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*A. A. Qureshi*

# PLANE SYMMETRIC SPACE-TIME WITH WET DARK ENERGY IN BIMETRIC RELATIVITY

**Abstract**: In this paper, plane symmetric space-time is studied with the matter wet dark energy in the context of Rosen's Bimetric Theory of Relativity. Here it is shown that only vacuum model can be obtained.

**Keywords**: Plane Symmetric, Wet Dark Energy, Bimetric Relativity, General Relativity.

**Mathematical Subject Classification No.:** 83C05.

## 1. Introduction

**A** new theory of gravitation called the Bimetric theory of gravitation, was proposed by Rosen [12, 13, 9] to modify the Einstein's general theory of relativity by assuming two metric tensors, viz., a Riemannian metric tensor $g_{ij}$ and a background metric tensor $\gamma_{ij}$ . The metric tensor $g_{ij}$ determines the Riemannian geometry of the curved space time which plays the same role as given in the Einstein's general relativity and it interacts with matter. The background metric tensor $\gamma_{ij}$ refers to the geometry of the empty (free from matter and radiation) universe and describes the inertial forces. This metric tensor $\gamma_{ij}$ has no direct physical significance but appears in the field equations. Therefore it interacts with $g_{ij}$ but not directly with matter. One can regard $\gamma_{ij}$ as describing the geometry that would exist if there were no matter. Moreover, the bimetric theory also satisfied the covariance and equivalence principles: the formation of general relativity. The theory agrees with the present observational facts pertaining to general relativity. Thus at every point of space-time there are two line elements:

$$ds^2 = g_{ij} dx^i dx^j \tag{1.1}$$

and

$$d\sigma^2 = \gamma_{ij} dx^i dx^j \tag{1.2}$$

Where $ds$ is the interval between two neighboring events as measured by means of a clock and a measuring rod. The interval $d\sigma$ is an abstract or geometrical quantity not directly measurable. One can regard it as describing the geometry that would exist if no matter were present.

Yilmaz (12-14), Israelit (6-7) have studied various aspects of bimetric theory of relativity. In continuation of this study Deo, S. D. (3), Deo and Ronghe (4-5), Deo and Qureshi (1-2 ), Deo and Suple (11) have studied several aspects of Bianchy Type model, Plane Symmetric model and Plane gravitational waves respectively in the context of bimetric theory of relativity with various source of matters like cosmic string, wet dark fluid, massive meson etc.

In this paper, a study of plane symmetric space-time with wet dark energy shall be undertaken and will observe the result in the context of Bimetric theory of relativity.

**2. Field Equations in Bimetric Relativity**

Rosen N. has proposed the field equations of Bimetric Relativity from variation principle as

$$K_i^j = N_i^j - \frac{1}{2} N g_i^j = -8\pi\kappa T_i^j \tag{2.1}$$

where

$$N_i^j = \frac{1}{2}\gamma^{\alpha\beta}\left[g^{hj} g_{hi}|_\alpha\right]|_\beta \tag{2.2}$$

$$N = N_\alpha^\alpha, \kappa = \sqrt{\frac{g}{\gamma}} \tag{2.3}$$

and

$$g = |g_{ij}|, \gamma = |\gamma_{ij}| \tag{2.4}$$

Where a vertical bar (|) denotes a covariant differentiation with respect to $\gamma_{ij}$ and $T_i^j$ the energy momentum tensor for wet dark energy is given by

$$T_i^j = T_{i\ wdf}^j = \left(\rho_{wdf} + p_{wdf}\right)u_i u^j - p_{wdf} g_i^j \tag{2.5}$$

together with $g_i^j u_i u^j = 1$, $u_4 u^4 = 1$, where $u_i$ is the four-velocity vector of the fluid having p and $\rho$ as proper pressure and energy density respectively.

In co-moving coordinate system we have

$$T_1^1 = T_2^2 = T_3^3 = -p_{wdf}\,,\quad T_4^4 = \rho_{wdf} \quad \text{and } T_{i\ wdf}^j = 0 \text{ for } i \neq j$$

**3. Plane Symmetric Space-Time with Wet Dark Energy**

We consider here the plane symmetric line element of the form

$$ds^2 = dt^2 - A^2(dx^2 + dy^2) - B^2 dz^2 \tag{3.1}$$

Where A and B are functions of t only.

Corresponding to equation (3.1), we consider the line element for background metric $\gamma_{ij}$ as

$$d\sigma^2 = dt^2 - (dx^2 + dy^2 + dz^2) \tag{3.2}$$

Since $\gamma_{ij}$ is the Lorentz metric i.e. (-1, -1, -1, 1), therefore $\gamma$-covariant derivative becomes the ordinary partial derivative.

Using equations (2.1) to (2.5) with (3.1) and (3.2), we get,

$$\left(\frac{B_4}{B}\right)_4 = -\,16\pi\kappa\, p_{wdf} \tag{3.3}$$

$$\left(\frac{B_4}{B}\right)_4 - 2\left(\frac{A_4}{A}\right)_4 = 16\pi\kappa\, p_{wdf} \tag{3.4}$$

$$\left(\frac{B_4}{B}\right)_4 + 2\left(\frac{A_4}{A}\right)_4 = 16\pi\kappa\rho_{wdf} \tag{3.5}$$

And hereafter the suffix 4 after field variables stands for ordinary differentiation with respect to coordinate t.

Using the field equations (3.3) – (3.5), we obtain

$$3p_{wdf} + \rho_{wdf} = 0 \tag{3.6}$$

In view of the reality conditions i.e., $p > 0, \quad \rho > 0$ must hold.

The above conditions (3.6) is satisfied only when

$$p_{wdf} = 0 = \rho_{wdf} . \tag{3.7}$$

This means that the physical parameters, viz. proper pressure ( $p_{wdf}$ ), energy density ( $\rho_{wdf}$ ) are identically zero.

Thus, plane symmetric space-time with wet dark energy in bimetric relativity does not survive and hence only vacuum model is obtained.

Using (3.7), the vacuum field equations are

$$\left(\frac{A_4}{A}\right)_4 = \left(\frac{B_4}{B}\right)_4 = 0 \tag{3.8}$$

On solving (3.8), we get

$$A = \exp\left(k_1 t + k_2\right) \tag{3.9}$$

and
$$B = \exp\left(k_3 t + k_4\right) \tag{3.10}$$

where, $k_1, k_2, k_3$ and $k_4$ are the constants of integration.

Thus, in view of equations (3.9) and (3.10) the metric (3.1) takes the form

$$ds^2 = dt^2 - \exp(\mathrm{k}_1\mathrm{t} + \mathrm{k}_2)(dx^2 + dy^2) - \exp(\mathrm{k}_3\mathrm{t} + \mathrm{k}_4)dz^2 \tag{3.11}$$

If $k_1 = k_3 = \alpha, \quad k_2 = k_4 = \beta$ then (3.11) reduces to

$$ds^2 = dt^2 - \exp(\alpha\, t + \beta)(dx^2 + dy^2 + dz^2) \tag{3.12}$$

## 4. Conclusion

In the study of plane symmetric space-time, there is nil contribution of wet dark energy in Bimetric theory of relativity. It is observed that the matter field like wet dark energy cannot be a source of gravitational field in the Rosen's bimetric theory but only vacuum model exists.

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*Thomas Koshy*

# A FAMILY OF SUMS OF JACOBSTHAL POLYNOMIAL PRODUCTS OF ORDER 6

**Abstract:** We explore the Jacobsthal and Jacobsthal-Lucas versions of the sums of gibonacci polynomialproducts of order 6 investigated in [8].

**Keywords:** Jacobsthal and Jacobsthal Polynomials, Gibonacci Polynomials, Vieta Lucas Polynomioals.

**Mathematical Subject Classification (2010) No.:** Primary 05A19, 11B37, 11B39, 11Cxx.

## 1. Introduction

*Extended gibonacci polynomials* $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where $x$ is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the $n$th *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the $n$th *Lucas polynomial*.

Clearly, $f_n(1) = F_n$, the $n$th Fibonacci number; and $l_n(1) = L_n$, the $n$th Lucas number [1, 4, 5].

*Pell polynomials* $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* $P_n$ and *Pell-Lucas numbers* $Q_n$ are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [5].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the $n$th *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the $n$th *Jacobsthal Lucas polynomial* [2, 4]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the $n$th Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$ and $j_n(1) = L_n$.

Let $a(x) = x$ and $b(x) = -1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = V_n(x)$, the $n$th *Vieta polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = v_n(x)$, the $n$th *Vieta-Lucas polynomial* [3, 4].

Finally, let $a(x) = 2x$ and $b(x) = -1$. When $z_0(x) = 1$ and $z_1(x) = x$ $v$, $z_n(x) = T_n(x)$, the $n$th *Chebyshev polynomial of the first kind*; and when $z_0(x) = 1$ and $z_1(x) = 2x$, $z_n(x) = U_n(x)$, the $n$th *Chebyshev polynomial of the second kind* [3, 4].

Table 1 shows the close relationships among the Jacobsthal, Vieta, and Chebyshev subfamilies, where $i = \sqrt{-1}$ [3, 4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so $z_n$ will mean $z_n(x)$. We also let $c_n = J_n(x)$ or $j_n(x)$; and

| | | |
|---|---|---|
| $A^* = J_{n+2}^6$ | $B^* = J_{n+2}^5 J_n$ | $C* = J_{n+2}^4 J_n^2$ |
| $D^* = J_{n+2}^4 J_n J_{n-2}$ | $E^* = J_{n+2}^3 J_n^3$ | $F^* = J_{n+2}^3 J_n^2 J_{n-2}$ |
| $G^* = J_{n+2}^3 J_n J_{n-2}^2$ | $H^* = J_{n+2}^2 J_n^4$ | $I^* = J_{n+2}^2 J_n^3 J_{n-2}$ |
| $J^* = J_{n+2}^2 J_n^2 J_{n-2}^2$ | $K^* = J_{n+2}^2 J_n J_{n-2}^3$ | $L^* = J_{n+2} J_n^5$ |

| | | |
|---|---|---|
| $M^* = J_{n+2}J_n^4J_{n-2}$ | $N^* = J_{n+2}f_n^3J_{n-2}^2$ | $O^* = J_{n+2}J_n^2J_{n-2}^3$ |
| $P^* = J_n^6$ | $Q^* = J_n^5J_{n-2}$ | $R^* = J_n^4J_{n-2}^2$ |
| $S^* = J_n^3J_{n-2}^3$ | $T^* = J_n^2J_{n-2}^4$; | |

Table 1: Links Among the Subfamilies

| | |
|---|---|
| $J_n(x) = x^{(n-1)/2}f_n(1/\sqrt{x})$ | $j_n(x) = x^{n/2}l_n(1/\sqrt{x})$ |
| $V_n(x) = i^{n-1}f_n(-ix)$ | $v_n(x) = i^n l_n(-ix)$ |
| $V_n(2x) = U_{n-1}(x)$ | $v_n(2x) = 2T_n(x)$. |

and $a^*$ through $t^*$ denote their numeric counterparts, respectively; and omit a lot of basic algebra.

A *Jacobsthal polynomial product of order* $m$ is a product of Jacobsthal polynomials $g_{n+k}$ of the form $\prod_{k\in Z} g_{n+k}^{s_j}$, where $\sum_{s_j\geq 1} s_j = m$ [6, 9].

**1.1 Sums of Gibonacci Polynomial Products of Order 4:** In [7], we studied the following sums of Jacobsthal polynomial products of order 4:

$$\begin{aligned} J_{4n} &= J_{n+2}^3J_n - 2xJ_{n+2}^2J_n^2 - x^2J_{n+2}^2J_nJ_{n-2} + 2(x^2+x)J_{n+2}J_n^3 \\ &\quad + x^4J_{n+2}J_nJ_{n-2}^2 - 2(x^4+x^3)J_n^3J_{n-2} + 2x^5J_n^2J_{n-2}^2 - x^6J_nJ_{n-2}^3; \end{aligned} \tag{1}$$

$$\begin{aligned} xJ_{4n-1} &= J_{n+2}^4 - 4(x+1)J_{n+2}^3J_n + (6x^2+13x+4)J_{n+2}^2J_n^2 \\ &\quad - (4x^3+10x^2+7x+1)J_{n+2}J_n^3 - 2(2x^3+x^2)J_{n+2}J_n^2J_{n-2} \\ &\quad + (x^4+2x^3+3x^2+x)J_n^4 + (2x^4+3x^3+x^2)J_n^3J_{n-2} + x^5J_n^2J_{n-2}^2; \end{aligned} \tag{2}$$

$$\begin{aligned} J_{4n+1} &= J_{n+2}^4 - 4xJ_{n+2}^3J_n + 2(3x^2+2x)J_{n+2}^2J_n^2 - (4x^3+6x^2+x)J_{n+2}J_n^3 \\ &\quad - 2x^3J_{n+2}J_n^2J_{n-2} + (x^2+x)^2J_n^4 + (2x^4+x^3)J_n^3J_{n-2}, \end{aligned} \tag{3}$$

where $c_n = c_n(x)$. They play a pivotal role in our explorations.

With this background, we can explore the Jacobasthal companions of the gibonacci sums studied in [8]. Although we can extract them using the gibonacci-Jacobsthal relationships in Table 1, we will employ the *Jacobsthal addition formula* $c_{a+b} = J_{a+1}c_b + xJ_a c_{b-1}$ to realize our goals. This technique will shorten our work considerably.

## 2. Sums of Jacobsthal Polynomial Products of Order 6

Our objective is to express $J_{6n+k}$ as sums of Jacobsthal polynomial products of order 6, where $0 \le k \le 5$. Our discourse hinges on the identities $J_{2n+1} = J_{n+1}^2 + xJ_n^2$, $J_{2n} = J_n j_n$, $J_{n+1} + xJ_{n-1} = j_n$, $J_{n+2} - x^2 J_{n-2} = j_n$, and the addition formula $J_{a+b} = J_{a+1}J_b + xJ_a J_{b-1}$.

We begin our explorations with $J_{6n}$.

**2.1 A Jacobsthal Sum for $J_{6n}$:** By the Jacobsthal addition formula, and identities (1) and (2), we have

$$\begin{aligned} J_{6n} &= J_{2n+1}J_{4n} + xJ_{2n}J_{4n-1} \\ &= J_{4n}\left(J_{n+1}^2 + xJ_n^2\right) + xJ_{4n-1}J_n j_n \\ &= J_{4n}\left[\left(J_{n+2} - xJ_n\right)^2 + xJ_n^2\right] + \left(xJ_{4n-1}\right)J_n\left(J_{n+2} - x^2 J_{n-2}\right) \\ &= V + W, \end{aligned}$$

where

$$\begin{aligned} V &= J_{4n}\left[\left(J_{n+2} - xJ_n\right)^2 + xJ_n^2\right] \\ &= J_{4n}\left[J_{n+2}^2 - 2xJ_{n+2}J_n + \left(x^2 + x\right)J_n^2\right] \\ &= \left[J_{n+2}^3 J_n - 2xJ_{n+2}^2 J_n^2 - x^2 J_{n+2}^2 J_n J_{n-2} \right. \\ &\qquad + 2\left(x^2 + x\right)J_{n+2}J_n^3 + x^4 J_{n+2}J_n J_{n-2}^2 \\ &\qquad \left. - 2\left(x^4 + x^3\right)J_n^3 J_{n-2} + 2x^5 J_n^2 J_{n-2}^2 - x^6 J_n J_{n-2}^3\right] \\ &\qquad \left[J_{n+2}^2 - 2xJ_{n+2}J_n + \left(x^2 + x\right)J_n^2\right] \\ &= B^* - 4xC^* - x^2 D^* + \left(7x^2 + 3x\right)E^* + 2x^3 F^* \end{aligned}$$

$$+ x^4G^* - 6(x^3 + x^2)H^* - 3(x^4 + x^3)I^*$$
$$- x^6K^* + 2(x^4 + 2x^3 + x^2)L^* + 4(x^5 + x^4)M^*$$
$$- (3x^6 - x^5)N^* + 2x^7O^* - 2(x^6 + 2x^5 + x^4)Q^*$$
$$+ 2(x^7 + x^6)R^* - (x^8 + x^7)S^*;$$

$$W = (xJ_{4n-1})(J_{n+2}J_n - x^2J_nJ_{n-2})$$
$$= \left[J_{n+2}^4 - 4(x+1)J_{n+2}^3J_n + (6x^2 + 13x + 4)J_{n+2}^2J_n^2\right.$$
$$- (4x^3 + 10x^2 + 7x + 1)J_{n+2}J_n^3 - 2(2x^3 + x^2)J_{n+2}J_n^2J_{n-2}$$
$$+ (x^4 + 2x^3 + 3x^2 + x)J_n^4 + (2x^4 + 3x^3 + x^2)J_n^3J_{n-2}$$
$$\left.+ x^5J_n^2J_{n-2}^2\right](J_{n+2}J_n - x^2J_nJ_{n-2})$$

$$= B^* - 4(x+1)C^* - x^2D^* + (6x^2 + 13x + 4)E^* + 4(x^3 + x^2)F^*$$
$$- (4x^3 + 10x^2 + 7x + 1)H^* - (6x^4 + 17x^3 + 6x^2)I^*$$
$$+ (x^4 + 2x^3 + 3x^2 + x)L^* + 2(2x^5 + 6x^4 + 5x^3 + x^2)M^*$$
$$+ (5x^5 + 2x^4)N^* - (x^6 + 2x^5 + 3x^4 + x^3)Q^*$$
$$- (2x^6 + 3x^5 + x^4)R^* - x^7S^*.$$

Consequently,

$$J_{6n} = 2B^* - 4(2x+1)C^* - 2x^2D^* + (13x^2 + 16x + 4)E^* + 2(3x^3 + 2x^2)F^*$$
$$+ x^4G^* - (10x^3 + 16x^2 + 7x + 1)H^*$$
$$- (9x^4 + 20x^3 + 6x^2)I^* - x^6K^* + (3x^4 + 6x^3 + 5x^2 + x)L^*$$
$$+ 2(4x^5 + 8x^4 + 5x^3 + x^2)M^* - (3x^6 - 6x^5 -^2 x^4)N^*$$
$$+ 2x^7O^* - (3x^6 + 6x^5 + 5x^4 + x^3)Q^*$$
$$+ (2x^7 - 3x^5 - x^4)R^* - (x^8 + 2x^7)S^*. \quad (4)$$

Next we investigate $J_{6n+1}$.

