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CONVOLUTIONS OF EXTENDED TRIBONACCI-GIBONACCI POLYNOMIAL PRODUCTS OF ORDERS 2 AND 3

Abstract: We develop convolutions of extended gibonacci and tribonacci polynomial products of order 2, and extract their numeric counterparts.

Keywords: Extended Gibonacci Polynomials, Fibonacci, Lucas Polynomials, Extended Tribonacci Polynomials, Extended Tribonacci Gibonacci Polynomials.

Mathematical Subject Classification (2020) No.: Primary 11B37, 11B39, 11B83, 11C08.

1. Introduction

1.1 Extended Gibonacci Polynomials: *Extended gibonacci polynomials* $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary copmlex variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary copmlex polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*.

Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 8, 10].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [8, 10].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal Lucas polynomial* [6, 10]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$ and $j_n(1) = L_n$.

1.2 Extended Tribonacci Polynomials: *Extended tribonacci polynomials* $w_n(x)$ are defined by the recurrence $w_{n+3}(x) = x^2 w_{n+2}(x) + x w_{n+1}(x) + w_n(x)$, where x is an arbitrary complex variable; $w_0(x)$, $w_1(x)$, and $w_2(x)$ are arbitrary complex polynomials; and $n \geq 0$.

Suppose $w_0(x) = 0$, $w_1(x) = 1$, and $w_2(x) = x^2$. Then $w_n(x) = t_n(x)$, the nth *tribonacci polynomial*; and when $w_0(x) = 3$, $w_1(x) = x^2$, and $w^2(x) = x^4 + 2x$, $w_n(x) = k_n(x)$, the nth *tribonacci-Lucas polynomial*.

Tribonacci polynomials $t_n(x)$ were originally studied by Hoggatt and Bicknell [5, 10], and tribonacci-Lucas polynomials $k_n(x)$ by Kose, Yilmaz, and Taskara [7]. Correspondingly, $t_n(1) = T_n$, the nth *tribonacci number*, first studied by M. Feinberg when he was a 14-year old; and $k_n(1) = K_n$, the nth *tribonacci-Lucas number*, originally studied by M. Catalani [2].

Both gibonacci and tribonacci polynomials can be extended to negative subscripts. For example, $t_{-1}(x) = 0$, $t_{-2}(x) = 1$, and $t_{-3}(x) = -x$; and $k_{-1}(x) = -x$, $k_{-2}(x) = -x^2$, and $k_{-3}(x) = 2x^3 + 3$.

As can be expected, there is an interesting relationship between tribonacci and tribonacci-Lucas polynomials; it can be confirmed using induction [7]:

$$k_n(x) = x^2 t_n(x) + 2x t_{n-1}(x) + 3t_{n-2}(x). \tag{1}$$

Consequently, formulas involving tribonacci-Lucas polynomials can be rewritten in terms of tribonacci polynomials. Formula (1) implies that

$$K_n = T_n + 2T_{n-1} + 3T_{n-2}. \tag{2}$$

Using the identity $J_{n+1}(x) + xJ_{n-1}(x) = j_n(x)$ [10], we can express formulas involving Jacobsthal-Lucas polynomials in terms of Jacobsthal polynomials.

In the interest of clarity, brevity, and convenience, we *omit* the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. We also omit a lot of basic algebra.

A *polynomial product of order* m is a product of polynomials z_{n+k} of the form $\prod_{k\geq 0} z_{n+k}^{s_j}$, where $\sum_{sj\geq 1} s_j = m$ [9, 12].

1.3 Generating Functions: Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas, tribonacci, and tribonacci-Lucas polynomials can be generated by generating functions [4, 7, 10, 11]:

$$f(z) = \sum_{n=0}^{\infty} f_n z^n = \frac{z}{1 - xz - z^2}; \qquad l(z) = \sum_{n=0}^{\infty} l_n z^n = \frac{2 - xz}{1 - xz - z^2};$$

$$p(z) = \sum_{n=0}^{\infty} p_n z^n = \frac{2z}{1 - 2xz - 4z^2}; \qquad q(z) = \sum_{n=0}^{\infty} q_n z^n = \frac{2 - 2xz}{1 - 2xz - 4z^2};$$

$$J(z) = \sum_{n=0}^{\infty} J_n(x) z^n = \frac{z}{1 - z - xz^2}; \qquad j(z) = \sum_{n=0}^{\infty} j_n(x) z^n = \frac{2 - z}{1 - z - xz^2};$$

$$u(z) = \sum_{n=0}^{\infty} t_n z^n = \frac{z}{1 - x^2 z - xz^2 - z^3}; \quad k(z) = \sum_{n=0}^{\infty} k_n z^n = \frac{3 - 2x^2 z - xz^2}{1 - x^2 z - xz^2 - z^3}.$$

When $x = 1$, they yield the corresponding numbers, except for Jacobsthal and Jacobsthal-Lucas numbers; they are generated when $x = 2$.

Formula (1) can easily be established using the generating functions *u* and *k*. To see this, we have

$$kz = (3 - 2x^2 z - xz^2)u$$

$$\sum_{n-1}^{\infty} k_{n-1}z^n = 3\sum_{n=0}^{\infty} t_n z^n - 2x^2 \sum_{n=1}^{\infty} t_{n-1}z^n - x\sum_{n=2}^{\infty} t_{n-2}z^n .$$

Consequently,

$$k_n = 3t_{n+1} - 2x^2 t_n - xt_{n-1} .$$

This, coupled with the tribonacci recurrence $t_{n+1} = x^2 t_n + xt_{n-1} + t_{n-2}$, yields the desired relationship.

With this background, we are now ready for the discourse on tribonacci-gibonacci convolutions.

2. Convolutions of Order 2

2.1 Tribonacci-Fibonacci Convolutions: It follows from the generating functions $u = u(z)$ and $f = f(z)$ that

$$1 - x\left(1 - \frac{z}{f}\right) - z^3 = \frac{z}{u}$$

$$z(xu - f) = \left[(x-1) + z^3\right] fu . \tag{3}$$

Rewriting this in terms of power series, we get

$$\sum_{n=1}^{\infty} (xt_{n-1} - f_{n-1})z^n = (x-1)\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n} t_r f_{n-r}\right)z^n + \sum_{n=3}^{\infty}\left(\sum_{r=0}^{n-3} t_r f_{n-r-3}\right)z^n .$$

This implies

$$xt_n - f_n = (x-1)\sum_{r=0}^{n+1} t_r f_{n-r+1} + \sum_{r=0}^{n-2} t_r f_{n-r-2}. \tag{4}$$

Consequently, we have [4]

$$T_n - F_n = \sum_{r=0}^{n-2} T_r F_{n-r-2}. \tag{5}$$

It also follows from (4) that

$$2xt_n(2x) - p_n = (2x-1)\sum_{r=0}^{n+1} t_r(2x)p_{n-r+1} + \sum_{r=0}^{n-2} t_r(2x)p_{n-r-2};$$

$$2t_n(2) - P_n = \sum_{r=0}^{n+1} t_r(2)P_{n-r+1} + \sum_{r=0}^{n-2} t_r(2)P_{n-r-2}.$$

2.2 Tribonacci-Lucas Convolutions: Using the generating functions $u = u(z)$ and $l = l(z)$, we have

$$xz + z^2 = 1 - \frac{2 - xz}{l}$$

$$1 - \frac{x(l - 2 + xz)}{l} - z^2 = \frac{z}{u}$$

$$2xu - x^2zu - zl = [(x-1) + z^3]ul. \tag{6}$$

Translating this into power series, we get

$$2x\sum_{n=0}^{\infty} t_n z^n - x^2\sum_{n=1}^{\infty} t_{n-1}\, z^n - \sum_{n=1}^{\infty} l_{n-1}\, z^n = (x-1)\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n} t_r l_{n-r}\right)z^n + \sum_{n=3}^{\infty}\left(\sum_{r=0}^{n-3} t_r l_{n-r-3}\right)z^n;$$

$$2xt_n - x^2 t_{n-1} - l_{n-1} = (x-1)\sum_{r=0}^{n} t_r l_{n-r} + \sum_{r=0}^{n-3} t_r l_{n-r-3}. \tag{7}$$

In particular, we have

$$2T_n - T_{n-1} - L_{n-1} = \sum_{r=0}^{n-3} T_r L_{n-r-3} \ ; \tag{8}$$

$$4xt_n(2x) - 4x2t_{n-1}(2x) - q_{n-1} = (2x-1)\sum_{r=0}^{n} t_r(2x)q_{n-r} + \sum_{r=0}^{n-3} t_r(2x)q_{n-r-3};$$

$$2t_n(2) - 2t_{n-1}(2) - Q_{n-1} = \sum_{r=0}^{n} t_r(2)Q_{n-r} + \sum_{r=0}^{n-3} t_r(2)Q_{n-r-3}.$$

Formula (8) also appears in [4].

2.3 Tribonacci-Jacobsthal Convolutions: Using the generating functions $u = u(z)$ and $J = J(z)$, we have

$$1 - z - xz^2 = \frac{z}{J};$$

$$1 - x^2 z - xz^2 - z^3 = \frac{z}{u}.$$

Subtracting, we get

$$u - J = [(x^2-1) + z^2]uJ. \tag{9}$$

This yields

$$\sum_{n=0}^{\infty} [t_n(x) - J_n(x)]z^n = (x^2-1)\sum_{n=0}^{\infty}\left[\sum_{r=0}^{n} t_r(x)J_{n-r}(x)\right]z^n + \sum_{n=2}^{\infty}\left[\sum_{r=0}^{n-2} t_r(x)J_{n-r-2}(x)\right]z^n \ ;$$

$$t_n(x) - J_n(x) = (x^2-1)\sum_{r=0}^{n} t_r(x)J_{n-r}(x) + \sum_{r=0}^{n-2} t_r(x)J_{n-r-2}(x). \tag{10}$$

Clearly, formula (5) follows from this. When $x = 2$, formula (10) yields

$$t_n(2) - J_n = 3\sum_{r=0}^{n} t_r(2)J_{n-r} + \sum_{r=0}^{n-2} t_r(2)J_{n-r-2}.$$

2.4 Tribonacci–Jacobsthal-Lucas Convolutions: It follows from the generating functions $u = u(z)$ and $j = j(z)$ that

$$1 - z - xz^2 = \frac{2 - z}{J};$$

$$1 - x^2 z - xz^2 - z^3 = \frac{z}{u}.$$

Subtracting, we get

$$(2 - z)u - zj = [(x^2 - 1)z + z^3]uj. \tag{11}$$

Now replace the generating functions with their power series expansions:

$$2\sum_{n=0}^{\infty} t_n z^n - \sum_{n=1}^{\infty} t_{n-1}\, z^n - \sum_{n=1}^{\infty} j_{n-1}(x)\, z^n = (x^2 - 1)\sum_{n=1}^{\infty}\left[\sum_{r=0}^{n-1} t_r l_{n-r-1}(x)\right] z^n$$

$$+ \sum_{n=3}^{\infty}\left[\sum_{r=0}^{n-3} t_r l_{n-r-3}(x)\right] z^n$$

This gives the summation formula

$$2t_n - t_{n-1} - j_{n-1}(x) = (x^2 - 1)\sum_{r=0}^{n-1} t_r j_{n-r-1}(x) + \sum_{r=0}^{n-3} t_r j_{n-r-3}(x). \tag{12}$$

Clearly, this yields formula (8). In addition, we have

$$2t_n(2) - t_{n-1}(2) - j_{n-1} = 3\sum_{r=0}^{n-1} t_r(2) j_{n-r-1} + \sum_{r=0}^{n-3} t_r(2) j_{n-r-3}.$$

Next we investigate convolutions of order 2 involving tribonacci-Lucas polynomials and the extended gibonacci subfamilies. The steps involved remain basically the same; so, in the interest of brevity, we *omit* the basic algebra.

2.6 Tribonacci-Lucas–Fibonacci Convolutions: It follows by the generating functions $k = k(z)$ and $f = f(z)$ that

$$xzk - (3 - 2x^2z - xz^2)f = [(x-1) + z^3]kf. \tag{13}$$

This implies

$$x\sum_{n=0}^{\infty} k_{n-1}z^n - 3\sum_{n=0}^{\infty} f_n z^n + 2x^2\sum_{n=1}^{\infty} f_{n-1}z^n + x\sum_{n=2}^{\infty} f_{n-2}z^n = (x-1)\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n} k_r f_{n-r}\right)z^n$$

$$+ \sum_{n=3}^{\infty}\left(\sum_{r=0}^{n-3} k_r f_{n-r-3}\right)z^n .$$

Since, $2x^2 f_{n-1} + xf_{n-2} = xf_n + x^2 f_{n-1}$ and $P_n + Q_n = P_{n+1}$ [8, 10], we then have

$$xk_{n-1} + (x-3)f_n + x^2 f_{n-1} = (x-1)\sum_{r=0}^{n} k_r f_{n-r} + \sum_{r=0}^{n-3} k_r f_{n-r-3}; \tag{14}$$

$$2k_n(2) + P_n + Q_{n-1} = \sum_{r=0}^{n+1} k_r f_{n-r+1} + \sum_{r=0}^{n-2} k_r(2) f_{n-r-2}.$$

Since $2F_n - F_{n-1} = L_{n-1}$, it follows from formula (14) that [4]

$$K_n - L_n = \sum_{r=0}^{n-2} K_r F_{n-r-2}. \tag{15}$$

That is

$$T_n + 2T_{n-1} + 3T_{n-2} - L_n = \sum_{r=0}^{n-2} (T_r + 2T_{r-1} + 3T_{r-2})F_{n-r-2}.$$

2.6 Tribonacci-Lucas–Lucas Convolutions: Employing the generating functions $k = k(z)$ and $l = l(z)$, we get

$$(2x - x^2z)k - (3 - 2x^2z - xz^2)\,l = [(x-1) + z^3]kl. \tag{16}$$

This yields

$$2x\sum_{n=0}^{\infty} k_n z^n - x^2 \sum_{n=1}^{\infty} k_{n-1} z^n - 3\sum_{n=0}^{\infty} l_n z^n + 2x^2 \sum_{n=1}^{\infty} l_{n-1} z^n$$

$$+ x\sum_{n=2}^{\infty} l_{n-2} z^n = (x-1)\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n} k_r l_{n-4}\right) z^n$$

$$+ \sum_{n=3}^{\infty}\left(\sum_{r=0}^{n-3} k_r l_{n-r-3}\right) z^n.$$

Using the recurrence $l_n = x l_{n-1} + l_{n-2}$, this yields

$$2xk_n - x^2 k_{n-1} + (x-3) l_n + x^2 l_{n-1} = (x-1)\sum_{r=0}^{n} k_r l_{n-r} + \sum_{r=0}^{n-3} k_r l_{n-r-3}. \quad (17)$$

It follows from this equation that

$$2xk_n(2x) - 4x^2 k_{n-1}(2x) + (2x-3) q_n + 4x^2 q_{n-1} = (2x-1)\sum_{r=0}^{n} k_r(2x) q_{n-r} + \sum_{r=0}^{n-3} k_r(2x) q_{n-r-3}\ ;$$

$$k_n(2) - 2k_{n-1}(2) - Q_n + 4Q_{n-1} = \sum_{r=0}^{n} k_r(2) Q_{n-r} + \sum_{r=0}^{n-3} k_r(2) Q_{n-r-3}.$$

Since, $2L_n - L_{n-1} = 5F_{n-1}$, it also follows from formula (17) that [4]

$$2K_n - K_{n-1} - 5F_{n-1} = \sum_{r=0}^{n-3} K_r L_{n-r-3}. \quad (18)$$

This implies

$$2T_n + 3T_{n-1} + 4T_{n-2} - 3T_{n-3} - 5F_{n-1} = \sum_{r=0}^{n-3} (T_r + 2T_{r-1} + 3T_{r-2}) L_{n-r-3}.$$

2.7 Tribonacci-Lucas–Jacobsthal Convolutions: Using the generating functions $k = k(z)$ and $J = J(z)$, we get

$$zk - (3 - 2x^2 z - xz^2) J = [(x^2-1)z + z^3] kJ. \quad (19)$$

Consequently,

$$\sum_{n=1}^{\infty} k_{n-1} z^n - 3\sum_{n=0}^{\infty} J_n(x) z^n + 2x^2 \sum_{n=1}^{\infty} J_{n-1} z^n$$

$$+ x \sum_{n=2}^{\infty} J_{n-2}(x) z^n = (x^2-1) \sum_{n=1}^{\infty} \left[\sum_{r=0}^{n-1} k_r j_{n-r-1}(x) \right] z^n$$

$$+ \sum_{n=3}^{\infty} \left[\sum_{r=0}^{n-3} k_r j_{n-r-3}(x) \right] z^n.$$

This implies

$$k_{n-1} - 3J_n(x) + 2x^2 J_{n-1}(x) + xJ_{n-2}(x) = (x^2-1) \sum_{r=0}^{n-1} k_r J_{n-r-1}(x) + \sum_{r=0}^{n-3} k_r J_{n-r-3}(x). \tag{20}$$

Since, $3F_{n-2}F_{n-1} - F_{n-2} = L_{n-1}$, formula (18) follows from this. In addition, since $3J_n - 8J_{n-1} - 2J_{n-2} = 2J_n - 7J_{n-1}$, it also follows that

$$k_{n-1}(2) - 2J_n + 7J_{n-1} = 3\sum_{r=0}^{n-1} k_r(2) J_{n-r-1} + \sum_{r=0}^{n-3} k_r(2) J_{n-r-3}.$$

2.8 Tribonacci-Lucas–Jacobsthal-Lucas Convolutions: Using the generating functions $k = k(z)$ and $j = j(z)$, we get

$$(2-z)k - (3 - 2x^2 z - xz^2)j = [(x^2-1)z + z^3]kj. \tag{21}$$

It then follows that

$$2\sum_{n=0}^{\infty} k_n z^n - \sum_{n=1}^{\infty} k_{n-1} z^n - 3\sum_{n=0}^{\infty} j_n(x) z^n$$

$$+ 2x^2 \sum_{n=1}^{\infty} j_{n-1} z^n + x \sum_{n=2}^{\infty} j_{n-2}(x) z^n = (x^2-1) \sum_{n=1}^{\infty} \left[\sum_{r=0}^{n-1} k_r j_{n-r-1}(x) \right] z^n$$

$$+ \sum_{n=3}^{\infty} \left[\sum_{r=0}^{n-3} k_r j_{n-r-3}(x) \right] z^n;$$

$$2k_n - k_{n-1} - 3j_n(x) + 2x^2 j_{n-1}(x) + xj_{n-2}(x) = (x^2 - 1)\sum_{r=0}^{n-1} k_r j_{n-r-1}(x) + \sum_{r=0}^{n-3} k_r j_{n-r-3}(x).$$

When $x = 1$, this yields formula (18).

