{1+3^n+3^{2n}}{3^{3n}} Q_n = \frac{54,354}{10,447}; \qquad \sum_{n=1}^{\infty} \frac{1+3^n+3^{2n}+3^{3n}}{3^{4n}} Q_n = \frac{207,726,726}{33,419,953}$$

With $\lambda = 2$, they yield

$$\sum_{n=1}^{\infty} \frac{P_n}{5^n} = \frac{5}{14}; \qquad \sum_{n=1}^{\infty} \frac{1+5^n}{5^{2n}} P_n = \frac{115}{287};$$

$$\sum_{n=1}^{\infty} \frac{1+5^n+5^{2n}}{5^{3n}} P_n = \frac{1,803,885}{4,412,338}; \qquad \sum_{n=1}^{\infty} \frac{1+5^n+5^{2n}+5^{3n}}{5^{4n}} P_n = \frac{176,285,907,310}{429,512,424,103}.$$

$$\sum_{n=0}^{\infty} \frac{Q_n}{5^n} = \frac{10}{7}; \qquad \sum_{n=0}^{\infty} \frac{1+5^n}{5^{2n}} Q_n = \frac{710}{287};$$

$$\sum_{n=0}^{\infty} \frac{1+5^n+5^{2n}}{5^{3n}} Q_n = \frac{7,682,020}{2,206,169}; \qquad \sum_{n=0}^{\infty} \frac{1+5^n+5^{2n}+5^{3n}}{5^{4n}} Q_n = \frac{1,925,792,328,740}{429,512,424,103};$$

Finally, we pursue the Jacobsthal versions of formulas (3) and (4).

4. Jacobsthal Implications

Using the gibonacci-Jacobsthal relationships, we now explore the Jacobsthal consequences of formulas (3), and (4). In the interest of brevity and clarity, we let A denote the left side of the given equation and B its right side, and LHS and RHS the left-hand side and right-hand side of the corresponding Jacobsthal formula to be found.

4.1 Jacobsthal Version of Formula (3)**:** ***Proof*****:** Let $A = \sum_{r=1}^{k} \frac{f_n}{(1+\lambda x)^{rn}}$. Replacing x with $1/\sqrt{x}$ in A, and then multiplying the numerator and denominator of the resulting expression with $x^{(n-1)/2}$, we get

$$A = \sum_{r=1}^{k} \frac{x^{[(r-1)n+1]/2}\left[x^{(n-1)/2} f_n\right]}{x^{(n-1)/2}\left(\sqrt{x}+\lambda\right)^{rn}}$$

$$= \sum_{r=1}^{k} \frac{x^{[(r-1)n+1]/2} J_n}{\left(\sqrt{x}+\lambda\right)^{rn}};$$

$$\text{LHS} = \sum_{n=1}^{k} \left\{ \sum_{r=1}^{k} \frac{x^{[(r-1)n+1]/2} J_n}{\left(\sqrt{x}+\lambda\right)^{rn}} \right\},$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Next we let $B = \sum_{r=1}^{k} \frac{(1+\lambda x)^r}{(1+\lambda x)^{2r} - x(1+\lambda x)^r - 1}$. Replacing x with $1/\sqrt{x}$ in B, this yields

$$B = \sum_{r=1}^{k} \frac{x^{3/2}\left(\sqrt{x}+\lambda\right)^r}{x^{(r+1)/2}\left(\sqrt{x}+\lambda\right)^{2r} - x\left(\sqrt{x}+\lambda\right)^r - x^{(r+3)/2}}.$$

$$\text{RHS} = \sum_{r=1}^{k} \frac{\sqrt{x}\left(\sqrt{x}+\lambda\right)^r}{x^{(r-1)/2}\left(\sqrt{x}+\lambda\right)^{2r} - \left(\sqrt{x}+\lambda\right)^r - x^{(r+1)/2}}.$$

Equating the two sides gives the desired Jacobsthal version:

$$\sum_{n=1}^{\infty}\left[\sum_{r=1}^{k}\frac{x^{(r-1)n/2}}{(\sqrt{x}+\lambda)^{rn}}\right]J_n = \sum_{r=1}^{k}\frac{(\sqrt{x}+\lambda)^r}{x^{(r-1)/2}(\sqrt{x}+\lambda)^{2r}-(\sqrt{x}+\lambda)^r-x^{(r+1)/2}}. \tag{7}$$

where $J_n = J_n(x)$. □

In particular, this yields

$$\sum_{n=1}^{\infty}\frac{F_n}{2^n} = 2; \qquad \sum_{n=1}^{\infty}\frac{1+2^n}{2^{2n}}F_n = \frac{26}{11}; \text{ [2]}$$

$$\sum_{n=1}^{\infty}\frac{F_n}{3^n} = \frac{3}{5}; \qquad \sum_{n=1}^{\infty}\frac{1+3^n}{3^{2n}}F_n = \frac{258}{355}$$

as found earlier.

Next we explore the Jacobsthal counterpart of equation (4).

4.2 Jacobsthal Version of Formula (4)**:** ***Proof*****:** We have $A=\sum_{r=1}^{k}\frac{f_n}{(1+\lambda x)^{rn}}$.

Replace x with $1/\sqrt{x}$ in A, and then multiply the numerator and denominator of the resulting expression with $x^{(n-1)/2}$. This yields

$$A=\sum_{r=1}^{k}\frac{x^{(r-1)n/2}(x^{n/2}l_n)}{(\sqrt{x}+\lambda)^{rn}}$$

$$=\sum_{r=1}^{k}\frac{x^{(r-1)n/2}j_n}{(\sqrt{x}+\lambda)^{rn}};$$

$$\text{LHS}=\sum_{n=0}^{k}\left[\sum_{r=1}^{k}\frac{x^{(r-1)n/2}}{(\sqrt{x}+\lambda)^{rn}}\right]j_n$$

where $l_n = l_n(1/\sqrt{x})$ and $j_n = j_n(x)$.

We have $B = \sum_{r=1}^{k} \frac{2(1+\lambda x)^r - x}{(1+\lambda x)^{2r} - x(1+\lambda x)^r - 1}$ Replacing x with $1/\sqrt{x}$, this yields

$$\text{RHS} = \sum_{r=1}^{k} \frac{2(\sqrt{x}+\lambda)^{2r} - x^{(r-1)/2}}{(\sqrt{x}+\lambda)^{2r} - x^{(r-1)/2}(\sqrt{x}+\lambda)^r - x^{r/2}}.$$

Combining the two sides, we get the desired Jacobsthal counterpart:

$$\sum_{n=0}^{\infty}\left[\sum_{r=1}^{k} \frac{x^{(r-1)n/2}}{(\sqrt{x}+\lambda)^{rn}}\right] j_n = \sum_{r=1}^{k} \frac{2(\sqrt{x}+\lambda)^{2r} - x^{(r-1)/2}}{(\sqrt{x}+\lambda)^{2r} - x^{(r-1)/2}(\sqrt{x}+\lambda)^r - x^{r/2}}.$$

where $j_n = j_n(x)$. □

In particular, this yields

$$\sum_{n=0}^{\infty} \frac{L_n}{2^n} = 6; \qquad \sum_{n=0}^{\infty} \frac{1+2^n}{2^{2n}} L_n = \frac{94}{11}; \text{ [2]}$$

$$\sum_{n=0}^{\infty} \frac{L_n}{3^n} = 3; \qquad \sum_{n=0}^{\infty} \frac{1+3^n}{3^{2n}} L_n = \frac{366}{71},$$

as found earlier.

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GENERALIZED GIBONACCI POLYNOMIAL PRODUCTS WITH IMPLICATIONS

Abstract: We explore infinite products involving gibonacci polynomials, and their Pell and Pell-Lucas implications.

Keywords: Extended Gibonacci Polynomials, Pell Polynomials, Pell-Lucas Polynomials, Binet Formula.

Mathematical Subject Classification (2020) No.: 11B37, 11B39, 11B83, 11C08.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 4].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [3, 4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, and $b_n = p_n$ or q_n.

It follows by the Binet-like formulas that $\lim_{n\to\infty} \frac{f_n}{l_n} = \frac{1}{\Delta}$, $\lim_{n\to\infty} \frac{l_n}{f_n} = \Delta$, $\lim_{n\to\infty} \frac{p_n}{q_n} = \frac{1}{2E}$, and $\lim_{n\to\infty} \frac{q_n}{p_n} = 2E$, where $\Delta = \sqrt{x^2+4}$ and $E = \sqrt{x^2+1}$.

1.1 Two Fundamental Gibonacci Identities: It follows by the Binet-like formulas for f_n and l_n [4] that

$$f_{2n} + (-1)^{n+k} f_{2k} = f_{n+k} l_{n-k}; \tag{1}$$

$$f_{2n} - (-1)^{n+k} f_{2k} = f_{n-k} l_{n+k}. \tag{2}$$

These two results play a pivotal role in our discourse. With this background, we now begin our explorations.

2. Generalized Gibonacci Polynomial Products

We split our discussion into two cases, depending on the parity of k in identities (1) and (2).

2.1 A Generalization with *k* Odd: When k is odd, identities (1) and (2) yield

$$f_{2(2n)} + f_{2k} = f_{2n-k} l_{2n+k}; \tag{3}$$

$$f_{2(2n)} - f_{2k} = f_{2n+k} l_{2n-k}. \tag{4}$$

With these two identities, we now establish the first result.

Theorem 1: *Let k be an odd positive integer. Then*

$$\prod_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{\infty} \frac{f_{2(2n)} + f_{2k}}{f_{2(2n)} - f_{2k}} = \prod_{r=1}^{k} \frac{f_{2r-1}}{l_{2r-1}} \cdot \Delta^k \tag{5}$$

Proof: Using recursion [4], we will first establish that

$$\prod_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{m} \frac{f_{2(2n)} + f_{2k}}{f_{2(2n)} - f_{2k}} = \prod_{r=1}^{k} \frac{f_{2r-1}}{l_{2r-1}} \cdot \prod_{r=1}^{k} \frac{l_{2m+2r-k}}{f_{2m+2r-k}}. \tag{6}$$

To this end, we let $A_m = \text{LHS}$ and $B_m = \text{RHS}$. Using identities (3) and (4), we then have

$$\begin{aligned}
\frac{B_m}{B_{m-1}} &= \prod_{r=1}^{k} \frac{l_{2m+2r-k}}{f_{2m+2r-k}} \cdot \prod_{r=1}^{k} \frac{l_{2m+2(r-1)-k}}{f_{2m+2(r-1)-k}} \\
&= \frac{f_{2m-k} l_{2m+k}}{f_{2m+k} l_{2m-k}} \\
&= \frac{f_{2(2m)} + f_{2k}}{f_{2(2m)} - f_{2k}} \\
&= \frac{A_m}{A_{m-1}}.
\end{aligned}$$

Recursively, this implies that

$$\begin{aligned}
\frac{A_m}{B_m} = \frac{A_{m-1}}{B_{m-1}} = \cdots &= \frac{A_{(k+1)/2}}{B_{(k+1)/2}} \\
&= \frac{f_{2(k+1)} + f_{2k}}{f_{2(k+1)} - f_{2k}} \cdot \prod_{r=1}^{k} \frac{l_{2r-1}}{f_{2r-1}} \cdot \prod_{r=1}^{k} \frac{f_{(k+1)+2r-k}}{l_{(k+1)+2r-k}}
\end{aligned}$$

$$= \frac{f_1}{l_1} \cdot \frac{l_{2k+1}}{f_{2k+1}} \cdot \frac{l_1}{f_1} \cdot \frac{f_{2k+1}}{l_{2k+1}}$$

$$= 1.$$

Consequently, $A_m = B_m$, confirming formula (6).

Since $\lim_{n\to\infty} \frac{l_n}{f_n} = \Delta$, it follows from equation (6) that

$$\prod_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{\infty} \frac{f_{2(2n)} + f_{2k}}{f_{2(2n)} - f_{2k}} = \prod_{r=1}^{k} \frac{f_{2r-1}}{l_{2r-1}} \cdot \prod_{r=1}^{k} \Delta$$

$$= \prod_{r=1}^{k} \frac{f_{2r-1}}{l_{2r-1}} \cdot \Delta^k,$$

as desired. □

In particular, we have

$$\prod_{n=3}^{\infty} \frac{f_{2(2n)} + f_{10}}{f_{2(2n)} - f_{10}} = \frac{f_1 f_3 f_5 f_7 f_9}{l_1 l_3 l_5 l_7 l_9} \cdot \Delta^5;$$

$$\prod_{n=3}^{\infty} \frac{F_{2(2n)} + 55}{F_{2(2n)} - 55} = \frac{F_1 F_3 F_5 F_7 F_9}{L_1 L_3 L_5 L_7 L_9} \cdot 25\sqrt{5}.$$

With k odd, equations (1) and (2) also yield

$$f_{2(2n+1)} + f_{2k} = f_{2n+1+k} l_{2n+1-k}; \tag{7}$$

$$f_{2(2n+1)} - f_{2k} = f_{2n+1-k} l_{2n+1+k}. \tag{8}$$

They help us establish the next result.

Theorem 2: *Let k be an odd positive integer. Then*

$$\prod_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{\infty} \frac{f_{2(2n+1)} + f_{2k}}{f_{2(2n+1)} - f_{2k}} = \prod_{r=1}^{k} \frac{l_{2r}}{f_{2r}} \cdot \frac{1}{\Delta^k}. \tag{9}$$

Proof: Again, using recursion [4], we will first confirm that

$$\prod_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{m} \frac{f_{2(2n+1)} + f_{2k}}{f_{2(2n+1)} - f_{2k}} = \prod_{r=1}^{k} \frac{l_{2r}}{f_{2r}} \cdot \prod_{r=1}^{k} \frac{f_{2m+(2r+1)-k}}{l_{2m+(2r+1)-k}}. \tag{10}$$

Suppose $A_m = \text{LHS}$ and $B_m = \text{RHS}$. Using identities (7) and (8), we then get

$$\begin{aligned}
\frac{B_m}{B_{m-1}} &= \prod_{r=1}^{k} \frac{f_{2m+(2r+1)-k}}{l_{2m+(2r+1)-k}} \cdot \prod_{r=1}^{k} \frac{l_{2m+(2r-1)-k}}{f_{2m+(2r-1)-k}} \\
&= \frac{f_{2m+1+k} l_{2m+1-k}}{f_{2m+1-k} l_{2m+1+k}} \\
&= \frac{f_{2(2m+1)} + f_{2k}}{f_{2(2m+1)} - f_{2k}} \\
&= \frac{A_m}{A_{m-1}}.
\end{aligned}$$

Recursively, this yields

$$\begin{aligned}
\frac{A_m}{B_m} &= \frac{A_{m-1}}{B_{m-1}} = \cdots = \frac{A_{(k+1)/2}}{B_{(k+1)/2}} \\
&= \frac{f_{2(k+2)} + f_{2k}}{f_{2(k+2)} - f_{2k}} \cdot \prod_{r=1}^{k} \frac{f_{2r}}{l_{2r}} \cdot \prod_{r=1}^{k} \frac{l_{(k+1)+(2r+1)-k}}{f_{(k+1)+(2r+1)-k}}
\end{aligned}$$

$$= \frac{f_{2k+2} l_2}{f_2 l_{2k+2}} \cdot \prod_{r=1}^{k} \frac{f_{2r}}{l_{2r}} \cdot \prod_{r=1}^{k} \frac{l_{2k+2}}{f_{2k+2}}$$

$$= 1.$$

Consequently, $A_m = B_m$, confirming formula (10).