**2.2 A Jacobsthal Sum for** $J_{6n+1}$**:** By the addition formula, and identities (1) and (3), we have

$$\begin{aligned} J_{6n+1} &= J_{2n+1}J_{4n+1} + xJ_{2n}J_{4n} \\ &= J_{4n+1}(J_{n+1}^2 + xJ_n^2) + xJ_nJ_{4n}(J_{n+2} - x^2J_{n-2}) \\ &= J_{4n+1}\left[(J_{n+2} - xJ_n)^2 + xJ_n^2\right] + xJ_nJ_{4n}(J_{n+2} - x^2J_{n-2}) \\ &= X + Y\ , \end{aligned}$$

where

$$\begin{aligned} X &= J_{4n+1}\left[J_{n+2}^2 - 2xJ_{n+2}J_n + (x^2 + x)J_n^2\right] \\ &= \left[J_{n+2}^4 - 4xJ_{n+2}^3J_n + 2(3x^2 + 2x)J_{n+2}^2J_n^2 - (4x^3 + 6x^2 + x)J_{n+2}J_n^3\right. \\ &\qquad \left. - 2x^3J_{n+2}J_n^2J_{n-2} + (x^2 + x)^2J_n^4 + (2x^4 + x^3)J_n^3J_{n-2}\right] \\ &\qquad \left[J_{n+2}^2 - 2xJ_{n+2}J_n + (x^2 + x)J_n^2\right] \\ &= A^* - 6xB^* + 5(3x^2 + x)C^* - (20x^3 + 18x^2 + x)E^* - 2x^3F^* \\ &\qquad + (15x^4 + 24x^3 + 7x^2)H^* + (6x^4 + x^3)I^* \\ &\qquad - (6x^5 + 14x^4 + 9x^3 + x^2)L^* - 2(3x^5 + 2x^4)M^* \\ &\qquad + (x^2 + x)^3P^* + (2x^6 + 3x^5 + x^4)Q^*\ ; \end{aligned}$$

$$\begin{aligned} Y &= \left[J_{n+2}^3J_n - 2xJ_{n+2}^2J_n^2 - x^2J_{n+2}^2J_nJ_{n-2} + 2(x^2 + x)J_{n+2}J_n^3\right. \\ &\qquad + x^4J_{n+2}J_nJ_{n-2}^2 - 2(x^4 + x^3)J_n^3J_{n-2} + 2x^5J_n^2J_{n-2}^2 \\ &\qquad \left. - x^6J_nJ_{n-2}^3\right]xJ_n(J_{n+2} - x^2J_{n-2}) \\ &= xC^* - 2x^2E^* - 2x^3F^* + 2(x^3 + x^2)H^* + 2x^4I^* + 2x^5J^* \\ &\qquad - 4(x^5 + x^4)M^* + 2x^6N^* - 2x^7O^* \\ &\qquad + 2(x^7 + x^6)R^* - 2x^8S^* + x^9T^*. \end{aligned}$$

Thus,

$$\begin{aligned} J_{6n+1} = A^* - 6xB^* + 3(5x^2 + 2x)C^* - (20x^3 + 20x^2 + x)E^* - 4x^3F^* \\ + (15x^4 + 26x^3 + 9x^2)H^* + (8x^4 + x^3)I^* + 2x^5J^* \\ - (6x^5 + 14x^4 + 9x^3 + x^2)L^* - 2(5x^5 + 4x^4)M^* + 2x^6N^* \\ - 2x^7O^* + (x^2 + x)^3P^* + (2x^6 + 3x^5 + x^4)Q^* \\ + 2(x^7 + x^6)R^* - 2x^8S^* + x^9T^*. \qquad (5) \end{aligned}$$

Next we express $J_{6n+2}$ as a Jacobsthal sum using the Jacobsthal recurrence.

**2.3 A Jacobsthal Sum for $J_{6n+2}$:** Using identities (4) and (5), and the Jacobsthal recurrence, we get

$$\begin{aligned} J_{6n+2} &= J_{6n+1} + xJ_{6n} \\ &= \big[A^* - 6xB^* + 3(5x^2 + 2x)C^* - (20x^3 + 20x^2 + x)E^* \\ &\quad - 4x^3F^* + (15x^4 + 26x^3 + 9x^2)H^* + (8x^4 + x^3)I^* + 2x^5J^* \\ &\quad - (6x^5 + 14x^4 + 9x^3 + x^2)L^* - 2(5x^5 + 4x^4)M^* + 2x^6N^* \\ &\quad - 2x^7O^* + (x^2 + x)3P^* + (2x^6 + 3x^5 + x^4)Q^* \\ &\quad + 2(x^7 + x^6)R^* - 2x^8S^* + x^9T^*\big] + x\big[2B^* - 4(2x+1)C^* \\ &\quad - 2x^2D^* + (13x^2 + 16x + 4)E^* + 2(3x^3 + 2x^2)F^* + x^4G^* \\ &\quad - (10x^3 + 16x^2 + 7x + 1)H^* - (9x^4 + 20x^3 + 6x^2)I^* \\ &\quad - x^6K^* + (3x^4 + 6x^3 + 5x^2 + x)L^* \\ &\quad + 2(4x^5 + 8x^4 + 5x^3 + x^2)M^* - (3x^6 - 6x^5 - 2x^4)N^* \\ &\quad + 2x^7O^* - (3x^6 + 6x^5 + 5x^4 + x^3)Q^* \\ &\quad + (2x^7 - 3x^5 - x^4)R* - (x^8 + 2x^7)S^*\big] \\ &= A^* - 4xB^* + (7x^2 + 2x)C^* - 2x^3D^* - (7x^3 + 4x^2 - 3x)E^* + 6x^4F^* \end{aligned}$$

$$
\begin{aligned}
&+ x^5G^* + (5x^4 + 10x^3 + 2x^2 - x)H^* \\
&- (9x^5 + 12x^4 + 5x^3)I^* + 2x^5J^* - x^7K^* \\
&- (3x^5 + 8x^4 + 4x^3)L^* + 2(4x^6 + 3x^5 + x^4 + x^3)M^* \\
&- (3x^7 - 8x^6 - 2x^5)N^* + 2(x^8 - x^7)O^* + (x^2 + x)^3P^* \\
&- (3x^7 + 4x^6 + 2x^5)Q^* + (2x^8 + 2x^7 - x^6 - x^5)R^* \\
&- (x^9 + 4x^8)S^* + x^9T^*. \qquad (6)
\end{aligned}
$$

Next we express $J_{6n+3}$ as a Jacobsthal sum.

**2.4 A Jacobsthal Sum for $J_{6n+3}$:** Using identities (5) and (6), and the Jacobsthal recurrence, we have

$$
\begin{aligned}
J_{6n+3} &= J_{6n+2} + xJ_{6n+1} \\
&= \big[A^* - 4xB^* + (7x^2 + 2x)C^* - 2x^3D^* - (7x^3 + 4x^2 - 3x)E^* \\
&\quad + 6x^4F^* + x^5G^* + (5x^4 + 10x^3 + 2x^2 - x)H^* \\
&\quad - (9x^5 + 12x^4 + 5x^3)I^* + 2x^5J^* - x^7K^* \\
&\quad - (3x^5 + 8x^4 + 4x^3)L^* + 2(4x^6 + 3x^5 + x^4 + x^3)M^* \\
&\quad - (3x^7 - 8x^6 - 2x^5)N^* + 2(x^8 - x^7)O^* + (x^2 + x)^3P^* \\
&\quad - (3x^7 + 4x^6 + 2x^5)Q^* + 2(x^8 + 2x^7 - x^6 - x^5)R^* \\
&\quad - (x^9 + 4x^8)S^* + x^9T^*\big] + x\big[A^* - 6xB^* + 3(5x^2 + 2x)C^* \\
&\quad - (20x^3 + 20x^2 + x)E^* - 4x^3F^* + (15x^4 + 26x^3 + 9x^2)H^* \\
&\quad + (8x^4 + x^3)I^* + 2x^5J^* - (6x^5 + 14x^4 + 9x^3 + x^2)L^* \\
&\quad - 2(5x^5 + 4x^4)M^* + 2x^6N^* - 2x^7O^* + (x^2 + x)^3P^* \\
&\quad + (2x^6 + 3x^5 + x^4)Q^* + 2(x^7 + x^6)R^* - 2x^8S^* + x^9T^*\big] \\
&= (x + 1)A^* - 2(3x^2 + 2x)B^* + (15x^3 + 13x^2 + 2x)C^* - 2x^3D^* \\
&\quad - (20x^4 + 27x^3 + 5x^2 - 3x)E^* + 2x^4F^* + x^5G^*
\end{aligned}
$$

$$\begin{aligned}
&+ (15x^5 + 31x^4 + 19x^3 + 2x^2 - x)H^* \\
&- (x^5 + 11x^4 + 5x^3)I^* + 2(x^6 + x^5)J^* - x^7K^* \\
&- (6x^6 + 17x^5 + 17x^4 + 5x^3)L^* - 2(x^6 + x^5 - x^4 - x^3)M^* \\
&- (x^7 - 8x^6 - 2x^5)N^* - 2x^7O^* + (x+1)(x^2 + x)^3P^* \\
&- (x^7 + x^6 + x^5)Q^* + (4x^8 + 4x^7 - x^6 - x^5)R^* \\
&- (3x^9 + 4x^8)S^* + (x^{10} + x^9)T^*. \qquad (7)
\end{aligned}$$

We now explore a Jacobsthal sum for $J_{6n+4}$ .

**2.5 A Jacobsthal Sum for** $J_{6n+4}$**:** Using identities (6) and (7), and the Jacobsthal recurrence, we get

$$\begin{aligned}
J_{6n+4} &= J_{6n+3} + xJ_{6n+2} \\
&= \big[(x+1)A^* - 2(3x^2 + 2x)B^* + (15x^3 + 13x^2 + 2x)C^* \\
&\quad - 2x^3D^* - (20x^4 + 27x^3 + 5x^2 - 3x)E^* + 2x^4F^* \\
&\quad + x^5G^* + (15x^5 + 31x^4 + 19x^3 + 2x^2 - x)H^* \\
&\quad - (x^5 + 11x^4 + 5x^3)I^* + 2(x^6 + x^5)J^* - x^7K^* \\
&\quad - (6x^6 + 17x^5 + 17x^4 + 5x^3)L^* \\
&\quad - 2(x^6 + x^5 - x^4 - x^3)M^* - (x^7 - 8x^6 - 2x^5)N^* \\
&\quad - 2x^7O^* + (x+1)(x^2 + x)^3P^* - (x^7 + x^6 + x^5)Q^* \\
&\quad + (4x^8 + 4x^7 - x^6 - x^5)R^* - (3x^9 + 4x^8)S^* \\
&\quad + (x^{10} + x^9)T^*\big] + x\big[A^* - 4xB^* + (7x^2 + 2x)C^* - 2x^3D^* \\
&\quad - (7x^3 + 4x^2 - 3x)E^* + 6x^4F^* + x^5G^* \\
&\quad + (5x^4 + 10x^3 + 2x^2 - x)H^* - (9x^5 + 12x^4 + 5x^3)I^* \\
&\quad + 2x^5J^* - x^7K^* - (3x^5 + 8x^4 + 4x^3)L^*
\end{aligned}$$

$$
\begin{aligned}
&+\ 2(4x^6+3x^5+x^4+x^3)M^*-(3x^7-8x^6-2x^5)N^*\\
&+\ 2(x^8-x^7)O^*+(x^2+x)^3P^*-(3x^7+4x^6+2x^5)Q^*\\
&+\ 2(x^8+2x^7-x^6-x^5)R^*-(x^9+4x^8)S^*+x^9T^*\big]
\end{aligned}
$$

$$
\begin{aligned}
=&\ (2x+1)A^*-2(5x^2+2x)B^*+(22x^3+15x^2+2x)C^*\\
&-\ 2(x^4+x^3)D^*-(27x^4+31x^3+2x^2-3x)E^*\\
&+\ 2(3x^5+x^4)F^*+(x^6+x^5)G^*\\
&+\ (20x^5+41x^4+21x^3+x^2-x)H^*\\
&-(9x^6+13x^5+16x^4+5x^3)I^*+2(2x^6+x^5)J^*\\
&-\ (x^8+x^7)K^*-(9x^6+25x^5+21x^4+5x^3)L^*\\
&+\ 2(4x^7+2x^6+2x^4+x^3)M^*\\
&-(3x^8-7x^7-10x^6-2x^5)N^*+2(x^9-x^8-x^7)O^*\\
&+\ (2x+1)(x^2+x)^3P^*-(3x^8+5x^7+3x^6+x^5)Q^*\\
&+\ (2x^9+6x^8+3x^7-2x^6-x^5)R^*-(x^{10}+7x^9+4x^8)S^*\\
&+\ (2x^{10}+x^9)T^*. \qquad (8)
\end{aligned}
$$

Finally, we express $J_{6n+5}$ as a Jacobsthal sum.

**2.6 A Jacobsthal Sum for $J_{6n+5}$:** It follows by the identities (7) and (8), and the Jacobsthal recurrence that

$$
\begin{aligned}
J_{6n+5} &= J_{6n+4}+xJ_{6n+3}\\
&= \big[(2x+1)A^*-2(5x^2+2x)B^*+(22x^3+15x^2+2x)C^*\\
&\qquad -2((x^4+x^3)D^*-(27x^4+31x^3+2x^2-3x)E^*\\
&\qquad +2(3x^5+x^4)F^*+(x^6+x^5)G^*\\
&\qquad +(20x^5+41x^4+21x^3+x^2-x)H^*\\
&\qquad -\ (9x^6+13x^5+16x^4+5x^3)I^*+2(2x^6+x^5)J^*
\end{aligned}
$$

$$
\begin{aligned}
&- (x^8+x^7)K^* - (9x^6+25x^5+21x^4+5x^3)L^* \\
&+ 2(4x^7+2x^6+2x^4+x^3)M^* \\
&- (3x^8-7x^7-10x^6-2x^5)N^* + 2(x^9-x^8-x^7)O^* \\
&+ (2x+1)(x^2+x)^3P^* - (3x^8+5x^7+3x^6+x^5)Q^* \\
&+ (2x^9+6x^8+3x^7-2x^6-x^5)R^* - (x^{10}+7x^9+4x^8)S^* \\
&+ (2x^{10}+x^9)T^*\big] + x\big[(x+1)A^* - 2(3x^2+2x)B^* \\
&+ (15x^3+13x^2+2x)C^* - 2x^3D^* \\
&- (20x^4+27x^3+5x^2-3x)E^* + 2x^4F^* + x^5G^* \\
&+ (15x^5+31x^4+19x^3+2x^2-x)H^* \\
&- (x^5+11x^4+5x^3)I^* + 2(x^6+x^5)J^* - x^7K^* \\
&- (6x^6+17x^5+17x^4+5x^3)L^* \\
&- 2(x^6+x^5-x^4-x^3)M^* - (x^7-8x^6-2x^5)N^* - 2x^7O^* \\
&+ (x+1)(x^2+x)^3P^* - (x^7+x^6+x^5)Q^* \\
&+ (4x^8+4x^7-x^6-x^5)R^* - (3x^9+4x^8)S^* \\
&+ (x^{10}+x^9)T^*\big] \\
= {}& (x^2+3x+1)A^* - 2(3x^3+7x^2+2x)B^* \\
&+ (15x^4+35x^3+17x^2+2x)C^* - 2(2x^4+x^3)D^* \\
&- (20x^5+54x^4+36x^3-x^2-3x)E^* + 2(4x^5+x^4)F^* \\
&+ (2x^6+x^5)G^* + (15x^6+51x^5+60x^4+23x^3-x)H^* \\
&- (10x^6+24x^5+21x^4+5x^3)I^* + 2(x^7+3x^6+x^5)J^* \\
&- (2x^8+x^7)K^* - (6x^7+26x^6+42x^5+26x^4+5x^3)L^* \\
&+ 2(3x^7+x^6+x^5+3x^4+x^3)M^* \\
&- (4x^8-15x^7-12x^6-2x^5)N^* + 2(x^9-2x^8-x^7)O^* \\
&+ (x^2+3x+1)(x^2+x)^3P^* - (4x^8+6x^7+4x^6+x^5)Q^* \\
&+ (6x^9+10x^8+2x^7-3x^6-x^5)R^* \\
&- (4x^{10}+11x^9+4x^8)S^* + (x^{11}+3x^{10}+x^9)T^*. \qquad (9)
\end{aligned}
$$

Next we explore the Jacobsthal-Lucas counterparts of identities (4) through (9).

**3. Sums of Jacobsthal-Lucas Polynomial Products of Order 6**

For convenience, we begin with $j_{6n+1}$.

**3.1 A Jacobsthal Sum for $j_{6n+1}$:** Using the identity $j_n = J_{n+1} + xJ_{n-1}$, and equations (4) and (6), we get

$$
\begin{aligned}
j_{6n+1} &= J_{6n+2} + xJ_{6n} \\
&= \big[A^* - 4xB^* + (7x^2 + 2x)C^* - 2x^3 D^* - (7x^3 + 4x^2 - 3x)E^* \\
&\quad + 6x^4 F^* + x^5 G^* + (5x^4 + 10x^3 + 2x^2 - x)H^* \\
&\quad - (9x5 + 12x4 + 5x3)I^* + 2x^5 J^* - x^7 K^* \\
&\quad - (3x^5 + 8x^4 + 4x^3)L^* + 2(4x^6 + 3x^5 + x^4 + x^3)M^* \\
&\quad - (3x^7 - 8x^6 - 2x^5)N^* + 2(x^8 - x^7)O^* + (x^2 + x)^3 P^* \\
&\quad - (3x^7 + 4x^6 + 2x^5)Q^* + 2(x^8 + 2x^7 - x^6 - x^5)R^* \\
&\quad - (x^9 + 4x^8)S^* + x^9 T^*\big] + x\,\big[2B^* - 4(2x + 1)C^* - 2x^2 D^* \\
&\quad + (13x^2 + 16x + 4)E^* + 2(3x^3 + 2x^2)F^* + x^4 G^* \\
&\quad - (10x^3 + 16x^2 + 7x + 1)H^* - (9x^4 + 20x^3 + 6x^2)I^* \\
&\quad - x^6 K^* + (3x^4 + 6x^3 + 5x^2 + x)L^* \\
&\quad + 2(4x^5 + 8x^4 + 5x^3 + x^2)M^* - (3x^6 - 6x^5 - 2x^4)N^* \\
&\quad + 2x^7 O^* - (3x^6 + 6x^5 + 5x^4 + x^3)Q^* \\
&\quad + (2x^7 - 3x^5 - x^4)R^* - (x^8 + 2x^7)S^*\big] \\
&= A^* - 2xB^* - (x^2 + 2x)C^* - 4x^3 D^* + (6x^3 + 12x^2 + 7x)E^* \\
&\quad + 4(3x^4 + x^3)F^* + 2x^5 G^* - (5x^4 + 6x^3 + 5x^2 + 2x)H^* \\
&\quad - (18x^5 + 32x^4 + 11x^3)I^* + 2x^5 J^* - 2x^7 K^*
\end{aligned}
$$

$$
\begin{aligned}
&- (2x^4 - x^3 - x^2)L^* + 2(8x^6 + 11x^5 + 6x^4 + 2x^3)M^* \\
&- 2(3x^7 - 7x^6 - 2x^5)N^* + 2(2x^8 - x^7)O^* + (x^2 + x)^3 P^* \\
&- (6x^7 + 10x^6 + 7x^5 + x^4)Q^* \\
&+ 2(2x^8 + x^7 - 2x^6 - x^5)R^* - 2(x^9 + 3x^8)S^* + x^9 T^* . \quad (10)
\end{aligned}
$$

Next we investigate $j_{6n+2}$ .

**3.2 A Jacobsthal Sum for $j_{6n+2}$ :** Using the same technique as above with equations (5) and (7), we get

$$
\begin{aligned}
j_{6n+2} &= J_{6n+3} + xJ_{6n+1} \\
&= \big[(x+1)A^* - 2(3x^2 + 2x)B^* + (15x^3 + 13x^2 + 2x)C^* - 2x^3 D^* \\
&\quad - (20x^4 + 27x^3 + 5x^2 - 3x)E^* + 2x^4 F^* + x^5 G^* \\
&\quad + (15x^5 + 31x^4 + 19x^3 + 2x^2 - x)H^* \\
&\quad - (x^5 + 11x^4 + 5x^3)I^* + 2(x^6 + x^5)J^* - x^7 K^* \\
&\quad - (6x^6 + 17x^5 + 17x^4 + 5x^3)L^* - 2(x^6 + x^5 - x^4 - x^3)M^* \\
&\quad - (x^7 - 8x^6 - 2x^5)N^* - 2x^7 O^* + (x+1)(x^2 + x)^3 P^* \\
&\quad - (x^7 + x^6 + x^5)Q^* + (4x^8 + 4x^7 - x^6 - x^5)R^* \\
&\quad - (3x^9 + 4x^8)S^* + (x^{10} + x^9)T^*\big] + x\,\big[A^* - 6xB^* \\
&\quad + 3(5x^2 + 2x)C^* - (20x^3 + 20x^2 + x)E^* - 4x^3 F^* \\
&\quad + (15x^4 + 26x^3 + 9x^2)H^* + (8x^4 + x^3)I^* + 2x^5 J^* \\
&\quad - (6x^5 + 14x^4 + 9x^3 + x^2)L^* - 2(5x^5 + 4x^4)M^* + 2x^6 N^* \\
&\quad - 2x^7 O^* + (x^2 + x)^3 P^* + (2x^6 + 3x^5 + x^4)Q^* \\
&\quad + 2(x^7 + x^6)R^* - 2x^8 S^* + x^9 T^*\big] \\
&= (2x+1)A^* - 4(3x^2 + x)B^* + (30x^3 + 19x^2 + 2x)C^* - 2x^3 D^* \\
&\quad - (40x^4 + 47x^3 + 6x^2 - 3x)E^* - 2x^4 F^* + x^5 G^*
\end{aligned}
$$

$$\begin{aligned}
&+ (30x^5 + 57x^4 + 28x^3 + 2x^2 - x)H^* \\
&+ (7x^5 - 10x^4 - 5x^3)I^* + 2(2x^6 + x^5)J^* - x^7K^* \\
&- (12x^6 + 31x^5 + 26x^4 + 6x^3)L^* \\
&- 2(6x^6 + 5x^5 - x^4 - x^3)M^* + (x^7 + 8x^6 + 2x^5)N^* \\
&- (2x^8 + x^7)O^* + (2x+1)(x^2+x)^3P^* + (x^7 + 2x^6)Q^* \\
&+ (6x^8 + 6x^7 - x^6 - x^5)R^* - (5x^9 + 4x^8)S^* \\
&+ (2x^{10} + x^9)T^*. \qquad (11)
\end{aligned}$$

Knowing the sums for both $j_{6n+1}$ and $j_{6n+2}$, we can now find the remaining sums in the family.