Since $3j_n - 8j_{n-1} - 2j_{n-2} = 2j_n - 7j_{n-1}$, it also follows that

$$2k_n(2) - k_{n-1}(2) - 2j_n + 7j_{n-1} = 2\sum_{r=0}^{n-1} k_r(2) j_{n-r-1} + \sum_{r=0}^{n-3} k_r j_{n-r-3}.$$

Finally, we explore convolutions of order 2 involving tribonacci and tribonacci-Lucas polynomials.

2.9 Tribonacci–Tribonacci-Lucas Convolutions: It follows from the generating functions u and k that $kz = (3 - 2x^2z - xz^2)u$, so

$$(uk)z = (3 - 2x^2z - xz^2)u^2$$

$$\sum_{n=1}^{\infty}\left(\sum_{r=0}^{n-1} t_r k_{n-r-1}\right)z^n = 3\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n} t_r k_{n-r}\right)z^n - 2x^2\sum_{n=1}^{\infty}\left(\sum_{r=0}^{n-1} t_r k_{n-r-1}\right)z^n$$
$$- x\sum_{n=2}^{\infty}\left(\sum_{r=0}^{n-2} t_r k_{n-r-2}\right)z^n.$$

Consequently, we have

$$\sum_{r=0}^{n-1} t_r k_{n-r-1} = 3\sum_{r=0}^{n} t_r t_{n-r} - 2x^2\sum_{r=0}^{n-1} t_r t_{n-r-1} - x\sum_{r=0}^{n-2} t_r t_{n-r-2};$$

$$\sum_{r=0}^{n-1} T_r K_{n-r-1} = 3\sum_{r=0}^{n} T_r T_{n-r} - 2\sum_{r=0}^{n-1} T_r T_{n-r-1} - \sum_{r=0}^{n-2} T_r T_{n-r-2}.$$

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ADDITION FORMULAS FOR EXTENDED TRIBONACCI POLYNOMIALS WITH GRAPH-THEORETIC CONFIRMATIONS

Abstract: We explore the addition formulas for tribonacci and tribonacci-Lucas polynomials, and confirm them using graph-theoretic tools. We also investigate a special case for each family, and the corresponding summation formulas and their implications.

Keywords: Extended Tribonacci Polynomials, Extended Tribonacci-Lucas Polynomials, Fibonacci Polynomials, Graph Theoretic Confirmation.

Mathematical Subject Classification (2020) No.: Primary 11B37, 11B39, 11B83, 11C08.

1. Introduction

Extended tribonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+3}(x) = x^2 z_{n+2}(x) + x z_{n+1}(x) + z_n(x)$, where x is an arbitrary complex variable; $z_0(x)$, $z_1(x)$, and $z_2(x)$ are arbitrary complex polynomials; and $n \geq 0$. They can be extended to negative subscripts.

When $z_0(x) = 0$, $z_1(x) = 1$, and $z_2(x) = x^2$, $z_n(x) = t_n(x)$, the nth *tribonacci polynomial*; and when $z_0(x) = 3$, $z_1(x) = x^2$, and $z_2(x) = x^4 + 2x$, $z_n(x) = k_n(x)$, the nth *tribonacci-Lucas polynomial*. Tribonacci polynomials were originally studied by Hoggatt and Bicknell [3, 7], and tribonacci-Lucas polynomials by Yilmaz and Taskara [10, 4].

Tribonacci numbers $T_n = t_n(1)$ were originally studied by Feinberg in 1963 when he was a 14-year old ninth-grader [2, 7], and *tribonacci-Lucas numbers* $K_n = k_n(1)$ by Yilmaz and Taskara [10, 4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is no ambiguity; so z_n will mean $z_n(x)$. We also omit a lot of basic algebra.

The Q-matrix for Fibonacci polynomials [7] has a tribonacci counterpart [3]:

$$Q = \begin{bmatrix} x^2 & 1 & 0 \\ x & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

where $Q = Q(x)$. It then follows by induction that

$$Q^n = \begin{bmatrix} t_{n+1} & t_n & t_{n-1} \\ xt_n + t_{n-1} & xt_{n-1} + t_{n-2} & xt_{n-2} + t_{n-3} \\ t_n & t_{n-1} & t_{n-2} \end{bmatrix},$$

where $n \ge 1$; it can be used to generate tribonacci polynominals [3].

1.1 A Link Between Tribonacci and Tribonacci-Lucas Polynomials: The polynomials t_n and k_n are closely linked by the relationship

$$k_n = x^2 t_n + 2xt_{n-1} + 3t_{n-2}. \tag{1}$$

Although it follows by induction [4], it can easily be established using the generating functions for t_n and k_n [8].

Consequently,

$$\begin{aligned} K_n &= T_n + 2T_{n-1} + 3T_{n-2} \\ &= T_{n+1} + T_{n-1} + 2T_{n-2}. \end{aligned}$$

It follows by tribonacci recurrence and equation (1) that

$$\begin{aligned} k_n &= \left(x^2 t_n + x t_{n-1} + t_{n-2}\right) + x t_{n-1} + 2 t_{n-2} \\ &= t_{n+1} + x t_{n-1} + 2 t_{n-2} \\ &= \text{trace of } Q_n. \end{aligned}$$

Next we briefly cite combinatorial models for t_n and k_n.

2. Combinatorial Models

Suppose we would like to tile an $n \times 1$ board with $1 \times 1, 2 \times 1$, and 3×1 tiles. We call such a board an *n-board*, and such tiles as squares, dominoes, and *triminoes*. The *weight* of a square is x^2, that of a domino is x, and that of a trimino is 1. The *weight of a tiling* is the product of the weights of the tiles in the tiling. The weight of the *empty tiling* is defined as 1. The sum of weights of such tilings of an n-board is t_{n+1}, where $n \geq 0$ [6, 7]; there are T_{n+1} such tilings of an n-board.

Consequently, there are T_{n+1} compositions of a positive integer n using the summands 1, 2, and 3 [6, 7].

Using equation (1), we can now interpret k_n combinatorially:

$$\begin{aligned} k_{n+1} = x^2 &\begin{pmatrix} \text{sum of the weights of} \\ \text{tilings of an n} - \text{board} \end{pmatrix} + 2x \begin{pmatrix} \text{sum of the weights of} \\ \text{tilings of an } (n-1)\text{-board} \end{pmatrix} \\ &+ 3 \begin{pmatrix} \text{sum of the weights of} \\ \text{tilings of an } (n-2)\text{-board} \end{pmatrix}. \end{aligned}$$

3. Addition Formulas

As in the case of Fibonacci and Lucas polynomials, tribonacci and tribonacci-Lucas polynomials satisfy analogous addition formulas:

$$t_{m+n} = t_{m+1} t_n + x t_m t_{n-1} + t_m t_{n-2} + t_{m-1} t_{n-1}; \tag{2}$$

$$k_{m+n} = t_{m+1} k_n + x t_m k_{n-1} + t_m k_{n-2} + t_{m-1} k_{n-1}. \tag{3}$$

Using the concept of *breakability* [1, 7] in tilings, we can establish formula (2) [6, 7]. Formula (2), coupled with formula (1), yields formula (3); we omit the basic algebra for brevity.

Next we establish both addition formulas using graph-theoretic techniques.

4. Graph-theoretic Models

Consider the *weighted tribonacci digraph* D with vertices v_1, v_2, and v_3 in Figure 4, where a *weight* is assigned to each edge [6, 7]. Clearly, its *weighted adjacency matrix* is the Q-matrix.

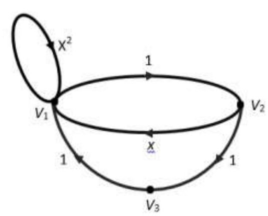

Figure 1: Tribonacci Digraph

A *walk* from vertex v_i to vertex v_j is a sequence $v_i - e_i - v_{i+1} - \cdots - v_{j-1} - e_{j-1} - v_j$ of vertices v_k and edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The walk is *closed* if $v_i = v_j$; otherwise, it is *open*. The *length* of a walk is the number of edges in the walk. The *weight* of a walk is the product of the weights of the edges along the walk.

We can employ the matrix Q^n to compute the weight of a walk of length n from any vertex v_i to any vertex v_j, as the following theorem shows [5, 6].

Theorem 1: *Let M be the weighted adjacency matrix of a weighted, connected digraph with vertices $v_1, v_2, \ldots, v_k$. Then the ijth entry of the matrix M^n gives the sum of the weights of all walks of length n from v_i to v_j, where $n \geq 1$.* □

The next result follows from this theorem.

Corollary 1: *The ijth entry of Q^n gives the sum of the weights of all walks of length n from v_i to v_j in the tribonacci digraph D, where $1 \le i, j \le n$.* □

It follows by the corollary that the sum of the weights of all closed walks of length n originating at v_1 is t_{n+1}; that from v_1 to v_2 is t_n; that from v_2 to v_1 is $xt_n + t_{n-1}$; and that from v_3 to v_1 is t_n. These results play a pivotal role in the confirmation proofs.

Proof of Formula (2): Let A be the set of closed walks of length $m + n - 1$ originating at v_1. The sum S of their weights is given by $S = t_{m+n}$.

We will now compute S in a different way. To this end, let w be an arbitrary walk in A.

Case 1: Suppose w lands at v_1 after m steps: $\underbrace{\text{subwalk from } v_1 \text{ to } v_1}_{\text{length } m}$ $\underbrace{\text{subwalk from } v_1 \text{ to } v_1}_{\text{length } n-1}$. The sum of the weights such walks is $t_{m+1}t_n$.

Case 2: Suppose w lands at v_2 after m steps: $\underbrace{\text{subwalk from } v_1 \text{ to } v_2}_{\text{length } m}$ $\underbrace{\text{subwalk from } v_2 \text{ to } v_1}_{\text{length } n-1}$. The sum of the weights such walks is $t_m(xt_{n-1} + t_{n-2})$.

Case 3: Suppose w lands at v_3 after m steps: $\underbrace{\text{subwalk from } v_1 \text{ to } v_3}_{\text{length } m}$ $\underbrace{\text{subwalk from } v_3 \text{ to } v_1}_{\text{length } n-1}$. The sum of the weights such walks is $t_{m-1}t_{n-1}$.

The cumulative sum S is then given by

$$S = t_{m+1}t_n + t_m(xt_{n-1} + t_{n-2}) + t_{m-1}t_{n-1}.$$

This, coupled with the earlier value of S, yields the desired result. □

Proof of Formula (3): The sum S_n of the weights of closed walks of length n originating at v_1 is given by $S_n = t_{n+1}$. It then follows by equation (1) that

$$k_{m+n} = x^2\begin{pmatrix}\text{sum of the weights of}\\ \text{closed walks of length}\\ m+n-1 \text{ originating at } v_1\end{pmatrix} + 2x\begin{pmatrix}\text{sum of the weights of}\\ \text{closed walks of length}\\ m+n-2 \text{ originating at } v_1\end{pmatrix}$$

$$+3\begin{pmatrix}\text{sum of the weights of}\\ \text{closed walks of length}\\ m+n-3 \text{ originating at } v_1\end{pmatrix}$$

$$= x^2S_{m+n-1} + 2xS_{m+n-2} + 3S_{m+n-3}. \tag{4}$$

We will now compute the sum $S^* = x^2S_{m+n-1} + 2xS_{m+n-2} + 3S_{m+n-3}$ in a different way. Let A, B, and C denote the sets of closed walks of length $m+n-1$, $m+n-2$, and $m+n-3$, all originating at v_1, respectively. Let w be an arbitrary walk in A.

Case 1: Suppose w begins with a loop: $\underbrace{\text{subwalk } v_1v_1}_{\text{length } 1}$ $\underbrace{\text{subwalk from } v_1 \text{ to } v_1}_{\text{length } m+n-2}$. The sum of the weights such walks is x^2t_{m+n-1}.

Case 2: Suppose w does not begin with a loop. If w lands at v_2 and returns to v_1, then w is of the form $\underbrace{\text{subwalk } v_1v_2v_1}_{\text{length } 2}$ $\underbrace{\text{subwalk from } v_1 \text{ to } v_1}_{\text{length } m+n-3}$. The sum of the weights such walks equals xt_{m+n-2}.

On the other hand, if w lands at v_2 and then at v_3, and returns home, then w is of the form $\underbrace{\text{subwalk } v_1v_2v_3v_1}_{\text{length } 3}\ \underbrace{\text{subwalk from } v_1 \text{ to } v_1}_{\text{length } m+n-4}$. The sum of the weights such walks equals t_{m+n-3}.

Combining the two cases, we get

$$\begin{aligned} S_{m+n-1} &= x^2 t_{m+n-1} + (xt_{m+n-2} + t_{m+n-3}) \\ &= t_{m+n}. \end{aligned}$$

This implies, $S_{m+n-2} = t_{m+n-1}$ and $S_{m+n-3} = t_{m+n-2}$. Thus,

$$\begin{aligned} S^* &= x^2 S_{m+n-1} + 2xS_{m+n-2} + 3S_{m+n-3} \\ &= x^2 t_{m+n} + 2xt_{m+n-1} + 3t_{m+n-2} \\ &= x^2(t_{m+1}t_n + xt_m t_{n-1} + t_m t_{n-2} + t_{m-1}t_{n-1}) \\ &\qquad + 2x(t_{m+1}t_n + xt_{m+1}t_{n-1} + t_m t_{n-3} + t_{m-1}t_{n-2}) \\ &\qquad + 3(t_{m+1}t_{n-2} + xt_m t_{n-3} + t_m t_{n-4} + t_{m-1}t_{n-3}) \\ &= t_{m+1}(x^2 t_n + 2xt_{n-1} + 3t_{n-2}) + xt_m(x^2 t_{n-1} + 2xt_{n-2} + 3t_{n-3}) \\ &\qquad + t_m(x^2 t_{n-2} + 2xt_{n-3} + 3t_{n-4}) + t_{m-1}(x^2 t_{n-1} + 2xt_{n-2} + 3t_{n-3}) \\ &= t_{m+1}k_n + xt_m k_{n-1} + t_m k_{n-2} + t_{m-1}k_{n-1}. \end{aligned} \tag{5}$$

Equating the two values of S^* in equations (4) and (5) yields the desired result. [As a byproduct, the above equations give an algebraic proof of formula (3)]. □

In particular, we have

$$K_{m+n} = T_{m+1}K_n + T_m K_{n-1} + T_m K_{n-2} + T_{m-1}K_{n-1}. \tag{6}$$

Next we develop the *Binet-like formulas* for t_n and k_n.

5. Binet-like Formulas

Let $(\alpha,\ \beta,\ \text{and}\ \gamma)$ be the solutions of the characteristic equation $z^3 = x^2z^2 + xz + 1$, and λ and an arbitrary solution. One of them is real and the other two are complex conjugates of each other.