The given result now follows from equation (10), as desired. □

Equation (9) yields

$$\prod_{n=3}^{\infty} \frac{f_{2(2n+1)} + f_{10}}{f_{2(2n+1)} - f_{10}} = \frac{l_2 l_4 l_6 l_8 l_{10}}{f_2 f_4 f_6 f_8 f_{10}} \cdot \frac{1}{\Delta^5};$$

$$\prod_{n=3}^{\infty} \frac{F_{2(2n+1)} + 55}{F_{2(2n+1)} - 55} = \frac{L_2 L_4 L_6 L_8 L_{10}}{F_2 F_4 F_6 F_8 F_{10}} \cdot \frac{\sqrt{5}}{125}.$$

2.2 A Gibonacci Delight: Equation (5), coupled with (9), yields a delightful consequence:

$$\prod_{\substack{n=k+1 \\ k\geq 1,\text{odd}}}^{\infty} \frac{f_{2n} + f_{2k}}{f_{2n} - f_{2k}} = \prod_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{\infty} \frac{f_{2(2n)} + f_{2k}}{f_{2(2n)} - f_{2k}} \cdot \prod_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{\infty} \frac{f_{2(2n+1)} + f_{2k}}{f_{2(2n+1)} - f_{2k}}$$

$$= \prod_{r=1}^{k} \frac{f_{2r-1}}{l_{2r-1}} \cdot \prod_{r=1}^{k} \frac{l_{2r}}{f_{2r}}. \tag{11}$$

This yields

$$\prod_{\substack{n=k+1 \\ k\geq 1,\text{odd}}}^{\infty} \frac{F_{2n} + F_{2k}}{F_{2n} - F_{2k}} = \prod_{r=1}^{k} \frac{F_{2r-1}}{L_{2r-1}} \cdot \prod_{r=1}^{k} \frac{L_{2r}}{F_{2r}}$$

$$\prod_{n=6}^{\infty} \frac{F_{2n} + 55}{F_{2n} - 55} = \frac{F_1 F_3 F_5 F_7 F_9}{L_1 L_3 L_5 L_7 L_9} \cdot \frac{L_2 L_4 L_6 L_8 L_{10}}{F_2 F_4 F_6 F_8 F_{10}}.$$

We now turn to the case k with even parity.

2.3 A Generalization with *k* even: When k is even, identities (1) and (2) yield

$$f_{2(2n)} + f_{2k} = f_{2n+k} l_{2n-k}; \tag{12}$$

$$f_{2(2n)} - f_{2k} = f_{2n-k} l_{2n+k}. \tag{13}$$

respectively.

With these two identities at our disposal, we now establish the next result.

Theorem 3: *Let k be an even positive integer. Then*

$$\prod_{\substack{n=k/2+1 \\ k\geq 2,\text{even}}}^{\infty} \frac{f_{2(2n)} + f_{2k}}{f_{2(2n)} - f_{2k}} = \prod_{r=1}^{k} \frac{l_{2r}}{f_{2r}} \cdot \frac{1}{\Delta^k} \tag{14}$$

***Proof*:** With recursion [4], we will now establish that

$$\prod_{\substack{n=k/2+1 \\ k\geq 2,\text{even}}}^{m} \frac{f_{2(2n)} + f_{2k}}{f_{2(2n)} - f_{2k}} = \prod_{r=1}^{k} \frac{l_{2r}}{f_{2r}} \cdot \prod_{r=1}^{k} \frac{f_{2m+2r-k}}{l_{2m+2r-k}}. \tag{15}$$

To begin with, $A_m = \text{LHS}$ and $B_m = \text{RHS}$. Using identities (12) and (13), we then have

$$\begin{aligned}
\frac{B_m}{B_{m-1}} &= \prod_{r=1}^{k} \frac{f_{2m+2r-k}}{l_{2m+2r-k}} \cdot \prod_{r=1}^{k} \frac{l_{2m+2(r-1)-k}}{f_{2m+2(r-1)-k}} \\
&= \frac{f_{2m+k} l_{2m-k}}{f_{2m-k} l_{2m+k}} \\
&= \frac{f_{2(2m)} + f_{2k}}{f_{2(2m)} - f_{2k}} \\
&= \frac{A_m}{A_{m-1}}.
\end{aligned}$$

Recursively, this yields

$$
\begin{aligned}
\frac{A_m}{B_m} &= \frac{A_{m-1}}{B_{m-1}} = \cdots = \frac{A_{(k/2+1)}}{B_{(k/2+1)}} \\
&= \frac{f_{2(k+2)} + f_{2k}}{f_{2(k+2)} - f_{2k}} \cdot \prod_{r=1}^{k} \frac{f_{2r}}{l_{2r}} \cdot \prod_{r=1}^{k} \frac{l_{2k-2r+4}}{f_{2k-2r+4}} \\
&= \frac{l_2}{f_2} \cdot \frac{f_{2k+2}}{l_{2k+2}} \cdot \prod_{r=1}^{k} \frac{f_{2r}}{l_{2r}} \cdot \prod_{r=1}^{k} \frac{l_{2r+2}}{f_{2r+2}} \\
&= 1.
\end{aligned}
$$

Thus, $A_m = B_m$, confirming formula (15).

Since $\lim_{n\to\infty} \frac{f_n}{l_n} = \frac{1}{\Delta}$, the given result follows from equation (15), as desired. □

In particular, we have

$$\prod_{n=4}^{\infty} \frac{f_{2(2n)} + f_{12}}{f_{2(2n)} - f_{12}} = \frac{l_2 l_4 l_6 l_8 l_{10} l_{12}}{f_2 f_4 f_6 f_8 f_{10} f_{12}} \cdot \frac{1}{\Delta^6}$$

$$\prod_{n=4}^{\infty} \frac{F_{2(2n)} + 144}{F_{2(2n)} - 144} = \frac{L_2 L_4 L_6 L_8 L_{10} L_{12}}{F_2 F_4 F_6 F_8 F_{10} F_{12}} \cdot \frac{1}{125}.$$

Next, we explore the counterpart of Theorem 2, using the identities

$$f_{2(2n+1)} + f_{2k} = f_{2n+1-k} l_{2n+1+k}; \qquad (16)$$

$$f_{2(2n+1)} - f_{2k} = f_{2n+1+k} l_{2n+1-k}. \qquad (17)$$

where k is an even positive integer.

Theorem 4: *Let k be an even positive integer. Then*

$$\prod_{\substack{n=k/2\\k\geq 2,\text{even}}}^{\infty} \frac{f_{2(2n+1)} + f_{2k}}{f_{2(2n+1)} - f_{2k}} = \prod_{r=1}^{k} \frac{f_{2r-1}}{l_{2r-1}} \cdot \Delta^k. \tag{18}$$

Proof: With recursion [4], we will first confirm that

$$\prod_{\substack{n=k/2\\k\geq 2,\text{even}}}^{m} \frac{f_{2(2n+1)} + f_{2k}}{f_{2(2n+1)} - f_{2k}} = \prod_{r=1}^{k} \frac{f_{2r-1}}{l_{2r-1}} \cdot \prod_{r=1}^{k} \frac{l_{2m+(2r+1)-k}}{f_{2m+(2r+1)-k}} \tag{19}$$

Once again, we let $A_m = \text{LHS}$ and $B_m = \text{RHS}$. Using identities (16) and (17), we then get

$$\begin{aligned}\frac{B_m}{B_{m-1}} &= \prod_{r=1}^{k} \frac{l_{2m+(2r+1)-k}}{f_{2m+(2r+1)-k}} \cdot \prod_{r=1}^{k} \frac{f_{2m+(2r-1)-k}}{l_{2m+(2r-1)-k}} \\ &= \frac{f_{2m+1-k}l_{2m+1+k}}{f_{2m+1+k}l_{2m+1-k}} \\ &= \frac{f_{2(2m+1)} + f_{2k}}{f_{2(2m+1)} - f_{2k}} \\ &= \frac{A_m}{A_{m-1}}.\end{aligned}$$

Recursively, this yields

$$\begin{aligned}\frac{A_m}{B_m} = \frac{A_{m-1}}{B_{m-1}} = \cdots &= \frac{A_{(k/2)}}{B_{(k/2)}} \\ &= \frac{f_{2(k+1)} + f_{2k}}{f_{2(k+1)} - f_{2k}} \cdot \prod_{r=1}^{k} \frac{l_{2r-1}}{f_{2r-1}} \cdot \prod_{r=1}^{k} \frac{f_{2r+1}}{l_{2r+1}} \\ &= \frac{f_1 l_{2k+1}}{f_{2k+1} l_1} \cdot \prod_{r=1}^{k} \frac{l_{2r-1}}{f_{2r-1}} \cdot \prod_{r=1}^{k} \frac{f_{2r+1}}{l_{2r+1}}\end{aligned}$$

$$= \frac{f_1}{l_1} \cdot \frac{l_{2k+1}}{f_{2k+1}} \cdot \frac{l_1}{f_1} \cdot \frac{f_{2k+1}}{l_{2k+1}}$$

$$= 1.$$

Consequently, $A_m = B_m$, confirming formula (19).

Since $\lim_{n\to\infty} \frac{f_n}{l_n} = \frac{1}{\Delta}$ the given result follows from equation (19).

In particular, we have

$$\prod_{n=3}^{\infty} \frac{f_{2(2n+1)} + f_{12}}{f_{2(2n+1)} - f_{12}} = \frac{f_1 f_3 f_5 f_7 f_9 f_{11}}{l_1 l_3 l_5 l_7 l_9 l_{11}} \cdot \Delta^6$$

$$\prod_{n=3}^{\infty} \frac{F_{2(2n+1)} + 144}{F_{2(2n+1)} - 144} = \frac{F_1 F_3 F_5 F_7 F_9 F_{11}}{L_1 L_3 L_5 L_7 L_9 L_{11}} \cdot 5^3.$$

2.4 A Second Gibonacci Delight: It follows from Theorems 3 and 4 that

$$\prod_{\substack{n=k+1 \\ k\geq 2,\text{even}}}^{\infty} \frac{f_{2n} + f_{2k}}{f_{2n} - f_{2k}} = \prod_{\substack{n=k/2+1 \\ k\geq 2,\text{even}}}^{\infty} \frac{f_{2(2n)} + f_{2k}}{f_{2(2n)} - f_{2k}} \cdot \prod_{\substack{n=k/2+1 \\ k\geq 2,\text{even}}}^{\infty} \frac{f_{2(2n+1)} + f_{2k}}{f_{2(2n+1)} - f_{2k}}$$

$$= \prod_{r=1}^{k} \frac{f_{2r-1}}{l_{2r-1}} \cdot \prod_{r=1}^{k} \frac{l_{2r}}{f_{2r}}. \tag{20}$$

This implies

$$\prod_{\substack{n=k+1 \\ k\geq 2,\text{even}}}^{\infty} \frac{F_{2n} + F_{2k}}{F_{2n} - F_{2k}} = \prod_{r=1}^{k} \frac{F_{2r-1}}{L_{2r-1}} \cdot \prod_{r=1}^{k} \frac{L_{2r}}{F_{2r}};$$

$$\prod_{n=7}^{\infty} \frac{F_{2n} + 144}{F_{2n} - 144} = \prod_{r=1}^{6} \frac{F_{2r-1}}{L_{2r-1}} \cdot \prod_{r=1}^{6} \frac{L_{2r}}{F_{2r}}.$$

2.5 A Gibonacci Treasure: Combining equations (11) and (20), we get

$$\prod_{\substack{n=k+1\\k\geq 1}}^{\infty} \frac{f_{2n}+f_{2k}}{f_{2n}-f_{2k}} = \prod_{r=1}^{k} \frac{f_{2r-1}}{l_{2r-1}} \cdot \prod_{r=1}^{k} \frac{l_{2r}}{f_{2r}}. \tag{21}$$

In particular, we have

$$\prod_{n=2}^{\infty} \frac{f_{2n}+f_2}{f_{2n}-f_2} = \frac{f_1}{l_1} \cdot \frac{l_2}{f_2}; \qquad \prod_{n=3}^{\infty} \frac{f_{2n}+f_4}{f_{2n}-f_4} = \frac{f_1 f_3}{l_1 l_3} \cdot \frac{l_2 l_4}{f_2 f_4};$$

$$\prod_{n=4}^{\infty} \frac{f_{2n}+f_6}{f_{2n}-f_6} = \frac{f_1 f_3 f_5}{l_1 l_3 l_5} \cdot \frac{l_2 l_4 l_6}{f_2 f_4 f_6}; \qquad \prod_{n=5}^{\infty} \frac{f_{2n}+f_8}{f_{2n}-f_8} = \frac{f_1 f_3 f_5 f_7}{l_1 l_3 l_5 l_7} \cdot \frac{l_2 l_4 l_6 l_8}{f_2 f_4 f_6 f_8}.$$

We also have

$$\prod_{\substack{n=k+1\\k\geq 1}}^{\infty} \frac{F_{2n}+F_{2k}}{F_{2n}-F_{2k}} = \prod_{r=1}^{k} \frac{F_{2r-1}}{L_{2r-1}} \cdot \prod_{r=1}^{k} \frac{L_{2r}}{F_{2r}},$$

and hence [2, 4]

$$\prod_{n=2}^{\infty} \frac{F_{2n}+1}{F_{2n}-1} = 3.$$

3. Alternate Forms

Using the identity $l_n^2 - \Delta^2 f_n^2 = 4(-1)^n$ [4], we can rewrite equations (11), (20), and (21) in terms of both f_n and l_n. For example, equation (21) yields

$$\prod_{\substack{n=k+1\\k\geq 1}}^{\infty} \frac{l_{2n}^2 - \Delta^2 f_{2k}^2 - 4}{\Delta^2 (f_{2n}-f_{2k})^2} = \prod_{\substack{n=k+1\\k\geq 1}}^{\infty} \frac{f_{2n}^2 - f_{2k}^2}{(f_{2n}-f_{2k})^2}$$

$$= \prod_{r=1}^{k} \frac{f_{2r-1}}{l_{2r-1}} \cdot \prod_{r=1}^{k} \frac{l_{2r}}{f_{2r}}. \tag{22}$$

This implies,

$$\prod_{\substack{n=k+1\\k\geq 1}}^{\infty} \frac{L_{2n}^2 - 5F_{2k}^2 - 4}{5(F_{2n} - F_{2k})^2} = \prod_{r=1}^{k} \frac{F_{2r-1}}{L_{2r-1}} \cdot \prod_{r=1}^{k} \frac{L_{2r}}{F_{2r}}.$$

In particular, we then have

$$\prod_{n=2}^{\infty} \frac{L_{2n}^2 - 9}{5(F_{2n} - 1)^2} = 3;$$

$$\prod_{n=3}^{\infty} \frac{L_{2n}^2 - 49}{5(F_{2n} - 3)^2} = \frac{7}{2}.$$

4. Pell Implications

Using the relationship $b_n(x) = g_n(2x)$, we can find the Pell versions of equations (4) through (22). In the interest of brevity, we will showcase only those of (21) and (22):

$$\prod_{\substack{n=k+1\\k\geq 1}}^{\infty} \frac{p_{2n} + p_{2k}}{p_{2n} - p_{2k}} = \prod_{r=1}^{k} \frac{p_{2r-1}}{q_{2r-1}} \cdot \prod_{r=1}^{k} \frac{q_{2r}}{p_{2r}};$$

$$\prod_{\substack{n=k+1\\k\geq 1}}^{\infty} \frac{q_{2n}^2 - 4E^2 p_{2k}^2 - 4}{4E^2(p_{2n} - p_{2k})^2} = \prod_{r=1}^{k} \frac{p_{2r-1}}{q_{2r-1}} \cdot \prod_{r=1}^{k} \frac{q_{2r}}{p_{2r}},$$

respectively.