We begin with $j_{6n}$.

**3.3 A Jacobsthal Sum for $j_{6n}$:** By the Jacobsthal recurrence, we have

$$\begin{aligned}
xj_{6n} &= j_{6n+2} - j_{6n+1} \\
&= \big[(2x+1)A^* - 4(3x^2 + x)B^* + (30x^3 + 19x^2 + 2x)C^* - 2x^3D^* \\
&\quad - (40x^4 + 47x^3 + 6x^2 - 3x)E^* - 2x^4F^* + x^5G^* \\
&\quad + (30x^5 + 57x^4 + 28x^3 + 2x^2 - x)H^* \\
&\quad + (7x^5 - 10x^4 - 5x^3)I^* + 2(2x^6 + x^5)J^* - x^7K^* \\
&\quad - (12x^6 + 31x^5 + 26x^4 + 6x^3)L^* \\
&\quad - 2(6x^6 + 5x^5 - x^4 - x^3)M^* + (x^7 + 8x^6 + 2x^5)N^* \\
&\quad - (2x^8 + x^7)O^* + (2x+1)(x^2+x)^3P^* + (x^7 + 2x^6)Q^* \\
&\quad + (6x^8 + 6x^7 - x^6 - x^5)R^* - (5x^9 + 4x^8)S^* \\
&\quad + (2x^{10} + x^9)T^*\big] - \big[A^* - 2xB^* - (x^2 + 2x)C^* - 4x^3D^* \\
&\quad + (6x^3 + 12x^2 + 7x)E^* + 4(3x^4 + x^3)F^* + 2x^5G^*
\end{aligned}$$

$$- (5x^4 + 6x^3 + 5x^2 + 2x)H^* - (18x^5 + 32x^4 + 11x^3)I^*$$
$$+ 2x^5J^* - 2x^7K^* - (2x^4 - x^3 - x^2)L^*$$
$$+ 2(8x^6 + 11x^5 + 6x^4 + 2x^3)M^* - 2(3x^7 - 7x^6 - 2x^5)N^*$$
$$+ 2(2x^8 - x^7)O^* + (x^2 + x)^3P^*$$
$$- (6x^7 + 10x^6 + 7x^5 + x^4)Q^* + 2(2x^8 + x^7 - 2x^6 - x^5)R^*$$
$$- 2(x^9 + 3x^8)S^* + x^9T^*\Big]$$

$$j_{6n} = 2A^* - 2(6x-1)B^* + (30x^2 + 4x - 4)C^* - 2x^2D^*$$
$$- (40x^3 + 27x^2 - 14x - 4)E^* - 2(x^3 - 2x^2)F^* + x^4G^*$$
$$+ (30x^4 + 42x^3 + 2x^2 - 7x - 1)H^* + (7x^4 - 18x^3 - 6x^2)I^*$$
$$+ 4x^5J^* - x^6K^* - (12x^5 + 25x^4 + 12x^3 - 3x^2 - x)L^*$$
$$- 2(6x^5 - 5x^3 - x^2)M^* + (x^6 + 6x^5 + 2x^4)N^* - 2x^7O^*$$
$$+ 2(x^2 + x)^3P^* + (x^6 - 3x^4 - x^3)Q^*$$
$$+ (6x^7 + 4x^6 - 3x^5 - x^4)R^* - (5x^8 + 2x^7)S^* + 2x^9T^*. \quad (12)$$

**3.4 A Jacobsthal Sum for $j_{6n+3}$:** Again, by the Jacobsthal recurrence, we get

$$j_{6n+3} = j_{6n+2} + xj_{6n+1}$$
$$= \Big[(2x+1)A^* - 4(3x^2 + x)B^* + (30x^3 + 19x^2 + 2x)C^*$$
$$- 2x^3D^* - (40x^4 + 47x^3 + 6x^2 - 3x)E^* - 2x^4F^* + x^5G^*$$
$$+ (30x^5 + 57x^4 + 28x^3 + 2x^2 - x)H^*$$
$$+ (7x^5 - 10x^4 - 5x^3)I^* + 2(2x^6 + x^5)J^* - x^7K^*$$
$$- (12x^6 + 31x^5 + 26x^4 + 6x^3)L^*$$
$$- 2(6x^6 + 5x^5 - x^4 - x^3)M^* + (x^7 + 8x^6 + 2x^5)N^*$$
$$- (2x^8 + x^7)O^* + (2x+1)(x^2 + x)^3P* + (x^7 + 2x^6)Q^*$$

$$
\begin{aligned}
&+ (6x^8 + 6x^7 - x^6 - x^5)R^* - (5x^9 + 4x^8)S^* \\
&+ (2x^{10} + x^9)T^*\big] + x\big[A^* - 2xB^* - (x^2 + 2x)C^* - 4x^3 D^* \\
&+ (6x^3 + 12x^2 + 7x)E^* + 4(3x^4 + x^3)F^* \\
&+ 2x^5 G^* - (5x^4 + 6x^3 + 5x^2 + 2x)H^* \\
&- (18x^5 + 32x^4 + 11x^3)I^* + 2x^5 J^* \\
&- 2x^7 K^* - (2x^4 - x^3 - x^2)L^* \\
&+ 2(8x^6 + 11x^5 + 6x^4 + 2x^3)M^* - 2(3x^7 - 7x^6 - 2x^5)N^* \\
&+ 2(2x^8 - x^7)O^* + (x^2 + x)^3 P^* \\
&- (6x^7 + 10x^6 + 7x^5 + x^4)Q^* + 2(2x^8 + x^7 - 2x^6 - x^5)R^* \\
&- 2(x^9 + 3x^8)S^* + x^9 T^*\big]
\end{aligned}
$$

$$
\begin{aligned}
= (3x+1)A^* &- 2(7x^2 + 2x)B^* \\
&+ (29x^3 + 17x^2 + 2x)C^* - 2(2x^4 + x^3)D^* \\
&- (34x^4 + 35x^3 - 2 - 3x)E^* + 2(6x^5 + x^4)F^* \\
&+ (2x^6 + x^5)G^* + (25x^5 + 51x^4 + 17x^3 - x)H^* \\
&- (18x^6 + 25x^5 + 21x^4 + 5x^3)I^* + 2(3x^6 + x^5)J^* \\
&- (2x^8 + x^7)K^* - (12x^6 + 33x^5 + 25x^4 + 5x^3)L^* \\
&+ 2(8x^7 + 5x^6 + x^5 + 3x^4 + x^3)M^* \\
&- (6x^8 - 15x^7 - 12x^6 - 2x^5)N^* + 2(2x^9 + x^7)O^* \\
&+ (3x+1)(x^2 + x)^3 P^* - (6x^8 + 9x^7 + 5x^6 + x^5)Q^* \\
&+ (4x^9 + 8x^8 + 2x^7 - 3x^6 - x^5)R^* \\
&- (2x^{10} + 11x^9 + 4x^8)S^* + (3x^{10} + x^9)T^*. \qquad (13)
\end{aligned}
$$

**3.5 A Jacobsthal Sum for $j_{6n+4}$:** Since $j_{6n+4} = j_{6n+3} + xj_{6n+2}$, it follows by equations (12) and (13) that

$$\begin{aligned}
j_{6n+4} = {} & (2x^2+4x+1)A^* - 2(6x^3+9x^2+2x)B^* \\
& + (30x^4+48x^3+19x^2+2x)C^* - 2(3x^4+x^3)D^* \\
& - (40x^5+81x^4+41x^3-4x^2-3x)E^* + 2(5x^5+x^4)F^* \\
& + (3x^6+x^5)G^* + (30x^6+82x^5+79x^4+19x^3-x^2-x)H^* \\
& - (11x^6+35x^5+26x^4+5x^3)I^* + 2(2x^7+4x^6+x^5)J^* \\
& - (3x^8+x^7)K^* - (12x^7+43x^6+59x^5+31x^4+5x^3)L^* \\
& + 2(2x^7+2x^5+4x^4+x^3)M^* - (5x^8-23x^7-14x^6-2x^5)N^* \\
& + 2(x^9-x^8+x^7)O^* + (2x^2+4x+1)(x^2+x)^3 P^* \\
& - (5x^8+7x^7+5x^6+x^5)Q^* + (10x^9+14x^8+x^7-4x^6-x^5)R^* \\
& - (7x^{10}+15x^9+4x^8)S^* + (2x^{11}+4x^{10}+x^9)T^*. \qquad (14)
\end{aligned}$$

Finally, we explore a sum for the final member of the family.

**3.6 A Jacobsthal Sum for $j_{6n+5}$:** It follows by the recurrence $j_{6n+5} = j_{6n+4} + xj_{6n+3}$, and equations (13) and (14) that

$$\begin{aligned}
j_{6n+5} = {} & (5x^2+5x+1)A^* - 2(13x^3+11x^2+2x)B^* \\
& + (59x^4+65x^3+21x^2+2x)C^* - 2(2x^5+4x^4+x^3)D^* \\
& - (74x^5+116x^4+40x^3-7x^2-3x)E^* \\
& + 2(6x^6+6x^5+x^4)F^* + (2x^7+4x^6+x^5)G^* \\
& + (55x^6+133x^5+96x^4+19x^3-2x^2-x)H^* \\
& - (18x^7+36x^6+56x^5+21x^4+5x^3)I^* \\
& + 2(5x^7+5x^6+x^5)J^* - (2x^9+4x^8+x^7)K^* \\
& - (24x^7+76x^6+84x^5+36x^4+5x^3)L^* \\
& + 2(8x^8+7x^7+x^6+5x^5+5x^4+x^3)M^* \\
& - (6x^9-10x^8-35x^7-16x^6-2x^5)N^*
\end{aligned}$$

$$+\ 2(2x^{10}+x^9+x^7)O^* + (5x^2+5x+1)(x^2+x)^3P^*$$
$$-\ (6x^9+14x^8+12x^7+6x^6+x^5)Q^*$$
$$+\ (4x^{10}+18x^9+16x^8-2x^7-5x^6-x^5)R^*$$
$$-\ (2x^{11}+18x^{10}+19x^9+4x^8)S^*$$
$$+\ (5x^{11}+5x^{10}+x^9)T^*. \qquad (15)$$

**4. Numeric Byproducts**

Since $J_n = J_n(2)$ and $j_n = j_n(2)$, equations (4) and (15) yield the following identities, where $a^*$ through $t^*$ denote the numeric counterparts of $A^*$ through $T^*$, respectively.

$$J_{6n} = 2b^* - 20c^* - 8d^* + 88e^* + 64f^* + 16g^* - 159h^* - 328i^* - 64k^* + 118l^*$$
$$+\ 600m^* + 32n^* + 256o^* - 472q^* + 144r^* - 512s^*;$$

$$J_{6n+1} = a^* - 12b^* + 72c^* - 242e^* - 32f^* + 484h^* + 136i^* + 64j^* - 492l^*$$
$$-\ 448m^* + 128n^* - 256o^* + 216p^* + 240q^* + 384r^* - 512s^* + 512t^*;$$

$$J_{6n+2} = a^* - 8b^* + 32c^* - 16d^* - 66e^* + 96f^* + 32g^* + 166h^* - 520i^* + 64j^* - 128k^*$$
$$-\ 256l^* + 752m^* + 192n^* + 256o^* + 216p^* - 704q^* + 672r^*$$
$$+1536s^* + 512t^*;$$

$$J_{6n+3} = 3a^* - 32b^* + 176c^* - 16d^* - 550e^* + 32f^* + 32g^* + 1134h^* - 248i^*$$
$$+\ 192j^* - 128k^* - 1240l^* - 144m^* + 448n^* - 256o^* + 648p^* - 224q^*$$
$$-\ 1440r^* - 2560s^* + 1536t^*;$$

$$J_{6n+4} = 5a^* - 48b^* + 240c^* - 48d^* - 682e^* + 224f^* + 96g^* + 1466h^* - 1288i^*$$
$$+\ 320j^* - 384k^* - 1752l^* + 1360m^* + 832n^* + 256o^* + 1080p^*$$
$$-\ 1632q^* + 2784r^* - 5632s^*;$$

$$J_{6n+5} = 11a^* - 112b^* + 592c^* - 80d^* - 1782e^* + 288f^* + 160g^* + 3734h^* - 1784i^* + 704j^* - 640k^* - 4232l^* + 1072m^* + 1728n^* - 256o^* + 2376p^* - 2080q^* + 5664r^* - 10752s^* + 5632t^*;$$

$$j_{6n} = 2a^* - 22b^* + 124c^* - 8d^* - 396e^* + 16g^* + 809h^* - 56i^* + 128j^* - 64k^* - 866l^* - 296m^* + 288n^* - 256o^* + 432p^* + 8q^* + 912r^* - 1536s^* + 1024t^*;$$

$$j_{6n+1} = a^* - 4b^* - 8c^* - 32d^* + 110e^* + 224f^* + 64g^* - 152h^* - 1176i^* + 64j^* - 256k^* - 20l^* + 1952m^* + 262n^* + 7560o^* + 216p^* - 1648q^* + 960r^* - 2560s^* + 512t^*;$$

$$j_{6n+2} = 5a^* - 56b^* + 320c^* - 16d^* - 1034e^* - 32f^* + 32g^* + 2102h^* + 24i^* + 320j^* - 128k^* - 2224l^* - 1040m^* + 704n^* - 768o^* + 1080p^* + 256q^* + 2208r^* - 3584s^* + 2560t^*;$$

$$j_{6n+3} = 7a^* - 64b^* + 304c^* - 80d^* - 814e^* + 416f^* + 160g^* + 1750h^* - 2328i^* + 448j^* - 640k^* - 2264l^* - 2864m^* + 1216n^* + 2304o^* + 1512p^* - 3040q^* + 4128r^* - 8704s^* + 3584t^*;$$

$$j_{6n+4} = 17a^* - 176b^* + 944c^* - 112d^* - 2882e^* + 352f^* + 224g^* + 5954h^* - 2280i^* + 1088j^* - 896k^* - 6712l^* + 784m^* + 624n^* + 768o^* + 3672p^* - 2528q^* + 8544r^* - 15872s^* + 8704t^*;$$

$$j_{6n+5} = 31a^* - 304b^* + 1552c^* - 272d^* - 4510e^* + 1184f^* + 544g^* + 9454h^* - 6776i^* + 1984j^* - 2176k^* - 11240l^* + 6512m^* + 5056n^* - 5376o^* + 6696p^* - 8608q^* + 16800r^* - 33280s^* + 15872t^*.$$

*An interesting observation.* Exactly one of the coefficients of the expressions $a^*$ through $t^*$ is odd.

This implies that $C_{6n+k}$ is odd, where $C_n = J_n$ or $j_n$, and $0 \le k \le 5$.

This is consistent with the fact that every $C_n$ is odd [5].

## 5. Pell, Vieta, and Chebyshev Implications

Using the gibonacci-Pell and gibonacci-Vieta relationships, we can find the Pell and Vieta counterparts of identities (4) through (15). Likewise, using the Vieta-Chebyshev relationships in Table 1, we can extract their Chebyshev counterparts. Again in the interest of brevity, we omit them.

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# A FOURTH-ORDER RECURRENCE FOR GIBONACCI POLYNOMIAL SQUARES

**Abstract:** We explore a fourth-order recurrence for gibonacci polynomial squares, and its Pell and Jacobsthal consequences, and confirm the gibonacci and Jacobthal versions using graph-theoretic techniques.

**Keywords:** Pell lucas Polynomials, Vieta Lucas Polynomioals, Gibonacci Polynomials Jacobsthal Polynomials.

**Mathematical Subject Classification (2010) No.:** Primary 11B39, 11B83, 05C20, 05C22, 05C08.

## 1. Introduction

*Extended gibonacci polynomials* $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where $x$ is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the $n$th *Fibonacci polynomials*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the $n$th *Lucas polynomials*. Clearly, $f_n(1) = F_n$, the $n$th Fibonacci number; and $l_n(1) = L_n$, the $n$th Lucas number [1, 6, 8].

*Pell polynomials* $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* $P_n$ and *Pell-Lucas numbers* $Q_n$ are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [8].

*Pell polynomials* $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. They also can be defined by Binet-like formulas. In particular, the *Pell numbers* $P_n$ and *Pell-Lucas numbers* $Q_n$ are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [8].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the $n$th *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the $n$th *Jacobsthal-Lucas polynomial*. They have their own *Binet-like* formulas. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the $n$th Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$ [4, 8].

Suppose $a(x) = x$ and $b(x) = -1$. When $z_0(x) = 0$ and $z_1(x) = 1$, then $z_n(x) = V_n(x)$, the $n$th *Vieta polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, then $z_n(x) = v_n(x)$, the $n$th *Vieta-Lucas polynomial* [5, 8].

Finally, suppose $a(x) = 2x$ and $b(x) = -1$. When $z_0(x) = 1$ and $z_1(x) = x$, then $z_n(x) = T_n(x)$, the $n$th *Chebyshev polynomial of the first kind*; and when $z_0(x) = 1$ and $z_1(x) = 2x$, then $z_n(x) = U_n(x)$, the $n$th *Chebyshev polynomial of the first kind* [5, 8].

**1.1 Links among the Extended Gibonacci Family:** Table 1 shows the relationships among the subfamilies of the extended gibonacci family [5, 8].

| | |
|---|---|
| $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ | $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ |
| $V_n(x) = i^{n-1} f_n(-ix)$ | $v_n(x) = i^n l_n(-ix)$ |
| $V_n(x) = U_{n-1}(x/2)$ | $v_n(x) = 2T_n(x/2)$, |

where $i = \sqrt{-1}$.

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so $z_n$ will mean $z_n(x)$. In addition, we let $g_n = f_n$ or $l_n$, $b_n = p_n$ or $q_n$, $c_n = J_n(x)$ or $j_n(x)$. Correspondingly, we denote their numeric counterparts with $G_n$, $B_n$, and $C_n$, respectively. We also omit a lot of basic algebra.

With this background, we now explore a recurrence for $g_{n+1}^2$ with three predecessors.

**2. A Recurrence for Gibonacci Squares**

First, we find a simple recurrence for $g_{n+1}^2$.

**Lemma 1:** Let $g_n = f_n$ or $l_n$. Then

$$g_{n+1}^2 = (x^2+1)g_n^2 + (x^2+1)g_{n-1}^2 - g_{n-2}^2$$

$$g_{n+1}^2 = (x^2+1)g_n^2 + (x^2+1)g_{n-1}^2 - g_{n-2}^2. \tag{1}$$

**Proof:** Using the gibonacci recurrence, we get

$$\begin{aligned} g_{n+1}^2 &= x^2 g_n^2 + g_n(g_n - g_{n-2}) + xg_{n-1}(xg_{n-1} + g_{n-2}) + g_{n-1}^2 \\ &= (x^2+1)g_n^2 + (x^2+1)g_{n-1}^2 - g_{n-2}(g_n - xg_{n-1}) \\ &= (x^2+1)g_n^2 + (x^2+1)g_{n-1}^2 - g_{n-2}^2, \end{aligned}$$

as desired. □

In particular, this yields

$$G_{n+1}^2 = 2G_n^2 + 2G_{n-1}^2 - G_{n-2}^2;$$

the case $G_n = F_n$ appears in [2, 7].