Clearly,

$$\alpha + \beta + \gamma = x^2$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = -x$$

$$\alpha\beta\gamma = 1.$$

The general solution of the tribonacci recurrence is $t_n = A\alpha^n + B\beta^n + C\gamma^n$, where the unknowns A, B, and C can be determined using the initial conditions $t_0 = 0$, $t_1 = 1$, and $t_2 = x^2$. Omitting a lot of basic algebra, we get the *Binet-like formula*

$$t_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}.$$

In particular

$$T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}. \tag{7}$$

where [4]

$$\alpha = \alpha(1) = \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3};$$

$$\beta = \beta(1) = \frac{1 + \omega\sqrt[3]{19 + 3\sqrt{33}} + \omega^2\sqrt[3]{19 - 3\sqrt{33}}}{3};$$

$$\gamma = \gamma(1) = \frac{1 + \omega^2 \sqrt[3]{19 + 3\sqrt{33}} + \omega^3 \sqrt[3]{19 - 3\sqrt{33}}}{3};$$

$$\omega = \frac{-1 + \sqrt{3}i}{2}.$$

Let $\mu = \lambda(1)$. Since $\mu^2 + \frac{1}{\mu^2} = 2\mu$, we have

$$\frac{(\alpha - \beta)(\alpha - \gamma)}{\alpha} = -\alpha^2 + 4\alpha - 1;$$

$$\frac{(\beta - \alpha)(\beta - \gamma)}{\beta} = -\beta^2 + 4\beta - 1;$$

$$\frac{(\gamma - \alpha)(\gamma - \beta)}{\gamma} = -\gamma^2 + 4\gamma - 1,$$

where $\alpha = \alpha(1)$, $\beta = \beta(1)$, and $\gamma = \gamma(1)$. Consequently, we can rewrite formula (7) in a slightly different form [9]:

$$T_n = \frac{\alpha^n}{-\alpha^2 + 4\alpha - 1} + \frac{\beta^n}{-\beta^2 + 4\beta - 1} + \frac{\gamma^n}{-\gamma^2 + 4\gamma - 1}. \tag{8}$$

5.1 A Formula for t_{3n}: Formula (8) has an interesting byproduct. It can be used to express t_{3n} as a sum of three binomial sums. To this end, let λ be an arbitrary solution of $z^3 = x^2 z^2 + xz + 1$. We then have [4]

$$\lambda^{3n} = [(\lambda^3 - 1) + 1]^n$$

$$= \sum_{i=0}^{n} \binom{n}{i} (\lambda^3 - 1)^i$$

$$= \sum_{i=0}^{n} \binom{n}{i} (x^2\lambda^3 - x\lambda)^i$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} (x\lambda)^{i+j}.$$

Using the Binet-like formula for t_n, this yields

$$t_{3n} = \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} x^{i+j} t_{i+j}.$$

Let

$$S_\mu = \frac{\mu^{3n}}{-\mu^2 + 4\mu - 1}$$

$$= \frac{1}{-\mu^2 + 4\mu - 1} \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} \mu^{i+j}.$$

Then $$T_{3n} = S_\alpha + S_\beta + S_\gamma.$$

5.2 An Alternate Formula for t_{3n}: Addition formula (2) also can be employed to develop a formula for t_{3n}. We have

$$t_{2n-1} = t_n^2 + x t_{n-1}^2 + 2t_{n-1}t_{n-2};$$

$$t_{2n} = t_{n+1}t_n + x t_n t_{n-1} + t_n t_{n-2} + t_{n-1}^2$$

$$t_{2n+1} = t_{n+1}^2 + x t_n^2 + 2t_n t_{n-1}.$$

Then

$$t_{3n} = t_{n+1}^2 t_n + x t_{2n} t_{n-1} + t_{2n} t_{n-2} + t_{2n-1} t_{n-1}$$

$$= t_{n+1}^2 t_n + x t_{n+1} t_n t_{n-1} + t_{n+1} t_n t_{n-2} + x t_n^3 + 3t_n^2 t_{n-1}$$

$$+ 2x t_n t_{n-1} t_{n-2} + x^2 t_n t_{n-1}^2 + t_n t_{n-2}^2 + 3t_{n-1}^2 t_{n-2} + 2t_{n-1}^3.$$

This implies

$$T_{3n} = T_{n+1}^2 T_n + T_{n+1}T_nT_{n-1} + T_{n+1}T_nT_{n-2} + T_n^3 + 3T_n^2T_{n-1}$$
$$+ 2T_nT_{n-1}T_{n-2} + T_nT_{n-1}^2 + T_nT_{n-2}^2 + 3T_{n-1}^2T_{n-2} + 2T_{n-1}^3.$$

5.3 A Summation Formula For k_{3n}: Using the *Binet-like formula* $k_n = A\alpha^n + B\beta^n + C\gamma^n$ and equation (9), it follows that

$$k_{3n} = \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}x^{i+j}k_{i+j},$$

where A, B, and C are rational functions and can be found using the initial conditions $k_0 = 3$, $k_1 = x^2$, and $k^2 = x^4 + 2x$ [4].

Using formula (3), we then have

$$t_{2n+1}k_n + xt_{2n}k_{n-1} + t_{2n}k_{n-2} + t_{2n-1}k_{n-1} = k_{3n}$$
$$= \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}k_{i+j}x^{i+j}.$$

Consequently, we have

$$T_{3n} + 2T_{3n-1} + 3T_{3n-2} = T_{3n+1} + T_{3n-1} + 2T_{3n-2}$$
$$= T_{2n+1}K_n + T_{2n}K_{n-1} + T_{2n}K_{n-2} + T_{2n-1}K_{n-1}$$
$$= K_{3n}$$
$$= \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}k_{i+j}.$$

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JABIONOMIAL COEFFICIENTS WITH DIVIDENDS

Abstract: We explore a new class of polynomial functions, called *jabinomial coefficients*, and develop some properties of these functions and then deduce their numeric counterparts.

Keywords: Jabinomial Coefficients, Fibonacci Lucas Polynomials, Jacobsthal-Lucas Polynomials.

Mathematical Subject Classification (2020) No.: 11B37, 11B39, 11B83, 11C08.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary complex variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary complex polynomials; and $n \geq 0$. Obviously, the definition can be extended to negative subscripts. [2, 3].

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. In particular, $f_n(1) = F_n$ and $l_n(1) = L_n$ are the nth Fibonacci and Lucas numbers, respectively.

On the other hand, let $a(x) = 1$ and $b(x) = x$. Suppose $z_0(x) = 0$ and $z_1(x) = 1$. Then $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and if $z_0(x) = 2$ and $z_1(x) = 1$. Then $z_n(x) = l_n(x)$, the nth *Jacobsthal-Lucas polynomial*. Their numeric counterparts are given by $J_n = J_n(2)$ and $j_n = j_n(2)$ [2, 3, 4].

Fibonacci and Lucas polynomials, and Jacobsthal and Jacobsthal-Lucas polynomials are linked by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(x)$ [2, 3, 4].

In the interest of brevity, clarity, and convenience, we often omit the argument in the functional notation; so z_n will mean $z_n(x)$.

2. Jabinomial Coefficients

Gibonomial coefficients $\left[\left[\begin{matrix} n \\ r \end{matrix}\right]\right]_f$, studied in [3] (notice the change in the notation), are defined by $\left[\left[\begin{matrix} n \\ r \end{matrix}\right]\right]_f = \dfrac{f_n^*}{f_r^* f_{n-r}^*}$,

where $f_k^* = f_k f_{k-1} \dots f_2 f_1$, $f_0^* = 1$, and $0 \le r \le n$. Correspondingly, we define the *jabinomial coefficients* (*Ja*cobsthal *binomial coefficients*) $\left[\left[\begin{matrix} n \\ r \end{matrix}\right]\right]_J$, by

$$\left[\left[\begin{matrix} n \\ r \end{matrix}\right]\right]_J = \frac{f_n^*(x)}{f_r^*(x) f_{n-r}^*(x)}, \tag{1}$$

where $J_n^*(x) = J_n(x) J_{n-1}(x) \dots J_2(x) J_1(x)$, $J_0^* * (x) = 1$, and $0 \le r \le n$.

Since $J_n(x) = x^{(n-1)/2} f_n(u)$, it follows that

$$J_n^*(x) = J_n(x) J_{n-1}(x) \dots J_2(x) J_1(x),$$

$$= \left[x^{(n-1)/2} f_n(u)\right]\left[x^{(n-2)/2} f_{n-1}(u)\right] \cdots \left[x^{(2-1)/2} f_2(u)\right]\left[x^{(1-1)/2} f_1(u)\right]$$

$$= x^{n(n-1)/4} f_n^*(u), \tag{2}$$

where $u = 1/\sqrt{x}$.

In particular, we then have

$$J_n^* = 2^{n(n-1)/4} f_n^*(1/\sqrt{2}).$$

For example, $2^5 f_5^*(1/\sqrt{2}) = 165 = J_5^*$.

It follows from equations (1) and (2) that

$$\left[\left[\begin{matrix} n \\ n-r \end{matrix}\right]\right]_J = \frac{x^{n(n-1)/4} f_n^*(u)}{x^{r(r-1)/4} f_r^*(u) x^{(n-r)(n-r-1)/4} f_{n-r}^*(u)}$$

$$= x^{n(n-r)r/2} \cdot \frac{f_n^*(u)}{f_r^*(u) f_{n-r}^*(u)}$$

$$= x^{(n-r)r/2} \left[\left[\begin{matrix} n \\ r \end{matrix}\right]\right]_f, \tag{3}$$

where $f_n = f_n(u)$.

Suppose $\left[\left[\begin{matrix} n \\ n-r \end{matrix}\right]\right]_J (2)$ denotes the value of the jabinomial coefficient at $x = 2$. Then

$$\left[\left[\begin{matrix} n \\ n-r \end{matrix}\right]\right]_J (2) = 2^{(n-r)r/2} \left[\left[\begin{matrix} n \\ r \end{matrix}\right]\right]_f (1/\sqrt{2}),$$

For example,

$$2^{(7-3)3/2}\left[\left[\begin{matrix}7\\3\end{matrix}\right]\right]_f(1/\sqrt{2}) = 2^6 \cdot \frac{f_7(1/\sqrt{2})\, f_6(1/\sqrt{2})\, f_5(1/\sqrt{2})}{f_3^*(1/\sqrt{2})},$$

$$= 3,311$$

$$= \left[\left[\begin{matrix}7\\3\end{matrix}\right]\right]_J(2).$$

It follows from equation (1) that $\left[\left[\begin{matrix}n\\r\end{matrix}\right]\right]_J = \left[\left[\begin{matrix}n\\n-r\end{matrix}\right]\right]_J$ and hence $\left[\left[\begin{matrix}n\\0\end{matrix}\right]\right]_J = \left[\left[\begin{matrix}n\\n\end{matrix}\right]\right]_J = 1$; and from equation (3) that $\left[\left[\begin{matrix}n\\r\end{matrix}\right]\right]_J(1) = \left[\left[\begin{matrix}n\\r\end{matrix}\right]\right]_f(1) = \frac{F_n^*}{F_r^* F_{n-r}^*}$.

Next we present a graph-theoretic interpretation of jabinomial coefficients.

2.1 Graph-theoretic Interpretation: Consider the weighted digraph D_1 in Figure 1 with vertices v_1 and v_2 [5]. The sum of the weights of closed walks of length n originating at v_1 is $J_{n+1}(x)$, and that of those originating at v_2 is $xJ_{n-1}(x)$. Consequently, the sum of the weights of all closed walks of length n in D_1 is $J_{n+1(x)} + xJ_{n-1}(x) = j_{n(x)}$ [4].

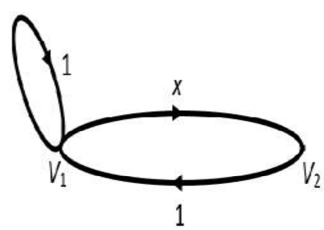

Figure 1: Weighted Jacobsthal Digraph D_1

Let W_n denote the product of the sums of weights w_k of closed walks of length k originating at v_1, where $1 \le k \le n$. Then

$$W_{n-1} = \prod_{k=1}^{n-1} w_k = \prod_{k=1}^{n-1} J_{k+1}(x) = J_n^*(x).$$

Thus,

$$\left[\!\left[\begin{matrix} n \\ r \end{matrix}\right]\!\right]_J = \frac{W_{n-1}}{W_{r-1}W_{n-r-1}}.$$

(We can employ the same technique to give a graph-theoretic interpretation of gibonomial coefficients using the weighted Fibonacci digraph D_2 in Figure 2 [5].)

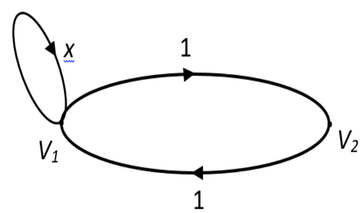

Figure 2: Weighted Fibonacci Digraph D_2

2.2 Jabinomial Recurrences: Jabinomial coefficients satisfy gibinomial-like recurrences:

$$\left[\!\left[\begin{matrix} n \\ r \end{matrix}\right]\!\right]_J = \left[\!\left[\begin{matrix} n-1 \\ r \end{matrix}\right]\!\right]_J J_{r+1} + x\left[\!\left[\begin{matrix} n-1 \\ r-1 \end{matrix}\right]\!\right]_J J_{n-r-1}; \tag{4}$$

$$= \left[\!\left[\begin{matrix} n-1 \\ r \end{matrix}\right]\!\right]_J xJ_{r-1} + \left[\!\left[\begin{matrix} n-1 \\ r-1 \end{matrix}\right]\!\right]_J J_{n-r+1}, \tag{5}$$

where $J_n = J_n(x)$.

We can establish these recurrences using the addition formula [4]

$$J_{m+n}(x) = J_{m+1}(x)J_n(x) + xJ_m(x)J_{n-1}(x).$$

For example,

$$\begin{aligned}
\left[\!\!\left[\begin{matrix} n \\ r \end{matrix}\right]\!\!\right]_J &= \frac{J^*_{n-1}J_n}{J^*_r J^*_{n-r}} \\
&= \frac{J^*_{n-1}}{J^*_r J^*_{n-r}}\left(J_{r+1}J_{n-r} + xJ_rJ_{n-r-1}\right) \\
&= \frac{J^*_{n-1}}{J^*_r J^*_{n-r-1}} J_{r+1} + x\frac{J^*_{n-1}}{J^*_{r-1} J^*_{n-r}} J_{n-r-1} \\
&= \left[\!\!\left[\begin{matrix} n-1 \\ r \end{matrix}\right]\!\!\right]_J J_{r+1} + x\left[\!\!\left[\begin{matrix} n-1 \\ r-1 \end{matrix}\right]\!\!\right]_J J_{n-r-1},
\end{aligned}$$

where $J_n = J_n(x)$. Figure 3 shows a pictorial representation of this recurrence.

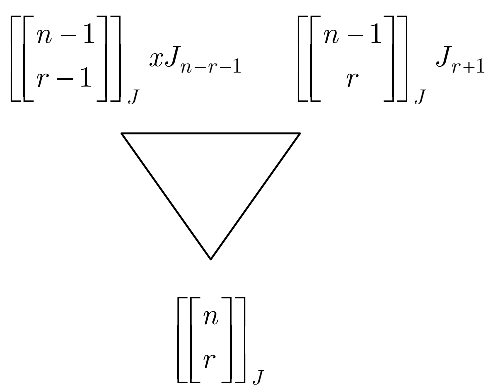

Figure 3

Since $J_{n+1}(x) + xJ_{n-1}(x) = j_n(x)$, it follows by recurrences (4) and (5) that

$$2\left[\!\!\left[\begin{matrix} n \\ r \end{matrix}\right]\!\!\right]_J = \left[\!\!\left[\begin{matrix} n-1 \\ r \end{matrix}\right]\!\!\right]_J (J_{r+1} + xJ_{r-1})\left[\!\!\left[\begin{matrix} n-1 \\ r-1 \end{matrix}\right]\!\!\right]_J (J_{n-r+1} + xJ_{n-r-1})$$

$$= \left[\left[\begin{matrix} n-1 \\ r \end{matrix}\right]\right]_J J_r + \left[\left[\begin{matrix} n-1 \\ r-1 \end{matrix}\right]\right]_J J_{n-r},$$

where $J_n = J_n(x)$ and $j_n = j_n(x)$. Consequently,

$$2\left[\left[\begin{matrix} n \\ r \end{matrix}\right]\right]_J (2) = \left[\left[\begin{matrix} n-1 \\ r \end{matrix}\right]\right]_J (2)J_r + \left[\left[\begin{matrix} n-1 \\ r-1 \end{matrix}\right]\right]_J (2)J_{n-r},$$

where $j_n = j_n(2)$.

For example,

$$\left[\left[\begin{matrix} 7 \\ 5 \end{matrix}\right]\right]_J (2)j_5 + \left[\left[\begin{matrix} 7 \\ 4 \end{matrix}\right]\right]_J (2)J_3 = 51,170 = 2\left[\left[\begin{matrix} 8 \\ 5 \end{matrix}\right]\right]_J (2).$$

Recurrence (4), coupled with the initial conditions $\left[\left[\begin{matrix} 0 \\ 0 \end{matrix}\right]\right] = 1 = \left[\left[\begin{matrix} 1 \\ 0 \end{matrix}\right]\right] = \left[\left[\begin{matrix} 1 \\ 1 \end{matrix}\right]\right]$, implies that every jabinomial coefficient is an integer-valued polynomial.

2.3 Jabinomial Polynomial Array: Both recurrences can used to construct a Pascal-like triangular array for jabinomial coefficients, as in Figure 4. Figure 5 shows its numeric version.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 1 & & 1 & & \\ & 1 & & x+1 & & x+1 & & 1 & \\ 1 & & 2x+1 & & 2x^2+3x+1 & & 2x+1 & & 1 \end{array}$$

Figure 4: Jabinomial Polynomial Array

1

1 1

1 1 1

1 3 3 1

1 5 15 5 1

1 11 55 55 11 1

Figure 5: Jabinomial Numeric Array

Next we explore the central jabinomial coefficients.

2.4 Central Jabinomial Coefficients: The *central jabinomial coefficients* $\left[\!\left[\begin{matrix} 2n \\ n \end{matrix} \right]\!\right]_J$ satisfy an intriguing property; see the circled numbers in Figure 5. By recurrence (4), we have

$$\left[\!\left[\begin{matrix} 2n \\ n \end{matrix} \right]\!\right]_J = \left[\!\left[\begin{matrix} 2n-1 \\ n \end{matrix} \right]\!\right]_J \left[J_{n+1}(x) + xJ_{n-1}(x)\right]$$

$$= \left[\!\left[\begin{matrix} 2n-1 \\ n \end{matrix} \right]\!\right]_J j_n(x).$$

This property has an interesting byproduct. Since

$$\left[\!\left[\begin{matrix} 2n \\ n \end{matrix} \right]\!\right]_J = \left[\!\left[\begin{matrix} 2n-1 \\ n \end{matrix} \right]\!\right]_J \cdot \frac{J_{2n}(x)}{J_n(x)},$$

it follows that $J_{2n}(x) = J_n(x) j_n(x)$ and hence $J_{2n} = J_n j_n$ [4].