They yield

$$\prod_{\substack{n=k+1\\k\geq 1}}^{\infty} \frac{P_{2n} + P_{2k}}{P_{2n} - P_{2k}} = \prod_{r=1}^{k} \frac{P_{2r-1}}{Q_{2r-1}} \cdot \prod_{r=1}^{k} \frac{Q_{2r}}{P_{2r}};$$

$$\prod_{\substack{n=k+1\\k\geq 1}}^{\infty} \frac{Q_{2n}^2 - 2P_{2k}^2 - 1}{2(P_{2n} - P_{2k})^2} = \prod_{r=1}^{k} \frac{P_{2r-1}}{Q_{2r-1}} \cdot \prod_{r=1}^{k} \frac{Q_{2r}}{P_{2r}},$$

respectively.

In particular, we then get

$$\prod_{n=2}^{\infty} \frac{P_{2n}+2}{P_{2n}-2} = \frac{3}{2};$$

$$\prod_{n=3}^{\infty} \frac{Q_{2n}^2-289}{2(P_{2n}-12)^2} = \frac{85}{56}.$$

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ON EQUI-ORDERED ELEMENTS IN A GROUP

Abstract: Taking p as the order of all elements, except the identity one, of some finite non-product groups $G_0, G_1, \ldots, G_n, \ldots$ arranged in successive order, a theorem is proved regarding the existence, order and abelianity of the groups and also the case of the product groups is discussedseparately.

Keywords: Klein 4-Group, Equi-Ordered Elements, Finite Non-Product and Product Group.

Mathematical Subject Classifications No.: 20D05.

1. Introduction

The concept of group theory was originated on the basis of axiomatic definition and first appeared through permutation groups in the 2nd half of the eighteenth century. In 1854, Cayley proved that a group need not be a permutation group or even finite group and the abstract notion of a group thus appeared first [4].

Many authors [1, 2, 4] had made several attempts to develop the subject. Euler considered algebraic operations on numbers modulo an integer and Gauss established some properties of cyclic and, in general, of abelian groups and also explicitly stated the associative law on composition. Galois [4, 6] was the first to use the word 'group' and his contribution on group theory was published in 1846 by Liouville. Now Galois group is also called the permutation group. Moreover, the author introduced the notion of normal subgroups and honoured as the first mathematician pioneering in group theory. In 1870, Kronecker [6, 7] gave the definition of abelian group to generalize Gauss's work, but it was not in conformity with the definition of permutation group. Weber also gave a similar definition in 1882 which involved the cancellation property. The formal axioms and the corresponding definitions are the basic algebraic structure of 'group' in modern algebra.

Klein [3] worked on the function theory and non-euclidean geometry connecting geometry and group theory and solved the general equations of the fifth degree using the group of the icosahedron for which the word 'vierergruppe (four-group)' was used here. Later vierergruppe was transformed into Klein 4-group.

Lagrange [5] investigated that the order of a subgroup of a finite group is a divisor of the groups order. However, Sylow [5] showed the inverse of Lagrange theorem for prime integer in 1959. The author illustrated that for a prime p, if p^m (m is natural number) is a divisor of the order of a group G, then G contains a subgroup of order p^m.

In the present paper we study some results on the order and abelianity of the groups containing equi-ordered elements, except the identity. The paper is an extension and generalization of Klein 4-group. From this we can obtain an idea about the matter whether it is possible to have the existence of a group with elements of order 2 or more like Klein 4 group or more ordered groups.

2. Notations

$(G_n^{(p)}, .)$, $(n = 0, 1, 2, \ldots)$, is a finite non-product group having all elements (except identity element) of order p satisfying $\circ(G_0^{(p)}) < \circ(G_1^{(p)}) < \ldots < \circ(G_k^{(p)}) < \ldots$ and there does not exist any group containing all elements of order p other than identity and having order lying between $G_k^{(p)}$ and $G_{k+1}^{(p)}$, $\forall k$. For our convenience, we shall use the notation G_n instead of $G_n^{(p)}$.

The identity element of the group G_n is denoted by 'e'.

The set $\{a, \ a^2, \ \ldots, \ a^{p-1}\}$ is denoted by $\langle a \rangle$.

3. Definitions

I. **1st category box:** The set consisting all elements, except e, of the group G_n is called the nth *order 1st category box* (*box*-I) of G_n and is denoted by $B^{(n)}$.

II. **2nd category box:** The composition of two different nth order box-I, say $B_i^{(n)}$ and $B_j^{(n)} (i \neq j)$ is called the nth *order 2nd category box* (*box*-II) and is denoted by $B_{i,j}^{(n)}$.

III. **Power box:** A box is said to be *power box* if all its elements can be expressed as some power $(< p)$ of an element of this box. For example, all $B^{(0)}$ type boxes or $\langle a \rangle$ $\forall a \in G_n$, are power boxes but any $B^{(k)},\ k > 1$ type box is not power box.

IV. **Component:** The disjoint $B^{(n-1)}$ type boxes in a $B^{(n)}$ type box are called the *components* of that box.

4. Lemmas

Lemma 1: p is prime.

Proof: If possible, let $p = mn(m, n < p)$ and order of all elements are p. Thus, for some $a \in G_n$, $a^p = e \Rightarrow a^{mn} = e \Rightarrow (a^m)^n = e \Rightarrow$ the order of a^m is $n(<p)$, a contradiction, as all elements are of order p. Therefore p is prime.

Lemma 2: G_0 is cyclic as well as abelian.

Proof: For $a \in G_0, \{a, a^2, \ \ldots, \ a^{p-1}, a^p = e\} = G_0$ and clearly G_0 is cyclic as well as abelian.

Lemma 3: All elements except e of G_0 , are generators of G_0.

Proof: For any $r(< p)$, r is not a divisor of p, so that for $a(\neq e) \in G_0$, the order of a^r is p. Therefore, the group $\{a^r, a^{2r}, \ldots, a^{(p-1)r}, a^{pr} = e\}$ is equivalent to $\{a, a^2, \ldots, a^{(p-1)}, a^p = e\}$ or G_0 for all $r(< p)$ and consequently a^r is a generator of G_0 for all $r(< p)$.

Lemma 4: Let the sets $\{a_i\}_i, \{b_i\}_i$ and $\{c_i\}_i$ indicate same ordered 1st category boxes. If $a_1 b_1 = c_1$ then, $a_t b_1 \neq c_s$ or $a_1 b_t \neq c_s$ for any $t, s \neq 1$.

Proof: There exist $c_j, c_k \in \{c_i\}_i$ such that $c_s = c_j c_1 = c_1 c_k$. Now if $a_t b_1 = c_s = c_j c_1 = c_j a_1 b_1$, then $a_t = c_j a_1$ which is a contradiction because every box is well composed (i.e. the composition of any two elements of a box is either an element of same box or e). Similarly $a_1 b_t = c_s$ is contradicted by $b_t = b_1 c_k$.

Lemma 5: Two 0th ordered boxes $B_i^{(0)}$ and $B_j^{(0)}$ are either disjoint or equivalent.

Proof: Each of the given two power boxes contain $(p-1)$ elements. These boxes with e form two cyclic groups. Therefore the groups are either disjoint or equivalent and consequently the boxes are of same type.

Lemma 6: $B^{(n)}(n > 0)$ is a union of some power boxes.

Proof: The order of all elements of $B^{(n)}$ are p and the number of elements of $B^{(0)}$ is $(p-1)$. $B^{(n)}$ and $\{e\}$ form a group with order greater than p so that the number of elements of $B^{(n)}$ are greater than $(p-1)$ as $n > 0$. By composition rule, for any $a(\neq e) \in B^{(n)}$, $a \in G_n$ and $\{a, a^2, \ldots, a^{p-1}\} \subset G_n$ and consequently $\{a, a^2, \ldots, a^{p-1}\}$ or $\langle a \rangle \subset B^{(n)}$.

Since $B^{(n)}$ has more than $(p-1)$ elements and a is arbitrary, so

$$B^{(n)} = \bigcup_a \langle a \rangle .$$

Lemma 7: G_n has many power boxes, say, $B^{(0)}$ or $\langle a_i \rangle$, where $i = 1, 2, 3, \ldots$. If we consider $a_j.a_1 = a_k$, $a_j.a_1^2 = a_l$, $\ldots, a_j.a_1^{p-1} = a_x$ (say) (with the help of Lemma 4) then,

$$a_k.a_1 = a_l, \quad \ldots, \quad \ldots, \quad \ldots, \quad \ldots, \quad a_k.a_1^{p-2} = a_x, \quad a_k.a_1^{p-1} = a_j;$$

$$\ldots, \quad \ldots, \quad \ldots, \quad \ldots, \quad a_l.a_1^{p-3} = a_x, \quad a_l.a_1^{p-2} = a_j, \quad a_l.a_1^{p-1} = a_k;$$

$$a_x.a_1 = a_j, \quad a_x.a_1^2 = a_k, \quad a_x.a_1^3 = a_l, \quad \ldots, \quad \ldots, \quad \ldots, \quad \ldots, \quad .$$

Similarly in case of the composition of $B^{(0)}.a_j$, the first and second positions of the elements are interchanged.

Proof: The proof is trivial. This result shows that if $a_j.B_1^{(0)}$ or $(B_1^{(0)}.a_j)$ is complete, then all those elements by which a_j is composed with $B^{(0)}$ are automatically completed corresponding to $B_1^{(0)}$.

5. Theorem

5.1 Theorem 1: A finite non-product group (G_n) containing all elements, except identity element, of order p is of order p^{2^n}, where n is a non negative integer. Moreover, the group is abelian.

Proof: We prove the theorem in the following steps.

Step 1: Existence and order of G_1

Noting that G_0 exists and is cyclic, so it is abelian and is of order p i.e. p^{2^0}. Thus, it contains only one power box.

Since $\circ(G_1) > p$, so G_1 must consist of at least two different power boxes, say, $B_1^{(0)} = \langle b_1 \rangle$ and $B_2^{(0)} = \langle b_2 \rangle$. By closer property, $B_1^{(0)}.B_2^{(0)}$ is in G_1. Now using

Lemma 4, $b_1.b_2, b_1.b_2^2, \ldots, b_1.b_2^{p-1}$ are all members of $(p-1)$ distinct power boxes. As the boxes are power boxes and each contains $(p-1)$ elements, so by any initial assumption for $b_1.b_2, b_1.b_2^2, \ldots, b_1.b_2^{p-1}$ satisfying Lemma 4 and Lemma 7, $B_1^{(0)}.B_2^{(0)}$ is complete if all elements are commutative. Therefore, we see that, if the elements are abelian, $B_1^{(0)}.B_2^{(0)}, B_1^{(0)}, B_2^{(0)}$ and $\{e\}$ form a group with order

$$\underbrace{(p+1)}_{\substack{\text{Number of}\\ \text{power boxes}}}\ \underbrace{(p-1)}_{\substack{\text{Number of}\\ \text{elements}\\ \text{in each box}}} + \underbrace{1}_{\substack{\text{For}\\ \text{identity}\\ \text{element}}} = p^2 - 1 + 1 = p^2 = p^{2^1}.$$

Thus, any group described above with order lying between p and p^{2^1} may not exist. Therefore $\circ(G_1) = p^{2^1}$.

Step 2: Abelianity of G_1

Now suppose, if possible, that there also exists a non-abelian G_1. Then in this G_1, $\exists$ at least two elements a, b such that $ab \neq ba$. Here two cases may arise.

Case 1: ab vand ba are elements of two different power boxes. Let $ab = c$, $ba = d$ and we have $ad^\gamma = c^\gamma a$, for $\gamma = 1, 2, \ldots, (p-1)$. Then by Lemma 4, the elements $ad, ad^2, \ldots, ad^{p-1}$ are members of different $(p-1)$ power boxes and not of $\langle a \rangle$, $\langle c \rangle$ and $\langle d \rangle$. Thus, the number of power boxes exceeds the number $(p+1)$. Again $ad^\gamma \neq d^\gamma a$ and $ac^\gamma \neq c^\gamma a$, $(\gamma = 1, 2, \ldots, (p-\ 1))$ and from these we get many unequal relations. Therefore, by our assumption the composition table cannot be completed by finite number of elements.

Case 2: ab and ba are elements of the same power box. Let $ab = c_1$, $ba = c_2$ where $c_1, c_2 \in \langle c \rangle$. Therefore, we have $c_2 b = bc_1 \neq bc_2$ and by Lemma 4, $c_2 b$ and bc_2 are members of two different power boxes. Thus, the case is transformed into Case 1.

Therefore, we conclude that G_1 is abelian.

Step 3: Abelianity of $G_n, (n > 1)$

We first show that if $G_n(n > 1)$ exists then it is abelian.

Consider two elements a, b in $G_n(n > 1)$. If a and b are elements of the same power box, then obviously they are commutative and if not, then $\langle a\rangle \cdot \langle b\rangle, \langle a\rangle, \langle b\rangle$ and $\{e\}$ form a G_1 type subgroup of G_n. This subgroup is abelian and contains a and b and so $ab = ba$ (by step 2). Since a and b are arbitrarily chosen from G_n, so all its elements are commutative and consequently G_n is abelian.

Step 4: Order and existence of $G_n, (n > 1)$

We have seen that $\circ(G_0) = p^{2^0}$ and $\circ(G_1) = p^{2^1}$.

Now suppose $\circ(G_k) = p^{2^k}$. Obviously we have $\circ(G_{k+1}) > \circ(G_k)$ and in G_{k+1} there must exist p^{2^k} elements which form a G_k type subgroup. So, by Lagrange's theory, $\circ(G_k) | \circ(G_{k+1})$. Again both of $\circ(G_{k+1}) > \circ(G_k)$ and $\circ(G_k) | \circ(G_{k+1})$ imply $\circ(G_{k+1}) > \circ(2p^{2^k} - 1)$. Thus, $\exists$ at least two distinct kth ordered boxes-I in G_{k+1}.

Using the closer property, the combination of these two boxes are also in G_{k+1}. So by Lemma 4, we see that another $(p^{2^k} - 1)$ number of disjoint kth ordered boxes-I are contained in G_{k+1}.

Since all these elements are commutative and all kth ordered boxes-I are the union of some power boxes (from Lemma 6), so these $(p^{2^k} + 1)$ number of kth ordered boxes along with $\{e\}$ form a abelian group with order

$$\underbrace{(p^{2^k}+1)}_{\substack{\text{Number of}\\ k\text{th order}\\ \text{boxes}}}\ \underbrace{(p^{2^k}-1)}_{\substack{\text{Number of}\\ \text{elements}\\ \text{in each box}}} + \underbrace{1}_{\substack{\text{For}\\ \text{identity}\\ \text{element}}} = (p^{2^k})^2 - 1 + 1 = p^{2^{k+1}}$$

and no group can be formed with order lying between p^{2^k} and $p^{2^{k+1}}$. Thus, by the principle of induction we get $\circ(G_n) = p^{2n}$.