With this lemma at our disposal, we now explore the fourth-order recurrence.

**Theorem 1:** *Let* $g_n = f_n$ or $l_n$, *and* $\Delta^2 = x^2 + 4$. *Then*

$$g_{n+1}^2 = x^2 g_n^2 + 2(x^2+1)g_{n-1}^2 + x^2 g_{n-2}^2 - g_{n-3}^2 . \tag{2}$$

**Proof:** Using Lemma 1, the *Cassin-like identities* $f_{n+k}f_{n-k} - f_n^2 = (-1)^{n-k+1} f_k^2$ and $l_{n+k}l_{n-k} - l_n^2 = (-1)^{n-k}\Delta^2 f_k^2$ [8], and the gibonacci recurrence, we have

$$\begin{aligned} g_{n+1}^2 &= (x^2+1)g_n^2 + (x^2+1)g_{n-1}^2 - g_{n-2}^2 \\ &= \left[x^2 g_n^2 + 2(x^2+1)g_{n-1}^2 + x^2 g_{n-2}^2 - g_{n-3}^2\right] + A , \end{aligned}$$

where

$$\begin{aligned} A &= g_n^2 - (x^2+1)g_{n-1}^2 - (x^2+1)g_{n-2}^2 + g_{n-3}^2 \\ &= \left(g_n^2 - x^2 g_{n-1}^2\right) - g_n^{2-1} - g_{n-2}^2 - \left(x^2 g_{n-2}^2 - g_{n-3}^2\right) \\ &= \left(g_n g_{n-2} - g_{n-1}^2\right) + \left(g_{n-1}g_{n-3} - g_{n-2}^2\right) = \\ &= \begin{cases} (-1)^{n-1} + (-1)^n & \text{if} \quad g_n = f_n \\ (-1)^{n-1}\Delta^2 + (-1)^{n-1}\Delta^2 & \text{otherwise} \end{cases} \\ &= 0 . \end{aligned}$$

Thus,

$$g_{n+1}^2 = x^2 g_n^2 + 2(x^2+1)g_{n-1}^2 + x^2 g_{n-2}^2 - g_{n-3}^2 ,$$

as expected. □

In particular, we have

$$G_{n+1}^2 = G_n^2 + 4G_{n-1}^2 + G_{n-2}^2 - G_{n-3}^2 . \tag{3}$$

The formula with $G_n = F_n$ appears in [3].

Figure 1 gives a geometric interpretation of identity (3).

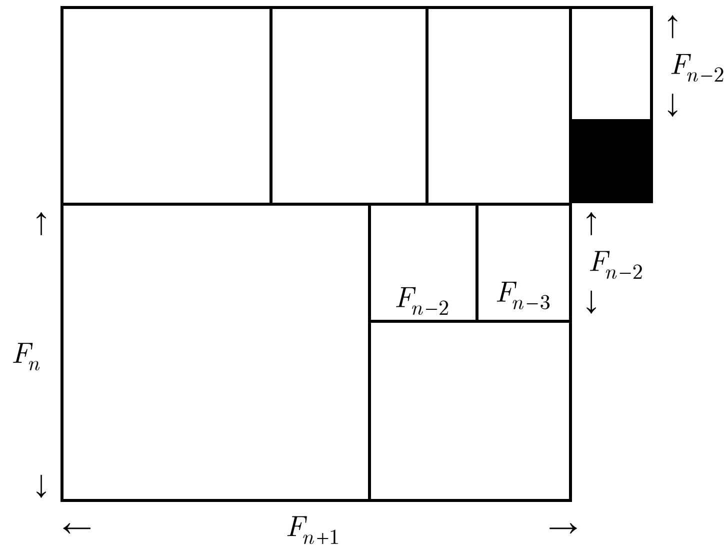


**Figure 1:** $F_{n+1}^2 = F_n^2 + 4F_{n-1}^2 + F_{n-2}^2 - F_{n-3}^2$

### 3. Pell Implications

It follows from equations (1) and (2) that

$$\begin{aligned} b_{n+1}^2 &= (4x^2+1)\,b_n^2 + (4x^2+1)\,b_{n-1}^2 - b_{n-2}^2; \\ &= 4x^2b_n^2 + 2\,(4x^2+1)\,b_{n-1}^2 + 4x^2b_{n-2}^2 - b_{n-3}^2; \end{aligned}$$

$$\begin{aligned} B_{n+1}^2 &= 5B_n^2 + 5B_{n-1}^2 - B_{n-2}^2; \\ &= 4B_n^2 + 10B_{n-1}^2 + 4B_{n-2}^2 - B_{n-3}^2. \end{aligned}$$

### 4. Jacobsthal Versions

Using the gibonacci-Jacobsthal relationships in Table 1, we now explore the Jacobsthal versions of identities (1) and (2).

**4.1 Jacobsthal Version of Identity (1):** Suppose $g_n = f_n$. Replacing $x$ with $1/\sqrt{x}$ in (1) and multiplying the resulting equation with $x^n$, we get

$$\left(x^{n/2}f_{n+1}\right)^2 = (x+1)\left[\left(x^{(n-1)/2}f_n\right]^2 + x(x+1)\right)\left[^{(x(n-2)/2}f_{n-1}\right]^2 - x^3\left[\left(x^{(n-3)/2}f_{n-2}\right]^2$$

$$J_{n+1}^2 = (x+1)J_n^2 + x(x+1)J_{n-1}^2 - x^3J_{n-2}^2,$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

On the other hand, let $g_n = l_n$. Replace $x$ with $1/\sqrt{x}$ and multiply the resulting equation with $x^{n+1}$. This yields

$$j_{n+1}^2 = (x+1)j_n^2 + x(x+1)j_{n-1}^2 - x^3 j_{n-2}^2.$$

Combining the two cases, we get

$$c_{n+1}^2 = (x+1)c_n^2 + x(x+1)c_{n-1}^2 - x^3c_{n-2}^2; \tag{4}$$

$$C_{n+1}^2 = 3C_n^2 + 6C_{n-1}^2 - 8C_{n-2}^2. \tag{5}$$

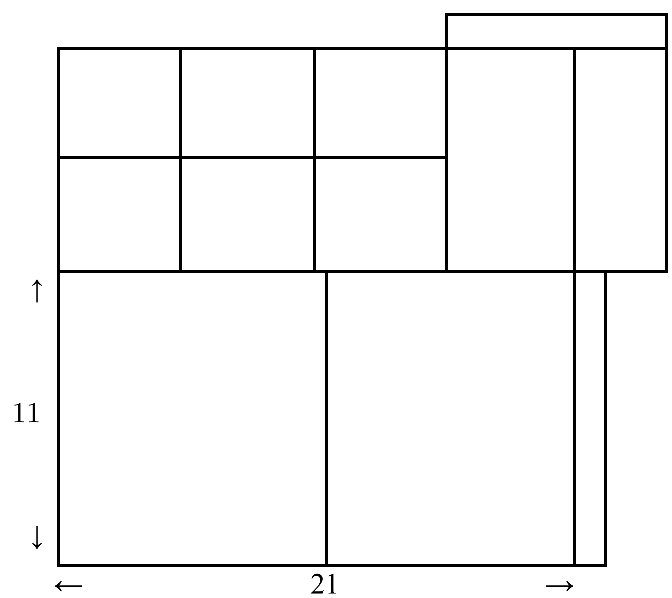


**Figure 2:** $J_6^2 = 3J_5^2 + 6J_4^2 - 8J_3^2$

Figure 2 shows a geometric illustration of identity (5) with $C_n = J_n$ and $n = 5$.

**4.2 Jacobsthal Version of Identity (2):** Let $g_n = f_n$. Replace $x$ with $1/\sqrt{x}$ in (2) and multiply the resulting equation with $x_n$. We then get

$$\begin{aligned}(x^{n/2} f_{n+1})^2 &= \left[(x^{(n-1)/2} f_n\right]^2 + 2x(x+1)\left[(x^{(n-2)/2} f_{n-1}\right]^2 + x^2\left[(x^{(n-3)/2} f_{n-2}\right]^2 \\ &\quad - x^4\left[(x^{(n-4)/2} f_{n-3}\right]^2 \\ J_{n+1}^2 &= J_n^2 + 2x(x+1)J_{n-1}^2 + x^2 J_{n-2}^2 - x^4 J_{n-3}^2,\end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Now let $g_n = l_n$. Replacing $x$ with $1/\sqrt{x}$ and multipling the resulting equation with $x^{n+1}$, similarly we get

$$j_{n+1}^2 = j_n^2 + 2x(x+1)j_{n-1}^2 + x^2 j_{n-2}^2 - x^4 j_{n-3}^2,$$

where $j_n = j_n(x)$.

Combining the two cases yields

$$c_{n+1}^2 = c_n^2 + 2x(x+1)c_{n-1}^2 + x^2 c_{n-2}^2 - x^4 c_{n-3}^2; \tag{6}$$

$$C_{n+1}^2 = C_{n+1}^2 2C_{n-1}^2 + 4C_{n-2}^2 - 8C_{n-3}^2. \tag{7}$$

**5. Vieta and Chebyshev Implications**

Using the Vieta-gibonacci and Vieta-Chebyshev relationships in Table 1, we can extract the Vieta and Chebyshev versions of identities (1) and (2). In the interest of brevity, we omit them.

Next we will confirm identities (1) and (2) using graph-theoretic techniques. To this end, first we present the needed graph-theoretic tools.

**6. Graph-Theoretic Tools**

Consider the Fibonacci digraph $D_1$ in Figure 1 with vertices $v_1$ and $v_2$, where a *weight* is assigned to each edge [8, 9]. It follows from its *weighted adjacency matrix* $Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ that

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $n \geq 1$ [8, 9].

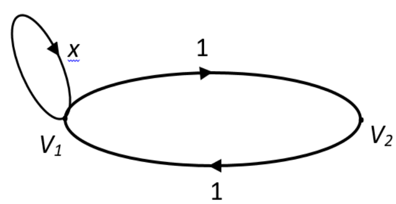


**Figure 1:** Weighted Fibonacci Digraph $D_1$

A *walk* from vertex $v_i$ to vertex $v_j$ is a sequence $v_i - e_i - v_{i+1} - \cdot \cdot \cdot - v_{j-1} - e_{j-1} - v_j$ of vertices $v_k$ and edges $e_k$, where edge $e_k$ is incident with vertices $v_k$ and $v_{k+1}$. We will denote the edge $v_i = v_j$ by the *word* $ij$ when there is *no* confusion. The walk is *closed* if $v_i = v_j$; otherwise, it is *open*. The *length* of a walk is the number of edges in the walk. The *weight* of a walk is the product of the weights of the edges along the walk.

The sum of the weights of closed walks of length $n$ originating at $v_1$ in the digraph is $f_{n+1}$ and that of those originating at $v_2$ is $f_{n-1}$. So, the sum of the weights of all closed walks of length $n$ in the digraph is $f_{n+1} + f_{n-1} = l_n$.

To confirm the Jacobsthal identities, we employ the *weighted Jacobsthal digraph* $D_2$ in Figure 2 with vertices $v_1$ and $v_2$ [8, 9].

*Its weighted adjacency matrix* $M = \begin{bmatrix} 1 & x \\ 1 & 0 \end{bmatrix}$ implies that

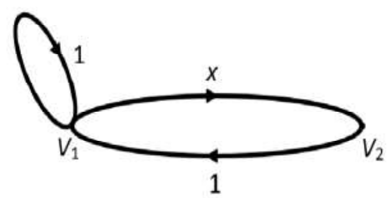


**Figure 2:** Weighted Jacobsthal Digraph $D_2$

$$M^n = \begin{bmatrix} J_{n+1} & xJ_n \\ J_n & xJ_{n-1} \end{bmatrix},$$

where $J_n = J_n(x)$ and $n \geq 1$.

The sum of the weights of closed walks of length $n$ originating at $v_1$ is $J_{n+1}$, and that of those originating at $v_2$ is $xJ_{n-1}$. So the sum of the weights of all closed walks of length $n$ in the digraph is $J_{n+1} + xJ_{n-1} = j_n$.

In both cases, suppose $A$ and $B$ denote the sets of walks of varying lengths originating at a vertex $v$. Then the sum of the weights of the elements $(a,b)$ in the product set $A \times B$ is *defined* as the product of the sums of weights from each component [9].

With these tools at our finger tips, we are now ready for the graph-theoretic proofs.

**7. Graph-Theoretic Confirmations**

**7.1 Confirmation of identity (1) with** $g_n = f_n$**: Proof:** Let A denote the set of closed walks of length $n$ originating at $v_1$ in digraph $D_1$. The sum of the weights all such walks is $f_{n+1}$. So the sum $S$ of the weights of all elements in the product set $A \times A$ is given by $S = f_{n+1}^2$.

We will now compute $S$ in a different way. Let $B$ be the set of closed walks of length $n-1$ in digraph $D_1$ originating at $v_1$. Let $w$ be an arbitrary walk in $B$. It can land at $v_1$ or $v_2$ at the $(n-2)$nd step: $w = \underbrace{v_1 - \ldots - v}_{\text{subwalk of length } n-2} - v_1$, where $v = v_1$ or $v_2$. If $v = v_1$, the sum of the weights of such walks is $f_{n-1} \cdot x = xf_{n-1}$. Otherwise, the sum of the walks in is $B$ $f_{n-2} \cdot 1 = f_{n-2}$. So the sum of the weights of the walks in $B$ is $xf_{n-1} + f_{n-2} = f_n$. Consequently, the sum $S_1$ of the weights of the elements in $B \times B$ is given by $S_1 = f_n^2$.

Now, let $C$ be the set of closed walks of length $n-2$ originating at $v_1$, and $w'$ an arbitrary element in $C$. It can land at $v_1$ or $v_2$ at step $(n-3)$: $w' = \underbrace{v_1 - \ldots - v}_{\text{subwalk of length } n-3} - v_1$, where $v = v_1$ or $v_2$. As above, the sum of the weights in $C$ is $xf_{n-2} + f_{n-3} = f_{n-1}$, and hence the sum $S_2$ of the weights of the elements in $C \times C$ is given by $S_2 = f_{n-1}^2$.

Finally, let $D$ denote the set of closed walks of length $n-3$ originating at $v_1$. The sum of the weights of the walks is $xf_{n-3} + f_{n-4} = f_{n-2}$; so the sum $S_3$ of the weights of the elements in $D \times D$ is given by $S_3 = f_{n-2}^2$.

We now let $S^* = (x^2+1)S_1 + (x^2+1)S_2$. Since the elements in $D$ appear in both $B$ and $C$ as subwalks, they contribute $S_3$ to both $S_1$ and $S_2$; so we discount it from $S^*$ to yield

$$
\begin{aligned}
S^* &= (x^2+1)S_1 + (x^2+1)S_2 - S_3 \\
&= (x^2+1)f_n^2 + (x^2+1)f_{n-1}^2 - f_{n-2}^2 \\
&= (x^2+1)f_n^2 + (x^2+1)f_{n-1}^2 - f_{n-2}(f_n - xf_{n-1}) \\
&= x^2 f_n^2 + f_{n-1}^2 + f_n(f_n - f_{n-2}) + xf_{n-1}(xf_{n-1} + f_{n-2}) \\
&= (xf_n + f_{n-1})^2
\end{aligned}
$$

$$= f_{n+1}^2$$
$$= S\,,$$

as desired. □

Next, we present an algorithm for extracting the elements of $B$ from those of $A$.

***Algorithm* 1:** Let $w$ be an arbitrary walk (word) in $A$. Let $w'$ be the subword obtained by deleting the rightmost 1 in $w$ when it ends in 11. Then $B$ consists of all such subwords $w'$.

The same algorithm works for constructing $C$ from $B$, and $D$ from $C$. Table 2 shows the walks in $A$, $B$, $C$, and $D$, weights of the elements in each, and their cumulative sums, when $n = 5$. Notice that the elements in $D$ are subwalks of the walks in both $B$ and $C$, as expected.

**Table 2:** Walks in Sets $A$, $B$, $C$, and $D$, Weights, and Cumulative Sums

| $A$ | | $B$ | | $C$ | | $D$ | |
|---|---|---|---|---|---|---|---|
| walks | weights | walks | weights | walks | weights | walks | weights |
| 111111 | $x^5$ | 11111 | $x^4$ | 1111 | $x^3$ | 111 | $x^2$ |
| 111121 | $x^3$ | • | • | • | • | • | • |
| 111211 | $x^3$ | 11121 | $x^2$ | • | • | • | • |
| 112111 | $x^3$ | 11211 | $x^2$ | 1121 | $x$ | . | . |
| 121111 | $x^3$ | 12111 | $x^2$ | 1211 | $x$ | 121 | 1 |
| 112121 | $x$ | • | • | • | • | • | • |
| 121121 | $x$ | • | • | • | • | • | • |
| 121211 | $x$ | 12121 | 1 | . | . | . | . |
| sum | $f_6$ | • | $f_5$ | • | $f_4$ | • | $f_3$ |

**7.2 Confirmation of identity (1) with $g_n = l_n$: Proof:** Let $A$ denote the set of all closed walks of length $n+1$ in digraph $D_1$. The sum of the weights all such walks is $l_{n+1}$ . So the sum $S$ of the weights of all elements in the product set $A \times A$ is given by $S = l_{n+1}^2$.

We will now compute $S$ in a different way. Let $B$ be the set of closed walks of length $n$ in digraph $D_1$, and $w$ an arbitrary element of $B$. The sum of the weights walks originating at $v_1$ is $f_{n+1}$ and that of those originating at $v_2$ is $f_{n-1}$. So the sum of the walks of all elements in $B$ is $f_{n+1} + f_{n-1} = l_n$, and the sum $S_1$ of the elements in $B \times B$ is given by $S_1 = l_n^2$.

Suppose $C$ denotes the set of all closed walks of length $n-1$. Clearly, the sum of the walks of all elements in $C$ is $f_n + f_{n-2} = l_{n-1}$, and hence the sum $S_2$ of those in $B \times B$ is given by $S_2 = l_{n-1}^2$.

Now let $D$ be the set of all closed walks of length $n-2$. It follows from above that the sum of the walks of all elements in $D$ is $l_{n-2}$ and the sum $S_3$ of those in $C \times C$ is given by $S_3 = l_{n-2}^2$.

We let $S^* = (x^2+1)S_1 + (x^2+1)S_2$. Since the walks in $D$ occur as subwalks of the elements in both $B$ and $C$, we discount their contributions to $S^*$:

$$\begin{aligned}
S^* &= (x^2+1)S_1 + (x^2+1)S_2 - S_3 \\
&= (x^2+1)l_n^2 + (x^2+1)l_{n-1}^2 - l_{n-2}^2 \\
&= (x^2+1)l_n^2 + (x^2+1)l_{n-1}^2 - l_{n-2}(l_n - xl_{n-1}) \\
&= x^2 l_n^2 + l_{n-1}^2 + l_n(l_n - l_{n-2}) + xl_{n-1}(xl_{n-1} + l_{n-2}) \\
&= (xl_n + l_{n-1})^2 \\
&= l_{n+1}^2 \\
&= S,
\end{aligned}$$

as expected. □

We now present an algorithm for extracting the elements of $B$ from those of $A$; it is an extension of Algorithm 1.

***Algorithm* 2:** Let $w$ be an arbitrary walk (word) in $A$. If it ends in 11, deleting the rightmost 1 yields a subword $w'$ of length $n-1$. Suppose $w$ ends in 112; deleting the rightmost 2 and replacing the rightmost 1 in the deleted word with 2 yield a subword $w'$ of length $n-1$. Then $B$ consists of all such subwords $w'$.