2.5 Star of David Property: Like gibonomial coefficients, jabionomial coefficients also satisfy the *Star of David property*:

$$\left[\!\left[\begin{matrix} n-1 \\ r-1 \end{matrix}\right]\!\right]_J \left[\!\left[\begin{matrix} n \\ r+1 \end{matrix}\right]\!\right]_J \left[\!\left[\begin{matrix} n+1 \\ r \end{matrix}\right]\!\right]_J = \left[\!\left[\begin{matrix} n-1 \\ r \end{matrix}\right]\!\right]_J \left[\!\left[\begin{matrix} n+1 \\ r+1 \end{matrix}\right]\!\right]_J \left[\!\left[\begin{matrix} n \\ r-1 \end{matrix}\right]\!\right]_J ;$$

see Figure 6.

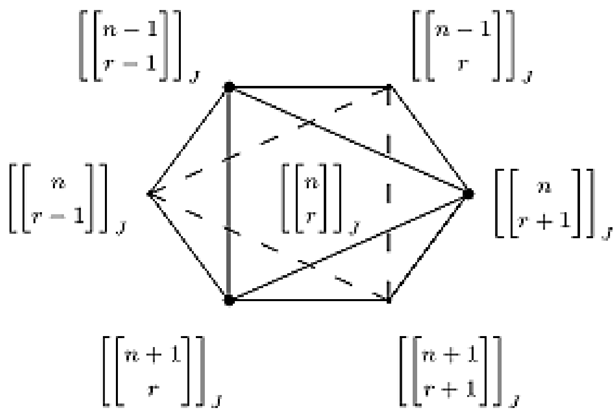

Figure 6

We can establish this property also algebraically:

$$\text{LHS} = \frac{J^*_{n-1}}{J^*_{r-1}J^*_{n-r}} \cdot \frac{J^*_{n}}{J^*_{r+1}J^*_{n-r-1}} \cdot \frac{J^*_{n+1}}{J^*_{r}J^*_{n-r+1}}$$

$$= \frac{J^*_{n-1}}{J^*_{r}J^*_{n-r-1}} \cdot \frac{J^*_{n+1}}{J^*_{r+1}J^*_{n-r}} \cdot \frac{J^*_{n}}{J^*_{r-1}J^*_{n-r+1}}$$

$$= \left[\!\left[\begin{matrix} n-1 \\ r \end{matrix}\right]\!\right]_J \left[\!\left[\begin{matrix} n+1 \\ r+1 \end{matrix}\right]\!\right]_J \left[\!\left[\begin{matrix} n \\ r-1 \end{matrix}\right]\!\right]_J$$

$$= \text{RHS}.$$

Next we present the occurrences of jabinomial coefficients in two different contexts and some special cases.

3. Jabinomial Occurrences

3.1 Product of Jacobsthal Recurrences: Jabinomial coefficients occur in the characteristic equation of the product of n polynomial recurrences $z_{n+2} = z_{n+1} + xz_n$:

$$\sum_{r=0}^{n+1} (-1)^{t_r} x^{t_{r-1}} = \left[\!\!\left[\begin{matrix} n+1 \\ r \end{matrix} \right]\!\!\right]_J z^{n-r+1} = 0,$$

where t_n denotes the nth *triangular number* $n(n+1)/2$. This follows by letting $h = -x$, $n = 0$, $j = r$, and $k = n+1$ in equation (3) in [7].

When $n = 1, 2, 3, 4$, and 5, it yields the following characteristic equations:

$$z^2 - z - x = 0;$$

$$z^3 - (x+1)z^2 - x(x+1)z + x^3 = 0;$$

$$z^4 - (2x+1)z^3 - x(x+1)(2x+1)z^2 + x^3(2x+1)z + x^6 = 0;$$

$$z^5 - (x^2+3x+1)z^4 - x(2x+1)(x^2+3x+1)z^3$$
$$+ x^3(2x+1)(x^2+3x+1)z^4 + x^6(x^2+3x+1)z - x^{10} = 0;$$

$$z^6 - (x+1)(3x+1)z^5 - x(x+1)(3x+1)(x^2+3x+1)z^4$$

$$z^6 - (x+1)(3x+1)z^5 - x(x+1)(3x+1)(x^2+3x+1)z^4$$

$$+ x^3(2x+1)(3x+1)(x^2+3x+1)z^3 + x^6(x+1)(3x+1)(x^2+3x+1)z^2$$
$$- x^{10}(x+1)(3x+1)z - x^{15} = 0.$$

respectively.

Correspondingly, we have the following recurrences for powers of Jacobsthal polynomials:

$$J_{n+2} = J_{n+1} + xJ_n;$$

$$J_{n+3}^2 = (x+1)J_{n+2}^2 + x(x+1)J_{n+2}^2 - x^3 J_n^2;$$

$$J_{n+4}^3 = (2x+1)J_{n+3}^3 + x(x+1)(2x+1)J_{n+2}^3 - x^3(2x+1)J_{n+1}^3 - x^6 J_n^3;$$

$$J_{n+5}^4 = (x^2+3x+1)J_{n+4}^4 + x(2x+1)(x^2+3x+1)J_{n+3}^4$$
$$- x^3(2x+1)(x^2+3x+3)J_{n+2}^4 - x^6(x^2+3x+1)J_{n+1}^4 + x^{10}J_n^4;$$

$$J_{n+6}^5 = (x+1)(3x+1)J_{n+5}^5 + x(x+1)(3x+1)(x^2+3x+1)J_{n+4}^5$$
$$- x^3(2x+1)(3x+1)(x^3+3x+1)J_{n+3}^5 - x^6(x+1)(3x+1)(x^2+3x+1)J_{n+2}^5$$
$$+ x^{10}(x+1)(3x+1)J_{n+1}^5 + x^{15}J_n^5,$$

where $J_n = J_n(x)$.

In particular, we then have

$$J_{n+2} = J_{n+1} + 2J_n;$$

$$J_{n+3}^2 = 3J_{n+2}^2 + 5J_{n+1}^2 - 8J_n^2;$$

$$J_{n+4}^3 = 5J_{n+3}^3 + 30J_{n+2}^3 - 40J_{n+1}^3 - 64J_n^3;$$

$$J_{n+5}^4 = 11J_{n+4}^4 + 110J_{n+3}^4 - 440J_{n+2}^4 - 704J_{n+1}^4 + 1024J_n^4;$$

$$J_{n+6}^5 = 21J_{n+5}^5 + 462J_{n+4}^5 - 3080J_{n+3}^5 - 14784J_{n+2}^5 + 21504J_{n+1}^5 + 32768J_n^5;$$

respectively.

3.2 Addition Formulas: Jabinomial coefficients also occur in the addition formula involving the sum of products of $m+1$ terms of sequences satisfying the Jacobsthal recurrence:

$$\sum_{r=0}^{m} (-1)^{t_r + r} x^{t_{r-1}} \left[\!\left[\begin{matrix} m \\ r \end{matrix}\right]\!\right]_J J_{n+m-r}^{m+1} = J_m^* J_{(m+1)(n+m/2)} \qquad (7)$$

where $J_n = J_n(x)$. This follows from formula (5) in [7] by letting $U_n = J_n(x)$, $k = m$, $j = r$, and $a_i = n$ for $1 \le i \le k$. Notice that the second subscript on the RHS can be expressed in terms of a triangular number: $(m+1)(n+m/2) = (m+1)n + t_m$.

As can be predicted, formula (7) also has interesting special cases. For example, when $m = 1, 2, 3, 4$, and 5, it yields the following addition formulas, where $J_n = J_n(x)$:

$$J_{n+1}^2 + xJ_n^2 = J_{2n+1};$$

$$J_{n+2}^3 + xJ_{n+1}^2 - x^3 J_n^3 = xJ_{3n+3};$$

$$J_{n+3}^4 + x(x+1)J_{n+2}^4 - x^3(x+1)J_{n-1}^4 - x^6 J_n^4 = (x+1)J_{4n+6};$$

$$J_{n+4}^5 + x(2x+1)J_{n+3}^5 - x^3(x+1)(2x+1)J_{n+2}^5 - x^6(2x+1)J_{n+1}^5 + x^{10}J_n^5 = (x+1)(2x+1)J_{5n+10};$$

$$J_{n+5}^6 + x(x^2+3x+1)J_{n+4}^6 - x^3(2x+1)(x^2+3x+1)J_{n+3}^6 - x^6(2x+1)(x^2+3x+1)J_{n+2}^6 + x^{10}(x^2+3x+1)J_{n+1}^6 + x^{15}J_n^6 = (x+1)(2x+1)(x^2+3x+1)J_{6n+15}.$$

In particular, we have

$$J_{n+1}^2 + 2J_n^2 = J_{2n+1};$$

$$J_{n+2}^3 + 2J_{n+1}^3 - 8J_n^3 = 2J_{3n+3};$$

$$J_{n+3}^4 + 6J_{n+2}^4 - 24J_{n+1}^4 - 64J_n^4 = 3J_{4n+6}$$

$$J_{n+4}^5 + 10J_{n+3}^5 - 120J_{n+2}^5 - 320J_{n+1}^5 + 1024J_n^5 = 15J_{5n+10};$$

$$J_{n+5}^6 + 22J_{n+4}^6 - 440J_{n+3}^6 - 3520J_{n+2}^6 + 11264J_{n+1}^6 + 32762J_n^6 = 165J6_{n+15}.$$

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A SUBFAMILY OF THE EXTENDED TRIBONACCI POLYNOMIALS FAMILY

Abstract: We present a new member of the extended tribonacci polynomial family, a link among the subfamilies, Binet-like and addition formulas for the tribonacci family and a summation formula for a special case. We also present combinatorial and graph-theoretic models for the subfamilies, and confirm the addition formula for the new subfamily using graph-theoretic tools.

Keywords: Extended Tribonacci Polynomials, Binet Addition Formula, Combinatorial Models.

Mathematical Subject Classification (2020) No.: Primary 05A19, 11B37, 11B39, 11Cxx.

1. Introduction

Extended tribonacci polynomials $z_n(x)$ are defined by the third-order recurrence $z_{n+3}(x) = x^2 z_{n+2}(x) + x_{zn+1}(x) + z_n(x)$, where x is an arbitrary complex variable; $z_0(x)$, $z_1(x)$, and $z_2(x)$ are arbitrary complex polynomials; and $n \geq 0$.

When $z_0(x) = 0$, $z_1(x) = 1$, and $z_2(x) = x^2$, $z_n(x) = t_n(x)$, the nth *tribonacci polynomial* [2, 6, 8]; when $z_0(x) = 3$, $z_1(x) = x^2$, and $z_2(x) = x^4 + 2x$, $z_n(x) = k_n(x)$, the nth *tribonacci-Lucas polynomial* [3, 8, 10]; and when

$z_0(x) = x^2 - 3$, $z_1(x) = x^4 - x^2 + x$, and $z_2(x) = x^6 - x^4 + 2x^3 -^2 x + 1$, $z_n(x) = d_n(x)$, a new member of the extended family. Table 1 shows the first six members of the subfamily $\{d_n(x)\}$.

Table 1: First Six Polynomials

n	$d_n(x)$
0	$x^2 - 3$
1	$x^4 - x^2 + x$
2	$x^6 - x^4 + 2x^3 - 2x + 1$
3	$x^8 - x^6 + 3x^5 - 3x^3 + 3x^2 - 3$
4	$x^{10} - x^8 + 4x^7 - 4x^5 + 6x^4 - 6x^2 + 2x$
5	$x^{12} - x^{10} + 5x^9 - 5x^7 + 10x^6 - 10x^4 + 7x^3 - 5x + 1$

Tribonacci numbers T_n and *tribonacci-Lucas numbers* K_n are given by $t_n(1)$ and $k_n(1)$, respectively [1, 3, 6, 8, 10]. Correspondingly, we have a new family of numbers D_n, defined by $D_n = d_n(1)$. Table 2 shows the first 10 numbers T_n, K_n, and D_n.

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is no ambiguity; so z_n will mean $z_n(x)$. We also omit a lot of basic algebra.

Table 2: Numbers T_n, K_n and D_n for $0 \le n \le 9$ –

n	0	1	2	3	4	5	6	7	8	9
T_n	0	1	1	2	4	7	13	24	44	81
K_n	3	1	3	7	11	21	39	71	131	241
D_n	-2	1	1	0	2	3	5	10	18	33

The polynomials t_n, k_n, and d_n, and hence their numeric counterparts can be extended to negative subscripts. For example, $t_{-1} = 0$, $t_{-2} = 1$, and $t_{-3} = -x$; $k_{-1} = -x$, $k_{-2} = -x^2$, and $k_{-3} = 2x^3 + 3$; and $d_{-1} = x+1$, $d_{-2} = x^2$, and $d_{-3} = -2x^3 - 3$.

There is a close relationship between tribonacci and tribonacci-Lucas polynomials:

$$k_n = x^2 t_n + 2x t_{n-1} + 3t_{n-2} \tag{1}$$

$$= t_{n+1} + x t_{n-1} + 2t_{n-2}. \tag{2}$$

Formula (1) can be confirmed using induction [3] or the generating functions $u = \dfrac{z}{1 - x^2 z - xz^2 - z^3}$ and $k = \dfrac{3 - 2x^2 z - xz^2}{1 - x^2 z - xz^2 - z^3}$ of tribonacci and tribonacci-Lucas polynomials, respectively [7].

Using the generating functions for d_n and t_n, we now develop a formula for d_n in terms of tribonacci polynomials.

2. A Formula For d_n

Let $d = \sum_{n=0}^{\infty} d_n z^n$. Using standard generating function techniques, we can show that

$$d = \frac{x^2 - 3 + (2x^2 + x)z + (x+1)z^2}{1 - x^2 z - xz^2 - z^3} .$$

By the generating functions d and u, we have

$$dz = [x^2 - 3 + (2x^2 + x)z + (x+1)z^2]\,u$$

$$\sum_{n=1}^{\infty} d_{n-1} z^n = (x^2 - 3)\sum_{n=0}^{\infty} t_n z^n + (2x^2 + x)\sum_{n=1}^{\infty} t_{n-1} z^n + (x+1)\sum_{n=2}^{\infty} t_{n-2} z^n$$

$$\begin{aligned} d_n &= (x^2 - 3)t_{n+1} + (2x^2 + x)t_n + (x+1)t_{n-1} \\ &= (x^2 t_{n+1} + xt_n + t_{n-1}) - 3t_{n+1} + 2x^2 t_n + xt_{n-1} \\ &= t_{n+2} - 3t_{n+1} - x^3 t_{n+1} + x(x^2 t_{n+1} + xt_n + t_{n-1}) + x^2 t_n \\ &= (x+1)t_{n+2} - (x^3 + 3)t_{n+1} + x^2 t_n. \end{aligned} \tag{3}$$

This yields

$$D_n = 2T_{n+2} - 4T_{n+1} + T_n.$$

Consequently, $D_n \equiv T_n \pmod 2$; so D_n and T_n have the same parity; see Table 2.

3. A Link among The Tribonacci Subfamilies

Using induction, we now establish a link among the three subfamilies of the extended tribonacci family.

Theorem 1: *Let* $n \geq 0$. *Then* $t_{n+2} = k_n + d_n$.

Proof: Clearly, the formula works when $n = 0$, 1, and 2.

Now assume it is true for n, $n-1$, and $n-2$, where $n \geq 2$. By the tribonacci recurrence, we then have

$$\begin{aligned} t_{n+3} &= x^2 t_{n+2} + xt_{n+1} + t_n \\ &= x^2(k_n + d_n) + x(k_{n-1} + d_{n-1}) + (k_{n-2} + d_{n-2}) \\ &= k_{n+1} + d_{n+1}. \end{aligned}$$

So the formula works for $n+1$ also. Induction then guarantees that the given formula works for all $n \geq 0$, as desired. □

It follows by the theorem that $T_{n+2} = K_n + D_n$.

Theorem 1, coupled with formula (2), can be used to develop an alternate formula for d_n in terms of tribonacci polynomials:

$$\begin{aligned} d_n &= t_{n+2} - k_n \\ &= (x^2 t_{n+1} + x t_n + t_{n-1}) - (t_{n+1} + x t_{n-1} + 2t_{n-2}) \\ &= (x^2 - 1) t_{n+1} + x t_n - (x-1) t_{n-1} - 2t_{n-2}. \end{aligned} \tag{4}$$

Consequently, $D_n = T_n - 2T_{n-2}$ and hence T_n and D_n have the same parity, as found earlier.

It follows by Theorem 1 and formula (2) that

$$\begin{aligned} t_{n+2} &= k_n + [(x^2-1) t_{n+1} + x t_n - (x-1) t_{n-1} - 2t_{n-2}]. \\ &= k_n + (x^2 t_{n+1} + 2x t_n + 3t_{n-1}) - t_{n+1} - x t_n - (x+2) t_{n-1} - 2t_{n-2} \\ &= k_{n+1} + k_n - t_{n+1} - x t_n - (x+2) t_{n-1} - 2t_{n-2}. \end{aligned}$$

In particular, we have

$$T_{n+2} = K_{n+1} + K_n - T_{n+1} - T_n - 3T_{n-1} - 2T_{n-2} \tag{5}$$

$$T_{n+3} = K_{n+1} + K_n - 3T_{n-1} - 2T_{n-2}.$$

It follows from formula (5) that

$$2T_{n+2} = K_{n+1} + K_n - 2(T_{n-1} + T_{n-2}).$$

Consequently, $K_{n+1} \equiv K_n \pmod 2$. Since both K_0 and K_1 are odd, this implies that every K_n is odd. (This follows by the tribonacci recurrence also, since K_0, K_1 and K_2 are all odd.)