We now consider the existence of $G_n, (n = 2, 3, \ldots)$. Forming a group by taking only p^{2^2} elements (putting $k = 1$, in the described $p^{2^{k+1}}$ elements of this part), the structure of G_2 is shown first.

There are (p^2+1) number of distinct $B^{(1)}$ type well-composed boxes say $B_1^{(1)}, B_2^{(1)}, \ldots, B_{p^2+1}^{(1)}$ each of which has $(p+1)$ disjoint power boxes. Suppose $B_i^{(1)} = \bigcup_{j=1}^{(p+1)} B_{ij}^{(0)} = \bigcup_{j=1}^{(p+1)} \langle a_{ij} \rangle$. We now construct $B_i^{(1)}, \forall\, i = 1, 2, \ldots, (p^2+1)$. Taking $a_{i1}, a_{i2} = a_{i3}, a_{i1}a_{i2}^2 = a_{i4}, \ldots, a_{i1}a_{i2}^{p-1} = a_{i(p+1)}$ (we may arrange another way following Lemma 4), $B_i^{(1)}$ is complete and the inner composition between the elements of $B_i^{(1)}$ are known for all $i = 1, 2, \ldots, (p^2+1)$. Now our interest is on $B_1^{(1)}.B_2^{(1)}$ and so following Lemma 4, select

$$\begin{array}{llll}
a_{11}a_{21} = a_{31}, & a_{11}a_{21}^2 = a_{41}, & \ldots, & a_{11}a_{21}^{p-1} = a_{(p+1)1}, \\
a_{11}a_{22} = a_{(p+2)1}, & a_{11}a_{22}^2 = a_{(p+3)1}, & \ldots, & a_{11}a_{22}^{p-1} = a_{(2p)1}, \\
\ldots & \ldots & \ldots & \ldots \\
a_{11}a_{2(p+1)} = a_{(p(p-1)+3)1}, & a_{11}a_{2(p+1)}^2 = a_{(p)p-1)+4)1}, & ,\ldots, & a_{11}a_{2(p+1)}^{p-1} = a_{(p^2+1)1}.
\end{array}$$

This choice directly completes $\langle a_{11} \rangle . B_2^{(1)}$. But $\langle a_{12} \rangle . B_2^{(1)}$ is not come out.

Again setting (using Lemma 4) $a_{12}a_{21} = a_{(p+2)2}$, $a_{12}a_{21}^2 = a_{(p+3)2}$, $\ldots$, $a_{12}a_{21}^{p-1} = a_{(2p)2}$, we immediately obtain $\langle a_{12}\rangle.\langle a_{21}\rangle, \langle a_{13}\rangle.\langle a_{23}\rangle, \langle a_{14}\rangle.\langle a_{2\frac{p+5}{2}}\rangle$ and other similar $(p-3)$ terms. The last selections give us many restrictions. For example,

$$a_{(p+2)3} = a_{13}a_{23} = (a_{11}a_{12})(a_{21}a_{22}) = (a_{11}a_{21})(a_{12}a_{22}) = (a_{31})(a_{12}a_{22}).$$

So that $a_{12}a_{22}$ is neither an element of $B_3^{(1)}$ nor of $B_{(p+2)}^{(1)}$. Following all the restrictions and Lemma 4 we can set $a_{12}.\langle a_{22}\rangle$, $a_{12}.\langle a_{23}\rangle$, ... $a_{12}.\langle a_{2(p+1)}\rangle$ and $B_1^{(1)}.B_2^{(1)}$ i.e. the box $B_{1,2}^{(1)}$ is automatically completed. Now completed. Now commutative property and Lemma 7 complete all boxes $B_{i,j}^{(1)}; i, j = 1, 2 \ldots, (p^2+1), i \neq j$. Therefore the formation of G_2 is shown and $\circ(G_2) = p^{2^2}$.

Similarly we can show a group G_3 such that $\circ(G_3) = p^{2^3}$. In this case, we have to form a group by $(p^{2^2}+1)$ well-composed disjoint $B^{(2)}$ type boxes each of which has (p^2+1) number of $B^{(1)}$ type boxes. Following the above rules and steps, at least a 2nd ordered 2nd category box can be filled up and that box completes all other. In the successive process we can form the groups $G_4, G_5, .., G_n, \ldots$ with above described order.

This proves the theorem.

5.2 Theorem 2: If $(G,.)$ is a finite product group, where $G \equiv G_i \times G_j \times \ldots \times G_x$ and each G_i contains all elements of order p except identity, defined as

$$a.b = (a_i, a_j, \ldots a_x).(b_i, b_j, \ldots b_x) = (a_i.b_i, a_j.b_j, \ldots a_x.b_x) \text{ for any } a, b \in G, a_i, b_i \in G_i, \forall_i,$$

then $\circ(G) = \circ(G_i) \times \circ(G_j) \times \ldots \times \circ(G_x)$.

Proof: Obviously G contains all elements of order p except identity and theorem 2 immediately follows from theorem 1.

6. Conclusions

Here we used four steps to prove the theorem 1. The first two steps show the existence, order and abelianity of (G_1), whereas last two steps are used to prove (with the help of previous steps) the same for $G_n, n > 1$, due to dissimilarities between (G_1) and $G_n, n > 1$.

Theorem 2 is elementary, as it is well known that the order of a product group is equal to the product of the order of each individual group, so its proof is obvious.

It is not possible to extend a group composing G_m and G_n for $m \neq n$. So any group having order p^m (p is prime), not of the form p^{2^n} is either a cyclic group or a product group like $G_i \times G_j \times \ldots \times G_x$. In both cases the group is abelian. This shows that Sylow subgroup is abelian.

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PAIR MEAN CORDIAL LABLEING OF SOME CORONA GRAPHS

Abstract: Let a graph $G = (V, E)$ be a (p, q) graph. Define $\rho = \begin{cases} \frac{p}{2} & p \text{ is even} \\ \frac{p-1}{2} & p \text{ is odd,} \end{cases}$ and $M = \{\pm 1, \pm 2, \ldots., \pm \rho\}$ called the set of labels. Consider a mapping $\lambda : V \to M$ by assigning different labels in M to the different elements of V when p is even and different labels in M to p 1 elements of V and repeating a label for the remaining one vertex when *p* is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge uv of G, there exists a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ if $\lambda(u) + \lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$ if $\lambda(u) + \lambda(v)$ is odd such that $\left|\overline{\mathbb{S}}_{\lambda_1} \quad \overline{\mathbb{S}}_{\lambda_1^c}\right| \leq 1$ where $\overline{\mathbb{S}}_{\lambda_1}$ and $\overline{\mathbb{S}}_{\lambda_1^c}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph G for which there exists a pair mean cordial labeling is called pair mean cordial graph. In this paper, we investigate the pair mean cordial labeling of some corona graphs like $L_n \odot K_1$, $L_n \odot 2K_1$, $L_n \odot K_2$, $W_n \odot K_1$ $W_n \odot 2K_1$, $W_n \odot K_2$, gear graph, $G_n \odot K_1$, $G_n \odot 2K_1$ and $G_n \odot K_2$.

Keywords: Ladder Graph, Wheel Graph and Gear Graph.

Mathematic Subject Classification No.: 05C78.

1. Introduction

In this paper, we consider only finite, simple and undirected graphs. For basic notation and terminology in graph theory we refer to F. Harary [3]. A detailed survey of various graph labeling is explained in Gallian [2]. The concept of cordial labeling was introduced by I. Cahit [1]. Ponraj *et al*. [6] discussed pair difference cordial labeling of some corona related graphs. We have been introduced the concept of pair mean cordial labeling in [4] and investigate pair mean cordiality of some snake graphs in [5]. In this paper, we investigate the pair man cordial labeling of some corona graphs like $L_n \odot K_1$, $L_n \odot 2K_1$, $L_n \odot K_2, W_n \odot K_1$, $W_n \odot 2K_1$, $W_n \odot K_2$, gear graph, $G_n \odot K_1$, $G_n \odot 2K_1$ and $G_n \odot K_2$.

2. Preliminaries

Definition 2.1: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The join $G_1 + G_2$ as $G_1 \cup G_2$. Together with all the edges joining vertices of V_1 to the vertices of V_2.

Definition 2.2: The corona graph $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and n copies of G_2 and joining i^t vertex of G_1 with an edge to every vertex in the i^t copy of G_2, where G_1 is graph of order n.

Definition 2.3: The ladder L_n is the product of $P_n \times K_2$ with $2n$ vertices and $3n$ 2 edges.

Definition 2.4: The graph $W_n = C_n + K_1$ is called the wheel graph.

Definition 2.5: The gear graph G_n is obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices of the cycle C_n.

3. Pair Mean Cordial Labeling

Definition 3.1: Let a graph $G = (V, E)$ be a (p, q) graph. Define $\rho = \begin{cases} \frac{p}{2} & p \text{ is even} \\ \frac{p-1}{2} & p \text{ is odd,} \end{cases}$ and $M = \{\pm 1, \pm 2, \dots, \pm \rho\}$ called the set of labels. Consider a mapping $\lambda: V \rightarrow M$ by assigning different labels in M to the different elements of V when p is even and different labels in M to p 1elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge uv of G, there exists a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ $if\ \lambda(u)+\lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$ $if\ \lambda(u)+\lambda(v)$ is odd such that $|\overline{\mathbb{S}}_{\lambda_1} \quad \overline{\mathbb{S}}_{\lambda_1^c}| \leq 1$ where $\overline{\mathbb{S}}_{\lambda_1}$ and $\overline{\mathbb{S}}_{\lambda_1^c}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph G for which there exists a pair mean cordial labeling is called pair mean cordial graph.

Theorem 3.2: The ladder graph L_n is pair mean cordial [4].

Theorem 3.3: The wheel graph W_n is pair mean cordial [4].

Theorem 3.4: The graph $L_n \odot K_1$ is pair mean cordial for all $n \geq 2$.

Proof: Let $V(L_n \odot K_1) = \{u_i, v_i, x_i, y_i: 1 \leq i \leq n\}$ and $E(L_n \odot K_1) = \{u_i u_{i+1}, v_i v_{i+1}: 1 \leq i \leq n \quad 1\} \quad \{u_i v_i, u_i x_i, v_i y_i: 1 \leq i \leq n\}$. Then there are $4n$ vertices and $5n$ 2 edges. This proof is divided into two cases:

Case (i) n is odd: First assign the labels 1, 5, …, $2n+1$ to the vertices $u_1, u_3, \ldots, u_n$ respectively. Then we assign the labels 4,8, …,$2n$ 2 respectively to the vertices $u_2, u_4, \ldots, u_{n-1}$. We assign the labels 2,6, …,$2n$ to the vertices $x_1, x_3, \ldots, x_n$ respectively. Also we assign the labels 3, 7, …, $2n+3$ respectively to the vertices $x_2, x_4, \ldots, x_{n-1}$. Next we give the labels 3,5, …,$2n$ 1 to the vertices $v_1, v_2, \ldots, v_{n-1}$ respectively. Assign the label 1 to the vertex v_n. Finally we give the labels 2, 4, …, $2n$ respectively to the vertices $y_1, y_2, \ldots, y_n$. Hence $\overline{\mathbb{S}}_{\lambda_1} = \frac{5n-3}{2}$ and $\overline{\mathbb{S}}_{\lambda_1^c} = \frac{5n-1}{2}$.

Case (ii) n is even: As in Case (i), assign the label to the vertices $v_i, y_i, 1 \leq i \leq n$. Assign the labels 1, 5, …, $2n+3$ to the vertices $u_1, u_3, \ldots, u_{n-1}$ respectively. Then we assign the labels 4,8, …,$2n$ respectively to the vertices $u_2, u_4, \ldots, u_n$. Also we assign the labels 2,6, …,$2n$ 2 to the vertices $x_1, x_3, \ldots, x_{n-1}$ respectively. Finally we assign the labels 3, 7, …, $2n+1$ respectively to the vertices $x_2, x_4, \ldots, x_n$. Hence, $\overline{\mathbb{S}}_{\lambda_1} = \frac{5n-2}{2} = \overline{\mathbb{S}}_{\lambda_1^c}$.

Theorem 3.5: The graph $L_n \odot 2K_1$ is pair mean cordial for all $n \geq 2$.

Proof: Let $V(L_n \odot 2K_1) = \{u_i, v_i, x_i, y_i, a_i, b_i : 1 \leq i \leq n\}$ and $(L_n \odot 2K_1) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n \quad 1\} \quad \{u_i v_i, u_i x_i, u_i y_i, v_i a_i, v_i b_i : 1 \leq i \leq n\}$.

Then there are $6n$ vertices and $7n \quad 2$ edges. This proof is divided into two cases:

Case (i) n is odd: First assign the labels $\quad 1, \quad 4, \ldots, \quad 3n + 2$ to the vertices $u_1, u_2, \ldots, u_n$ respectively. Then we assign the labels $4, 10, \ldots, 3n \quad 5$ respectively to the vertices $v_1, v_3, \ldots, v_{n-2}$. We assign the labels $\quad 5, \quad 11, \ldots, \quad 3n + 4$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively. Also we give the label 1 to the vertex v_n. Next we give the labels $2, 5, \ldots, 3n \quad 1$ respectively to the vertices $x_1, x_2, \ldots, x_n$. We assign the labels $3, 6, \ldots, 3n$ to the vertices $y_1, y_2, \ldots, y_n$ respectively. Then we give the labels $\quad 2, \quad 8, \ldots, \quad 3n + 1$ respectively to the vertices $a_1, a_3, \ldots, a_n$. We give the labels $\quad 6, \quad 12, \ldots, \quad 3n + 3$ to the vertices $a_2, a_4, \ldots, a_{n-1}$ respectively. Then we assign the labels $\quad 3, \quad 9, \ldots, \quad 3n$ respectively to the vertices $b_1, b_3, \ldots, b_n$. Finally assign the labels $7, 13, \ldots, 3n \quad 2$ to the vertices $b_2, b_4, \ldots, b_{n-1}$ respectively. Hence, $\bar{\mathbb{S}}_{\lambda_1} = \frac{7n-3}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{7n-1}{2}$.

Case (ii) n is even: As in case (i), Assign the label to the vertices $u_i, x_i, y_i, 1 \leq i \leq n$. Assign the labels $4, 10, \ldots, 3n \quad 2$ respectively to the vertices $v_1, v_3, \ldots, v_{n-1}$. Then we assign the labels $\quad 5, \quad 11, \ldots, \quad 3n + 1$ to the vertices $v_2, v_4, \ldots, v_n$ respectively. Now we give the labels $\quad 2, \quad 8, \ldots, \quad 3n + 4$ respectively to the vertices $a_1, a_3, \ldots, a_{n-1}$. We give the labels $\quad 6, \quad 12, \ldots, \quad 3n$ to the vertices $a_2, a_4, \ldots, a_n$ respectively. Then we assign the labels $\quad 3, \quad 9, \ldots, \quad 3n + 3$ respectively to the vertices $b_1, b_3, \ldots, b_{n-1}$. Finally assign the labels $7, 13, \ldots, 3n \quad 5$ to the vertices $b_2, b_4, \ldots, b_{n-2}$ respectively. Finally assign the label 1 to the vertex b_n. Hence, $\bar{\mathbb{S}}_{\lambda_1} = \frac{7n-2}{2} = \bar{\mathbb{S}}_{\lambda_1^c}$.

Theorem 3.6: The graph $L_n \odot K_2$ is pair mean cordial for all $n \geq 2$.