The same algorithm works for constructing $C$ from $B$, and $D$ from $C$.

**Table 3:** Walks in Sets $A$, $B$, $C$, and $D$, Weights, and Cumulative Sums

| $A$ | | $B$ | | $C$ | | $D$ | |
|---|---|---|---|---|---|---|---|
| walks | weights | walks | weights | walks | weights | walks | weights |
| 111111 | $x^5$ | 11111 | $x^4$ | 1111 | $x^3$ | 111 | $x^2$ |
| 111121 | $x^3$ | • | • | • | • | • | • |
| 111211 | $x^3$ | 11121 | $x^2$ | • | • | • | • |
| 112111 | $x^3$ | 11211 | $x^2$ | 1121 | $x$ | . | |
| 121111 | $x^3$ | 12111 | $x^2$ | 1211 | $x$ | 121 | 1 |
| 112121 | $x$ | • | • | • | • | • | • |
| 121121 | $x$ | • | • | • | • | • | • |
| 121211 | $x$ | 12121 | 1 | . | . | . | . |
| 211112 | $x^3$ | 21112 | $x^2$ | 2112 | $x$ | 212 | 1 |
| 211212 | $x$ | . | . | . | . | . | . |
| 212112 | $x$ | 21212 | 1 | . | . | . | . |
| sum | $l_5$ | • | $l_4$ | • | $l_3$ | • | $l_2$ |

Table 3 shows the walks in $A$, $B$, C, and $D$, weights of the elements in each, and their cumulative sums, when $n = 4$. Clearly, the elements in $D$ are subwalks of the walks in both $B$ and $C$, as expected.

Next we confirm identity (6) using digraph $D_2$.

**7.3 Confirmation of identity (6) with** $c_n = J_n$ **: Proof:** Let $A$ be the set of all closed walks of length $n$ originating at $v_1$ in digraph $D_2$. The sum of the weights such walks is $J_{n+1}$. So the sum $S$ of the weights of elements in the product set $A \times A$ is given by $S = J_{n+1}^2$.

We will now compute it in a different way. Let $w$ be an arbitrary element in the set $B$ of closed walks of length $n-1$ originating at $v_1$. It can land at $v_1$ or $v_2$ at step $(n-2)$: $w = \underbrace{v_1 - \ldots - v}_{\text{subwalk of length } n-2} - v_1$, where $v = v_1$ or $v_2$. If $v = v_1$, the sum of the weights of such walks is $J_{n-1} \times 1 = J_{n-1}$ and that of those with $v = v_2$ is $xJ_{n-2}$. Thus, the sum of the weights of the walks in $B$ is $J_{n-1} + xJ_{n-2} = J_n$. Consequently, the sum $S_1$ of the weights of the elements in $B \times B$ is given by $S_1 = J_n^2$.

Now let $C$ be the set of closed walks of length $n-2$ originating at $v_1$, and $w'$ an arbitrary element in $C$. It can land at $v_1$ or $v_2$ at step $n-3$: $w = \underbrace{v_1 - \ldots - v}_{\text{subwalk of length } n-3} - v_1$, where $v = v_1$ or $v_2$. Clearly, the sum of the weights of walks in $C$ is $J_{n-2} + xJ = J_{n-1}$. So the sum $S_2$ of the weights of the elements in $C \times C$ is given by $S_2 = J_{n-1}^2$.

Finally, let $D$ be the set of closed walks of length $n-3$ originating at $v_1$. The sum of the weights of walks in $D$ is $J_{n-2}$, and hence the sum $S_3$ of the weights of elements in $D \times D$ is given by $S_3 = J_{n-2}^2$.

Now let

$$\begin{aligned} S^* &= (x+1)S_1 + x(x+1)S_2 \\ &= (x+1)J_n^2 + x(x+1)J_{n-1}^2. \end{aligned}$$

Clearly, the walks in $D$ are subwalks of the elements in both $B$ and $C$. So their contributions to $S^*$ must be discounted once to eliminate duplicate counting. To this end, notice that $\deg J_n = \left\lfloor \frac{n-1}{2} \right\rfloor$, where $\deg J_n$ denotes the *degree* of $J_n(x)$, $\lfloor x \rfloor$ the *floor* of the real number $x$, and $\lfloor x + k \rfloor = \lfloor x \rfloor + k$, $k$ being an integer. We will now find the correct multiple $x^k$ of $S_3 = J_{n-2}^2$ that must be subtracted. Since $\deg (x+1)J_n^2 = 1 + 2\left\lfloor \frac{n-1}{2} \right\rfloor$, $\deg\ x(x+1)J_{n-1}^2 = 2\lfloor n/2 \rfloor = \deg\ J_{n+1}^2$, and

$$\deg\ J_{n+1}^2 = \begin{cases} \deg\ (x+1)J_n^2 - 1 & n \text{ is odd} \\ \deg\ (x+1)J_n^2 + 1 & \text{otherwise,} \end{cases}$$

we let $k$ be such that

$$k + \deg J_{n-2}^2 = \deg (x+1)J_n^2$$

$$k + 2\left\lfloor \frac{n-3}{2} \right\rfloor = 1 + 2\left\lfloor \frac{n-1}{2} \right\rfloor.$$

This yields $k = 3$. Discounting $x^3 J_{n-2}^2$ from $S^*$ then yields

$$\begin{aligned} S^* &= (x+1)J_n^2 + x(x+1)J_{n-1}^2 - x^3 J_{n-2}^2 \\ &= (x+1)J_n^2 + x(x+1)J_{n-1}^2 - x^2 J_{n-2}(J_n - J_{n-1}) \\ &= J_n^2 + x^2 J_{n-1}^2 + xJ_n(J_n - xJ_{n-2}) + xJ_{n-1}(J_{n-1} + xJ_{n-2}) \\ &= (J_n + xJ_{n-1})^2 \\ &= S, \end{aligned}$$

as desired. □

Table 4 shows the walks in $A$, $B$, $C$, and $D$, the weights of the elements in each, and their cumulative sums, where $n = 5$. Using Algorithm 1, we can get extract the walks in $B$, $C$, and $D$, as before. The walks are the same as those in Table 2, but the weights are different. Again, notice that the elements in $D$ are subwalks of the walks in both $B$ and $C$, as expected.

**Table 4:** Walks in Sets $A$, $B$, $C$, and $D$, Weights, and Cumulative Sums

| $A$ | | $B$ | | $C$ | | $D$ | |
|---|---|---|---|---|---|---|---|
| walks | weights | walks | weights | walks | weights | walks | weights |
| 111111 | 1 | 11111 | 1 | 1111 | 1 | 111 | 1 |
| 111121 | $x$ | • | • | • | • | • | • |
| 111211 | $x$ | 11121 | $x$ | • | • | • | • |
| 112111 | $x$ | 11211 | $x$ | 1121 | $x$ | . | |
| 121111 | $x$ | 12111 | $x$ | 1211 | $x$ | 121 | $x$ |
| 112121 | $x^2$ | • | • | • | • | • | • |
| 121121 | $x^2$ | • | • | • | • | • | • |
| 121211 | $x^2$ | 12121 | $x^2$ | . | . | . | . |
| sum | $J_6$ | | $J_5$ | | $J_4$ | | $J_3$ |

**7.4 Confirmation of identity (6) with $c_n = j_n$: Proof:** Let $A$ be the set of all closed walks of length $n+1$ in digraph $D_2$. The sum of the weights of such walks originating at $v_1$ is $J_{n+2}$ and that of those originating at $v_2$ is $xJ_n$. So the sum of the weights of all such closed walks in $D_2$ is $J_{n+2} + xJ_n = j_{n+1}$, and hence the sum $S$ of the weights of elements in the product set $A \times A$ is given by $S = j_{n+1}^2$.

To compute $S$ in a different way, first consider the set $B$ of closed walks of length $n$ in $D_2$. The sum of the weights of such walks originating at $v_1$ is $J_{n+1}$ and that of those originating at $v_2$ is $xJ_{n-1}$. Thus, the sum of the weights of the walks in $B$ is $J_{n+1} + xJ_{n-1} = j_n$. Consequently, the sum $S_1$ of the weights of the elements in B × B is given by $S_1 = j_n^2$.

Now let $C$ denote the set of closed walks of length $n-1$. The sum of the weights of such walks originating at $v_1$ is $J_n$ and that of those originating at $v_2$ is $xJ_{n-2}$. So the sum of the walks in $C$ is $J_n + xJ_{n-2} = j_{n-1}$, and hence the sum $S_2$ of the weights of the elements in $C \times C$ is given by $S_2 = j_{n-1}^2$.

Finally, let $D$ be the set of closed walks of length $n-2$ in $D_2$. Clearly, the sum of the walks in $D$ is $J_{n-1} + xJ_{n-3} = j_{n-2}$, and hence the sum $S_3$ of the weights of the elements in $D \times D$ is given by $S_3 = j_{n-2}^2$.

We now let

$$\begin{aligned} S^* &= (x+1)S_1 + x(x+1)S_2 \\ &= (x+1)j_n^2 + x(x+1)j_{n-1}^2. \end{aligned}$$

Since the walks in $D$ can be subwalks of elements in $B$ or $C$, their contributions to $S^*$ must be discounted to eliminate duplication. As before, discounting $x^3 j_{n-3}^2$ from $S^*$ yields

$$\begin{aligned} S^* &= (x+1)j_n^2 + x(x+1)j_{n-1}^2 - x^3 j_{n-3}^2 \\ &= (x+1)j_n^2 + x(x+1)j_{n-1}^2 - x^2 j_{n-3}(j_n - j_{n-1}) \\ &= j_n^2 + x^2 j_{n-1}^2 + xj_n(j_n - xj_{n-2}) + xj_{n-1}(j_{n-1} + xj_{n-2}) \\ &= (j_n + xj_{n-1})^2 \\ &= S, \end{aligned}$$

as expected. □

Table 5 shows the walks in $A$, $B$, $C$, and $D$, the weights of the elements in them, and their cumulative sums, where $n = 5$.

Using Algorithm 2, we can obtain all elements in $B$, $C$, and $D$ from $A$. Notice that the elements in $D$ are subwalks of the walks in both $B$ and $C$.

**Table 5:** Walks in Sets $A$, $B$, $C$, and $D$, Weights, and Cumulative Sums

| $A$ | | $B$ | | $C$ | | $D$ | |
|---|---|---|---|---|---|---|---|
| walks | weights | walks | weights | walks | weights | walks | weights |
| 111111 | 1 | 11111 | 1 | 1111 | 1 | 111 | 1 |
| 111121 | $x$ | · | · | · | · | · | · |
| 111211 | $x$ | 11121 | $x$ | · | · | · | · |
| 112111 | $x$ | 11211 | $x$ | 1121 | $x$ | . | |
| 121111 | $x$ | 12111 | $x$ | 1211 | $x$ | 121 | $x$ |
| 112121 | $x^2$ | · | · | · | · | · | · |
| 121121 | $x^2$ | · | · | · | · | · | · |
| 121211 | $x^2$ | 12121 | $x^2$ | . | . | . | . |
| 211112 | $x$ | 21112 | $x$ | 2112 | $x$ | 212 | $x$ |
| 211212 | $x^2$ | . | . | . | . | . | . |
| 212112 | $x^2$ | 21212 | $x^2$ | . | . | . | . |
| sum | $j_5$ | · | $j_4$ | · | $j_3$ | · | $j_2$ |

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# GRAPH-THEORETIC CONFIRMATIONS OF SUMS OF JACOBSTHAL POLYNOMIAL PRODUCTS OF ORDER 6

**Abstract:** Using graph-theoretic tools, we confirm four identities involving sums of Jacobsthal polynomial products of order 6, investigated in [5].

**Keywords:** Graph-Theoretic Techniques, Jacobsthal and Jacobsthal Polynomials, Gibonacci Polynomials.

**Mathematical Subject Classification (2010) No.:** 05A19, 11B39, 11Cxx.

## 1. Introduction

*Extended gibonacci polynomials* $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where $x$ is an integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$ [1, 2, 5].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the $n$th *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the $n$th *Jacobsthal Lucas polynomial* [1, 2]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the $n$th Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$ and $j_n(1) = L_n$.

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so $z_n$ will mean $z_n(x)$. We also let $c_n(x) = J_n(x)$ or $j_n(x)$;

| | | |
|---|---|---|
| $A^* = J_{n+2}^6$ | $B^* = J_{n+2}^5 J_n$ | $C^* = J_{n+2}^4 J_n^2$ |
| $D^* = J_{n+2}^4 J_n J_{n-2}$ | $E^* = J_{n+2}^3 J_n^3$ | $F^* = J_{n+2}^3 J_n^2 J_{n-2}$ |
| $G^* = J_{n+2}^3 J_n J_{n-2}^2$ | $H^* = J_{n+2}^2 J_n^4$ | $I^* = J_{n+2}^2 J_n^3 J_{n-2}$ |
| $J^* = J_{n+2}^2 J_n^2 J_{n-2}^2$ | $K^* = J_{n+2}^2 J_n J_{n-2}^3$ | $L^* = J_{n+2} J_n^5$ |
| $M^* = J_{n+2} J_n^4 J_{n-2}$ | $N^* = J_{n+2} f_n^3 J_{n-2}^2$ | $O^* = J_{n+2} J_n^2 J_{n-2}^3$ |
| $P^* = J_n^6$ | $Q^* = J_n^5 J_{n-2}$ | $R^* = J_n^4 J_{n-2}^2$ |
| $S^* = J_n^3 J_{n-2}^3$ | $T^* = J_n^2 J_{n-2}^4$ ; | |

and omit a lot of basic algebra.

It is well known that $J_{n+1} + xJ_{n-1} = j_n$, $J_{2n} = J_n j_n$, $J_{2n+1} = J_{n+1}^2 + xJ_n^2$, $J_{n+2} + x^2 J_{n-2} = (2x+1)J_n$, $J_{n+2} - x^2 J_{n-2} = j_n$, $J_{2n+2} = J_{n+2}^2 - x^2 J_n^2$, and the *Jacobsthal addition formula* $J_{m+n} = J_{m+1}J_n + xJ_m J_{n-1}$ [3].

**1.1 Sums of Jacobsthal Polynomial Products of Order 4:** Sums of gibonacci polynomial products of order 4 are explored in [5]. Four of them form the basis of our discourse:

$$\begin{aligned} J_{4n} &= J_{n+2}^3 J_n - 2xJ_{n+2}^2 J_n^2 - x^2 J_{n+2}^2 J_n J_{n-2} + 2(x^2+x)J_{n+2} J_n^3 \\ &\quad + x^4 J_{n+2} J_n J_{n-2}^2 - 2(x^4+x^3)J_n^3 J_{n-2} + 2x^5 J_n^2 J_{n-2}^2 - x^6 J_n J_{n-2}^3; \quad (1) \end{aligned}$$

$$\begin{aligned} J_{4n+1} &= J_{n+2}^4 - 4xJ_{n+2}^3 J_n + 2(3x^2+2x)J_{n+2}^2 J_n^2 - (4x^3+6x^2+x)J_{n+2} J_n^3 \\ &\quad - 2x^3 J_{n+2} J_n^2 J_{n-2} + (x^2+x)^2 J_n^4 + (2x^4+x^3)J_n^3 J_{n-2}; \quad (2) \end{aligned}$$

$$J_{4n+2} = J_{n+2}^4 - 3x^2 J_{n+2}^2 J_n^2 + 2x^4 J_{n+2} J_n^2 J_{n-2} + x^4 J_n^4 - x^6 J_n^2 J_{n-2}^2; \quad (3)$$

$$\begin{aligned} J_{4n+3} &= (x+1)J_{n+2}^4 - 4x^2 J_{n+2}^3 J_n + (6x^3+x^2)J_{n+2}^2 J_n^2 - (4x^4+6x^3+x^2)J_{n+2} J_n^3 \\ &\quad + (x^5+3x^4+x^3)J_n^4 + (2x^5+x^4)J_n^3 J_{n-2} - x^6 J_n^2 J_{n-2}^2, \quad (4) \end{aligned}$$

where $c_n = c_n(x)$.

**1.2 Sums of Jacobsthal Polynomial Products of Order 6:** We investigated several sums of Jacobsthal polynomial products of order 6 in [6]. Six of them are the following:

$$\begin{aligned} J_{6n} &= 2B^* - 4(2x+1)C^* - 2x^2D^* + (13x^2+16x+4)E^* \\ &+ 2(3x^3+2x^2)F^* + x^4G^* - (10x^3+16x^2+7x+1)H^* \\ &- (9x^4+20x^3+6x^2)I^* - x^6K^* + (3x^4+6x^3+5x^2+x)L^* \\ &+ 2x^7O^* - (3x^6+6x^5+5x^4+x^3)Q^* + (2x^7-3x^5-x^4)R^* \\ &- (x^8+2x^7)S^*. \end{aligned} \tag{5}$$

$$\begin{aligned} J_{6n+2} &= A^* - 4xB^* + (7x^2+2x)C^* - 2x^3D^* - (7x^3+4x^2-3x)E^* \\ &+ 6x^4F^* + x^5G^* + (5x^4+10x^3+2x^2-x)H^* \\ &- (9x^5+12x^4+5x^3)I^* + 2x^5J^* - x^7K^* \\ &- (3x^5+8x^4+4x^3)L^* + 2(4x^6+3x^5+x^4+x^3)M^* \\ &- (3x^7-8x^6-2x^5)N^* + 2(x^8-x^7)O^* + (x^2+x)^3P^* \\ &- (3x^7+4x^6+2x^5)Q^* + (2x^8+2x^7-x^6-x^5)R^* \\ &- (x^9+4x^8)S^* + x^9T^*. \end{aligned} \tag{6}$$

$$\begin{aligned} J_{6n+3} &= (x+1)A^* - 2(3x^2+2x)B^* + (15x^3+13x^2+2x)C^* - 2x^3D^* \\ &- (20x^4+27x^3+5x^2-3x)E^* + 2x^4F^* + x^5G^* \\ &+ (15x^5+31x^4+19x^3+2x^2-x)H^* - (x5+11x4+5x^3)I^* \\ &+ 2(x^6+x^5)J^* - x^7K^* - (6x^6+17x^5+17x^4+5x^3)L^* \\ &- 2(x^6+x^5-x^4-x^3)M^* - (x^7-8x^6-2x^5)N^* - 2x^7O^* \\ &+ (x+1)(x^2+x)^3P^* - (x^7+x^6+x^5)Q^* \\ &+ (4x^8+4x^7-x^6-x^5)R^* - (3x^9+4x^8)S^* + (x^{10}+x^9)T^*. \end{aligned} \tag{7}$$

$$\begin{aligned}
J_{6n+4} &= (2x+1)A^* - 2(5x^2+2x)B^* + (22x^3+15x^2+2x)C^* \\
&- 2(x^4+x^3)D^* - (27x^4+31x^3+2x^2-3x)E^* + 2(3x^5+x^4)F^* \\
&+ (x^6+x^5)G^* + (20x^5+41x^4+21x^3+x^2-x)H^* \\
&- (9x^6+13x^5+16x^4+5x^3)I^* + 2(2x^6+x^5)J^* - (x^8+x^7)K^* \\
&- (9x^6+25x^5+21x^4+5x^3)L^* + 2(4x^7+2x^6+2x^4+x^3)M^* \\
&- (3x^8-7x^7-10x^6-2x^5)N^* + 2(x^9-x^8-x^7)O^* \\
&+ (2x+1)(x^2+x)^3P^* - (3x^8+5x^7+3x^6+x^5)Q^* \\
&+ (2x^9+6x^8+3x^7-2x^6-x^5)R^* - (x^{10}+7x^9+4x^8)S^* \\
&+ (2x^{10}+x^9)T^*. \qquad (8)
\end{aligned}$$

$$\begin{aligned}
j_{6n+1} &= A^* - 2xB^* - (x^2+2x)C^* - 4x^3D^* + (6x^3+12x^2+7x)E^* \\
&+ 4(3x^4+x^3)F^* + 2x^5G^* - (5x^4+6x^3+5x^2+2x)H^* \\
&- (18x^5+32x^4+11x^3)I^* + 2x^5J^* - 2x^7K^* - (2x^4-x^3-x^2)L^* \\
&+ 2(8x^6+11x^5+6x^4+2x^3)M^* - 2(3x^7-7x^6-2x^5)N^* \\
&+ 2(2x^8-x^7)O^* + (x^2+x)^3P^* - (6x^7+10x^6+7x^5+x^4)Q^* \\
&+ 2(2x^8+x^7-2x^6-x^5)R^* - 2(x^9+3x^8)S^* + x^9T^*. \qquad (9)
\end{aligned}$$

$$\begin{aligned}
j_{6n+5} &= (5x^2+5x+1)A^* - 2(13x^3+11x^2+2x)B^* \\
&+ (59x^4+65x^3+21x^2+2x)C^* - 2(2x^5+4x^4+x^3)D^* \\
&- (74x^5+116x^4+40x^3-7x^2-3x)E^* \\
&+ 2(6x^6+6x^5+x^4)F^* + (2x^7+4x^6+x^5)G^* \\
&+ (55x^6+133x^5+96x^4+19x^3-2x^2-x)H^* \\
&- (18x^7+36x^6+56x^5+21x^4+5x^3)I^* + 2(5x^7+5x^6+x^5)J^* \\
&- (2x^9+4x^8+x^7)K^* - (24x^7+76x^6+84x^5+36x^4+5x^3)L^*
\end{aligned}$$

$$+\ 2(8x^8+7x^7+x^6+5x^5+5x^4+x^3)M^*$$
$$-\ (6x^9-10x^8-35x^7-16x^6-2x^5)N^*+2(2x^{10}+x^9+x^7)O^*$$
$$+\ (5x^2+5x+1)(x^2+x)^3P^*-(6x^9+14x^8+12x^7+6x^6+x^5)Q^*$$
$$+\ (4x^{10}+18x^9+16x^8-2x^7-5x^6-x^5)R^*$$
$$-\ (2x^{11}+18x^{10}+19x^9+4x^8)S^*+(5x^{11}+5x^{10}+x^9)T^*. \qquad (10)$$

Our goal is to confirm the Jacobsthal identities (7) through (10) using graph-theoretic techniques.