Using Theorem 1, we now present an alternate proof of property (2).

3.1 An Alternate Proof of Property (2): It follows by the generating functions d and k that

$$\frac{3 - 2x^2 z - xz^2}{k} = \frac{x^2 - 3 + (2x^2 + x)z + (x+1)z^2}{d}$$

$$(3 - 2x^2 z - xz^2)d = [x^2 - 3 + (2x^2 + x)z + (x+1)z^2]k$$

$$3\sum_{n=0}^{\infty} d_n z^n - 2x^2 \sum_{n=1}^{\infty} d_{n-1} z^n - x\sum_{n=0}^{\infty} d_{n-2} z^n = (x^2 - 3)\sum_{n=0}^{\infty} k_n z^n + (2x^2 + x)\sum_{n=1}^{\infty} k_{n-1} z^n + (x+1)\sum_{n=2}^{\infty} k_{n-2} z^n$$

$$3d_n - 2x^2 d_{n-1} - xd_{n-2} = (x^2 - 3)k_n + (2x^2 + x)k_{n-1} + (x+1)k_{n-2}. \tag{6}$$

By Theorem 1, this yields

$$\begin{aligned} x^2 k_n + xk_{n-1} + k_{n-2} &= 3(d_n + k_n) - 2x^2(d_{n-1} + k_{n-1}) - x(d_{n-2} + k_{n-2}) \\ k_n &= 3t_{n+1} - 2x^2 t_n - xt_{n-1} \\ &= t_{n+1} + 2(t_{n+1} - x^2 t_n - xt_{n-1}) + xt_{n-1} \\ &= t_{n+1} + xt_{n-1} + 2t_{n-2}, \end{aligned}$$

as desired.

Next we explore an addition formula for d_n.

4. Addition Formulas

Using Theorem 1, and the addition formulas [8]

$$t_{m+n} = t_{m+1}t_n + xt_m t_{n-1} + t_m t_{n-2} + t_{m-1}t_{n-1};$$

$$k_{m+n} = t_{m+1}k_n + xt_m k_{n-1} + t_m k_{n-2} + t_{m-1}k_{n-1},$$

we have

$$\begin{aligned} d_{m+n} &= t_{m+n+2} - k_{m+n} \\ &= (t_{m+1}t_{n+2} + xt_m t_{n+1} + t_m t_n + t_{m-1}t_{n+1}) \\ &\quad - (t_{m+1}k_n + xt_m k_{n-1} + t_m k_{n-2} + t_{m-1}k_{n-1}) \\ &= t_{m+1}d_n + xt_m d_{n-1} + t_m d_{n-2} + t_{m-1}d_{n-1}. \end{aligned} \tag{7}$$

Combining the three addition formulas, we get

$$z_{m+n} = t_{m+1}z_n + xt_m z_{n-1} + t_m z_{n-2} + t_{m-1}z_{n-1}, \tag{8}$$

where $z_i = t_i,\ k_i,$ or d_i.

In particular,

$$Z_{m+n} = T_{m+1}Z_n + T_m Z_{n-1} + T_m Z_{n-2} + T_{m-1}Z_{n-1},$$

where $Z_i = T_i,\ K_i$, or D_i.

It follows from formula (8) that

$$z_{3n} = t_{2n+1}z_n + xt_{2n}z_{n-1} + t_{2n}z_{n-2} + t_{2n-1}z_{n-1};$$

$$Z_{3n} = T_{2n+1}Z_n + T_{2n}Z_{n-1} + T_{2n}Z_{n-2} + T_{2n-1}Z_{n-1}.$$

We will revisit these formulas later.

5. Binet-Like Formulas

Next we explore the Binet-like formula for the extended tribonacci polynomial z_n. To this end, let $\alpha = \alpha(x)$, $\beta = \beta(x)$, and $\gamma = \gamma(x)$ be the solutions of the characteristic equation $z^3 - x^2z^2 - xz - 1 = 0$. The general solution of the tribonacci recurrence is $z_n = A\alpha^n + B\beta^n + C\gamma^n$, where the unknowns $A,\ B$, and C can be found using the initial conditions. This yields the desired *Binet-like formula*

$$z_n = \frac{z_2 - z_1(\beta+\gamma) + z_0\beta\gamma}{(\alpha-\beta)(\alpha-\gamma)}\alpha^n + \frac{z_2 - z_1(\gamma+\alpha) + z_0\gamma\alpha}{(\beta-\alpha)(\beta-\gamma)}\beta^n + \frac{z_2 - z_1(\alpha+\beta) + z_0\alpha\beta}{(\gamma-\alpha)(\gamma-\beta)}\gamma^n. \tag{9}$$

Let λ be an arbitrary solution of $z^3 - x^2z^2 - xz - 1 = 0$.

It then follows from equation (9) that

$$t_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}$$

$$= A_\alpha \alpha^n + B_\beta \beta^n + C_\gamma \gamma^n, \tag{10}$$

where $A_\lambda = \dfrac{\lambda^2}{x^2\lambda^2 + 2x\lambda + 3}$.

It also follows from equation (9) that

$$T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \tag{11}$$

$$K_n = \alpha^n + \beta^n + \gamma^n$$

$$D_n = C_\alpha \alpha^n + C_\beta \beta^n + C_\gamma \gamma^n,$$

where $\alpha = \alpha(1)$, $\beta = \beta(1)$, and $\gamma = \gamma(1)$; $\lambda = \alpha$, β, or γ; and $C_\lambda = \dfrac{\lambda^2 - 2}{\lambda^2 + 2\lambda + 3}$.

Since $\mu = \lambda(1)$ is a solution of $z^3 - z^2 - z - 1 = 0$, $\mu^2 + \dfrac{1}{\mu^2} = 2\mu$.

Consequently, we can rewrite formula (11) in a slightly different form [9]:

$$T_n = \frac{\alpha^n}{-\alpha^2 + 4\alpha - 1} + \frac{\beta^n}{-\beta^2 + 4\beta - 1} + \frac{\gamma^n}{-\gamma^2 + 4\gamma - 1}.$$

Next we show that z_{3n} can be expressed as a double summation.

6. A Summation Formula for z_{3n}

Recall that λ is an arbitrary solution of the equation $z^3 = x^2z^2 + xz + 1$.

Then [3]

$$\begin{aligned}\lambda^{3n} &= \left[(\lambda^3 - 1) + 1\right]^n \\ &= \sum_{i=0}^{n} \binom{n}{i} (\lambda^3 - 1)^i \\ &= \sum_{i=0}^{n} \binom{n}{i} (x^2\lambda^3 - x\lambda)^i \\ &= \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} (x\lambda)^{i+j},\end{aligned}$$

It then follows by formula (9) that

$$z_{3n} = \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} x^{i+j} z_{i+j},$$

where $z_n = t_n$, k_n, or d_n. Consequently, by equation (8), we have

$$\begin{aligned}t_{2n+1}z_n + xt_{2n}z_{n-1} + t_{2n}z_{n-2} + t_{2n-1}z_{n-1} &= z_{3n} \\ &= \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} x^{i+j} z_{i+j}.\end{aligned}$$

This implies

$$\begin{aligned}T_{2n+1}Z_n + T_{2n}Z_{n-1} + T_{2n}Z_{n-2} + T_{2n-1}Z_{n-1} &= Z_{3n} \\ &= \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} Z_{i+j}.\end{aligned}$$

where $Z_n = T_n$, K_n, or D_n.

Next we briefly cite combinatorial models for z_n.

7. Combinatorial Models

Suppose we would like to tile an $n \times 1$ board with 1×1, 2×1, and 3×1 tiles. We call such a board an *n-board*, and such tiles as squares, dominoes, and *triminoes*. The *weight* of a square is x^2, that of a domino is x, and that of a trimino is 1. The *weight of a tiling* is the product of the weights of the tiles in the tiling. The weight of the *empty tiling* is defined as 1. Then the sum of weights of such tilings of an n-board is t_{n+1}, where $n \geq 0$ [5, 6]; there are T_{n+1} such tilings of an n-board.

As a result, there are T_{n+1} compositions of a positive integer n using the summands 1, 2, and 3 [5, 6].

Using equations (1) and (2), we can now interpret k_n combinatorially [8]:

$$k_n = x^2\begin{pmatrix}\text{sum of the weights of}\\ \text{tilings of an } (n-1)\text{-board}\end{pmatrix} + 2x\begin{pmatrix}\text{sum of the weights of}\\ \text{tilings of an } (n-2)\text{-board}\end{pmatrix} + 3\begin{pmatrix}\text{sum of the weights of}\\ \text{tilings of an } (n-3)\text{-board}\end{pmatrix}$$

$$= \begin{pmatrix}\text{sum of the weights of}\\ \text{tilings of an } n\text{-board}\end{pmatrix} + x\begin{pmatrix}\text{sum of the weights of}\\ \text{tilings of an } (n-2)\text{-board}\end{pmatrix} + 2\begin{pmatrix}\text{sum of the weights of}\\ \text{tilings of an } (n-3)\text{-board}\end{pmatrix}$$

There are $K_n = T_n + 2T_{n-1} + 3T_{n-2} = T_{n+1} + T_{n-1} + 2T_{n-2}$ such tilings.

It then follows by Theorem 1 that

$$d_n = \begin{pmatrix}\text{sum of the weights of}\\ \text{tilings of an } (n+1)\text{-board}\end{pmatrix} - \begin{pmatrix}\text{sum of the weights of}\\ \text{tilings of an } n\text{-board}\end{pmatrix} - x\begin{pmatrix}\text{sum of the weights of}\\ \text{tilings of an } (n-2)\text{-board}\end{pmatrix} - 2\begin{pmatrix}\text{sum of the weights of}\\ \text{tilings of an } (n-3)\text{-board}\end{pmatrix}.$$

There are $D_n = T_{n+2} - K_n$ such tilings.

Next we establish addition formula (7) using graph-theoretic techniques.

7. Graph-Theoretic Models

Consider the *weighted tribonacci digraph* D with vertices v_1, v_2 and v_3 in Figure 1, where a *weight* is assigned to each edge [5, 6]. Clearly it *weighted adjacency matrix* is the Q-matrix.

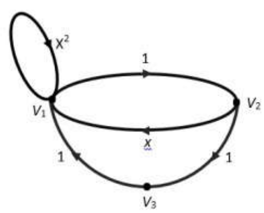

Figure 1: Tribonacci Digraph

A *walk* from vertex v_i to vertex v_j is a sequence $v_i - e_i - v_{i+1} - \cdots - v_{j-1} - e_{j-1} - v_j$ of vertices v_k and edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The walk is *closed* if $v_i = v_j$; otherwise, it is *open*. The *length* of a walk is the number of edges in the walk. The *weight* of a walk is the product of the weights of the edges along the walk.

Its *weighted adjacency matrix* is the Q-matrix [2, 6, 8]:

$$Q = \begin{bmatrix} x^2 & 1 & 0 \\ x & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

Where $Q = Q(x)$. It then follows by induction that

$$Q^n = \begin{bmatrix} t_{n+1} & t_n & t_{n-1} \\ xt_n + t_{n-1} & xt_{n-1} + t_{n-2} & xt_{n-2} + t_{n-3} \\ t_n & t_{n-1} & t_{n-2} \end{bmatrix}.$$

The ij-th entry of Q^n gives the sum of the weights of all walks of length n from any vertex v_i to any vertex v_j [4, 5, 6, 8], where $1 \le i,\ j \le 3$.

Consequently, by Theorem 1, we have

$$t_{n+2} = \begin{pmatrix} \text{sum of the weights of closed walks} \\ \text{of length } n+1 \text{ originating at } v_1 \end{pmatrix}$$

$$k_n = x^2 \begin{pmatrix} \text{sum of the weights of closed walks} \\ n-1 \text{ of length } n \text{ originating at } v_1 \end{pmatrix}$$

$$+2x \begin{pmatrix} \text{sum of the weights of closed walks} \\ \text{of length } n-2 \text{ originating at } v_1 \end{pmatrix}$$

$$+3 \begin{pmatrix} \text{sum of the weights of closed walks} \\ \text{of length } n-3 \text{ originating at } v_1 \end{pmatrix}$$

$$d_n = \begin{pmatrix} \text{sum of the weights of closed walks} \\ \text{of length } n+1 \text{ originating at } v_1 \end{pmatrix}$$

$$-x^2 \begin{pmatrix} \text{sum of the weights of closed walks} \\ \text{of length } n-1 \text{ originating at } v_1 \end{pmatrix}$$

$$-2x \begin{pmatrix} \text{sum of the weights of closed walks} \\ \text{of length } n-2 \text{ originating at } v_1 \end{pmatrix}$$

$$-3 \begin{pmatrix} \text{sum of the weights of closed walks} \\ \text{of length } n-3 \text{ originating at } v_1 \end{pmatrix}$$

With this background, we are now ready for the confirmation of formula (7).

Proof: Let S_n denote the sum of the weights of closed walks of length n originating at v_1. Then $S_n = t_{n+1}$. We then have

$$k_{m+n} = x^2 S_{m+n-1} + 2xS_{m+n-2} + 3S_{m+n-3}$$

$$d_{m+n} = S_{m+n+1} - x^2 S_{m+n-1} - 2xS_{m+n-2} - 3S_{m+n-3} \,. \tag{12}$$

We will now compute the sum $S = S_{m+n+1} - x^2 S_{m+n-1} - 2xS_{m+n-2} - 3S_{m+n-3}$ in a different way. Let A, B, C, and D denote the sets of closed walks of length $m+n+1$, $m+n-1$, $m+n-2$ and $m+n-3$ all originating at v_1, respectively.

Let w be an arbitrary walk in A.

Case 1: Suppose w begins with a loop: $\underbrace{\text{subwalk } v_1 v_1}_{\text{length } 1}\underbrace{\text{subwalk from } v_1 \text{ to } v_1}_{\text{length } m+n}$.

The sum of the weights such walk is $x^2 t_{m+n+1}$.

Case 2: Suppose w does *not* begin with a loop. If w lands at v_2 and returns home, then w is of the form $\underbrace{\text{subwalk } v_1 v_2 v_1}_{\text{length } 2}\underbrace{\text{subwalk from } v_1 \text{ to } v_1}_{\text{length } m+n+1}$. The sum of the weights such walks equals xt_{m+n}.

On the other hand, if w lands at v_2, then v_3, and then returns home, then w is of the form $\underbrace{\text{subwalk } v_1 v_2 v_3 v_1}_{\text{length } 3}\underbrace{\text{subwalk from } v_1 \text{ to } v_1}_{\text{length } m+n-2}$. The sum of the weights such walks equals t_{m+n-1}.

Combining the two cases, we get

$$\begin{aligned} S_{m+n+1} &= x^2 t_{m+n+1} + (xt_{m+n} + t_{m+n-1}) \\ &= t_{m+n+2} \,. \end{aligned}$$

It then follows that $S_{m+n-1} = t_{m+n}$, $S_{m+n-2} = t_{m+n-1}$, and $S_{m+n-3} = t_{m+n-2}$.

Thus, by the addition formulas for t_n and k_n, and Theorem 1, we have

$$S = t_{m+n+2} - x^2 t_{m+n} - 2x t_{m+n-1} - 3t_{m+n-2}$$

$$= t_{m+(n+2)} - k_{m+n}$$

$$= (t_{m+1}t_{n+2} + x t_m t_{n+1} + t_m t_n + t_{m-1}t_{n+1})$$

$$-(t_{m+1}k_n + x t_m k_{n-1} + t_m k_{n-2} + t_{m-1}k_{n-1})$$

$$= t_{m+1}d_n + x t_m d_{n-1} + t_m d_{n-2} + t_{m-1}d_{n-1}.$$

Equating this value of S with that in equation (12) yields the desired result. □

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SOME GENERALIZED DIFFERENCE SEQUENCE SPACES DEFINED BY SEQUENCE OF MODULI

Abstract: In this paper we define some generalized difference sequence spaces by using a sequence of moduli. Some topological results and inclusion relations of such spaces have also been discussed.

Keywords and phrases: Difference Sequence Spaces, Sequence of Moduli.

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1. Introduction

Let ω be the set of all sequences, real or complex numbers and l_∞, c and c_0 respectively be the Banach spaces of bounded, convergent and null sequences $x = (x_k)$, normed by $\|x\| = \sup_k |x_k|$, where $k \in N$, the set of positive integers.

Recently, Kizmaz [6] defined the sequence spaces

$$l_\infty(\Delta) = \{x = (x_k) : \Delta x \in l_\infty\},$$

$$c(\Delta) = \{x = (x_k) : \Delta x \in c\},$$

and

$$c_0(\Delta) = \{x = (x_k) : \Delta x \in c_0\},$$

where $\Delta x = x_k - x_{k+1}$.

Subsequently difference sequence spaces have been studied by several authors ([3], [4], [11], [13], [16], [17]).

Definition 1.1: *A function* $f:[0,\infty)\to[0,\infty)$ *is called a modulus if*

(i) $f(x)=0$ *if and only if* $x=0$,

(ii) $f(x+y)\le f(x)+f(y)$, *for all* $x,y\ge 0$,

(iii) f *is increasing*,

(iv) f *is continuous from the right at* 0.

it follows from (2) *and* (4) *that f must be continuous everywhere on* $[0,\infty)$.

Let X be a sequence space. Then the sequence space $X(f)$ is defined as

$$X(f)=\{x=(x_k):(f(|x_k|))\in X\}$$

for a modulus *f* ([10], [12], 15]).