Proof: Let $V(L_n \odot K_2) = \{u_i, v_i, x_i, y_i, a_i, b_i: 1 \leq i \leq n\}$ and $(L_n \odot K_2) = \{u_iu_{i+1}, v_iv_{i+1}: 1 \leq i \leq n \quad 1\} \cup \{u_iv_i, u_ix_i, u_iy_i, x_iy_i, v_ia_i, v_ib_i, a_ib_i: 1 \leq i \leq n\}$. Then there are $6n$ vertices and $9n \quad 2$ edges. This proof is divided into two cases:

Case (i) n is odd: First assign the labels $3, 6, \dots, 3n$ to the vertices $u_1, u_2, \dots, u_n$ respectively. Then we assign the labels $2, 8, \dots, 3n+1$ respectively to the vertices $v_1, v_3, \dots, v_n$. We assign the labels $7, 13, \dots, 3n \quad 2$ to the vertices $v_2, v_4, \dots, v_{n-1}$ respectively. Next we give the labels $1, 4, \dots, 3n+2$ respectively to the vertices $x_1, x_2, \dots, x_n$. We assign the labels $2, 5, \dots, 3n \quad 1$ to the vertices $y_1, y_2, \dots, y_n$ respectively. Then we give the labels $3, 9, \dots, 3n$ respectively to the vertices $a_1, a_3, \dots, a_n$. We give the labels $5, 11, \dots, 3n+4$ to the vertices $a_2, a_4, \dots, a_{n-1}$ respectively. Then we assign the labels $4, 10, \dots, 3n \quad 5$ respectively to the vertices $b_1, b_3, \dots, b_{n-2}$. Now assign the labels $6, 12, \dots, 3n+3$ to the vertices $b_2, b_4, \dots, b_{n-1}$ respectively. Finally assign the label 1 to the vertex b_n. Hence, $\overline{\mathbb{S}}_{\lambda_1} = \frac{9n-3}{2}$ and $\overline{\mathbb{S}}_{\lambda_1^c} = \frac{9n-1}{2}$.

Case (ii) n is even: As in case (i), Assign the label to the vertices $u_i, x_i, y_i, 1 \leq i \leq n$. Assign the labels $2, 8, \dots, 3n+4$ respectively to the vertices $v_1, v_3, \dots, v_{n-1}$. Then we assign the labels $7, 13, \dots, 3n \quad 5$ to the vertices $v_2, v_4, \dots, v_{n-2}$ respectively. We assign the label $3n+1$ to the vertex v_n. Now we give the labels $3, 9, \dots, 3n+3$ respectively to the vertices $a_1, a_3, \dots, a_{n-1}$. We give the labels $5, 11, \dots, 3n+7$ to the vertices $a_2, a_4, \dots, a_{n-2}$ respectively. Give the label 1 to the vertex a_n. Then we assign the labels $4, 10, \dots, 3n \quad 2$ respectively to the vertices $b_1, b_3, \dots, b_{n-1}$. Finally assign the labels $6, 12, \dots, 3n$ to the vertices $b_2, b_4, \dots, b_n$ respectively. Hence, $\overline{\mathbb{S}}_{\lambda_1} = \frac{9n-2}{2} = \overline{\mathbb{S}}_{\lambda_1^c}$.

Theorem 3.7: The graph $W_n \odot K_1$ is pair mean cordial for all $n \geq 3$.

Proof: Let $V(W_n \odot K_1) = \{u, u', v_i, v_i': 1 \leq i \leq n\}$ and $(W_n \odot K_1) = \{uu', uv_i, v_iv_i': 1 \leq i \leq n\} \quad \{v_iv_{i+1}: 1 \leq i \leq n \quad 1\} \cup \{v_nv_1\}$. Then there are $2n+2$ vertices and $3n+1$ edges. This proof is divided into two cases:

Case (i) n is odd: First assign the labels $n, 1$ to the vertices u, u' respectively. Then we assign the labels $1, 3, \dots, n+2$ respectively to the vertices $v_1, v_3, \dots, v_{n-2}$. We assign the labels $3, 5, \dots, n \quad 2$ to the vertices $v_2, v_4, \dots, v_{n-3}$ respectively. Also we give the labels $n+1, n+1$ respectively to the vertices v_{n-1}, v_n. Next we give the labels $2,4,\ldots,n \quad 1$ to the vertices $v_1', v_3', \ldots, v_{n-2}'$ respectively. We assign the labels $\quad 2, \quad 4, \ldots, \quad n+3$ to the vertices $v_2', v_4', \ldots, v_{n-3}'$ respectively. Finally we give the labels $\quad n \quad 1, \quad n$ to the vertices v_{n-1}', v_n' respectively. Hence, $\overline{\mathbb{S}}_{\lambda_1} = \frac{3n+1}{2} = \overline{\mathbb{S}}_{\lambda_1^c}$.

Case (ii) $\boldsymbol{n}$ is even: First assign the labels $-n \quad 1,1$ to the vertices u, u' respectively. Then we assign the labels $2,4,\ldots,n$ respectively to the vertices$v_1, v_3, \ldots, v_{n-1}$. We assign the labels $\quad 2, \quad 4, \ldots, \quad n$ to the vertices $v_2, v_4, \ldots, v_n$ respectively. Next we give the labels $\quad 1, \quad 3, \ldots, \quad n+1$ to the vertices $v_1', v_3', \ldots, v_{n-1}'$ respectively. Finally we assign the labels $3,5,\ldots,n+1$ to the vertices $v_2', v_4', \ldots, v_n'$ respectively. Hence, $\overline{\mathbb{S}}_{\lambda_1} = \frac{3n}{2}$ and $\overline{\mathbb{S}}_{\lambda_1^c} = \frac{3n+2}{2}$.

Theorem 3.8: The graph $W_n \odot 2K_1$ is pair mean cordial for all $n \geq 3$.

Proof: Let $V(W_n \odot 2K_1) = \{u, u', u'', v_i, v_i', v_i'': 1 \leq i \leq n\}$ and $(W_n \odot 2K_1) = \{uu', uu'', uv_i, v_iv_i', v_iv_i'' : 1 \leq i \leq n\} \quad \{v_iv_{i+1}: 1 \leq i \leq n \quad 1\} \cup \{v_nv_1\}$. Then there are $3n+3$ vertices and $4n+2$ edges. This proof is divided into two cases:

Case (i) $\boldsymbol{n}$ is odd: First assign the labels $\frac{3n+3}{2}, 1, \frac{-3n-1}{2}$ to the vertices u, u', u'' respectively. Then we assign the labels $\quad 1, \quad 4, \ldots, \frac{-3n+1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_n$. We assign the labels $4,7,\ldots,\frac{3n-1}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively. Next we give the labels $2,5,\ldots,\frac{3n+1}{2}$ respectively to the vertices $v_1', v_3', \ldots, v_n'$. We assign the labels $\quad 2, \quad 5, \ldots, \frac{-3n+5}{2}$ to the vertices $v_2', v_4', \ldots, v_{n-1}'$ respectively. Now we assign the labels $3,6,\ldots,\frac{3n-3}{2}$ respectively to the vertices $v_1'', v_3'', \ldots, v_{n-2}''$. Also we give the labels $\quad 3, \quad 6, \ldots, \frac{-3n+3}{2}$ to the vertices $v_2'', v_4'', \ldots, v_{n-1}''$ respectively. Finally we give the label $\frac{-3n-3}{2}$ to the vertex v_n''. Hence, $\overline{\mathbb{S}}_{\lambda_1} = 2n+1 = \overline{\mathbb{S}}_{\lambda_1^c}$.

Case (ii) $\boldsymbol{n}$ is even: First assign the labels $1,1,\frac{-3n-2}{2}$ to the vertices u, u', u'' respectively. Then we assign the labels $\quad 1, \quad 4, \ldots, \frac{-3n+4}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-1}$. Also we assign the labels $4,7, \ldots, \frac{3n+2}{2}$ to the vertices $v_2, v_4, \ldots, v_n$ respectively. Next we give the labels $2,5, \ldots, \frac{3n-2}{2}$ respectively to the vertices $v_1', v_3', \ldots, v_{n-1}'$. We assign the labels $2,\ 5, \ldots, \frac{-3n+2}{2}$ to the vertices $v_2', v_4', \ldots, v_n'$ respectively. Now we assign the labels $3,6, \ldots, \frac{3n}{2}$ respectively to the vertices $v_1'', v_3'', \ldots, v_{n-1}''$. Finally we give the labels $3,\ 6, \ldots, \frac{-3n}{2}$ to the vertices $v_2'', v_4'', \ldots, v_n''$ respectively. Hence, $\overline{\mathbb{S}}_{\lambda_1} = 2n + 1 = \overline{\mathbb{S}}_{\lambda_1^c}$.

Theorem 3.9: The graph $W_n \odot K_2$ is pair mean cordial if n is even.

Proof: Let $V(W_n \odot K_2) = \{u, u', u'', v_i, v_i', v_i'' : 1 \leq i \leq n\}$ and $(W_n \odot K_2) = \{uu', uu'', uv_i, v_iv_i', v_iv_i'' : 1 \leq i \leq n\}$ $\{v_iv_{i+1} : 1 \leq i \leq n \quad 1\} \cup \{v_nv_1\}$. Then there are $3n + 3$ vertices and $4n + 2$ edges. This proof is divided into two cases:

Case (i) n is odd: Suppose λ is a pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is $\frac{5n+1}{2}$. That is $\overline{\mathbb{S}}_{\lambda_1} \leq \frac{5n+1}{2}$. Then $\overline{\mathbb{S}}_{\lambda_1^c} \geq \frac{5n+5}{2}$. Therefore, $\overline{\mathbb{S}}_{\lambda_1^c} \quad \overline{\mathbb{S}}_{\lambda_1} \geq \frac{5n+5}{2} \quad \left(\frac{5n+1}{2}\right) = 2 > 1$, a contradiction.

Case (ii) n is even: Assign the labels $1, 1, \frac{-3n-2}{2}$ to the vertices u, u', u'' respectively. Then we assign the labels $3,6, \ldots, \frac{3n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-1}$. We assign the labels $2,\ 5, \ldots, \frac{-3n+2}{2}$ to the vertices $v_2, v_4, \ldots, v_n$ respectively. Next we give the labels $1,\ 4, \ldots, \frac{-3n+4}{2}$ respectively to the vertices $v_1', v_3', \ldots, v_{n-1}'$. Also we assign the labels $3,\ 6, \ldots, \frac{-3n}{2}$ to the vertices $v_2', v_4', \ldots, v_n'$ respectively. Now we assign the labels $2,5, \ldots, \frac{3n-2}{2}$ respectively to the vertices $v_1'', v_3'', \ldots, v_{n-1}''$. Finally we give the labels $4,7, \ldots, \frac{3n+2}{2}$ to the vertices $v_2'', v_4'', \ldots, v_n''$ respectively. Hence, $\overline{\mathbb{S}}_{\lambda_1} = \frac{5n+2}{2}$ and $\overline{\mathbb{S}}_{\lambda_1^c} = \frac{5n+4}{2}$.

Theorem 3.10: The gear graph G_n is not pair mean cordial for all $n \geq 3$.

Proof: Let $V(G_n) = \{u, v_i, v_i' : 1 \leq i \leq n\}$ and $(G_n) = \{uv_i, v_i v_i' : 1 \leq i \leq n\}$ $\{v_i v_{i+1}, v_i' v_{i+1} : 1 \leq i \leq n \quad 1\} \cup \{v_n' v_1, v_n v_1\}$.
Then there are $2n + 1$ vertices and $4n$ edges. Suppose λ is a pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is $2n \quad 1$. That is $\overline{\mathbb{S}}_{\lambda_1} \leq 2n \quad 1$. Then $\overline{\mathbb{S}}_{\lambda_1^c} \geq 2n + 1$. Therefore, $\overline{\mathbb{S}}_{\lambda_1^c} \quad \overline{\mathbb{S}}_{\lambda_1} \geq 2n + 1 \quad (2n \quad 1) = 2 > 1$, a contradiction.

Theorem 3.11: The graph $G_n \odot K_1$ is pair mean cordial for all $n \geq 3$.

Proof: Let $V(G_n \odot K_1) = \{u, u', v_i, v_i', w_i, w_i' : 1 \leq i \leq n\}$ and $\mathrm{E}(G_n \odot K_1) = \{uv_i, v_i v_i', w_i w_i', v_i w_i : 1 \leq i \leq n\}$ $\{v_i v_{i+1}, w_i v_{i+1} : 1 \leq i \leq n \quad 1\} \cup \{uu', v_n v_1, w_n v_1\}$. Then there are $4n + 2$ vertices and $6n + 1$ edges.

First assign the labels $1, \quad 2n \quad 1$ to the vertices u, u' respectively. Then we assign the labels $3, 5, \ldots, 2n + 1$ respectively to the vertices $v_1, v_2, \ldots, v_n$. We assign the labels $\quad 2, \quad 4, \ldots, \quad 2n$ to the vertices $v_1', v_2', \ldots, v_n'$ respectively. Also we give the labels $\quad 1, \quad 3, \ldots, \quad 2n + 1$ respectively to the vertices $w_1, w_2, \ldots, w_n$. Finally we give the labels $2, 4, \ldots, 2n$ to the vertices $w_1', w_2', \ldots, w_n'$ respectively. Hence, $\overline{\mathbb{S}}_{\lambda_1} = 3n$ and $\overline{\mathbb{S}}_{\lambda_1^c} = 3n + 1$.

Theorem 3.12: The graph $G_n \odot 2K_1$ is pair mean cordial for all $n \geq 3$.

Proof: Let $V(G_n \odot 2K_1) = \{u, u', u'', v_i, v_i', v_i'', w_i, w_i', w_i'' : 1 \leq i \leq n\}$ and $(G_n \odot 2K_1) =$ $\{uv_i, v_i v_i', v_i v_i'', w_i w_i', w_i w_i'', v_i w_i : 1 \leq i \leq n\}$ $\{v_i v_{i+1}, w_i v_{i+1} : 1 \leq i \leq n$ $1\} \cup \{uu', uu'', v_n v_1, w_n v_1\}$. Then there are $6n + 3$ vertices and $8n + 2$ edges.

First assign the labels $1, 1, \quad 3n \quad 1$ to the vertices u, u', u'' respectively. Then we assign the labels $4, 7, \ldots, 3n + 1$ respectively to the vertices $v_1, v_2, \ldots, v_n$. We assign the labels $\quad 2, \quad 5, \ldots, \quad 3n + 1$ to the vertices $v_1', v_2', \ldots, v_n'$ respectively. Next we assign the labels $\quad 3, \quad 6, \ldots, \quad 3n$ respectively to the vertices $v_1'', v_2'', \ldots, v_n''$. Also we give the labels $\quad 1, \quad 4, \ldots, \quad 3n + 2$ to the vertices $w_1, w_2, \ldots, w_n$ respectively.

Now we give the labels $2,5,\ldots,3n \quad 1$ respectively to the vertices $w_1{}', w_2{}', \ldots, w_n{}'$. Finally we give the labels $3,6,\ldots,3n$ to the vertices $w_1{}'', w_2{}'', \ldots, w_n{}''$ respectively. Hence $\overline{\mathbb{S}}_{\lambda_1} = 4n + 1 = \overline{\mathbb{S}}_{\lambda_1^c}$.

Theorem 3.13: The graph $G_n \odot K_2$ is pair mean cordial for all $n \geq 3$.