**2. Graph-Theoretic Tools**

To confirm these Jacobsthal results, consider the *weighted Jacobsthal digraph* $D$ in Figure 1 with vertices $v_1$ and $v_2$ [3, 4]. It follows from its *weighted adjacency matrix* $M=\begin{bmatrix}1 & x\\ 1 & 0\end{bmatrix}$ that

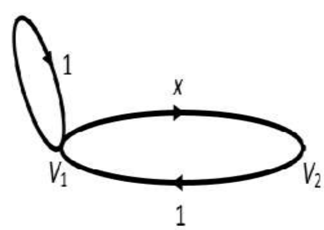


**Figure 1:** Weighted Digraph $D$

$$M^n=\begin{bmatrix}J_{n+1} & xJ_n\\ J_n & xJ_{n-1}\end{bmatrix},$$

where $J_n=J_n(x)$ and $n\geq 1$ .

The sum of the weights of closed walks of length $n$ originating at $v_1$ is $J_{n+1}$, and that of those originating at $v_2$ is $xJ_{n-1}$. So the sum of the weights of all closed walks of length $n$ in the digraph is $J_{n+1}+xJ_{n-1}=j_n$. These facts play a major role in the graph-theoretic proofs.

Let $A$, $B$, and $C$ denote the sets of closed walks of varying lengths originating at vertex $v$. Then the sum of the weights of the elements in the product set $A \times B \times C$ is *defined* as the product the sums of the walks in each component [4].

With these tools at our convenience, we are now ready to explore the graph-theoretic proofs.

## 3. Graph-Theoretic Confirmations

We begin our explorations with identity (7)

**3.1 Confirmation of Identity (7): Proof:** Let $S$ denote the sum of the weights of closed walks of length $6n+2$ originating at $v_1$. Clearly, $S = J_{6n+3}$.

We will now compute the sum $S$ in a different way. To this end, let $w$ be an arbitrary closed walk of length $6n+2$ originating at $v_1$. It can land at $v_1$ or $v_2$ at the $(2n+1)\text{st}$ and $(4n+2)\text{nd}$ steps:

$$w = \underbrace{v_1 - \ldots - v}_{\text{subwalk of length } 2n+1} \quad \underbrace{v - \ldots - v}_{\text{subwalk of length } 2n+1} \quad \underbrace{v - \ldots - v_1}_{\text{subwalk of length } 2n},$$

where $v = v_1$ or $v_2$.

**Table 1:** Sums of the Weights of Closed Walks Originating at $v_1$

| $w$ lands at $v_1$ at the $(2n+1)$st step? | $w$ lands at $v_1$ at the $(4n+2)$nd step? | $w$ lands at $v_1$ at the $(6n+2)$nd step? | sum of the weights of walks $w$ |
|---|---|---|---|
| yes | yes | yes | $J_{2n+2}^2 J_{2n+1}$ |
| yes | no | yes | $xJ_{2n+2}J_{2n+1}J_{2n}$ |
| no | yes | yes | $xJ_{2n+1}^3$ |
| no | no | yes | $x^2 J_{2n+1} J_{2n}^2$ |

Table 1 shows the possible cases and the sums of weights of the corresponding walks $w$, where $J_n = J_n(x)$. Using equations (3) and (4), it follows from the table that the sum $S$ of the weights of such walks $w$ is given by

$$
\begin{aligned}
S &= J_{2n+2}^2 J_{2n+1} + xJ_{2n+2}J_{2n+1}J_{2n} + xJ_{2n+1}^3 + x^2 J_{2n+1}J_{2n}^2 \\
&= J_{2n+1}(J_{2n+2}^2 + xJ_{2n+1}^2) + xJ_{2n+1}J_{2n}(J_{2n+2} + xJ_{2n}) \\
&= J_{4n+3}J_{2n+1} + xJ_{4n+2}J_{2n} \\
&= J_{4n+3}(J_{n+1}^2 + xJ_n^2) + J_{4n+2}(xJ_n)j_n \\
&= J_{4n+3}\left[J_{n+2}^2 - 2xJ_{n+2}J_n + (x^2+1)J_n^2\right] + J_{4n+2}(xJ_n)(J_{n+2} - x^2 J_{n-2}) \\
&= \left[(x+1)J_{n+2}^4 - 4x^2 J_{n+2}^3 Jn + (6x^3 + x^2)J_{n+2}^2 J_n^2\right. \\
&\qquad - (4x^4 + 6x^3 + x^2)J_{n+2}J_n^3 \\
&\qquad + (x^5 + 3x^4 + x^3)J_n^4 + (2x^5 + x^4)J_n^3 J_{n-2} \\
&\qquad \left. - x^6 J_n^2 J_{n-2}^2\right]\left[J_{n+2}^2 - 2xJ_{n+2}J_n + (x^2+1)J_n^2\right. \\
&\qquad + \left[J_{n+2}^4 - 3x^2 J_{n+2}^2 J_{n+2}^2 x^4 J_{n+2} J_n^2 J_{n-2}\right. \\
&\qquad \left. + x^4 J_n^4 - x^6 J_n^2 J_{n-2}^2\right](xJ_n)(J_{n+2} - x^2 J_{n-2}) \\
&= (x+1)A^* - 2(3x^2 + 2x)B^* + (15x^3 + 13x^2 + 2x)C^* - 2x^3 D^* \\
&\qquad - (20x^4 + 27x^3 + 5x^2 - 3x)E^* + 2x^4 F^* + x^5 G^* \\
&\qquad + (15x^5 + 31x^4 + 19x^3 + 2x^2 - x)H^* - (x^5 + 11x^4 + 5x^3)I^* \\
&\qquad + 2(x^6 + x^5)J^* - x^7 K^* - (6x^6 + 17x^5 + 17x^4 + 5x^3)L^* \\
&\qquad - 2(x^6 + x^5 - x^4 - x^3)M^* - (x^7 - 8x^6 - 2x^5)N^* - 2x^7 O^* \\
&\qquad + (x+1)(x^2 + x)^3 P^* - (x^7 + x^6 + x^5)Q^* \\
&\qquad + (4x^8 + 4x^7 - x^6 - x^5)R^* - (3x^9 + x^8)S^* + (x^{10} + x^9)T^*,
\end{aligned}
$$

where $J_n = J_n(x)$.

This value of $S$, coupled with its earlier value, yields identity (7), as desired. □

**3.2 Confirmation of Identity (8): Proof:** Let $S'$ denote the sum of the weights of closed walks of length $6n+3$ originating at $v_1$ in the digraph.

Then $S' = J_{6n+4}$.

To compute $S'$ in a different way, we first let $w$ be an arbitrary closed walk of length $6n+3$ originating at $v_1$. It can land at $v_1$ or $v_2$ at the $(2n+1)$st and $(4n+2)$nd steps:

$$w = \underbrace{v_1 - \ldots - v}_{\text{subwalk of length } 2n+1} \quad \underbrace{v - \ldots - v}_{\text{subwalk of length } 2n+1} \quad \underbrace{v - \ldots - v_1}_{\text{subwalk of length } 2n+1},$$

where $v = v_1$ or $v_2$.

**Table 2:** Sums of the Weights of Closed Walks Originating at $v_1$

| $w$ lands at $v_1$ at the $(2n+1)$st step? | $w$ lands at $v_1$ at the $(4n+2)$nd step? | $w$ lands at $v_1$ at the $(6n+3)$rd step? | sum of the weights of walks $w$ |
|---|---|---|---|
| yes | yes | yes | $J_{2n+2}^3$ |
| yes | no | yes | $xJ_{2n+2}J_{2n+1}^2$ |
| no | yes | yes | $xJ_{2n+2}J_{2n+1}^2$ |
| no | no | yes | $x^2J_{2n+1}^2J_{2n}$ |

Table 2 summarizes the possible cases and the sums of the weights of the respective walks $w$, where $J_n = J_n(x)$. It follows by the table, and equations (3) and (4) that

$$\begin{aligned}
S' &= J_{2n+2}^3 + 2xJ_{2n+2}J_{2n+1}^2 + x^2J_{2n+1}^2J_{2n} \\
&= J_{2n+2}\left(J_{2n+2}^2 + xJ_{2n+1}^2\right) + xJ_{2n+1}^2\left(J_{2n+2} + xJ_{2n}\right) \\
&= J_{4n+3}J_{2n+2} + xJ_{4n+2}J_{2n+1} \\
&= J_{4n+3}\left(J_{n+2}^2 - x^2J_n^2\right) + xJ_{4n+2}\left[J_{n+2}^2 - 2xJ_{n+2}J_n + (x^2+1)J_n^2\right] \\
&= (2x+1)A^* - 2(5x^2+2x)B^* + (22x^3+15x^2+2x)C^* - 2(x^4+x^3)D^* \\
&\quad - (27x^4+31x^3+2x^2-3x)E^* + 2(3x^5+x^4)F^* + (x^6+x^5)G^* \\
&\quad + (20x^5+41x^4+21x^3+x^2-x)H^* - (9x^6+13x^5+16x^4+5x^3)I^* \\
&\quad + 2(2x^6+x^5)J^* - (x^8+x^7)K^* - (9x^6+25x^5+21x^4+5x^3)L*
\end{aligned}$$

$$\begin{aligned}
&+ 2(4x^7 + 2x^6 + 2x^4 + x^3)M^* - (3x^8 - 7x^7 - 10x^6 - 2x^5)N^* \\
&+ 2(x^9 - x^8 - x^7)O^* + (2x+1)(x^2+x)^3 P^* \\
&- (3x^8 + 5x^7 + 3x^6 + x^5)Q^* + (2x^9 + 6x^8 + 3x^7 - 2x^6 - x^5)R^* \\
&- (x^{10} + 7x^9 + 4x^8)S^* + (2x^{10} + x^9)T^* .
\end{aligned}$$

By equating the two values of $S'$, we get the desired result, as expected. □

**3.3 Confirmation of Identity (9): Proof:** Let $S^*$ denote the sum of the weights of all closed walks of length $6n+1$ in the digraph. Clearly, $S^* = j_{6n+1}$.

To compute $S^*$ in a different way, we let $w$ be an arbitrary closed walk of length $6n+1$.

**Case 1:** Suppose $w$ originates at $v_1$. It can land at $v_1$ or $v_2$ at the $2n$th and $4n$th steps:

$$w = \underbrace{v_1 - \ldots - v}_{\text{subwalk of length } 2n} \quad \underbrace{v - \ldots - v}_{\text{subwalk of length } 2n} \quad \underbrace{v - \ldots - v_1}_{\text{subwalk of length } 2n+1},$$

where $v = v_1$ or $v_2$.

**Table 4:** Sums of the Weights of Closed Walks Originating at $v_1$

| $w$ lands at $v_1$ at $2n$th step? | $w$ lands at $v_1$ at the $4n$th step? | $w$ lands at $v_1$ at the $(6n+1)$st step? | sum of the weights of walks $w$ |
|---|---|---|---|
| yes | yes | yes | $J_{2n+2}J_{2n+1}^2$ |
| yes | no | yes | $xJ_{2n+1}^2J_{2n}$ |
| no | yes | yes | $xJ_{2n+2}J_{2n}^2$ |
| no | no | yes | $x^2J_{2n+1}J_{2n}J_{2n-1}$ |

It follows from Table 4 that the sum $S_1^*$ of the weights of all such walks $w$ is given by

$$\begin{aligned} S_1^* &= J_{2n+2}J_{2n+1}^2 + xJ_{2n+1}^2J_{2n} + xJ_{2n+2}J_{2n}^2 + x^2J_{2n+1}J_{2n}J_{2n-1} \\ &= J_{2n+2}\left(J_{2n+1}^2 + xJ_{2n}^2\right) + xJ_{2n+1}J_{2n}\left(J_{2n+1} + xJ_{2n-1}\right) \\ &= J_{4n+1}J_{2n+2} + xJ_{4n}J_{2n+1} \\ &= J_{6n+2}. \end{aligned}$$

**Case 2:** Suppose $w$ originates at $v_2$. Then also $w$ can land at $v_1$ or $v_2$ at the $2n$th and $4n$th steps:

$$w = \underbrace{v_2 - \ldots - v}_{\text{subwalk of length } 2n} \quad \underbrace{v - \ldots - v}_{\text{subwalk of length } 2n} \quad \underbrace{v - \ldots - v_2}_{\text{subwalk of length } 2n+1},$$

where $v = v_1$ or $v_2$.

**Table 5:** Sums of the Weights of Closed Walks Originating at $v_2$

| $w$ lands at $v_1$ at $2n$th step? | $w$ lands at $v_1$ at the $4n$th step? | $w$ lands at $v_1$ at the $(6n+1)$st step? | sum of the weights of walks $w$ |
|---|---|---|---|
| yes | yes | yes | $xJ_{2n+1}^2J_{2n}$ |
| yes | no | yes | $x^2J_{2n}^3$ |
| no | yes | yes | $x^2J_{2n+1}J_{2n}J_{2n-1}$ |
| no | no | yes | $x^3J_{2n}J_{2n-1}^2$ |

It follows from Table 5 that the sum $S_2^*$ of the weights of all such walks $w$ is given by

$$\begin{aligned} S_2^* &= xJ_{2n+1}^2J_{2n} + x^2J_{2n}^3 + x^2J_{2n+1}J_{2n}J_{2n-1} + x^3J_{2n}J_{2n-1}^2 \\ &= xJ_{2n}\left(J_{2n+1}^2 + xJ_{2n}^2\right) + x^2J_{2n}J_{2n-1}\left(J_{2n+1} + xJ_{2n-1}\right) \\ &= x\left(J_{4n+1}J_{2n} + xJ_{4n}J_{2n-1}\right) \\ &= xJ_{6n}. \end{aligned}$$

Combining the two cases, and using identities (5) and (6), we get

$$
\begin{aligned}
S^* &= S_1^* + S_2^* \\
&= J_{6n+2} + xJ_{6n} \\
&= \big[A^* - 4xB^* + (7x^2 + 2x)C^* - 2x^3 D^* - (7x^3 + 4x^2 - 3x)E^* \\
&\quad + 6x^4 F^* + x^5 G^* + (5x^4 + 10x^3 + 2x^2 - x)H^* \\
&\quad - (9x^5 + 12x^4 + 5x^3)I^* + 2x^5 J^* - x^7 K^* \\
&\quad - (3x^5 + 8x^4 + 4x^3)L^* + 2(4x^6 + 3x^5 + x^4 + x^3)M^* \\
&\quad - (3x^7 - 8x^6 - 2x^5)N^* + 2(x^8 - x^7)O^* + (x^2 + x)^3 P^* \\
&\quad - (3x^7 + 4x^6 + 2x^5)Q^* + (2x^8 + 2x^7 - x^6 - x^5)R^* \\
&\quad - (x^9 + 4x^8)S^* + x^9 T^*\big] + x\big[2B^* - 4(2x+1)C^* - 2x^2 D^* \\
&\quad - (10x^3 + 16x^2 + 7x + 1)H * -(9x^4 + 20x^3 + 6x^2)I^* - x6K^* \\
&\quad + (3x^4 + 6x^3 + 5x^2 + x)L^* + 2(4x^5 + 8x^4 + 5x^3 + x^2)M^* \\
&\quad - (3x^6 - 6x^5 - 2x^4)N^* + 2x^7 O^* - (3x^6 + 6x^5 + 5x^4 + x^3)Q^* \\
&\quad + (2x^7 - 3x^5 - x^4)R^* - (x^8 + 2x^7)S^*\big] \\
&= A^* - 2xB^* - (x^2 + 2x)C^* - 4x^3 D^* + (6x^3 + 12x^2 + 7x)E^* \\
&\quad + 4(3x^4 + x^3)F^* + 2x^5 G^* - (5x^4 + 6x^3 + 5x^2 + 2x)H^* \\
&\quad - (18x^5 + 32x^4 + 11x^3)I^* + 2x^5 J^* - 2x^7 K^* \\
&\quad - (2x^4 - x^3 - x^2)L^* + 2(8x^6 + 11x^5 + 6x^4 + 2x^3)M^* \\
&\quad - 2(3x^7 - 7x^6 - 2x^5)N^* + 2(2x^8 - x^7)O * + (x^2 + x)^3 P^* \\
&\quad - (6x^7 + 10x^6 + 7x^5 + x^4)Q^* \\
&\quad + 2(2x^8 + x^7 - 2x^6 - x^5)R^* - 2(x^9 + 3x^8)S^* + x^9 T^*.
\end{aligned}
$$

Equating this value of $S^*$ with its earlier value yields identity (9), as desired. □

Finally, we explore the confirmation of identity (10).

**3.4 Confirmation of Identity (10): Proof:** Let $S$ denote the sum of the weights of all closed walks of length $6n + 5$ in the digraph. Then $S = j_{6n+5}$.

We will now compute $S$ in a different way. To this end, let $w$ be an arbitrary walk of length $6n + 5$.

**Case 1:** Suppose $w$ originates at $v_1$. It can land at $v_1$ or $v_2$ at the $(2n + 2)$nd and $(4n + 4)$th steps:

$$w = \underbrace{v_1 - \ldots - v}_{\text{subwalk of length } 2n+2} \quad \underbrace{v - \ldots - v}_{\text{subwalk of length } 2n+2} \quad \underbrace{v - \ldots - v_1}_{\text{subwalk of length } 2n+1},$$

where $v = v_1$ or $v_2$.