Kolk [7], [8] gave an extension of $X(f)$ by considering a sequence of moduli $F=(f_k)$ i.e.

$$X(F)=\{x=(x):(f(|x|))\in X\}$$

After then Gaur and Mursaleen [5] defined the following sequence spaces

$$l_\infty(F,\Delta)=\{x=(x_k):\Delta x\in l_\infty(F)\}\ ,$$

$$c_0(F,\Delta)=\{x=(x_k):\Delta x\in c_0(F)\}$$

for a sequence of moduli $F=(f_k)$.

The notion of sequence of moduli was further generalized in [1] and [2].

For a sequence of moduli $F=(f_k)$ we give following conditions

$$(T_1) \sup_k f_k(t) < \infty \text{ for all } t > 0;$$

$$(T_2) \lim_{t\to 0} f_k(t) = 0 \text{ uniformly in } k \geq 1,$$

When $f_k = f(k \geq 1)$, where f is modulus, then condition (T_1) and (T_2) are automatically fulfilled.

The following inequality will be used throughout the paper.

If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq K(|a_k|^{p_k} + |b_k|^{p_k}) \tag{1.1}$$

for all k and $a_k, b_k \in C$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in C$. Throughout the paper summation without limits run from 1 to ∞.

In the present paper we define some generalized difference sequence spaces by using a sequence of moduli. Also we discuss some topological properties and inclusion relation between these spaces.

Definition 1.2: *Let $F = (f_k)$ be a sequence of moduli and suppose $A = (a_{nk}(i))$ be a nonnegative regular matrix, we define*

$$W_0[A, \Delta, F, p, s] = \left\{x \in \omega : \lim_n \sum_k a_{nk}(i) k^{-s} [f_k(|\Delta x_k|)]^{p_k} = 0, s \geq 0, \text{ uniformly in } i\right\},$$

$$W[A, \Delta, F, p, s] = \left\{x \in \omega : \lim_n \sum_k a_{nk}(i) k^{-s} [f_k(|\Delta x_k - L|)]^{p_k} = 0, s \geq 0, \text{ uniformly in } i\right\}$$

for some L}

$$W_\infty[A, \Delta, F, p, s] = \left\{x \in \omega : \sup_n \sum_k a_{nk}(i) k^{-s} [f_k(|\Delta x_k|)]^{p_k} \leq \infty,\right\}.$$

When $f_k = f$ *and* $p_k = 1$ *for all k, we have the following sequence spaces*;

$$W[A, \Delta, F, p, s] = W[A, \Delta, f, s]$$
$$= \left\{x \in \omega : \lim_n \sum_k a_{nk}(i)k^{-s}[f(|\Delta x_k - L|)] = 0, s \geq 0, \ uniformly \ in \ i\right\}.$$

Similarly

$$W_0[A, \Delta, F, p, s] = W_0[A, \Delta, f, s] \text{ and } W_\infty[A, \Delta, F, p, s] = W_\infty[A, \Delta, f, s].$$

If $x \in W[A, \Delta, f, s]$ we say that x is strongly $A(i)$ summable to L with respect to the modulus f.

When $f_k = f$, $\Delta x = x$, $s = 0$, $p_k = 1$ for all k,

and
$$a_{nk}(i) = \begin{cases} \frac{1}{n}, & (i + 1 \leq k \leq i + n) \\ 0, & \text{otherwise.} \end{cases}.$$

We have the following sequence spaces which were defined by Pehlivan [14],

$$W[A, \Delta, F, p, s] = \widehat{W}[f]$$
$$= \left\{x \in \omega : \lim_n \sum_{k=i+1}^{i+n} [f(|x_k - L|)] = 0, \text{ uniformly in } i \text{ for some } L\right\}$$

Similarly $W_0[A, \Delta, F, p, s] = \widehat{W_0}[f]$ and $W_\infty[A, \Delta, F, p, s] = \widehat{W_\infty}[f]$.

If $x \in \widehat{W}[f]$ we say that x is almost convergentto L.

Further if we take $f_k(x) = x$, for all k, in the above sequence spaces we have the sequence spaces which were defined by Lorentz [9].

If we set $A = (a_{nk}(\mathrm{i})) = (C, 1)$ Cesaro matrix for all i and $f_k = f$, $s = 0$, $\Delta x = x$, $p_k = 1$ for all k, we have the following generalized sequence spaces due to Maddox [10].

$$W(f) = \left\{ x \in \omega : \lim_{n} \frac{1}{n} \sum_{k=1}^{n} f(| x_k - L |)] = 0 \right\}$$

Similarly we can define $W_0(f)$ and $W_\infty(f)$.

If $x \in W(f)$, we say that x is strongly summable to L with respect to the modulus f.

2. Main Results

Theorem 2.1: *Let* $A = (a_{nk}(i))$ *be a nonnegative regular matrix and* $F = (f_k)$ *be a sequence of moduli,* $p = (p_k)$ *be a bounded sequence of positive real numbers. Then the sequence spaces* $W_0[A, \Delta, F, p, s]$, $W[A, \Delta, f, p, s]$ *and* $W_\infty[A, \Delta, F, p, s]$ *are linear spaces over the complex field* C.

Proof: We consider only $W_0[A, \Delta, F, p, s]$. Others can be treated similarly. Let $x, y \in W_0[A, \Delta, F, p, s]$ and $\lambda, \mu \in C$, there exists M_λ and N_μ integers such that $|\lambda| \le M_\lambda$ and $|\mu| \le N_\mu$. Since f_k is subadditive and is linear, we have

$$\sum_k a_{nk}(i) k^{-s} [fk(\Delta((\lambda x_k + \mu y_k))]^{pk}$$

$$\le \sum_k a_{nk}(i) k^{-s} [f_k(| \lambda | (\Delta x_k) + f_k(| \mu | (\Delta y_k))]^{pk}$$

$$\le K(M_\lambda)^H \sum_k a_{nk}(i) k^{-s} [f_k(\Delta x_k)]^{pk}$$

$$+ K(N_\mu)^H \sum_k a_{nk}(i) k^{-s} [f_k(\Delta y_k)]^{pk} \quad \text{By (1.1)}$$

For $n \to \infty$, we have $\lambda x + \mu y \in W_0[A, \Delta, F, p, s]$.

Therefore, this is a linear space over C. □

Theorem 2.2: *Let* $A = (a_{nk}(i))$ *be a nonnegative regular matrix and* $F = (f_k)$ *be sequence of moduli, if* (T_1) *holds then*

$$W_0[A, \Delta, F, p, s] \subset W[A, \Delta, F, p, s] \subset W_\infty[A, \Delta, F, p, s].$$

Proof: It is clear that $W_0[A, \Delta, F, p, s] \subset W[A, \Delta, F, p, s]$ and $W_0[A, \Delta, F, p, s] \subset W_\infty[A, \Delta, F, p, s]$. For second inclusion let $x \in W[A, \Delta, F, p, s]$ then by definition, we have

$$\sum_k a_{nk}(i)k^{-s}[f_k(|\ \Delta x_k\ |)]^{p_k} = \sum_k a_{nk}(i)k^{-s}[f_k(|\ \Delta x_k - L + L\ |)]^{p_k}$$

$$\leq K \sum_k a_{nk}\ (i)k^{-s}[f_k(|\ \Delta x_k - L\ |)]^{p_k} + K \sum_k a_{nk}(i)k^{-s}[f_k(|\ L\ |)]^{p_k}$$

There exists an integer D such that $|L| \leq D$. Hence we have

$$\sum_k a_{nk}(i)k^{-s}[f_k(|\ \Delta x_k\ |)]^{p_k} \leq K \sum_k a_{nk}(i)k^{-s}[f_k(|\ \Delta x_k - L\ |)]^{p_k} + K[Df_k(1)]^H \sum_k a_{nk}(i)k^{-s}$$

Since A is regular and $x \in W[A, \Delta, F, p, s]$ we get $x \in W_\infty[A, \Delta, F, p, s]$ and this completes the proof. □

Theorem 2.3: *Let* $A = (a_{nk}(i))$ *be a nonnegative regular matrix and* $F = (f_k)$ *be sequence of moduli,* $p = (p_k)$ *be a bounded sequence of positive real numbers. Then* $W_0[A, \Delta, F, p, s]$ *and* $W[A, \Delta, F, p, s]$ *are linear topological space paranormed by* g *defined by*

$$g(x) = \sup_{n,i} \left\{ \sum_k a_{nk}(i)k^{-s}[f_k(\Delta x_k)]^{p_k} \right\}^{\frac{1}{M}}$$

where $M = \max(1, H = \sup p_k)$.

Proof: Clearly $g(x) = g(-x)$, $g(0) = 0$ and by Minkowski's inequality, $g(x+y) \leq g(x) + g(y)$. We now show that the scalar multiplication is continuous.

Whence $\lambda \to 0$, $x \to 0$ imply $g(\lambda x) \to 0$ and also $x \to 0$, λ fixed imply $g(\lambda x) \to 0$. We now show that $\lambda \to 0$, x fixed imply $g(\lambda x) \to 0$.

Let $x \in W[A, \Delta, F, p, s]$, then as $n \to \infty$,

$$b_{n,i} = \sum_k a_{nk}(i) k^{-s} [f_k(|\Delta x_k - L|)] \to 0 \text{ uniformly in } i$$

for $|\lambda| < 1$, we have

$$\left\{\sum_k a_{nk}(i) k^{-s} [f_k(|\lambda \Delta x_k|)]^{p_k}\right\}^{\frac{1}{M}} \leq \left\{\sum_k a_{nk}(i) k^{-s} [f_k(|\lambda \Delta x_k - \lambda L + \lambda L|)]^{p_k}\right\}^{\frac{1}{M}}$$

$$\leq \left\{\sum_k a_{nk}(i) k^{-s} [f_k(|\lambda \Delta x_k - \lambda L|)]^{p_k}\right\}^{\frac{1}{M}}$$

$$+ \left\{\sum_k a_{nk}(i) k^{-s} [f_k(|\lambda L|)]^{p_k}\right\}^{\frac{1}{M}}, \text{ By Minkowski's}$$

inequality

$$\leq \left\{\sum_k a_{nk}(i) k^{-s} [f_k(|\Delta x_k - L|)]^{p_k}\right\}^{\frac{1}{M}} + \left\{\sum_{k \geq n} a_{nk}(i) k^{-s} [f_k(|\lambda \Delta x_k - \lambda L|)]^{p_k}\right\}^{\frac{1}{M}}$$

$$+ \left\{\sum_k a_{nk}(i) k^{-s} [f_k(|\lambda L|)]^{p_k}\right\}^{\frac{1}{M}},$$

Let $\epsilon > 0$ and choose N such that for each n, i and $D > N$ implies $b_{n,i} < \epsilon/2$. For each N, by continuity of f_k for all k, as $\lambda \to 0$,

$$\left\{\sum_{k \leq N} a_{nk}(i) k^{-s} [f_k(|\lambda \Delta x_k - L|)]^{p_k}\right\}^{\frac{1}{M}} + \left\{\sum_k a_{nk}(i) k^{-s} [f_k(|\lambda L|)]^{p_k}\right\}^{\frac{1}{M}} \to 0$$

then choose $\delta < 1$ such that $|\lambda| < \delta$ implies

$$\left\{\sum_{k\le N} a_{nk}(i)k^{-s}[f_k(|\lambda(\Delta x_k - L)|)]^{pk}\right\}^{\frac{1}{M}} + \left\{\sum_k a_{nk}(i)k^{-s}[f_k(|\lambda L|)]^{pk}\right\}^{\frac{1}{M}} < \epsilon/2$$

Hence we have $$\left\{\sum_k a_{nk}(i)k^{-s}[f_k(|\lambda\Delta x_k|)]^{pk}\right\}^{\frac{1}{M}} < \epsilon/2 = \epsilon$$

and $g(\lambda, x) \to 0$. Thus, $W[A, \Delta, F, p, s]$ is a paranormed linear topological space paranormed by g. □

Theorem 2.4: *Let* $A = (a_{nk}(i))$ *be a nonnegative regular matrix and* $F = (f_k)$ *be sequence of moduli then*

(i) $s > 0$ *implies* $l_\infty(\Delta) \subset W_0[A, \Delta, F, p, s]$,

(ii) $x_k \to L$ *implies* $x_k \to L[W(A, \Delta, F, p, s)]$,

(iii) $s_1 \le s_2$ *implies* $W_0[A, \Delta, F, p, s_1] \subset W_0[A, \Delta, F, p, s_2]$.

Proof: (i) Let $x \in l_\infty(\Delta)$ and $s > 0$. For regularity of A and (Δx_k) is bounded we have $f_k(\Delta x_k)$ is bounded so that $\sum_k a_{nk}(i)k^{-s}[f_k(|\Delta x_k|)]^{pk} \to (n \to \infty)$ uniformly in i so that $x \in W_0[A, \Delta, F, p, s]$.

(ii) Suppose that $x_k \to L$. Then

$$\lim_{k\to\infty}[f_k(|\Delta x_k - L|)]^{pk} = f_k[\lim_{k\to\infty}(|\Delta x_k - L|])^{pk} = 0$$

Since, (k^{-s}) is bounded, we write $\lim_{k\to\infty} k^{-s}[f_k(|\Delta x_k - L|)]^{pk} = 0$.

From regularity of A, we have $\lim_{n\to\infty}\sum_k a_{nk}(i)k^{-s}[f_k(|\Delta x_k - L|)]^{pk} = 0$ uniformly in i. So that $x \in W[A, \Delta, F, p, s]$.

(iii) Let $s_1 \le s_2$. Then $k^{-s_2} \le k^{-s_1}$ for all $k \in N$. Since,

$$k^{-s_2} f_k[(|\Delta xk|)]^{pk} \le k^{-s_1} f_k[(|\Delta x_k|)]^{pk}$$

Hence, we have

$$\sum_k a_{nk}(i) k^{-s_2}[f_k(|\Delta x_k - L|)]^{pk} \le \sum_k a_{nk}(i) k^{-s_1}[f_k(|\Delta x_k - L|)]^{pk}$$

Since, $x \in W_0[A, \Delta, F, p, s_1]$, we get $x \in W_0[A, \Delta, F, p, s_2]$.

The next theorem shows that the relation between $W[A, \Delta, F, p, s]$ and $W[A, \Delta, G, p, s]$ for sequence of moduli F and G. □

Theorem 2.5: *Suppose that* $F = (f_k)$ *and* $G = (g_k)$ *be sequence of moduli and* $g_k \ge f_k$ *, for all k. Then*

$$\lim_{x \to \infty} \frac{f_k(x)}{g_k(x)} < \infty$$

implies $W[A, \Delta, G, p, s] \subset W[A, \Delta, F, p, s]$.

Proof: It is trivial. □

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AN ITERATIVE METHOD FOR SOLVING SYSTEM OF NONLINEAR ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

Abstract: Arranging into a specific form under a certain condition, a general system of nonlinear algebraic or tran-scendental equations is solved by the use of an iterative method which is proved mathematically. Geometrical interpretation has also been given. The method has been enlightened by some examples and its disadvantages are also noted.

Keyword: Iterative Method, Nonlinear Simultaneous Equations, Geometrical Interpretation.

Mathematical Subject Classification No.: 65H10.

1. Introduction

Linear or nonlinear systems of equations arise in practical problems in various branches of science, e.g. physics, chemistry, engineering, mechanics and so on. Application of the analytic mathematical method for solving those problems becomes sometimes complicated or even impossible. Numerical process serves as an important tool to overcome the difficulties to a greater extent.

Numerical solutions for system of equations was initiated by Jacobi [7], who devised an iterative process to solve linear simultaneous equations. Gauss and Seidel modified this method (Gauss-Seidel iteration) [7], when the iteration converges rapidly. In 1949, Reich [12] described the necessary and sufficient condition for the convergence of the Jacobi's method for a particular model having symmetric real coefficient matrix with positive diagonal elements. Later, a sufficient condition for convergence of both Jacobi and Gauss-Seidel iterative method, the absolute value of each diagonal element of the coefficient matrix of which being greater than the sum of the absolute values of other elements in a particular row was considered by Collatz [4].

Now, to solve nonlinear equations with a single variable bisection, secant, false position, Newton-Raphson etc. methods are generally used. Ridders [13] modified the false position method using a quadric curve (formed by two given points and their midpoint) instead of a straight line. Also many researchers studied the same to improve these methods.

The solution of nonlinear system of equations is much difficult in comparison to single variable and numerical processes are generally used to solve it. A very important tool is Newton's method [3, 11] which is also an iterative process. In the method, the author used Taylor's expansion theorem to obtain the new point at each step. In addition quasi-Newton, secant, Broyden's methods are also found in literature [11]. Using quasi-Newton technique, Broyden [2] attempted to find a new method, each step of which needed a matrix (similar to Jacobian matrix in Newton's method) obtained by Sherman-Morrison formula. Moreover many other methods e.g. fixed point (for functions of several variables), steepest descent, homotopy and continuation etc. methods for solving system of nonlinear equations are also in use [3].

Bader [1] pointed out the disadvantages of Newton's method for multivariable problems in computation. To solve the problems, the author used tensor technique and Krylov subspace method. Effati and Nazemi [5] proposed a very effective process for solving nonlinear system of equations. Introducing a norm function the authors transformed the system into a minimization problem. Transforming the system of equations into a constraint optimization problem, a new technique was developed by Nie [10]. At each step, the author considered some equations satisfing the current point as constraints and other as objective function. Grosan *et al*. [6] studied nonlinear system of equations and proposed a method by transforming the system into a multiobjective optimization problem. Here the authors used Pareto dominance relationship between actual solutions and some random solutions evolved by iterative strategy searching for optimal solutions. Khirallah *et al*. [9] prepared a paper for solving a system of nonlinear equations using fourth order Jarratt method of iteration. Izadian *et al*. [8] proposed another method to solve the nonlinear system of equations. Here the authors used Newton method and homotopy analysis method to solve the problem.