Proof: Let $V(G_n \odot K_2) = \{u, u', u'', v_i, v_i{}', v_i'', w_i, w_i{}', w_i{}'': 1 \leq i \leq n\}$ and $(G_n \odot K_2) = \{uv_i, v_iv_i{}', v_iv_i{}'', w_iw_i{}', w_iw_i{}'', v_iw_i, v_i{}'v_i{}'', w_i{}'w_i{}'': 1 \leq i \leq n\} \quad \{v_iv_{i+1}, w_iv_{i+1}: 1 \leq i \leq n \quad 1\} \cup \{uu', uu'', u'u'', v_nv_1, w_nv_1\}$. Then there are $6n + 3$ vertices and $10n + 3$ edges.

First assign the labels $1, 1, \quad 3n \quad 1$ to the vertices u, u', u'' respectively. Then we assign the labels $2, 5, \ldots, 3n + 1$ respectively to the vertices $v_1, v_2, \ldots, v_n$. We assign the labels $3, 6, \ldots, 3n$ to the vertices $v_1{}', v_2{}', \ldots, v_n{}'$ respectively.

Next we assign the labels $4,7,\ldots,3n + 1$ respectively to the vertices $v_1{}'', v_2{}'', \ldots, v_n{}''$. Also we give the labels $3,6,\ldots,3n$ to the vertices $w_1, w_2, \ldots, w_n$ respectively. Now we give the labels $1, 4, \ldots, 3n + 2$ respectively to the vertices $w_1{}', w_2{}', \ldots, w_n{}'$.

Finally we give the labels $2,5,\ldots,3n + 1$ to the vertices $w_1{}'', w_2{}'', \ldots, w_n{}''$ respectively. Hence, $\overline{\mathbb{S}}_{\lambda_1} = 5n + 1$ and $\overline{\mathbb{S}}_{\lambda_1^c} = 5n + 2$.

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GENERALIZED JACOBSTHAL POLYNOMIAL PRODUCTS WITH IMPLICATIONS

Abstract: We explore the Jacobsthal versions of the generalized gibonacci polynomial products (5), (9), (11), (14), (18), (20), and (21), investigated in [6].

Keywords: Generalized Jacobsthal Polynomial Products, Binet Formulas, Jacobsthal-Lucas Polynomials, Fibonacci Polynomials, Lucas Polynomials.

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1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \ge 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 4].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x)=0$ and $z_1(x)=1$, $z_n(x)= J_n(x)$, the nth *Jacobsthal polynomial* ; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. They too can be defined by the *Binet-like* formulas

$$J_n(x) = \frac{u^n(x) - v^n(x)}{D} \quad \text{and} \quad j_n(x) = u^n(x) + v^n(x),$$

where $D = \sqrt{4x+1}$, $2u(x) = 1 + D$, and $2v(x) = 1 - D$. It follows by the Binet-like formulas that $\lim_{n\to\infty} \frac{j_n}{J_n} = D$. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$ [4].

Fibonacci and Jacobsthal polynomials, and Lucas and Jacobsthal-Lucas polynomials are closely related by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [2, 3, 4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $c_n = J_n$, j_n, $\Delta\sqrt{x^2+4}$, $D = \sqrt{4x+1}$.

1.1 Generalized Gibonacci Polynomial Products: Using the gibonacci identities [6]

$$f_{2n} + (-1)^{n+k} f_{2k} = f_{n+k} l_{n-k};$$

$$f_{2n} - (-1)^{n+k} f_{2k} = f_{n-k} l_{n+k},$$

we established the following infinite Fibonacci products [6]:

$$\prod_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{\infty} \frac{f_{2(2n)} + f_{2k}}{f_{2(2n)} - f_{2k}} = \prod_{r=1}^{k} \frac{f_{2r-1}}{l_{2r-1}} \cdot \Delta^k; \tag{1}$$

$$\prod_{\substack{n=(k+1)/2\\k\ge1,\text{odd}}}^{\infty} \frac{f_{2(2n+1)}+f_{2k}}{f_{2(2n+1)}-f_{2k}} = \prod_{r=1}^{k} \frac{l_{2r}}{f_{2r}} \cdot \frac{1}{\Delta^k}; \tag{2}$$

$$\prod_{\substack{n=k+1\\k\ge1,\text{odd}}}^{\infty} \frac{f_{2n}+f_{2k}}{f_{2n}-f_{2k}} = \prod_{r=1}^{k} \frac{f_{2r-1}}{l_{2r-1}} \cdot \prod_{r=1}^{k} \frac{l_{2r}}{f_{2r}}, \tag{3}$$

$$\prod_{\substack{n=k/2+1\\k\ge2,\text{even}}}^{\infty} \frac{f_{2(2n)}+f_{2k}}{f_{2(2n)}-f_{2k}} = \prod_{r=1}^{k} \frac{l_{2r}}{f_{2r}} \cdot \frac{1}{\Delta^k}; \tag{4}$$

$$\prod_{\substack{n=k/2\\k\ge2,\text{even}}}^{\infty} \frac{f_{2(2n+1)}+f_{2k}}{f_{2(2n+1)}-f_{2k}} = \prod_{r=1}^{k} \frac{f_{2r-1}}{l_{2r-1}} \cdot \Delta^k; \tag{5}$$

$$\prod_{\substack{n=k+1\\k\ge2,\text{even}}}^{\infty} \frac{f_{2n}+f_{2k}}{f_{2n}-f_{2k}} = \prod_{r=1}^{k} \frac{f_{2r-1}}{l_{2r-1}} \cdot \prod_{r=1}^{k} \frac{l_{2r}}{f_{2r}}; \tag{6}$$

$$\prod_{\substack{n=k+1\\k\ge1}}^{\infty} \frac{f_{2n}+f_{2k}}{f_{2n}-f_{2k}} = \prod_{r=1}^{k} \frac{f_{2r-1}}{l_{2r-1}} \cdot \prod_{r=1}^{k} \frac{l_{2r}}{f_{2r}}. \tag{7}$$

2. Generalized Jacobsthal Polynomial Products

Using the gibonacci-Jacobsthal relationships, we will now explore the Jacobsthal versions of the above equations. In the interest of clarity and convenience, we let A denote the fractional expression on the left side of each equation and B the corresponding right side, and LHS and RHS the corresponding sides of the Jacobsthal equation to be found.

2.1 Jacobsthal Version of Equation (1)**:** Let $A = \dfrac{f_{2(2n)}+f_{2k}}{f_{2(2n)}-f_{2k}}$ where $k \ge 1$ and odd. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and the denominator with $x^{(4n-1)/2}$, we get

$$A = \frac{f_{2(2n)} + f_{2k}}{f_{2(2n)} - f_{2k}}$$

$$= \frac{x^{(4n-1)/2} f_{2(2n)} + x^{2n-k}[x^{(2k-1)/2} f_{2k}]}{x^{(4n-1)/2} f_{2(2n)} - x^{2n-k}[x^{(2k-1)/2} f_{2k}]}$$

$$= \frac{J_{2(2n)} + x^{2n-k} J_{2k}}{J_{2(2n)} - x^{2n-k} J_{2k}}$$

$$\text{LHS} = \prod_{\substack{n=(k+1)/2 \\ k\ge 1,\text{odd}}}^{\infty} \frac{J_{2(2n)} + x^{2n-k} J_{2k}}{J_{2(2n)} - x^{2n-k} J_{2k}},$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Now, let $B = \frac{f_{2r-1}}{l_{2r-1}} \cdot \Delta$. Replace x with $1/\sqrt{x}$, and then multiply the numerator and the denominator with $x^{(2r-1)/2}$. This yields

$$B = \frac{f_{2r-1}}{l_{2r-1}} \cdot \frac{D}{x^{1/2}}$$

$$= \frac{x^{1/2}[x^{(2r-2)/2} f_{2r-1}]}{x^{(2r-1)/2} l_{2r-1}} \cdot \frac{D}{x^{1/2}}$$

$$= x^{1/2}\, \frac{J_{2r-1}}{j_{2r-1}} \cdot \frac{D}{x^{k/2}}$$

$$\text{RHS} = x^{k/2} \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot \frac{D^k}{xk/2}$$

$$= \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot D^k,$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Equating the two sides, we get

$$\prod_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{\infty} \frac{J_{2(2n)} + x^{2n-k} J_{2k}}{J_{2(2n)} - x^{2n-k} J_{2k}} = \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot D^k . \tag{8}$$

It then follows that [6]

$$\prod_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{\infty} \frac{F_{2(2n)} + F_{2k}}{F_{2(2n)} - F_{2k}} = \prod_{r=1}^{k} \frac{F_{2r-1}}{L_{2r-1}} \cdot 5^{k/2} ;$$

$$\prod_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{\infty} \frac{J_{2(2n)} + 2^{2n-k} J_{2k}}{J_{2(2n)} - 2^{2n-k} J_{2k}} = \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot 3^k$$

2.2 Jacobsthal Version of Equation (2)**:** Let $A = \dfrac{f_{2(2n+1)} + f_{2k}}{f_{2(2n+1)} - f_{2k}}$**.** Replace x with $1/\sqrt{x}$, and then multiply the numerator and the denominator with $x^{(4n+1)/2}$. This yields

$$A = \frac{f_{2(2n+1)} + f_{2k}}{f_{2(2n+1)} - f_{2k}}$$

$$= \frac{x^{(4n+1)/2} f_{2(2n+1)} + x^{2n-k+1}[x^{(2k-1)/2} f_{2k}]}{x^{(4n+1)/2} f_{2(2n+1)} - x^{2n-k+1}[x^{(2k-1)/2} f_{2k}]}$$

$$= \frac{J_{2(2n+1)} + x^{2n-k+1} J_{2k}}{J_{2(2n+1)} - x^{2n-k+1} J_{2k}}$$

$$\text{LHS} = \prod_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{\infty} \frac{J_{2(2n+1)} + x^{2n-k+1} J_{2k}}{J_{2(2n+1)} - x^{2n-k+1} J_{2k}},$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Now, let $B = \frac{l_{2r-1}}{f_{2r-1}} \cdot \frac{1}{\Delta}$. Replace x with $1/\sqrt{x}$, and then multiply the numerator and the denominator with $x^{2r/2}$. This yields

$$B = \frac{l_{2r}}{f_{2r}} \cdot \frac{x^{1/2}}{D}$$

$$= \frac{x^{2r/2} l_{2r}}{x^{1/2}[x^{(2r-1)/2} f_{2r}]} \cdot \frac{x^{1/2}}{D}$$

$$= \frac{j_{2r}}{x^{1/2} J_{2r}} \cdot \frac{x^{1/2}}{D}$$

$$\text{RHS} = \prod_{r=1}^{k} \frac{j_{2r}}{x^{1/2} J_{2r}} \cdot \frac{x^{k/2}}{D^k}$$

$$= \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}} \cdot \frac{1}{D^k},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Combining the two sides, we get

$$\prod_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{\infty} \frac{J_{2(2n+1)} + x^{2n-k+1} J_{2k}}{J_{2(2n)} - x^{2n-k+1} J_{2k}} = \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}} \cdot \frac{1}{D^k}. \tag{9}$$

In particular, this yields [6]

$$\prod_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{\infty} \frac{F_{2(2n+1)} + F_{2k}}{F_{2(2n+1)} - F_{2k}} = \prod_{r=1}^{k} \frac{L_{2r}}{F_{2r}} \cdot \frac{1}{5^{k/2}};$$

$$\prod_{\substack{n=(k+1)/2\\k\geq 1,\text{odd}}}^{\infty} \frac{J_{2(2n+1)} + 2^{2n-k+1} J_{2k}}{J_{2(2n+1)} - 2^{2n-k+1} J_{2k}} = \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}} \cdot \frac{1}{3^k}$$

2.3 Jacobsthal Version of Equation (3): Equation (8), coupled with equation (9), yields the Jacobsthal version of equation (3):

$$\prod_{\substack{n=k+1\\k\geq 1,\text{odd}}}^{\infty} \frac{J_{2n} + x^{n-k} J_{2k}}{J_{2n} - x^{n-k} J_{2k}} = \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot D^k \cdot \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}} \cdot \frac{1}{D^k}$$

$$= \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}}. \tag{10}$$

In particular, we then have [6]

$$\prod_{\substack{n=k+1\\k\geq 1,\text{odd}}}^{\infty} \frac{F_{2n} + F_{2k}}{F_{2n} - F_{2k}} = \prod_{r=1}^{k} \frac{F_{2r-1}}{L_{2r-1}} \cdot \prod_{r=1}^{k} \frac{L_{2r}}{F_{2r}}$$

$$\prod_{\substack{n=k+1\\k\geq 1,\text{odd}}}^{\infty} \frac{J_{2n} + 2^{n-k} J_{2k}}{J_{2n} - 2^{n-k} J_{2k}} = \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}}.$$

2.4 Jacobsthal Version of Equation (4): With $A = \dfrac{f_{2(2n)} + f_{2k}}{f_{2(2n)} - f_{2k}}$, as Subsection 2.1, we get

$$\text{LHS} = \prod_{\substack{n=k/2+1\\k\geq 2,\text{even}}}^{\infty} \frac{J_{2(2n)} + x^{2n-k} J_{2k}}{J_{2(2n)} - x^{2n-k} J_{2k}},$$

where $J_n = J_n(x)$.

With $B = \frac{l_{2r}}{f_{2r}} \cdot \frac{1}{\Delta}$, as in Subsection 2.2, we get

$$\text{RHS} = \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}} \cdot \frac{1}{D^k},$$

where $c_n = c_n(x)$.

Equating the two sides then yields

$$\prod_{\substack{n=k/2+1 \\ k\geq 2,\text{even}}}^{\infty} \frac{J_{2(2n)} + x^{2n-k} J_{2k}}{J_{2(2n)} - x^{2n-k} J_{2k}} = \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}} \cdot \frac{1}{D^k}. \tag{11}$$

This implies [6],

$$\prod_{\substack{n=k/2+1 \\ k\geq 2,\text{even}}}^{\infty} \frac{F_{2(2n)} + F_{2k}}{F_{2(2n)} - F_{2k}} = \prod_{r=1}^{k} \frac{L_{2r}}{F_{2r}} \cdot \frac{5^{k/2}}{5^k}$$

$$\prod_{\substack{n=k/2+1 \\ k\geq 2,\text{even}}}^{\infty} \frac{J_{2(2n)} + 2^{2n-k} J_{2k}}{J_{2(2n)} - 2^{2n-k} J_{2k}} = \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}} \cdot \frac{1}{3^k}.$$

2.5 Jacobsthal Version of Equation (5): With $A = \frac{f_{2(2n+1)} + f_{2k}}{f_{2(2n+1)} - f_{2k}}$, as Subsection 2.2, with $B = \frac{f_{2r-1}}{l_{2r-1}} \cdot \Delta$, as in Subsection 2.1, we get

$$\text{LHS} = \prod_{\substack{n=k/2 \\ k\geq 2,\text{even}}}^{\infty} \frac{J_{2(2n+1)} + x^{2n-k+1} J_{2k}}{J_{2(2n+1)} - x^{2n-k+1} J_{2k}}$$

$$\text{RHS} = \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot D^k$$

respectively, where $c_n = c_n(x)$.