**Table 6:** Sums of the Weights of Closed Walks Originating at $v_1$

| $w$ lands at $v_1$ at the $(2n+2)$nd step? | $w$ lands at $v_1$ at the $(4n+4)$th step? | $w$ lands at $v_1$ at the $(6n+5)$th step? | sum of the weights of walks $w$ |
|---|---|---|---|
| yes | yes | yes | $J^2_{2n+3}J_{2n+2}$ |
| yes | no | yes | $xJ_{2n+3}J_{2n+2}J_{2n+1}$ |
| no | yes | yes | $xJ^3_{2n+1}$ |
| no | no | yes | $x^2J_{2n+2}J^2_{2+1}$ |

It follows from Table 6 that the sum $S_1$ of the weights of such walks $w$ is given by

$$\begin{aligned} S_1 &= J^2_{2n+3}J_{2n+2} + xJ_{2n+3}J_{2n+2}J_{2n+1} + xJ^3_{2n+2} + x^2J_{2n+2}J^2_{2n+1} \\ &= J_{2n+2}(J^2_{2n+3} + xJ^2_{2n+2}) + xJ_{2n+2}J_{2n+1}(J_{2n+3} + xJ_{2n+1}) \\ &= J_{4n+5}J_{2n+2} + xJ_{4n+4}J_{2n+1} \\ &= J_{6n+6} \\ &= (x+1)J_{6n+4} + xJ_{6n+3}. \end{aligned}$$

**Case 2:** Suppose $w$ originates at $v_2$. It also can land at $v_1$ or $v_2$ at the $(2n+2)$nd and $(4n+4)$th steps:

$$w = \underbrace{v_2 - \ldots - v}_{\text{subwalk of length } 2n+2} \quad \underbrace{v - \ldots - v}_{\text{subwalk of length } 2n+2} \quad \underbrace{v - \ldots - v_2}_{\text{subwalk of length } 2n+1},$$

where $v = v_1$ or $v_2$.

**Table 7:** Sums of the Weights of Closed Walks Originating at $v_2$

| $w$ lands at $v_1$ at the $(2n+2)$nd step? | $w$ lands at $v_1$ at the $(4n+4)$th step? | $w$ lands at $v_1$ at the $(6n+5)$th step? | sum of the weights of walks $w$ |
|---|---|---|---|
| yes | yes | yes | $xJ_{2n+3}J_{2n+2}J_{2n+1}$ |
| yes | no | yes | $x^2J_{2n+2}^2J_{2n}$ |
| no | yes | yes | $x^2J_{2n+2}J_{2n+1}^2$ |
| no | no | yes | $x^3J_{2n+1}^2J_{2n}$ |

It follows from Table 7 that the sum $S_2$ of the weights of closed walks $w$ originating at $v_2$ is given by

$$\begin{aligned} S_2 &= xJ_{2n+3}J_{2n+2}J_{2n+1} + x^2J_{2n+2}^2J_{2n} + x^2J_{2n+2}J_{2n+1}^2 + x^3J_{2n+1}^2J_{2n} \\ &= xJ_{2n+2}J_{2n+1}(J_{2n+3} + xJ_{2n+1}) + x^2J_{2n}(J_{2n+2}^2 + xJ_{2n+1}^2) \\ &= x(J_{4n+4}J_{2n+1} + xJ_{4n+3}J_{2n}) \\ &= xJ_{6n+4}. \end{aligned}$$

Using equations (7) and (8), we then get

$$\begin{aligned} S &= S_1 + S_2 \\ &= (2x+1)J_{6n+4} + xJ_{6n+3} \\ &= (2x+1)\big[(2x+1)A^* - 2(5x^2+2x)B^* + (22x^3+15x^2+2x)C^* \\ &\qquad - 2(x^4+x^3)D^* - (27x^4+31x^3+2x^2-3x)E^* + 2(3x^5+x^4)F^* \end{aligned}$$

$$
\begin{aligned}
&+ (x^6+x^5)G^* + (20x^5+41x^4+21x^3+x^2-x)H^* \\
&- (9x^6+13x^5+16x^4+5x^3)I^* + 2(2x^6+x^5)J * - (x^8+x^7)K^* \\
&- (9x^6+25x^5+21x^4+5x^3)L^* + 2(4x^7+2x^6+2x^4+x^3)M^* \\
&- (3x^8-7x^7-10x^6-2x^5)N^* + 2(x^9-x^8-x^7)O * \\
&+ (2x+1)(x^2+x)^3P^* - (3x^8+5x^7+3x^6+x^5)Q^* \\
&+ (2x^9+6x^8+3x^7-2x^6-x^5)R^* - (x^{10}+7x^9+4x^8)S^* \\
&+ (2x^{10}+x^9)T^*\big] + x\big[(x+1)A^* - 2(3x^2+2x)B^* \\
&+ (15x^3+13x^2+2x)C^* - 2x^3D^* - (20x^4+27x^3+5x^2-3x)E^* \\
&+ 2x^4F^* + x^5G^* + (15x^5+31x^4+19x^3+2x^2-x)H^* \\
&- (x^5+11x^4+5x^3)I^* + 2(x^6+x^5)J^* - x^7K^* \\
&- (6x^6+17x^5+17x^4+5x^3)L^* - 2(x^6+x^5-x^4-x^3)M^* \\
&- (x^7-8x^6-2x^5)N^* - 2x^7O^* + (x+1)(x^2+x)^3P^* \\
&- (x^7+x^6+x^5)Q^* + (4x^8+4x^7-x^6-x^5)R^* - (3x^9+4x^8)S^* \\
&+ (x^{10}+x^9)T^*\big] \\
= {} & (5x^2+5x+1)A^* - 2(13x^3+11x^2+2x)B^* \\
&+ (59x^4+65x^3+21x^2+2x)C^* - 2(2x^5+4x^4+x^3)D^* \\
&- (74x^5+116x^4+40x^3-7x^2-3x)E^* + 2(6x^6+6x^5+x^4)F^* \\
&+ (2x^7+4x^6+x^5)G^* \\
&+ (55x^6+133x^5+96x^4+19x^3-2x^2-x)H^* \\
&- (18x^7+36x^6+56x^5+21x^4+5x^3)I^* + 2(5x^7+5x^6+x^5)J^* \\
&- (2x^9+4x^8+x^7)K^* - (24x^7+76x^6+84x^5+36x^4+5x^3)L^* \\
&+ 2(8x^8+7x^7+x^6+5x^5+5x^4+x^3)M^* \\
&- (6x^9-10x^8-35x^7-16x^6-2x^5)N^* + 2(2x^{10}+x^9+x^7)O^* \\
&+ (5x^2+5x+1)(x^2+x)^3P^* - (6x^9+14x^8+12x^7+6x^6+x^5)Q^* \\
&+ (4x^{10}+18x^9+16x^8-2x^7-5x^6-x^5)R^* \\
&- (2x^{11}+18x^{10}+19x^9+4x^8)S^* + (5x^{11}+5x^{10}+x^9)T^*.
\end{aligned}
$$

This value of $S$, coupled with the earlier value, yields the desired result, as expected. □

In conclusion, we add that the graph-theoretic confirmations of the numeric versions of the Jacobsthal identities (7) through (10) follow from the above arguments.

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# SOME DOUBLE SEQUENCE SPACES OF FUZZY NUMBERS DEFINED BY A MODULUS

**Abstract:** In this paper, we introduce some double sequence spaces of fuzzy numbers defined by a modulus function. Some topological results and inclusion relations have been discussed in this paper.

**Keywords:** Fuzzy Numbers, Modulus Function, Paranorm.

**Mathematical Subject Classification No.:** 40A05, 40D25, 46A45.

## 1. Introduction

The concept of fuzzy sets and fuzzy set operation was first introduced by Zadeh [14] and subsequently several authors have discussed various aspects of the theory and application of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events and fuzzy mathematical programming. The basic arithmetic structure for fuzzy numbers was later developed by Mizumoto and Tanaka [4], Diamond and Kloeden [3]. Matloka [5] introduced bounded and convergent sequence of fuzzy numbers, studied some of their properties and showed that every convergent sequence of fuzzy numbers is bounded. For sequences of fuzzy numbers, Nanda [6] studied sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzy numbers forms a complete matric space. In addition, sequences of fuzzy numbers have been discussed by Nuray and Savas [7], Savas [10], Mursaleen and M. Basarir [2], Y. Altin, M. Et and M. Basarir [1], B.C. Tripathy and B. Sarma [12], B. C. Tripathy, A. J. Dutta [13], N. Subramanian [11].

The main purpose of this paper is to introduce some double sequence spaces of fuzzy numbers defined by a modulus function.

**2. Definitions and Preliminaries**

Let $D$ denote the set of all closed bounded intervals $A = [\underline{A}, \overline{A}]$ on the real line $R$. For $A, B \in D$ we define,

$$A \leq B \text{ iff } \underline{A} \leq \underline{B} \text{ and } \overline{A} \leq \overline{B},$$

$$d(A, B) = \max(|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|)$$

Then it can be easily seen that $d$ defines a metric on $D$ and $(D, d)$ is a complete metric space [3]. Also it is easy to see that $\leq$ defined above is a partial order relation in $D$.

A fuzzy number is a fuzzy subset of the real line $R$ which is bounded, convex and Normal. Let $L(R)$ denote the set of all fuzzy numbers which are upper semicontinuous and have compact support i.e. if $X \in L(R)$ then for any $\alpha \in [0,1], X^{\alpha}$ is compact set in $R$, where

$$X^{\alpha} = \begin{cases} t : X(t) \geq \alpha & \text{if } \alpha \in (0,1] \\ t : X(t) > 0 & \text{if } \alpha = 0 \end{cases}$$

For each $0 < \alpha \leq 1$, the $\alpha$-level set $X^{\alpha}$ is a nonemtpy compact subset of $R$. The linear structure of $L(R)$ induces addition $X + Y$ and scalar multiplication $\lambda X, \lambda \in R$, in terms of $\alpha$-level sets by

$$[X + Y]^{\alpha} = [X]^{\alpha} + [Y]^{\alpha} \text{ and } [\lambda X]^{\alpha} = \lambda [X]^{\alpha}$$

for each $0 < \alpha \leq 1$.

The absolute value $|X|$ of $X \in L(R)$ is defined by

$$|X|(t) = \begin{cases} \max\{X(t), X(-t) & \text{if } t \geq 0 \\ 0, & \text{if } t < 0. \end{cases}$$

Define a map $\overline{d} : L(R) \times L(R) \to R$ by

$$\overline{d}(X,Y) = \sup_{0 \le \alpha \le 1} d(X^{\alpha}, Y^{\alpha}).$$

For $X, Y \in L(R)$, define $X \le Y$ iff $X^{\alpha} \le Y^{\alpha}$ for any $\alpha \in [0,1]$.

It is known that $(L(R), \overline{d})$, is a complete metric space [6].

A metric $\overline{d}$ on $L(R)$ is said to be translation invariant metric if

$$d(X + Z, Y + Z) = d(X,Y) \text{ for } X, Y, Z \in L(R):$$

A subset $E$ of $L(R)$ is said to be bounded above if there exists a fuzzy number $C$, called an upper bound of $E$, such that $X \le C$ for every $X \in E$. $C$ is called the least upper bound or sup of $E$ if $C$ is an upper bound and is the smallest of all upper bounds. A lower bound and the greatest lower bound or infimum are defined similarly.

$E$ is said to bounded if it is both bounded above and bounded below.

A sequence $X = (X_k)$ of fuzzy numbers is a function from $X$ into the set $N$ of all positive integers into $L(R^n)$. Thus, a sequence of fuzzy numbers $X$ is a correspondence from the set of positive integers to a set of fuzzy numbers i.e. to each positive integer $k$ there correspondence a fuzzy number $X(k)$. It is more common to write $X_k$ rather than $X(k)$ and to denote the sequence of $(X_k)$ rather than $X$. The fuzzy number $X(k)$ is called the $k$th term of the sequence.

By the convergence of a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence $x = (x_{kl})$ has Pringsheim limit $L$ (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathrm{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$ [9].

A fuzzy real valued double sequence is a double infinite array of fuzzy real numbers.

We denote a fuzzy real valued double sequence by $(X_{mn})$, where $X_{mn}$ are fuzzy real numbers for each $m, n \in N$.

We now give the following definitions of double sequences of fuzzy numbers which will be needed in the sequel ([10], [11]).

**Definition 2.1:** A double sequence $X = (X_{mn})$ of fuzzy numbers $X$ from $N \times N$ ($N$ is the set of all positive integers) into $L(R)$. The fuzzy number $X_{mn}$ denotes the value of the function at a point $(m, n) \in N \times N$ and is called the $(m, n)$ th term of the sequence.

**Definition 2.2:** A double sequence $X = (X_{mn})$ of fuzzy numbers is said to be convergent in Pringsheim's sense if there exists a fuzzy number $X_0$ such that $X_{mn}$ converges to $X_0$ as both $m$ and $n$ tend to $\infty$, independently of one another; $\lim_{m,n} X_{mn} = X_0$.

It is almost trivial that $X = (X_{mn})$ converges in Pringsheim's sense if and only if for every $\epsilon > 0$ there exists an integer $N = N(\epsilon)$ such that $\overline{d}(X_{jk}, X_{mn}) \leq \epsilon$ whenever $\min\left(j, k, m, n\right) \geq N$.

The crucial difference between the convergent of single sequence of fuzzy numbers and the convergence in Pringsheim's sense of double sequences of fuzzy numbers is that latter does not imply the boundedness of the terms of the double sequence of fuzzy numbers.

Let $F^2$ denote the set of all double convergent sequence of fuzzy numbers. In [ ], it was shown that $F^2$ is a complete metric space.

**Definition 2.3:** A double sequence $X = (X_{mn})$ of fuzzy numbers is said to be Cauchy sequence if for every $\epsilon > 0$ there exists $i_0 \in N$ such that

$$\overline{d}(X^i_{mn},\ X^j_{mn}) \leq \epsilon \text{ if } \min(i, j) \geq i_0 .$$

**Definition 2.4:** A double sequence $X = (X_{mn}))$ of fuzzy numbers is bounded if there exists a positive integer $M$ such that $\overline{d}(X_{mn}, X_0) < M$ for all $m$ and $n$,

$$\|x\|_{(\infty,2)} = \sup_{m,n} \overline{d}(X_{mn}, X_0) < \infty.$$

we will denote the set of all bounded double sequences by $F_\infty^2$.

**Definition 2.5:[10]** Let $X = (X_{kl})$ be a double sequence of fuzzy numbers. The space of strongly double cesaro summable sequences [*C*, 1, 1] (*F*) defined as follows:

$$P - \lim_{mn} \frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} \overline{d}(X_{kl}, X_0) = 0 .$$

**Definition 2.6:** A function $f : [0,\infty) \to [0,\infty)$ is called a modulus if

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x+y) \leq f(x) + f(y)$, for all $x \geq 0, y \geq 0$,

(iii) $f$ is increasing, and

(iv) *f* is continuous from the right at $0$. Since $|f(x) - f(y)| \leq f(|x-y|)$, it follows from here that $f$ is continuous on $[0,\infty)$.

By a paranorm we mean a function $g : E \to R$ (where $E$ is a real or complex linear space) which satisfies the following conditions;

(i) $g(\theta) = 0$, where $\theta = (0, 0, \cdots)$

(ii) $g(x) \geq 0$, for all $x \in E$,

(iii) $g(x) = g(-x)$,

(iv) $g(x+y) \leq g(x) + g(y)$ for all $x, y \in E$,

(v) If $(\lambda_n)$ is a sequence of scalars with $\lambda_n \to \lambda (n \to \infty)$ and $(x_n)$ is a sequence of the elements of $E$ with $g(x_n - x) \to 0 (n \to \infty)$, then $g(\lambda_n x_n - \lambda x) \to 0 (n \to \infty)$.

Then the pair $(E, g)$ is called a paranormed space and $g$ is a paranorm for $E$.

Let $A = (a_{k,l}^{mn})$ denote a four dimensional summability method that maps the complex double sequences $x$ into the double sequence $Ax$ where the $k$, $l$-th term to $Ax$ is as follows:

$$(Ax)_{kl} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{kl}^{mn} x_{mn}$$

such transformation is said to be non negative if $a_{kl}^{mn}$ is non negative.

The notion of regularity for two dimensional matrix transformations was presented by Silverman and Toeplitz. Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analog of regularity for double sequences in which they both added an additional assumption of boundedness. This assumption was made because a double sequence which is *P*-convergent is not necessarily bounded.

**3. Some new sequence spaces**

Recently, Mursaleen and M. Basarir [2] have defined the following spaces of sequences of fuzzy numbers as follows:

Let $A = (a_{nk})(n, k = 1, 2, \cdots)$ be a non negative regular matrix. We define

$$F_0[A, p] = \{X = (X_k) : \sum_k a_{nk}[\overline{d}(X_k, 0)]^{pk} \to 0\,(n \to \infty)\},$$

$$F[A, p] = \{X = (X_k) : \sum_k a_{nk}[\overline{d}(X_k, X_0)]^{pk} \to 0(n \to \infty)\},$$

$$F_\infty[A, p] = \{X = (X_k) : \sup_n\left(\sum_k a_{nk}[\overline{d}(X_k, 0)]^{pk}\right) < \infty\},$$

and call them respectively the spaces of strongly A-convergent to zero, strongly Aconvergent to $X_0$ and strognly A-bounded sequences of fuzzy numbers $X = (X_k)$.

In the present paper, we extend above spaces for double sequences with respect to a modulus *f*.

Let $A = (a_{kl}^{mn})(m, n = 1, 2, 3, \cdots)$ be a non negative regular matrix and $p = (p_{mn})$ is the double sequence of strictly positive real numbrs $p_{mn}$ for all $m, n \in N$. We define

$$F_0^2[A, f, p] = \{X = (X_{mn}) : P - \lim_{m,n} \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl}^{mn} [\overline{d}(f(X_{mn}, 0))]^{p_{mn}} \to 0(m, n \to \infty)\},$$

$$F^2[A, f, p] = \{X = (X_{mn}) : P - \lim_{m,n} \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl}^{mn} [\overline{d}(f(X_{mn}, X_0))]^{p_{mn}} \to 0(m, n \to \infty)\},$$

$$F_\infty^2[A, f, p] = \{X = \left(X_{mn}\right) : \sup_{m,n} \left( \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl}^{mn} [\overline{d}(f(X_{mn}, 0))]^{p_{mn}} \right) < \infty\}.$$

and call them respectively the spaces of strongly A-double convergent to zero, strongly A-double convergent to $X_0$ and strongly A-double bounded sequences of fuzzy numbers $X = (X_{mn})$ with respect to the modulus *f*.

If $A = [C, 1, 1](F)$ then we have the following new sequence spaces:

$$F_0^2[A, f, p] = F_0^2[f, p] = \{X = (X_{mn}) : P - \lim_{m,n} \frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} [\overline{d}(f(X_{mn}, 0))]^{p_{mn}} \to 0(m, n \to \infty)\},$$

$$F^2[A, f, p] = F^2[f, p] = \{X = (X_{mn}) : P - \lim_{m,n} \frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} [\overline{d}(f(X_{mn}, X_0))]^{p_{mn}} \to 0(m, n \to \infty)\},$$

$$F_\infty^2[A, f, p] = F_\infty^2[f, p] = \{X = (X_{mn}) : \sup_{m,n} \frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} [\overline{d}(f(X_{mn}, 0))]^{p_{mn}} < \infty\}.$$

If we take $p_{mn} = 1$ for all $m$, $n$, these spaces are reduced to the following new sequence spaces:

$$F_0^2[A, f, p] = F_0^2[A, f] = \{X = (X_{mn}) : P - \lim_{m,n} \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl}^{mn} [\overline{d}(f(X_{mn}, 0))] \to 0(m, n \to \infty)\},$$

$$F^2[A, f, p] = F^2[A, f] = \{X = (X_{mn}) : P - \lim_{m,n} \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl}^{mn} [\overline{d}(f(X_{mn}, X_0))] \to 0(m, n \to \infty)\},$$

$$F_\infty^2[A,f,p] = F_\infty^2[A,f] = \{X = (X_{mn}) : \sup_{m,n} \sum_{k=1}^{m}\sum_{l=1}^{n} a_{kl}^{mn}[\bar{d}(f(X_{mn},0))] < \infty\}.$$

If we take $f(x) = x$ and $p_{mn} = 1$ for all $m$, $n$ we have

$$F_0^2[A,f,p] = F_0^2[A] = \{X = (X_{mn}) : P - \lim_{m,n} \sum_{k=1}^{m}\sum_{l=1}^{n} a_{kl}^{mn}[\bar{d}(f(X_{mn},0))] \to 0(m,n \to \infty)\},,$$

$$F^2[A,f,p] = F^2[A] = \{X = (X_{mn}) : P - \lim_{m,n} \sum_{k=1}^{m}\sum_{l=1}^{n} a_{kl}^{mn}[\bar{d}(f(X_{mn},X_0))] \to 0(m,n \to \infty)\},$$

$$F_\infty^2[A,f,p] = F_\infty^2[A] = \{X = (X_{mn}) : \sup_{m,n} \sum_{k=1}^{m}\sum_{l=1}^{n} a_{kl}^{mn}[\bar{d}(f(X_{mn},0))] < \infty\}.$$

Now we have

**Proposition 3.1:** If $\bar{d}$ is a translation invariant metric of $L(R)$ then

(i) $\bar{d}(X+Y,0) \le \bar{d}(X,0) + \bar{d}(Y,0)$,

(ii) $\bar{d}(\lambda X,0) \le |\lambda|\,\bar{d}(X,0)$, $|\lambda| > 1$.

where $X = (X_{mn})$ and $Y = (Y_{mn})$ are double sequences of fuzzy numbers.