In the present paper we study a fixed point iterative method for solving nonlinear simultaneous equations. The theory is described with a sufficient condition for convergence and it is also pointed out that, under certain condition, the convergence of the iteration depends on the arrangement of the explicit variables or equations of the system on the iteration table. The worthnoting of this method is that it covers many other iterative processes for solving linear and nonlinear system of equations and its computation is rather easy. The method has been interpreted geometrically and also illustrated by some examples. Disadvantages of this tool are also pointed out.

2. Theorem

Consider a system of nonlinear equations

$$\begin{cases} f_1(\mathbf{X}) = 0 \\ f_2(\mathbf{X}) = 0 \\ \dots \ \dots \ \dots \ \dots \\ f_m(\mathbf{X}) = 0, \end{cases} \tag{1}$$

where $\mathbf{X} = (x_1, x_2, \dots, x_m)$ and suppose that a solution of the system (1) is $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ which is contained in or a limit point of a given region σ_m (neither containing any other solution of (1) nor as a limit point), and each equation of (1) possesses its first order bounded partial derivative with respect to the arguments.

In the region σ_m, for any $j(1 \le j \le m)$, suppose

$$\max_i \left\{\left|\frac{f_{i,j}}{\left|\nabla f_i\right|}\right|\right\} = \left|\frac{f_{l,j}}{\left|\nabla f_l\right|}\right|, \text{ (say)}, \tag{2}$$

for a unique $l(1 \le l \le m)$ depending on j, where $\left|\nabla f_i\right| = \sqrt{f_{i,1}^2 + f_{i,2}^2 + \dots + f_{i,m}^2}$ and $f_{i,j} = \dfrac{\partial f_i}{\partial x_j}$.

We now select $f_l(\mathbf{X}) = F_j(\mathbf{X})$. If $F_j(\mathbf{X}) = 0$ is possible to express in the explicit form as $x_j = \phi_j(x_1,\ x_2,\ ..\ x_{j-1},\ x_{j+1},\ ..\ x_m), \forall j$, then using (2) the system of equations (1) is transformed into

$$\begin{cases} x_1 = \phi_1(x_2, x_3, \cdots x_m), \\ x_2 = \phi_2(x_1, x_3, \cdots x_m), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_m = \phi_m(x_1, x_2, \cdots x_{m-1}), \end{cases} \tag{3}$$

with

$$\frac{\phi_{i,j}}{\left|1 - \nabla\phi_i\right|} < \frac{1}{\left|1 - \nabla\phi_j\right|}, \forall j \text{ and } i, i \neq j. \tag{4}$$

Now the system of equations (3) can be solved sufficiently with solution $\mathbf{X} = \alpha$ by iterative method taking any initial approximation in σ_m.

3. Method

First of all, the given system of equations is to be arranged into the form as mentioned above. Choose the initial approximation in σ_m as $x_1 = x_1^{(0)}$, $x_2 = x_2^{(0)}, \ldots, x_m = x_m^{(0)}$. Next putting this set of values in the first equation $x_1 = \phi_1(x_2, x_3, \ldots x_m)$ of (3) we obtain the value of x_1 as $x_1^{(1)}$. Now to get $x_2^{(1)}$, we use the set $(x_1^{(1)}, \_, x_3^{(0)}, \cdots, x_m^{(0)})$ of other variables in the 2nd equation of (3) and continue the process till we obtain the set $(x_1^{(1)}, x_2^{(1)}, \ldots, x_m^{(1)})$. This set is to be used as a starting value to get a new valued set and repeat the process.

To obtain the value of a variable from an equation, we use the set of all other immediate past valued variables. One such process of calculation is said to be a step. Any successive m steps using all m equations of the system form a cycle.

4. Proof

First we prove the theorem for two variables and then using this rule we shall consider the general case. For two-variable problem, the system is

$$\begin{cases} x_1 = \phi_1(x_2), \\ x_2 = \phi_2(x_1), \end{cases} \tag{5}$$

with the conditions

$$\frac{|\phi_{1,2}(x_2)|}{\sqrt{1+\phi_{1,2}^2(x_2)}} < \frac{1}{\sqrt{1+\phi_{2,1}^2(x_1)}}$$

and

$$\frac{|\phi_{2,1}(x_1)|}{\sqrt{1+\phi_{2,1}^2(x_1)}} < \frac{1}{\sqrt{1+\phi_{1,2}^2(x_2)}} \tag{6}$$

in the region σ_2, which contains only a solution $\alpha = (\alpha_1, \alpha_2)$, ($\alpha$ may be a limit point of σ_2) so that

$$\begin{cases} \alpha_1 = \phi_1(\alpha_2), \\ \alpha_2 = \phi_2(\alpha_1), \end{cases} \tag{7}$$

Now conditions (6) give us

$$|\phi_{1,2}(x_2)| . |\phi_{2,1}(x_1)| < 1, \forall (x_1, x_2) \in \sigma_2. \tag{8}$$

If at the nth cycle iteration, the approximate solution and the corresponding error are $(x_1^{(n)}, x_2^{(n)})$ and $(\xi_1^{(n)}, \xi_2^{(n)})$ respectively, then by the above described process we get,

$$\begin{cases} x_1^{(n)} = \phi_1(x_2^{(n-1)}), \\ x_2^{(n)} = \phi_2(x_1^{(n-1)}). \end{cases} \tag{9}$$

Subtracting (9) from (7) and applying the mean-value theorem we obtain,

$$\begin{cases} \xi_1^{(n)} = \xi_2^{(n-1)} \phi_{1,2}(x_{21}^{(n-1)}), \\ \xi_2^{(n)} = \xi_1^{(n)} \phi_{2,1}(x_{12}^{(n)}). \end{cases} \tag{10}$$

where $x_{12}^{(n)}$ lies in α_1 and $x_1^{(n)}, x_{21}^{(n-1)}$ lies in α_2 and $x_2^{(n-1)}$ and so that $(x_{12}^{(n)}, x_{21}^{(n-1)}) \in \sigma_2$.

Now using the imposed conditions (8), we get from (10),

$$\frac{\left|\xi_2^{(n)}\right|}{\left|\xi_2^{(n-1)}\right|} = \left|\phi_{1,2}(x_{21}^{(n-1)})\right| . \left|\phi_{2,1}(x_{12}^{(n)})\right| < 1. \tag{11}$$

Similarly, in view of the first equation of nth cycle and the last equation of $(n-1)$ th cycle we have

$$\frac{\left|\xi_1^{(n)}\right|}{\left|\xi_1^{(n-1)}\right|} < 1. \tag{12}$$

The conditions (11) and (12) show that $x_1^{(n)}, x_2^{(n)}$ converges to the exact solution (α_1, α_2).

The general case: Suppose that the approximate solution of the system of equations (3) at the nth cycle of the iterative method is $(x_1^{(n)}, x_2^{(n)}, \ldots, x_m^{(n)})$ and the corresponding error is $(\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_m^{(n)})$. For this system, if at any cycle, we proceed from one step to the next, $(m-1)$ variables remain unaltered and only one variable is changed and the change occurs for only two equations amongst all.

As only one variable varies and all others remain unchanged, so, for our convenience, we may consider for that particular step of the cycle, the problem is of two-variable of which one is changed and the other is chosen arbitrarily from the rest and remaining all other variables unaltered.

If we are interested in obtaining the value of x_i, say, after obtaining the value of $x_j, (j \neq i)$ at nth cycle, then we may assume for this particular step that the ith and jth equations of (3) are of x_i and x_j variables only. Using the concept of (11) and (12) we see that, $x_i^{(n)}$ is more closer to the component α_i of the solution α than $x_i^{(n-1)}$, i.e., we obtain a better result corresponding to the variable x_i.

Since i and j are arbitrary, so the rule is satisfied for all variables and the result obtained in a cycle is obviously closer to α than the previous cycle and if we continue the process, then the result must converge to the exact solution.

5. Geometrical Interpretation

We discuss the method geometrically for three variables and take the system as

$$F_1(\mathbf{X}) = 0,\ F_2(\mathbf{X}) = 0,\ F_3(\mathbf{X}) = 0,$$

with the conditions $|\psi_{ij}| < |\psi_{jj}|^{\mathbf{1}}$, $\forall i,\ j(i \neq j)$, in the region σ_3, where $\mathbf{X} = (x_1, x_2, x_3)$ and $\psi_{ij} = \dfrac{F_{i,j}}{|\nabla F_i|}$.

We know that ψ_{ij} represents the component of the unit normal to the surface $F_i(\mathbf{X}) = 0$ along x_j at the point under consideration. The increase or decrease of the absolute value of this component indicates respectively the approach or departure in angular distance of the normal from any side of the corresponding axis (the angle is supposed to lie between 0° and 90°). When the normal approaches more towards the axis, the corresponding tangent plane at that point departs more from the axis, that means if $|\psi_{ij}| > |\psi_{kj}|$, then the surface $F_i(\mathbf{X}) = 0$ departs more from x_j-axis than the surface $F_k(\mathbf{X}) = 0$.

Consider, for instant the x_1-axis. We choose an arbitrary point $P(x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$ in σ_3 and draw a plane Pox_1 through the point P and x_1-axis. The intersection of this plane and the surface of the equations $F_j(\mathbf{X}) = 0$, $(j = 1, 2, 3)$ are the curves lying on the plane. Sketch a straight line PK through P and parallel to x_1-axis on the plane.

Since $|\psi_{i1}| < |\psi_{11}|$, ψ_{11} i.e. $F_{1,1}$ unalters its sign and PK must intersect the curve of $F_1(\mathbf{X}) = 0$. Suppose the solution point is S. Now take a projection of the curves lying on PKS, on the plane Pox_1 and let S' be the projected point of S. As $|\psi_{11}| > |\psi_{j1}|$, $(j = 2, 3)$, in σ_3, so, in the given region, the surface corresponding to the function F_1 is angularly furthest from x_1-axis compared to any other surfaces; consequently, the projected curve of $F1$ is angularly furthest from x_1-axis compared to other projected curves on Pox_1.

Thus, the value of x_1 obtained by the intersection of PK and curve of F_1 is nearest to α_1, the component of the solution α along x_1-axis, compared to other.

Similar results can also be found for any choice of x_2 or x_3-axis and after completion of a cycle it can be found that we come toward the solution than previous cycle. This concept helps to think the general case.

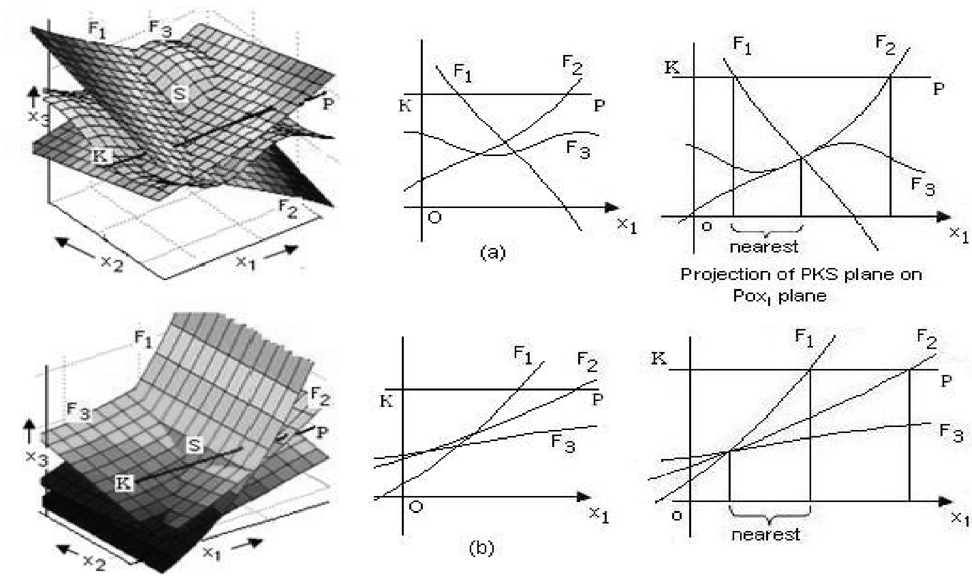

Figure 1

6. Discussions

Fixed point iterative method is well known. In this paper the described condition of convergence extends the field of application in various branches of science. The technique is very much easy and effective for solving linear and nonlinear simultaneous equations. Some examples are analyzed to show the importance of the method.

Example 1: Let us take the system of equations as

$$e^x + y = 1$$
$$x + e^y = 1.$$

It is clear that both the curves corresponding to the equations do not exist in the region $x, y > 0$ and $x, y < 0$. Now in the region $x < 0$, $y > 0$ or $x > 0$, $y < 0$ we see that

$$\left|\frac{e^x}{(1+e^{2x})^{\frac{1}{2}}}\right| < \left|\frac{1}{(1+e^{2y})^{\frac{1}{2}}}\right|$$

and

$$\left|\frac{1}{(1+e^{2x})^{\frac{1}{2}}}\right| < \left|\frac{e^y}{(1+e^{2y})^{\frac{1}{2}}}\right|. \tag{13}$$

so we need to transform the system into

$$\begin{aligned} y &= 1 - e^x \\ x &= 1 - e^y. \end{aligned} \tag{14}$$

Taking the initial value $x = 2$, $y = -2$ iterative calculation is shown in Table 1.

Table 1

n	y	x
1	-6.3891	.9983
2	-1.7137	.8198
5	-.9175	.6005
10	-.6142	.4589
20	-.4199	.3429
50	-.2581	.2275

Clear that the result converges to $x = 0$, $y = 0$.

Example 2: Choose a system of linear simultaneous equations as

$$10x + 4y + z = 15,$$

$$x + 8y + 2z = 11,$$

$$5x + 4y + 3z = 12.$$

For this problem, the condition asserted above is satisfied in $\mathbf{R}^3$ and we need to transform the system into

$$x = (15 - 4y - z) / 10$$

$$y = (11 - x - 2z) / 8$$

$$z = (12 - 5x - 4y) / 3.$$

Taking the initial approximation as $x = 2$, $y = 2$, $z = 2$, the calculation of iteration is shown in Table 2 and we see that the result converges to the exact solution $x = y = z = 1$.

Table 2

n	x	y	z
1	.5	.8125	2.0833
2	.96667	.73333	1.4111
3	1.0656	.88903	1.0387
4	1.0405	.98526	.95212
5	1.0107	1.0106	.96801

There are many problems, where we can't apply this technique directly but after a suitable substitution we can apply it (example 3).

Example 3: Consider a system as

$$3e(x^2 - y^2) + ln(xy)\cos(xy) + 2z = 4,$$

$$\sin(x^2 - y^2) + ln(2x^5y^5z) = 0,$$

$$\sin(xy(x^2 - y^2)) + \cos(x^2 - y^2) + 8z = 5.$$

All the equations of this system can't be expressed in explicit form. Now putting $x^2 - y^2 = u$, $ln(xy) = v$ and $2z = w$ we see that, that can be done and the system satisfies the asserted condition in $-1 \le u \le 1$, $-1 \le v \le 1$ and $.5 \le w \le 2$ which contains a solution and transforming the system in explicit form we have

$$u = ln\left(\frac{4 - v\cos(e^v) - w}{3}\right),$$

$$v = \frac{\sin(u) + ln(w)}{5},$$

$$w = \frac{5 - \cos(u) - \sin(ue^v)}{4}.$$

Choosing the initial approximation $u = 1$, $v = 1$ and $w = .5$, the table of iteration is shown bellow.

Table 3

No.	u	v	w
1	.23742	9.1591×10^{2}	.94270
2	1.4345×10^{2}	8.9333×10^{3}	.99641
3	6.6849×10^{4}	5.8609×10^{4}	.99983
4	2.0577×10^{5}	2.9326×10^{5}	.99999
5	-5.177×10^{8}	1.0356×10^{6}	1

After fifth cycle, the approximate result is $u = -5.177 \times 10^{-8}$, $v = 1.0356 \times 10^{-6}$, $w = 1$. Therefore the approximate solution of the given problem is $x = 1.0000$, $y = 1.0000$, $z = .5$, which is exact.

The rate of convergence of this method would be much speedy if for some j, $|\psi_{i,j}| \ll |\psi_{j,j}|$, $\forall i,\ i \ne j$. In example 1, we see that the result converges very slowly, because the relation $|\psi_{i,j}| < |\psi_{j,j}|$ holds very lightly as it towards the solution whereas an opposition occur in example 3 and the result converges rapidly.

The most advantage of the present iterative method is that, its algorithm and computation are easier than others. The number of computations for m variables to complete a cycle is only m.

7. Disadvantages

It is to be pointed out that, for a given j, if $\max_i \{|\psi_{ij}|\}$ occurs for some or all i in the region σ_m of a system, then the result obtained by the iterative method does not necessarily converge. For example, consider a system where $|\psi_{ij}| = |\psi_{jj}|$, $\forall j$ and i. If this case occurs for two variables problem, then the obtained values of the variables repeat after finite cycles (example 4), since the curve of any equation is a reflection of the other with respect to a line through the solution point and parallel to any axis. This may not arise for three or more variables problems, in such cases the value of the variables either repeat or diverge (example 5).