Combining the two sides then yields

$$\prod_{\substack{n=k/2 \\ k\geq 2,\text{even}}}^{\infty} \frac{J_{2(2n+1)} + x^{2n-k+1} J_{2k}}{J_{2(2n+1)} - x^{2n-k+1} J_{2k}} = \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot D^k \tag{12}$$

This yields [6]

$$\prod_{\substack{n=k/2 \\ k\geq 2,\text{even}}}^{\infty} \frac{F_{2(2n+1)} + F_{2k}}{F_{2(2n+1)} - F_{2k}} = \prod_{r=1}^{k} \frac{F_{2r-1}}{L_{2r-1}} \cdot 5^k ;$$

$$\prod_{\substack{n=k/2 \\ k\geq 2,\text{even}}}^{\infty} \frac{J_{2(2n+1)} + 2^{2n-k+1} J_{2k}}{J_{2(2n+1)} - 2^{2n-k+1} J_{2k}} = \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot 3^k .$$

2.6 Jacobsthal Version of Equation (6)**:** It follows by equations (10) and (12) that

$$\prod_{\substack{n=k+1 \\ k\geq 2,\text{even}}}^{\infty} \frac{J_{2n} + x^{n-k} J_{2k}}{J_{2n} - x^{n-k} J_{2k}} = \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}} . \tag{13}$$

This yields [6]

$$\prod_{\substack{n=k+1 \\ k\geq 2,\text{even}}}^{\infty} \frac{F_{2n} + F_{2k}}{F_{2n} - F_{2k}} = \prod_{r=1}^{k} \frac{F_{2r-1}}{L_{2r-1}} \cdot \prod_{r=1}^{k} \frac{L_{2r}}{F_{2r}} .$$

Additionally, we have

$$\prod_{\substack{n=k+1\\k\geq 2,\text{even}}}^{\infty} \frac{J_{2n}+2^{n-k}J_{2k}}{J_{2n}-2^{n-k}J_{2k}} = \prod_{r=1}^{k}\frac{J_{2r-1}}{j_{2r-1}}\cdot\prod_{r=1}^{k}\frac{j_{2r}}{J_{2r}}.$$

Finally, we explore the Jacobsthal version of the Fibonacci delight in equation (7).

2.7 Jacobsthal Version of Equation (7)**:** It follows by equations (11) and (13) that

$$\prod_{\substack{n=k+1\\k\geq 2}}^{\infty} \frac{J_{2n}+x^{n-k}J_{2k}}{J_{2n}-x^{n-k}J_{2k}} = \prod_{r=1}^{k}\frac{J_{2r-1}}{j_{2r-1}}\cdot\prod_{r=1}^{k}\frac{j_{2r}}{J_{2r}}. \tag{14}$$

This yields [6]

$$\prod_{\substack{n=k+1\\k\geq 1}}^{\infty} \frac{F_{2n}+F_{2k}}{F_{2n}-F_{2k}} = \prod_{r=1}^{k}\frac{F_{2r-1}}{L_{2r-1}}\cdot\prod_{r=1}^{k}\frac{L_{2r}}{F_{2r}};$$

$$\prod_{\substack{n=k+1\\k\geq 1}}^{\infty} \frac{J_{2n}+2^{n-k}J_{2k}}{J_{2n}-2^{n-k}J_{2k}} = \prod_{r=1}^{k}\frac{J_{2r-1}}{j_{2r-1}}\cdot\prod_{r=1}^{k}\frac{j_{2r}}{J_{2r}}.$$

3. Alternate Form

Using the identity $j_n^2 - D^2J_n^2 = 4(-x)^n$ [4], we can rewrite equation (14) in terms of both J_n and j_n:

$$\prod_{\substack{n=k+1\\k\geq 1}}^{\infty} \frac{j_{2n}^2 - x^{2n-2k}D^2J_{2k}^2 - 4x^{2n}}{D^2(J_{2n}-x^{n-k}J_{2k})^2} = \prod_{\substack{n=k+1\\k\geq 1}}^{\infty} \frac{J_{2n}^2 - x^{2n-2k}J_{2k}^2}{(J_{2n}-x^{n-k}J_{2k})^2}$$

$$= \prod_{r=1}^{k}\frac{J_{2r-1}}{j_{2r-1}}\cdot\prod_{r=1}^{k}\frac{j_{2r}}{J_{2r}}.$$

It then follows that [6]

$$\prod_{\substack{n=k+1 \\ k\geq 1}}^{\infty} \frac{L_{2n}^2 + 5F_{2k}^2 - 4}{5(F_{2n} - F_{2k})^2} = \prod_{r=1}^{k} \frac{F_{2r-1}}{L_{2r-1}} \cdot \prod_{r=1}^{k} \frac{L_{2r}}{F_{2r}}.$$

In particular, we have [4, 5]

$$\prod_{n=2}^{\infty} \frac{L_{2n}^2 - 9}{5(F_{2n} - 1)^2} = 3 \ ; \qquad \prod_{n=3}^{\infty} \frac{L_{2n}^2 - 49}{5(F_{2n} - 3)^2} = \frac{7}{2}.$$

Additionally, we have

$$\prod_{\substack{n=k+1 \\ k\geq 1}}^{\infty} \frac{j_{2n}^2 - 9.4^{n-k} J_{2k}^2 - 4^{n+1}}{9(J_{2n} - 2^{n-k} J_{2k})^2} = \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}}.$$

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GENERALIZED JACOBSTHAL POLYNOMIAL PRODUCTS: GRAPH-THEORETIC CONFIRMATIONS

Abstract: We confirm seven generalized Jacobsthal polynomial products using graph-theoretic techniques.

Keywords: Generalized Jacobsthal Polynomials, Extended Gibonacci Polynomials Lucas Number, Jacobsthal-Lucas Polynomials.

Mathematic Subject Classification (2020) No.: Primary 05C20, 05C22, 11B39, 11B83, 11C08.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*.

Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 4].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial* ; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. They too can be defined by the *Binet-like* formulas

$$J_n(x) = \frac{u^n(x) - v^n(x)}{D} \quad \text{and} \quad j_n(x) = u^n(x) + v^n(x),$$

where $D = \sqrt{4x+1}$, $2u(x) = 1 + D$, and $2v(x) = 1$. It follows by the Binet-like formulas that $\lim_{n\to\infty} \frac{j_n}{J_n} = D$. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$ [4].

Fibonacci and Jacobsthal polynomials, and Lucas and Jacobsthal-Lucas polynomials are closely related by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$[2, 3, 4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $c_n = J_n$ or j_n, and $\Delta = \sqrt{x^2 + 4}$.

It follows from the Binet-like formulas that $\lim_{n\to\infty} \frac{J_n}{j_n} = \frac{1}{D}$ and $\lim_{n\to\infty} \frac{j_n}{j_n} = D$.

1.1 Some Fundamental Jacobsthal Identities: Using the gibonacci-Jacobsthal relationships, it follows from the gibonacci identities [6]

$$f_{2n} + (-1)^{n+k} f_{2k} = f_{n+k} l_{n-k};$$

$$f_{2n} - (-1)^{n+k} f_{2k} = f_{n-k} l_{n+k},$$

that

$$J_{2n} + (-1)^{n+k} x^{n-k} J_{2k} = J_{n+k} j_{n-k}; \tag{1}$$

$$J_{2n} - (-1)^{n+k} x^{n-k} J_{2k} = J_{n-k} j_{n+k}, \tag{2}$$

where $g_n = g_n(x)$ and $c_n = c_n(x)$.

1.2 Generalized Jacobsthal Polynomial Products: In [7], we established the following infinite Jacobsthal products:

$$\prod_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{\infty} \frac{J_{2(2n)} + x^{2n-k} J_{2k}}{J_{2(2n)} - x^{2n-k} J_{2k}} = \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot D^k ; \tag{3}$$

$$\prod_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{\infty} \frac{J_{2(2n+1)} + x^{2n+1-k} J_{2k}}{J_{2(2n+1)} - x^{2n+1-k} J_{2k}} = \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}} \cdot \frac{1}{D^k} . \tag{4}$$

$$\prod_{\substack{n=k+1 \\ k\geq 1,\text{odd}}}^{\infty} \frac{J_{2n} + x^{n-k} J_{2k}}{J_{2n} - x^{n-k} J_{2k}} = \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}}; \tag{5}$$

$$\prod_{\substack{n=k/2+1 \\ k\geq 2,\text{even}}}^{\infty} \frac{J_{2(2n)} + x^{2n-k} J_{2k}}{J_{2(2n)} - x^{2n-k} J_{2k}} = \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}} \cdot \frac{1}{D^k} ; \tag{6}$$

$$\prod_{\substack{n=k/2 \\ k\geq 2,\text{even}}}^{\infty} \frac{J_{2(2n+1)} + x^{2n+1-k} J_{2k}}{J_{2(2n+1)} - x^{2n+1-k} J_{2k}} = \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot D^k ; \tag{7}$$

$$\prod_{\substack{n=k+1 \\ k\geq 2,\text{even}}}^{\infty} \frac{J_{2n} + x^{n-k} J_{2k}}{J_{2n} - x^{n-k} J_{2k}} = \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}} . \tag{8}$$

$$\prod_{\substack{n=k+1 \\ k\geq 1}}^{\infty} \frac{J_{2n} + x^{n-k} J_{2k}}{J_{2n} - x^{n-k} J_{2k}} = \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}}. \tag{9}$$

Our goal in this discourse is to confirm the validity of formulas (3) through (8) and hence (9), using graph-theoretic techniques. To this end, we first present a brief introduction to the needed graph-theoretic tools.

2. Graph-Theoretic Tools

To confirm these Jacobsthal results, consider the *weighted Jacobsthal digraph* in Figure 1 with vertices v_1 and v_2 [4, 5]. It follows from its *weighted adjacency matrix* $M = \begin{bmatrix} 1 & x \\ 1 & 0 \end{bmatrix}$ that

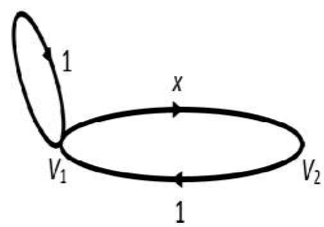

Figure 1: Weighted Jacobsthal Digram

$$M^n = \begin{bmatrix} J_{n+1} & xJ_n \\ J_n & xJ_{n-1} \end{bmatrix},$$

where $J_n = J_n(x)$ and $n \geq 1$.

Let A be the set of closed walks of length n originating at v_1, and B the set of those of length n in the digraph. Let T_n denote the sum of the weights of the elements in A, and U_n the sum of those in B. Then $T_n = J_{n+1}$ and $U_n = J_{n+1} + xJ_{n-1} = j_n$, where $c_n = c_n(x)$.

Let A, B, and C denote the sets of closed walks of varying lengths originating at vertex v. Then the sum of the weights of the elements in the product set $A \times B \times C$ is *defined* as the product the sums of the walks in each component [5]. Obviously, this definition can be extended to any finite number of components in the product. These facts play a major role in the graph-theoretic proofs.

With these tools at our disposal, we are now ready for the graph-theoretic explorations.

3. Graph-Theoretic Confirmations

We begin our explorations with equation (3).

3.1 Confirmation of Formula (3)**:** ***Proof*:** Let k be an odd positive integer and w the weight of the edge v_1v_2. Consider the product

$$P_m = \prod_{\substack{n=(k+1)/2 \\ k\ge 1,\text{odd}}}^{m} \frac{T_{4n-1} + w^{2n-k}T_{2k-1}}{T_{4n-1} - w^{2n-k}T_{2k-1}}$$

$$= \prod_{\substack{n=(k+1)/2 \\ k\ge 1,\text{odd}}}^{m} \frac{J_{2(2n)} + x^{2n-k}J_{2k}}{J_{2(2n)} - x^{2n-k}J_{2k}}.$$

We will now compute the product P_m in a different way. To this end, we let

$$P_m^* = \prod_{\substack{r=1 \\ k\ge 1,\text{odd}}}^{k} \frac{T_{2r-2}}{U_{2r-1}} \cdot \prod_{\substack{r=1 \\ k\ge 1,\text{odd}}}^{k} \frac{U_{2m+2r-k}}{T_{2m+2r-1-k}}$$

$$= \prod_{\substack{r=1 \\ k\ge 1,\text{odd}}}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{\substack{r=1 \\ k\ge 1,\text{odd}}}^{k} \frac{j_{2m+2r-k}}{J_{2m+2r-k}}$$

where $m \ge 1$. Using identities (1) and (2), we then get

$$P_1^* = \prod_{r=1}^{1} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{r=1}^{1} \frac{j_{2r+1}}{J_{2r+1}}$$

$$= \frac{J_1 j_3}{J_3 j_1} = \frac{J_4 + xJ_2}{J_4 - xJ_2}$$

$$= P_1;$$

$$P_2^* = \prod_{r=1}^{1} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{r=1}^{1} \frac{j_{2r+1}}{J_{2r+1}} \cdot \prod_{r=1}^{3} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{r=1}^{3} \frac{j_{2r+1}}{J_{2r+1}}$$

$$= \frac{J_1 j_3}{J_3 j_1} = \frac{J_1 j_7}{J_7 j_1}$$

$$= \frac{J_4 + xJ_2}{J_4 - xJ_2} \cdot \frac{J_8 + xJ_6}{J_8 - xJ_6}$$

$$= P_2 .$$

Based on these two initial values of P_m^*, we conjecture that $P_m^* = P_m$ where $m \geq 1$.

We will now establish this conjecture using recursion [4, 6]. To this end, we have

$$\frac{P_m^*}{P_{m-1}^*} = \prod_{\substack{r=1 \\ k\geq 1,\text{odd}}}^{k} \frac{j_{2m+2r-k}}{J_{2m+2r-k}} \cdot \prod_{\substack{r=1 \\ k\geq 1,\text{odd}}}^{k} \frac{J_{2m+2(r-1)-k}}{j_{2m+2(r-1)-k}}$$

$$= \frac{J_{2m-k} j_{2m+k}}{J_{2m+k} j_{2m-k}}$$

$$= \frac{J_{2(2m)} + x^{2m-k} J_{2k}}{J_{2(2m)} - x^{2m-k} J_{2k}}$$

$$= \frac{P_m}{P_{m-1}} .$$

Recursively, this implies that

$$\frac{P_m}{P_m^*} = \frac{P_{m-1}}{P_{m-1}^*} = \cdots = \frac{P_1}{P_1^*} = 1 .$$

Thus, $P_m^* = P_m$, as conjectured

$$\prod_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{m} \frac{J_{2(2n)} + x^{2n-k} J_{2k}}{J_{2(2n)} - x^{2n-k} J_{2k}} = \prod_{\substack{r=1 \\ k\geq 1,\text{odd}}}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{\substack{r=1 \\ k\geq 1,\text{odd}}}^{k} \frac{j_{2m+2r-k}}{J_{2m+2r-k}}. \qquad (10)$$

Since $\lim_{n\to\infty} \frac{j_n}{J_n} = D$, this yields formula (3) as desired. □

3.2 Confirmation of Formula (4): ***Proof*:** With k an odd positive integer, and w, T_n, and U_n as before, we let

$$Q_m = \prod_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{m} \frac{T_{4n+1} + w^{2n+1-k} T_{2k-1}}{T_{4n+1} - w^{2n+1-k} T_{2k-1}}$$

$$= \prod_{\substack{n=(k+1)/2 \\ k\geq,\text{odd}}}^{m} \frac{J_{2(2n+1)} + x^{2n+1-k} J_{2k}}{J_{2(2n+1)} - x^{2n+1-k} J_{2k}}.$$

We will now compute this product in a different way. To achieve this, consider the product

$$Q_m^* = \prod_{\substack{r=1 \\ k\geq 1,\text{odd}}}^{k} \frac{U_{2r}}{T_{2r-1}} \cdot \prod_{\substack{r=1 \\ k\geq 1,\text{odd}}}^{k} \frac{T_{2m+2r-k}}{U_{2m+(2r+1)-k}}$$

$$= \prod_{\substack{r=1 \\ k\geq 1,\text{odd}}}^{k} \frac{j_{2r}}{J_{2r}} \cdot \prod_{\substack{r=1 \\ k\geq 1,\text{odd}}}^{k} \frac{J_{2m+(2r+1)-k}}{j_{2m+(2r+1)-k}},$$

where $m \geq 1$. With identities (1) and (2), we then get

$$Q_1^* = \prod_{r=1}^{1} \frac{j_{2r}}{J_{2r}} \cdot \prod_{r=1}^{1} \frac{J_{2r+2}}{j_{2r+2}}$$

$$= \frac{J_4 j_2}{J_2 j_4} = \frac{J_6 + x^2 J_2}{J_6 - x^2 J_2}$$

$$= Q_1 ;$$

$$Q_2^* = \prod_{r=1}^{1} \frac{j_{2r}}{J_{2r}} \cdot \prod_{r=1}^{1} \frac{J_{2r+2}}{j_{2r+2}} \cdot \prod_{r=1}^{3} \frac{j_{2r}}{J_{2r}} \cdot \prod_{r=1}^{3} \frac{J_{2r+2}}{j_{2r+2}}$$

$$= \frac{J_4 j_2}{J_2 j_4} = \frac{J_8 j_2}{J_2 j_8}$$

$$= \frac{J_6 + x^2 J_2}{J_6 - x^2 J_2} \cdot \frac{J_{10} + x^2 J_6}{J_{10} - x^2 J_6}$$

$$= Q_2 .$$

Using these two initial values of Q_m^*, we conjecture that $Q_m^* = Q_m$, where $m \geq 1$.