Again, if for any $j(1 \le j \le m)$, $\max\{|\psi_{ij}|\}$ occurs for many i but not for all, then sometime the result may converge (example 6). If we can arrange the equations in a sequence satisfying $|\psi_{(i-1)i}| < |\psi_{ii}| \forall i$, there is a chance for convergence otherwise it diverges. Also using same logic some arrangement gives undesired result. In that case it jumps the solution point and tends to diverge.

Example 4: Let the system is $2x + y = 3, 2x - y = 1$. Arranging $x = (3 - y)/2$ and $y = 2x - 1$, we see from Table 4 that the values of x and y repeat and its cause is discussed above.

Table 4

n	x	y
0	2	-2
1	2.5	4
2	-.5	-2
3	2.5	4

Example 5: Choose the system as $4x - 2y + z = 3$, $4x + 2y - z = 5$, $4x + 2y + z = 7$.

Table 5

n	x	y	z
1	1.25	1	0
2	1.25	0	2
3	.25	3	0
4	2.25	-2	2
5	-.75	5	0

Table 6

n	x	z	y
0	2	2	2
1	0.75	0	0
2	1.25	2	2
3	0.75	0	0

Arranging $x = (3 + 2y - z)/4$, $y = (5 - 4x + z)/2$ and $z = 7 - 4x - 2y$, the Table 5 shows that the result diverges.

Again setting $x = (5 - 2y + z)/4$, $z = 7 - 4x - 2y$ and $y = (4x + z - 3)/2$, we see from the Table 6 that the value of all variables repeat.

Both cases of the problem are already discussed above in detail.

Example 6: Consider the system as

$$4x + 3y + 2z = 9,$$

$$4x - 2y - 3z = -1,$$

$$x + y + z = 3.$$

We choose $x = 2$, $y = 2$, $z = 2$ as initial guess. Arranging in given sequence we see that $|\psi_{11}| = |\psi_{21}| = \max\{|\psi_{j1}|\}$, $|\psi_{32}| = \max\{|\psi_{j2}|\}$ and $|\psi_{33}| = \max\{|\psi_{j3}|\}$.

Now if z is implicated from 3rd equation, then y have to implicate from rest. As $|\psi_{12}| > |\psi_{22}|$, so implicating y from first equation and x from remain the iterative calculation is shown in Table 7.

Table 7

n	x	y	z
1	2.2500	-1.3333	2.0833
2	0.6458	0.7500	1.6042
4	0.9640	0.7066	1.3294
6	1.0134	0.8138	1.1731
10	1.0084	0.9455	1.0461
20	1.0003	0.9980	1.0016

Also implicating y from 3^{rd}, z from 2^{nd} and x from rest the calculation is shown in Table 8.

Table 8

n	x	z	y
1	-0.2500	-1.3333	4.5833
2	-0.5208	-3.4167	6.9375
3	-1.2448	-5.9514	10.1962
4	-2.4214	-9.6927	15.1141
5	-4.2393	-15.3951	22.6344
6	-7.0282	-24.1272	34.1554

We see that for the first arrangement of equations the result converges but it diverges for second arrangement.

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END NOTE

1. This condition does not differ by condition (4) as F_j (**X**) = 0 and $x_j = \varphi_j$ represent a single surface.

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EXPANSION OF GENERATING FUNCTIONS INVOLVING VARIOUS POLYNOMIALS

Abstract: In this paper, we discuss the generating functions involving the product of various polynomials like modified Laguerre polynomials $L_n^{(\alpha-n)}(x)$, modified Jacobi polynomials $P_m^{(\alpha,\ \beta-m)}(r)$, modified Bessel polynomials $Y_p^{(\alpha+n)}(u)$, and the confluent hypergeometric functions ${}_1F_1[.]$ and then obtain generating functions by group-theoretic method. Also discuss their applications. Earlier Chandel, Kumar and Senger [1] introduce the generating functions involving the product of modified Bessel polynomials $Y_p^{(\alpha+n)}(u)$ and the confluent hypergeometric functions ${}_1F_1[.]$. In the present paper, we have introduced three linear partial differential operators R_1, R_2 and R_3 to obtained the generating relation.

Keywords: Modified Laguerre Polynomials, Modified Bessel Polynomials, Modified Jacobi Polynomials, Confluent Hypergeometric Functions, Group-Theoretic Method, Generating function.

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1. Introduction

Generating functions play an important role in the study of special functions. Group theoretic method have been mostly used by researchers in the derivation of generating functions of special functions comparison to other methods. Therefore in the present paper, group-theoretic method has been adopted to obtain the results of generating functions involving modified Laguerre polynomials $L_n^{(\alpha-n)}(x)$, modified Jacobi polynomials $P_m^{(\alpha,\ \beta-m)}(r)$, and modified Bessel polynomials $Y_p^{(\alpha+n)}(u)$ are defined by Srivastava and Manocha [7] as:

$$L_n^{(\alpha-n)}(x) = \frac{\Gamma(1+\alpha)}{\Gamma(1+n)\Gamma(1+\alpha-n)}\ {}_1F_1[\quad n; 1+\alpha \quad n; x] \tag{1.1}$$

$$P_n^{(\alpha,\ \beta)}(r) = \frac{(1+\alpha)_n}{n!}\ {}_2F_1\left[-n, 1+\alpha+\beta+n; 1+\alpha;\ \frac{1-r}{2}\right] \tag{1.2}$$

and

$$Y_p^{(\alpha+n)}(u) = \ {}_2F_0\left[\quad p,\ p+n+\alpha \quad 1\,; \quad ; \quad \frac{u}{\beta}\right] \tag{1.3}$$

In fact, while constructing the partial differential operators for the polynomials $L_n^{(\alpha-n)}(x)$, $P_m^{(\alpha,\ \beta-m)}(r)$ and $Y_p^{(\alpha+n)}(u)$, we have adopted the group-theoretic method as introduced by Weisner [6].

The confluent hypergeometric functions ${}_1F_1[.]$ can be replaced by many special functions. Srivastava and Manocha [7] defined and studied various bilinear, bilateral and multilinear generating functions.

In this paper, we introduce the following new general class of generating functions:

$$G(x,r,u,q,w) = \sum_{n=0}^{\infty} a_n\ L_n^{(\alpha-n)}(x)\ P_m^{(\alpha,\ \beta-m)}(r) Y_p^{(\alpha+n)}(u)\,{}_1F_1[-n; m+1; q]\, w^n \tag{1.4}$$

Where a_n is any arbitrary sequence independent of x, u, q and w.

Again in (1.4) setting various values of a_n , we may find several results on generating functions involving different special functions, hence (1.4) is a general class of generating functions.

In this paper, we evaluate some more general class of generating functions and finally discuss their applications.

2. Group-Theoretic Operators

In our investigations, we use the following group-theoretic operators:

The operators R_1 due to Majumdar [4] is given by

$$R_1 = xyz\,\frac{\partial}{\partial x} \quad y^2 z\,\frac{\partial}{\partial y} \quad (x \quad \alpha)yz \tag{2.1}$$

Such that

$$R_1\left[L_n^{(\alpha-n)}(x)y^n z^\alpha\right] = (n+1)\,L_{n+1}^{(\alpha-n-1)}(x)y^{n+1}z^{\alpha+1} \tag{2.2}$$

The operators R_2 due to Chongdar [2] is given by

$$R_2 = (1 \quad r^2)s\frac{\partial}{\partial r} \quad 2s^2\frac{\partial}{\partial s} \quad [(1+\alpha+\beta+p)(1+r) \quad 2\beta]s \tag{2.3}$$

Such that

$$R_2\left[P_m^{(\alpha,\ \beta-m)}(r)\,s^m\right] = \quad 2(m+1)P_{m+1}^{(\alpha,\ \beta-m-1)}(r)\,s^{m+1} \tag{2.4}$$

The operators R_3 due to Chongdar [3] is given by

$$R_3 = \ u^2t^{-1}v\,\frac{\partial}{\partial u} + uv\,\frac{\partial}{\partial t} + ut^{-1}v^2\frac{\partial}{\partial v} + \ t^{-1}v\,(\beta \quad u) \tag{2.5}$$

Such that

$$R_3\left[Y_p^{(\alpha+n)}(u)\,t^n v^p\right] = \ \beta\,Y_{p+1}^{(\alpha+n-1)}(u)\,t^{n-1}v^{p+1} \tag{2.6}$$

The operator R_4 due to Miller Jr. [5] is given by

$$R_4 = j\,\frac{\partial}{\partial k} + jqk^{-1}\frac{\partial}{\partial q} \quad jq\,k^{-1} \tag{2.7}$$

Such that

$$R_4[{}_1F_1[\quad n; m+1; q]j^n k^m] = m\ {}_1F_1[-n \quad 1; m; q]j^{n+1}\,k^{m-1} \tag{2.8}$$

The actions of R_1, R_2, R_3 and R_4 on function f are obtained as follows:

$$e^{wR_1}f(x,y,z) = (1+wyz)^\alpha\,exp(\quad wxyz)\,f\left[x(1+wyz), \frac{y}{1+wyz}, z\right] \tag{2.9}$$

(cf. Majumdar [4])

$$e^{wR_2}\,F(r,s) = \ \{1+ws(1+r)\}^{-1-\alpha-\beta-c}\,(1+2ws)^\beta\,F\left[\frac{r+ws(1+r)}{1+ws(1+r)}, \frac{s}{1+2ws}\right] \tag{2.10}$$

(cf. Chongdar [2])

$$e^{wR_3}F(u,t,v) = (1 \quad wut^{-1}v)\, exp(\beta wt^{-1}v)\, F\left[\frac{u}{1-wut^{-1}v}, \frac{t}{1-wut^{-1}v}, \frac{v}{1-wut^{-1}v}\right] \tag{2.11}$$

(cf. Chongdar [3])

and

$$e^{wR_4}f(j,k,q) = \exp\left(\frac{-qjw}{k}\right) f\left[j, k+wj, \; q\left(1+\frac{wj}{k}\right)\right] \tag{2.12}$$

(cf. Miller Jr. [5])

3. Main Result

Theorem: If there exists a generating functions involving the triple product of modified Laguerre polynomials $L_n^{(\alpha-n)}(x)$, modified Jacobi polynomials $P_m^{(\alpha,\ \beta-m)}(r)$, modified Bessel polynomials $Y_m^{(\alpha+m)}(u)$ and the confluent hypergeometric functions ${}_1F_1[\ \ n;\ m+1; q]$ given by

$$G(x,r,u,q,w) = \sum_{n=0}^{\infty} a_n\, L_n^{(\alpha-n)}(x)\, P_m^{(\alpha,\ \beta-m)}(r)\, Y_p^{(\alpha+m)}(u)\, {}_1F_1[-n; m+1; q]\, w^n \tag{3.1}$$

Then the following more general class of generating functions holds:

$$(1+wy)^{\alpha+m}\{1+ws(1+r)\}^{-1-\alpha-\beta-c}\,(1+2ws)^{\beta}(1 \quad \mathrm{wut}^{-1}v)^{1-\mathrm{p}}\ \exp[\ \ w(x \quad \beta \mathrm{t}^{-1}v+q)].\, G\left[x(1+wy), \frac{r+ws(1+r)}{1+ws(1+r)}, \frac{u}{1-wu}, q(1+w), \frac{wyv}{1+w}\right]$$

$$= \sum_{n,b,c,d,e=0}^{\infty} \frac{a_n\,(-2)_c (n+1)_b\,(m+1)_c}{b!\ \ c!\, d!\, e!} . L_{n+b}^{(\alpha-n-b)}(x)\, P_{m+c}^{(\alpha,\ \beta-m-c)}(r)\, Y_{p+d}^{(\alpha+n-d)}(u)$$

$${}_1F_1[-n \quad e;\ m \quad e+\ 1;\ q]\, (wy)^b\, (w\beta t^{-1}v)^d (mw)^e (wyj)^n \tag{3.2}$$

The importance of the above theorem lies in the fact that whenever one knows a generating relation of type (3.1), the corresponding general class of generating relation can at once be written down from (3.2). Thus a large number of bilateral generating relations can be obtained by attributing different values to a_n in (3.1).

Proof of the theorem: In the general class of generating functions (3.1), replacing w by $wytj$ and then multiplying by $z^{\alpha}\ s^m v^p\ k^m$ on both sides, we get

$$G(x,r,u,q,wytj)\, z^{\alpha}\ s^m v^p\ k^m = \sum_{n=0}^{\infty}\ a_n\ L_n^{(\alpha-n)}(x)\, P_m^{(\alpha,\ \beta-m)}(r)\ \ Y_p^{(\alpha+n)}(u).$$

$${}_1F_1\left[-n; m+1; q\right]\ y^n\ t^n\ j^n\ \ z^{\alpha}\ s^m v^p\ k^m w^n \tag{3.3}$$

Now, operating both the sides of (3.3) with $e^{wR_1}e^{wR_2}e^{wR_3}$, we obtain

$$e^{wR_1}e^{wR_2}e^{wR_3}G(x,r,u,q,wytj)\, z^{\alpha}\, s^m v^p\, k^m = e^{wR_1}e^{wR_2}e^{wR_3}\sum_{n=0}^{\infty} a_n\, L_n^{(\alpha-n)}(x)$$

$$y^n\;\; z^{\alpha}\, P_m^{(\alpha,\;\beta-m)}(r)s^m\;\; Y_p^{(\alpha+n)}(u)\, t^n v^p\; {}_1F_1[-n;m+1;q]\,.\; j^n\, k^m w^n \qquad (3.4)$$

The left hand side of (3.4) becomes

$$z^{\alpha}\,(1+wyz)^{\alpha}\,\{1+ws(1+r)\}^{-1-\alpha-\beta-c}(1+2ws)^{\beta}(1 \quad wut^{-1}v)\left(\frac{v}{1 \quad wut^{-1}v}\right)^p$$

$$(k+wj)^m\; \exp\!\left(\quad wxyz+\beta wt^{-1}v \quad \tfrac{qjw}{k}\right).$$

$$G\left[x(1+wyz)\,,\frac{r+ws(1+r)}{1+ws(1+r)},\frac{u}{1-wut^{-1}v}\,,q\left(1+\tfrac{wj}{k}\right),\frac{wyv}{1+wyz}\right] \qquad (3.5)$$

And the right hand side of (3.4) becomes

$$\sum_{n,b,c,d,e=0}^{\infty}\frac{a_n\,(-2)_c(n+1)_b\,(m+1)_c\,\beta^d\, m^e w^{n+i+j+k}}{b!\;\; c!\, d!\, e!}\; L_{n+b}^{(\alpha-n-b)}(x)y^{n+b}\;\; z^{\alpha+b}P_{m+c}^{(\alpha,\;\beta-m-c)}(r)s^{m+c}$$

$$Y_{p+d}^{(\alpha+n-d)}(u)t^{n-d}v^{p+d}\;\; {}_1F_1[-n \quad e;\; m \quad e+\;1;\; q]\; j^{n+e}\, k^{m-e} \qquad (3.6)$$

Now equating (3.5) and (3.6), and setting $\mathrm{j}=\mathrm{k}$ and $\mathrm{z}=1$, we obtain

$$(1+\mathrm{wy})^{\alpha+\mathrm{m}}\,\{1+ws(1+r)\}^{-1-\alpha-\beta-c}\,(1+2\mathrm{ws})^{\beta}(1 \quad \mathrm{wut}^{-1}v)^{1-\mathrm{p}}$$

$$\exp(\quad \mathrm{w}(x \quad \beta\mathrm{t}^{-1}v+\;\mathrm{q})\,.\,\mathrm{G}\left[\mathrm{x}(1+\mathrm{wy})+\frac{\mathrm{u}}{1 \quad \mathrm{wut}^{-1}v},\mathrm{q}(1+\mathrm{w}),\frac{\mathrm{wy}v}{1+\mathrm{w}}\right]$$

$$=\sum_{n,b,c,d,e=0}^{\infty}\frac{a_n\,(-2)_c(n+1)_b\,(m+1)_c}{b!\;\; c!\, d!\, e!}\,.\,L_{n+b}^{(\alpha-n-b)}(x)\; P_{m+c}^{(\alpha,\;\beta-m-c)}(r)\; Y_{p+d}^{(\alpha+n-d)}(u)$$

$${}_1F_1[-n \quad e;\; m \quad e+\;1;\; q]\,(wy)^b\,(w\beta t^{-1}v)^d(mw)^e(wyj)^n \qquad (3.7)$$

which is the required result.

4. Special Cases

Taking $r=o,\ u=o,\ q=0$ in given theorem and proceeding as the proof of the main theorem, we get

$$(1+\mathrm{wy})^{\alpha+\mathrm{m}} \exp(\quad wx).\mathrm{G}\left[\mathrm{x}(1+\mathrm{wy}),\ \frac{\mathrm{wy}v}{1+\mathrm{w}}\right]=$$

$$\sum_{n,b=0}^{\infty} \frac{a_n\,(n+1)_b\,(wy)^b}{b!}.L_{n+b}^{(\alpha-n-b)}(x)\ \ (wy)^n$$

$$\sum_{n=0}^{\infty}\sum_{b=0}^{n} \frac{a_{n-b}\,(n\quad b+1)_b\,w^n}{b!}\ L_n^{(\alpha-n)}(x)\,y^n$$

$$=\sum_{n=0}^{\infty}\ \sigma_n(x,y).w^n \tag{4.1}$$

Where $\sigma_n(x,y)=\sum_{b=0}^{n} \frac{a_{n-b}\,(n-b+1)_b}{p!}\ L_n^{(\alpha-n)}(x)\,y^n$ (4.2)

Which is given by Majumdar [4].

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