We will now establish this using recursion [4, 6]. We have

$$\frac{Q_m^*}{Q_{m-1}^*} = \prod_{\substack{r=1 \\ k \geq 1, \text{odd}}}^{k} \frac{J_{2m+(2r+1)-k}}{j_{2m+(2r+1)-k}} \cdot \prod_{\substack{r=1 \\ k \geq 1, \text{odd}}}^{k} \frac{j_{2m+(2r-1)-k}}{J_{2m+(2r-1)-k}}$$

$$= \frac{J_{2m+1+k} j_{2m+1-k}}{J_{2m+1-k} j_{2m+1+k}}$$

$$= \frac{J_{2(2m+1)} + x^{2m+1-k} J_{2k}}{J_{2(2m+1)} - x^{2m+1-k} J_{2k}}$$

$$= \frac{Q_m}{Q_{m-1}} .$$

Recursively, this implies that

$$\frac{Q_m}{Q_m^*} = \frac{Q_{m-1}}{Q_{m-1}^*} = \cdots = \frac{Q_1}{Q_1^*} = 1$$

This confirms the conjecture:

$$\prod_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{m} \frac{J_{2(2n+1)} + x^{2n+1-k} J_{2k}}{J_{2(2n+1)} - x^{2n+1-k} J_{2k}} = \prod_{\substack{r=1 \\ k\geq 1,\text{odd}}}^{k} \frac{j_{2r}}{J_{2r}} \cdot \prod_{\substack{r=1 \\ k\geq 1,\text{odd}}}^{k} \frac{j_{2m+(2r+1)-k}}{J_{2m+(2r+1)-k}}. \tag{11}$$

Since $\lim_{n\to\infty} \frac{J_n}{j_n} = \frac{1}{D}$, this yields formula (4) as desired. □

3.3 Confirmation of Formula (5)**:** ***Proof*****:** It follows by equations (10) and (11) that

$$\prod_{\substack{n=k+1 \\ k\geq 1,\text{odd}}}^{2m+1} \frac{J_{2n} + x^{n-k} J_{2k}}{J_{2n} - x^{n-k} J_{2k}} = \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot D^k \cdot \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}} \cdot \frac{1}{D^k};$$

$$\prod_{\substack{n=k+1 \\ k\geq 1,odd}}^{\infty} \frac{J_{2n} + x^{n-k} J_{2k}}{J_{2n} - x^{n-k} J_{2k}} = \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}},$$

as expected. □

Consequently, we have [7]

$$\prod_{\substack{n=k+1 \\ k\geq 1,\text{odd}}}^{\infty} \frac{F_{2n} + F_{2k}}{F_{2n} - F_{2k}} = \prod_{r=1}^{k} \frac{F_{2r-1}}{L_{2r-1}} \cdot \prod_{r=1}^{k} \frac{L_{2r}}{F_{2r}};$$

$$\prod_{\substack{n=k+1 \\ k\geq 1,\text{odd}}}^{\infty} \frac{J_{2n} + 2^{n-k} J_{2k}}{J_{2n} - 2^{n-k} J_{2k}} = \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}}.$$

3.4 Confirmation of Formula (6)**:** ***Proof*****:** Let k be an even positive integer. With w, T_n, and U_n as before, we let

$$R_m = \prod_{\substack{n=k/2+1\\k\geq 2,\text{even}}}^{m} \frac{T_{4n-1} + w^{2n-k}T_{2k-1}}{T_{4n-1} - w^{2n-k}T_{2k-1}}$$

$$= \prod_{\substack{n=k/2+1\\k\geq 2,\text{even}}}^{m} \frac{J_{2(2n)} + x^{2n-k}J_{2k}}{J_{2(2n)} - x^{2n-k}J_{2k}}.$$

We will now compute this product in a different way. To accomplish this, consider the product

$$R_m^* = \prod_{\substack{r=1\\k\geq 2,\text{even}}}^{k} \frac{U_{2r}}{T_{2r-1}} \cdot \prod_{\substack{r=1\\k\geq 2,\text{even}}}^{k} \frac{T_{2m+(2r-1)-k}}{U_{2m+2r-k}}$$

$$= \prod_{\substack{r=1\\k\geq 2,\text{even}}}^{k} \frac{j_{2r}}{J_{2r}} \cdot \prod_{\substack{r=1\\k\geq 2,\text{even}}}^{k} \frac{J_{2m+2r-k}}{j_{2m+2r-k}},$$

where $m \geq 2$.

We then have

$$R_2^* = \prod_{r=1}^{2} \frac{j_{2r}}{J_{2r}} \cdot \prod_{r=1}^{2} \frac{J_{2r+2}}{j_{2r+2}}$$

$$= \frac{J_6 j_2}{J_2 j_6} = \frac{J_8 + x^2 J_4}{J_8 - x^2 J_4}$$

$$= R_2;$$

$$R_3^* = \prod_{r=1}^{2} \frac{j_{2r}}{J_{2r}} \cdot \prod_{r=1}^{2} \frac{J_{2r+2}}{j_{2r+2}} \cdot \prod_{r=1}^{4} \frac{j_{2r}}{J_{2r}} \cdot \prod_{r=1}^{4} \frac{J_{2r+2}}{j_{2r+2}}$$

$$= \frac{J_6 j_2}{J_2 j_6} = \frac{J_{10} j_2}{J_2 j_{10}}$$

$$= \frac{J_8 + x^2 J_4}{J_8 - x^2 J_4} \cdot \frac{J_{12} + x^2 J_8}{J_{12} - x^2 J_8}$$

$$= R_3 .$$

Using these two initial values of R_m^*, we predict that $R_m^* = R_m$ where $m \geq 2$.

We will now confirm this conjecture using recursion [4, 6]. We have

$$\frac{R_m^*}{R_{m-1}^*} = \prod_{\substack{r=1 \\ k \geq 2, \text{even}}}^{k} \frac{J_{2m+2r-k}}{j_{2m+2r-k}} \cdot \prod_{\substack{r=1 \\ k \geq 2, \text{even}}}^{k} \frac{j_{2m+2(r-1)-k}}{J_{2m+2(r-1)-k}}$$

$$= \frac{J_{2m+k} j_{2m-k}}{J_{2m-k} j_{2m+k}}$$

$$= \frac{J_{2(2m)} + x^{2m-k} J_{2k}}{J_{2(2m)} - x^{2m-k} J_{2k}}$$

$$= \frac{R_m}{R_{m-1}} .$$

This implies

$$\frac{R_m}{R_m^*} = \frac{R_{m-1}}{R_{m-1}^*} = \cdots = \frac{R_2}{R_2^*} = 1.$$

Consequently $R_m^* = R_m$ as desired:

$$\prod_{\substack{n=k/2+1 \\ k \geq 2, \text{even}}}^{m} \frac{J_{2(2n)} + x^{2n-k} J_{2k}}{J_{2(2n)} - x^{2n-k} J_{2k}} = \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}} \cdot \prod_{r=1}^{k} \frac{J_{2m+2r-k}}{j_{2m+2r-k}} . \qquad (12)$$

This yields formula (6), as desired. □

3.5 Confirmation of Formula (7) ***Proof*****:** With k, w, T_n, and U_n as before, we let

$$S_m = \prod_{\substack{n=k/2 \\ k\geq 2,\text{even}}}^{m} \frac{T_{4n+1} + w^{2n+1-k}T_{2k-1}}{T_{4n+1} - w^{2n+1-k}T_{2k-1}}$$

$$= \prod_{\substack{n=k/2 \\ k\geq 2,\text{even}}}^{m} \frac{J_{2(2n+1)} + x^{2n+1-k}J_{2k}}{J_{2(2n+1)} - x^{2n+1-k}J_{2k}}.$$

We will now compute S_m in a different way. To this end, consider the product

$$S_m^* = \prod_{\substack{r=1 \\ k\geq 2,\text{even}}}^{k} \frac{T_{2r-2}}{U_{2r-1}} \cdot \prod_{\substack{r=1 \\ k\geq 2,\text{even}}}^{k} \frac{U_{2m+(2r+1)-k}}{T_{2m+2r-k}}$$

$$= \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{r=1}^{k} \frac{j_{2m+(2r+1)-k}}{J_{2m+(2r+1)-k}}$$

where $c_n = c_n(x)$ and $m \geq 1$.

We then get

$$S_1^* = \prod_{r=1}^{2} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{r=1}^{2} \frac{j_{2r+1}}{J_{2r+1}}$$

$$= \frac{J_1 j_5}{J_5 j_1} = \frac{J_6 + xJ_4}{J_6 - xJ_4}$$

$$= S_1;$$

$$S_2^* = \prod_{r=1}^{2} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{r=1}^{2} \frac{j_{2r+1}}{J_{2r+1}} \cdot \prod_{r=1}^{4} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{r=1}^{4} \frac{j_{2r+1}}{J_{2r+1}}$$

$$= \frac{J_1 j_5}{J_5 j_1} = \frac{J_1 j_9}{J_9 j_1}$$

$$= \frac{J_6 + xJ_4}{J_6 - xJ_4} \cdot \frac{J_{10} + xJ_8}{J_{10} - xJ_8}$$

$$= S_2 .$$

Based on these two initial values of S_m^*, we conjecture that $S_m^* = S_m$, where $m \geq 1$. We will now confirm this using recursion [4, 6].

We have

$$\frac{S_m^*}{S_{m-1}^*} = \prod_{r=1}^{k} \frac{j_{2m+(2r+1)-k}}{J_{2m+(2r+1)-k}} \cdot \prod_{r=1}^{k} \frac{J_{2m+(2r-1)-k}}{j_{2m+(2r-1)-k}}$$

$$= \frac{J_{2m+1-k} j_{2m+1+k}}{J_{2m+1+k} j_{2m+1-k}}$$

$$= \frac{J_{2(2m+1)} + x^{2m+1-k} J_{2k}}{J_{2(2m+1)} - x^{2m+1-k} J_{2k}}$$

$$= \frac{S_m}{S_{m-1}} .$$

This implies,

$$\frac{S_m}{S_m^*} = \frac{S_{m-1}}{S_{m-1}^*} = \cdots = \frac{S_2}{S_2^*} = 1.$$

Consequently $S_m^* = S_m$ confirming the conjecture:

$$\prod_{\substack{n=k/2 \\ k \geq 2,\text{even}}}^{m} \frac{J_{2(2n+1)} + x^{2n+1-k} J_{2k}}{J_{2(2n+1)} - x^{2n+1-k} J_{2k}} = \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{r=1}^{k} \frac{j_{2m+(2r+1)-k}}{J_{2m+(2r+1)-k}} . \quad (13)$$

where $c_n = c_n(x)$. Clearly, this yields formula (7), as desired. □

3.6 Confirmation of Formula (8)**:** ***Proof*****:** It follows by equations (12) and (13) that

$$\prod_{\substack{n=k+1\\k\geq 2,\text{even}}}^{2m+1} \frac{J_{2n}+x^{n-k}J_{2k}}{J_{2n}-x^{n-k}J_{2k}} = \prod_{\substack{n=k/2+1\\k\geq 2,\text{even}}}^{m} \frac{J_{2(2n)}+x^{2n-k}J_{2k}}{J_{2(2n)}-x^{2n-k}J_{2k}} \cdot \prod_{\substack{n=k/2\\k\geq 2,\text{even}}}^{m} \frac{J_{2(2n+1)}+x^{2n+1-k}J_{2k}}{J_{2(2n+1)}-x^{2n+1-k}J_{2k}}$$

$$\prod_{\substack{n=k+1\\k\geq 2,\text{even}}}^{\infty} \frac{J_{2n}+x^{n-k}J_{2k}}{J_{2n}-x^{n-k}J_{2k}} = \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}}, \tag{14}$$

as desired. □

Finally, we explore the graph-theoretic proof of formula (9).

3.7 Confirmation of Formula (9): ***Proof*****:** It follows by equations (12) and (14) that

$$\prod_{\substack{n=k+1\\k\geq 1}}^{\infty} \frac{J_{2n}+x^{n-k}J_{2k}}{J_{2n}-x^{n-k}J_{2k}} = \prod_{\substack{n=k+1\\k\geq 1,\text{odd}}}^{\infty} \frac{J_{2n}+x^{n-k}J_{2k}}{J_{2n}-x^{n-k}J_{2k}} \cdot \prod_{\substack{n=k+1\\k\geq 2,\text{even}}}^{\infty} \frac{J_{2n}+x^{n-k}J_{2k}}{J_{2n}-x^{n-k}J_{2k}}$$

$$= \prod_{r=1}^{k} \frac{J_{2r-1}}{j_{2r-1}} \cdot \prod_{r=1}^{k} \frac{j_{2r}}{J_{2r}}, \tag{15}$$

as desired. □

It follows from formula (15) that [6]

$$\prod_{\substack{n=k+1\\k\geq 1}}^{\infty} \frac{F_{2n}+F_{2k}}{F_{2n}-F_{2k}} = \prod_{r=1}^{k} \frac{F_{2r-1}}{L_{2r-1}} \cdot \prod_{r=1}^{k} \frac{L_{2r}}{F_{2r}};$$

$$\prod_{\substack{n=k+1\\k\geq 1}}^{\infty} \frac{L_{2n}^2-5F_{2k}^2-4}{5(F_{2n}-F_{2k})^2} = \prod_{r=1}^{k} \frac{F_{2r-1}}{L_{2r-1}} \cdot \prod_{r=1}^{k} \frac{L_{2r}}{F_{2r}}$$

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