**Proof:** This can be proved by using the same technique in [ ] and hence we omit the proof.

If $\bar{d}$ is a translation invariant, we have the following straight forward results.

**Proposition 3.2:** Let $p = (p_{mn})$ be a bounded sequence of strictly positive real numbers. Then $F_0^2[A,f,p]$, $F^2[A,f,p]$ and $F_\infty^2[A,f,p]$ are linear spaces of all double sequences of fuzzy numbers over the complex field.

**Proposition 3.3:** $F_0^2[A,f,p]$, $F^2[A,f,p]$ and $F_\infty^2[A,f,p]$ are absolutely convex subsets of the space of all double sequences of fuzzy numbers, where $0 \le p_{mn} \le 1$.

**4. Main Results**

**Theorem 4.1:** Let $p = (p_{mn})$ be a bounded sequence of strictly positive real numbers. Then $F_0^2[A, f, p]$ and $F^2[A, f, p]$ are complete paranormed spaces with the paranorm $g$ defined by

$$g(X) = \sup_{mn}\left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{kl}^{mn}[\overline{d}(f(X_{mn},0))]^{p_{mn}}\right)^{\frac{1}{M}}$$

where $M = \max(1, \sup_{mn} p_{mn})$ and $\overline{d}$ is a translation invariant.

**Proof:** We consider the case $F_0^2[A, f, p]$. Other can be treated similarly. Clearly $g(\theta) = 0$ and $g(X) = g(-X)$. Also we have $g(X+Y) \leq g(X) + g(Y)$ for $X = (X_{mn}), Y = (Y_{mn})$, in $F_0^2[A, f, p]$. Now for any scalar $\lambda$, we have

$$|\lambda|^{p_{mn}} < \max(1, |\lambda|^H),$$

where

$$H = \sup_{mn} p_{mn} < \infty,$$

so

$$g(\lambda X) < (\sup_{mn})(|\lambda|^{p_{mn}})^{\frac{1}{M}}.g(X) \text{ on } F_0^2[A, f, p]$$

Hence $\lambda \to 0$, $X \to \theta$ implies $\lambda X \to \theta$ and also $X \to \theta, \lambda$ fixed implies $\lambda X \to \theta$. Now let $\lambda \to 0$, $X$ fixed. For $|\lambda| < 1$ we have

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{kl}^{mn}[\overline{d}(f(\lambda X_{mn}, 0))]^{p_{mn}} < \epsilon \text{ for } m, n > N = N(\epsilon).$$

Also for $1 \leq m, n \leq N$, Since $\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{kl}^{mn}[\overline{d}(f(X_{mn}, 0))]^{p_{mn}} < \infty$, there exist $m, n$ such that $\sum_{k=m}^{\infty}\sum_{l=n}^{\infty} a_{kl}^{mn}[\overline{d}(f(\lambda X_{mn}, 0))]^{p_{mn}} < \epsilon$

Taking $\lambda$ small enough, since $f$ is continuous we have

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{kl}^{mn}[\overline{d}(f(\lambda X_{mn}, 0))]^{p_{mn}} < 2\epsilon \text{ for all } m,\ n$$

Hence, $g(\lambda X) \to 0$ as $\lambda \to 0$. Therefore $g$ is a paranorm on $F_0^2[A, f, p]$. Completeness can be proved by using the technique in [13] for ${}_2l_F^p$.

Similarly we can prove the following:

**Theorem 4.2:** If $0 < \inf_{mn} p_{mn} \le \sup_{mn} p_{mn} < \infty$ then $F_\infty^2[A, f, p]$ is a paranormed space with the above paranorm.

**Theorem 4.3:** Let $0 < p_{mn} \le q_{mn}$ and $(q_{mn} / p_{mn})$ be bounded. Then

$$F^2[A, f, q] \subseteq F^2[A, f, p].$$

**Proof:** Let $X = (X_{mn}) \in F^2[A, f, q]$. Put $t_{mn} = [\overline{d}(f(X_{mn}, X_0))]^{q_{mn}}$ and $\lambda_{mn} = \frac{q_{mn}}{p_{mn}}$. Of course $0 < \lambda_{mn} \le 1$ . Take $0 < \lambda < \lambda_{mn}$ . Define

$$u_{mn} = \begin{cases} t_{mn}, & t_{mn} \ge 1 \\ 0, & t_{mn} < 1 \end{cases}$$

and

$$u_{mn} = \begin{cases} 0, & t_{mn} \ge 1 \\ t_{mn}, & t_{mn} < 1 \end{cases}.$$

Then we have $t_{mn} = u_{mn} + v_{mn}$ and $t_{mn}^{\lambda_{mn}} = u_{mn}^{\lambda_{mn}} + v_{mn}^{\lambda_{mn}}$ and it follows that $u_{mn}^{\lambda_{mn}} \le u_{mn} \le t_{mn}$ and $v_{mn}^{\lambda_{mn}} \le v_{mn}^{\lambda}$. Therefore

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{kl}^{mn}[\overline{d}(f(X_{mn}, X_0))]^{p_{mn}} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{kl}^{mn} t_{mn}^{\lambda_{mn}} = \sum_{k=l}^{\infty}\sum_{l=1}^{\infty} a_{kl}^{mn}\left(u_{mn}^{\lambda_{mn}} + v_{mn}^{\lambda_{mn}}\right)$$

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{kl}^{mn} t_{mn} + \sum_{k=l}^{\infty}\sum_{l=1}^{\infty} a_{kl}^{mn} v_{mn}^{\lambda} \to 0(\mathrm{m,n} \to \infty).$$

Since $X \in F^2[A, f, q]$, $\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{kl}^{mn} t_{mn}$ is convergent and since $v_{mn} < 1$, $A$ is regular, $\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{kl}^{mn} v_{mn}^{\lambda}$ is also convergent.

Hence, $X \in F^2[A, f, p]$ i.e. $F^2[A, f, q] \subseteq F^2[A, f, p]$.

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# INVARIANT MEANS AND SOME CLASSES OF MATRICES

**Abstract:** In this paper, we investigate necessary and sufficient conditions to characterize the classes of matrices $(cs, V_\sigma(p))$ and $(cs, V_{0\sigma}(p))$. Finally some known and unknown results have been derived as corollaries.

**Keywords**: Invariant Means, Sequence Space.

**Mathematical Subject Classification (2000) No.:** 40C05, 40H05.

## 1. Introduction

Let $\sigma$ be a mapping of the set of positive integers into itself. A continuous linear functional $\phi$ and $\ell_\infty$ the space of bounded sequences, is said to be an invariant mean or a $\sigma$-mean, if and only if

(i) $\phi(x) \geq 0$ when the sequence $x = (x_n)$ has $(x_n \geq 0)$ for all $n$,

(ii) $\phi(e) = 1$ where, $e = (1, 1, \cdots)$,

(iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$.

In case, $\sigma$ is the translating mapping $n \to n+1$, a $\sigma$-mean is often called a Banach limit [1], and $v_\sigma$, the set of bounded sequences all of whose invariant means are equal, is the set $f$ of almost convergent sequences [5].

Let $f_0$ denote the space of almost convergent null sequences. If $x = (x_n)$, we write $Tx = (Tx_n) = (x_{\sigma(n)})$. It is known that [13],

$$V_\sigma = \{x \in \ell_\infty : \lim_{m\to\infty} t_{mn}(x) = Le, \text{uniformly in } n, \text{and } L = \sigma - \lim x\}$$

where

$$\ell_{mn}(x) = \frac{1}{m+1}\sum_{j=o}^{m} T^j x_n$$

Let $V_{0\sigma}$ denote the set of all bounded sequences which are $\sigma$-convergent to zero.

Recently, in [10] and [12] the spaces $V_\sigma$, $V_{0\sigma}$, $f$ and $f_0$ were extended to $V_\sigma(p), V_{0\sigma}(p)$, $f(p)$ and $f_0(p)$ in the following manner:

If $p = (p_m)$ is a sequence of real numbers such that $p_m > 0$ and $\sup_m p_m < \infty$, we define

$$V_{0\sigma}(p) = \left\{x : \lim_{m\to\infty} \left|t_{mn}(x)\right|^{pm} = 0, \quad \text{uniformly in } n\right\},$$

$$V_{0\sigma}(p) = \left\{x : \lim_{m\to\infty} \left| t_{mn}(x - Le) \right|^{pm} = 0, \quad \text{uniformly in } n, \sigma - \lim x = L\right\},$$

$$f_0(p) = \left\{x : \lim_{m\to\infty} \left| \frac{1}{m+1}\sum_{i=0}^{m} x_{i+n} \right|^{pm} = 0, \quad \text{uniformly in } n\right\},$$

$$f_0(p) = \left\{x : \lim_{m\to\infty} \left| \frac{1}{m+1}\sum_{i=0}^{m} x_{i+n} \right|^{pm} = 0, \quad \text{uniformly in } n\right\},$$

$$f(p) = \left\{x : \lim_{m\to\infty} \left| \frac{1}{m+1}\sum_{i=0}^{m} (x_{i+n} - L) \right|^{pm} = 0, \text{for some } L, \text{ uniformly in } n\right\},$$

In particular, if $p_m = p > 0$ for all $m$, we have $V_{0\sigma}(p) = V_{0\sigma}$ and $V_\sigma(p) = V_\sigma$. If $\sigma(n) = n + 1$, we get $V_\sigma(p) = f(p)$ and $V_{0\sigma}(p) = f_0(p)$.

## 2. Preliminaries

By $N$ and $C$ we shall denote the set of positive integers and the fields of complex numbers, respectively. Let $A = (a_{nk})$ be an infinite matrix of complex numbers $a_{nk}(n, k = 1, 2 \cdots)$ and $X$, $Y$ be any two subsets of the space of complex sequences. By $(X, Y)$ we mean the class of matrices $A$ such that for each $x \in X, A_n(x) \sum_k a_{nk} x_k$ converges for each $n$, and $Ax = (A_n(x)) \in Y$.

Schaefer [13] has defined the concept of $\sigma$-conservative, $\sigma$-regular and $\sigma$-coercive matrices and characterized the classes of these matrices ie $(c, V_\sigma)$, $(c, V_\sigma)$ reg and $(\ell; V_\sigma)$. Recently, the several authors such as Metin Basarir and Ekrem Savas [6], Mursaleen [8, 9, 10] Sirajudeen [14], Mushir A. Khan [11] and Husamettin Coskun [2] have characterized some matrix classes concerning $V_\sigma$. The main purpose of this paper is to determine necessary and sufficient conditions to characterize the classes $(cs, V_\sigma(p))$ and $(cs, V_{0\sigma}(p))$ which will fill up a gap in the existing literature. Where $cs$ is the space of convergent series.

If $X$ is a subset of the space of complex sequences, then we write $X^+$ for the generalized Köthe-Toeplitz dual of $X$, i.e.

$$X^+ = \{a : \sum_k a_k x_k \text{ converges for every } x \in X\} :$$

$X^+$ denotes the dual space of the continuous functional of $X$.

It is well known $cs^+ = bv$ and $cs^* = bv$ (linearly isomorphic) (see [4], p. 55 and [3]) where $bv$ is the space of bounded variation sequences.

Throughout this paper the sums without limits rum from $k = 1$ to $k = \infty$. We write for all integers $m, n \geq 1$

$$t_{mn} = t_m(Ax) = \sum_k a(n, k, m) x_k ,$$

where

$$a(n, k, m) = \frac{1}{m+1} \sum_{j=0}^{m} a(\sigma^j(n), k) .$$

Now we quote some known results which will be useful in the proof of our results.

**Lemma A[6]:** Let $X$ be a complete paranormed space with Schander basis $(b_k)$, and $(A_n)$ a sequence of elements of $X^*$ with $A_n(x) = \sum_k a_{nk}x_k$ for all $x \in X$ and $n \in N$.

Furthermore, let $q = (q_k)$ be a bounded sequence. Then

$A \in (X, V_{0\sigma}(q)) \Leftrightarrow$ (1) $(t_{mn}(b_k)) \in V_{0\sigma}(q)$ for all $k$,

(2) $\lim_{M\to\infty} \lim\sup_m (\|t_{mn}\|_M)^{qm} = 0$.

**Lemma B[6]:** Let $X$ be a complete paranormed space with Schander basis $(b_k)$, and $(A_n)$ a sequence of elements of $X^*$ with $A_n(x) = \sum_k a_{nk}x_k$ for all $x \in X$ and $n \in N$. Furthermore, let $q = (q_k)$ be a bounded sequence. Then

$A \in (X, V_\sigma(q)) \Leftrightarrow$ (1) there exists an $L \in X^*$ with $(t_{mn}(b_k) - L(b_k)) \in V_{0\sigma}(q)$ for all $k$,

(2) $\lim_{M\to 1} \lim\sup_m (\|t_{mn}\|_M)^{qm} = 0$.

**Lemma C[3]:** $cs^* = bv$ (linearly isomorphic).

**3. Main Results**

**Theorem 1:** Let $p \in \ell_\infty$. Then $A \in (cs, V_\sigma(p))$ if and only if

(i) $\sup_{m,n}, \sum_k |\Delta a(n,k,m)| < \infty$; where $\Delta a(n,k,m) = a_{n,k,m} - a_{n,k+1,m}$,

(ii) there exist $\alpha_1, \alpha_2 \cdots \in C$ with $|a(n,k,m) - \alpha_k|^{pm} \to 0$, as $m \to 1$, uniformly in $n$, for each $k$,

(iii) $\lim_{M\to 1}\limsup_{m}\left[M^{-1}(\sum_k \left|\Delta(a(n,k,m)-\alpha_k)\right| + \lim_k \left|a(n,k,m)-\alpha_k)\right|\right]^{pm} = 0.$

**Proof:** Let (i), (ii) and (iii) hold. Then, from (i) and (ii), for $n,\ m$ we have

$$\sum_k \left|\Delta\alpha k\right| \le \sum_k \left|\Delta(a(n,k,m)-\alpha k)\right| + \sum_k \left|\Delta a(n,k,m)\right| < \infty .$$

Hence, $a(n,k,m)\in b\nu$. Therefore, by Lemma $C$, there is an $L\in cs^*$ with $L(x)=\sum_k \alpha_k x_k$ for all $x\in cs$. By (i), since $a(n,k,m)\in b\nu$ and $t_{mn}\in cs^*$ for all $m,\ n$ so that $t_{mn}-L\in cs^*$ with

$$\left\|t_{mn}-L\right\| = \sum_k \left|\Delta(a(n,k,m)-\alpha k)\right| + \lim_k \left|a(n,k,m)-\alpha_k\right| .$$

for all $m,\ n$. By (ii), $(t_{mn}(e^k - L(e^k)) \in V_{0\sigma}(p)$ for all $k$, and by (iii)

$$\lim_M \limsup_m (\left\|t_{mn}-L\right\| M)^{pm} = 0.$$

Thus, since $(e^{(k)})$ is a fundamental set in $cs$, it follows by Lemma $B$ that $A\in(cs,V_\sigma(p))$.

Conversely, suppose that $A\in(cs,V_\sigma(p))$. Then $t_{mn}(Ax)=\sum_k a(n,k,m)x_k$ is defined for all $x\in cs$, $m$ and $n$. Clearly (i) must be satisfied or else those series $\sum_k a(n,k,m)x_k$ diverges for at least one $m\in N$, i.e. $A\notin 2(cs,V_\sigma(p))$.

Then $t_{mn}\in cs^*$ for all $m,\ n$. By Lemma $B$ there is an $L\in cs^*$ such that (1) and (2) hold. Since $L$ may be written as $L(x)=\sum_k \alpha_k x_k$ on $cs$, by Lemma $C$, and $(e(k))$ is a fundamental set in $cs$, (1) and (2) give us (ii) and (iii) respectively.

Hence, the proof is completed.

**Theorem 2:** Let $p \in \ell_\infty$. Then $A \in (cs, V_{0\sigma}(p))$ if and only if

(i) $\left|a(n,k,m)\right|^{pm} \to 0$ as $m \to \infty$, uniformly in $n$, for each $k$,

(ii) $\lim_M \lim\sup_m [M^{-1}(\sum_k \left|\Delta a(n,k,m)\right| + \lim_k \left|a(n,k,m)\right|)]^{pm} = 0$.

**Proof:** Let $A \in (cs, V_{0\sigma}(p))$. Then, since $V_{0\sigma}(p) \subset V_\sigma(p)$, we have $A \in (cs, V_\sigma(p))$. Hence, (i) and (ii) follow from the conditions (ii) and (iii) of Theorem 1 with $\alpha_k = 0$ $(k = 1, 2, \cdots)$.

Now, let (i) and (ii) hold. Then from (ii), we have $a(n,k,m) \in bv$ for all $m$, $n$ and by Lemma $C$, $t_{mn} \in cs^*$ with $\left\|t_{mn}\right\| = \sum_k \left|\Delta a(n,k,m)\right| + \lim_k \left|a(n,k,m)\right|$ for all $m$, $n$. By our choice of fundamental set in $cs$, (i) and (ii) are respectively equivalent to (1) and (2) of Lemma A. Hence, by Lemma A, $A \in (cs, V_{0\sigma}(p))$ and this completes the proof.

## 4. Corollaries

**Corollary 1[14]:** $A \in (cs, V_\sigma)$ if and only if

(i) $\sup_m (\sum_k \left|\Delta a(n,k,m)\right|) < \infty$; for all $n$,

(ii) $a_{(k)} = (a_{nk})_n \in V_\sigma$ for each $k$, i.e. $\lim_m a(n,k,m) = u_k$ uniformly in $n$.

In this case, the $\sigma$-limit of $Ax$ is $\sum u_k x_k$ for each $x \in cs$.

**Proof:** Take $p_m = 1$ for all $m$ in Theorem 1.

**Corollary 2:** $A \in (cs, V_{0\sigma})$ if and only if

(i) $a(n,k,m) \to 0$ as $m \to \infty$, uniformly in $n$, for each $k$,

(ii) $\lim_M \lim\sup_m [M^{-1}(\sum_k \left|\Delta a(n,k,m)\right| + \lim_k \left|a(n,k,m)\right|)] = 0$.

**Proof:** Take $p_m = 1$ for all $m$ in Theorem 2.

**Corollary 3:** Let $p \in \ell_\infty$. Then $A \in (cs, f(p))$ if and only if

(i) $\sup_{m,n} \sum_k \left|\Delta b(n,k,m)\right| < \infty$; where $\Delta b(n,k,m) = b_{n,k,m} - b_{n,k+1,m}$,

(ii) there exist $\alpha_1, \alpha_2, \cdots \in C$ with $\left|b(n,k,m) - \alpha_k\right|^{pm} \to 0$, as $m \to \infty$ uniformly in $n$, for each $k$,

(iii) $\lim_{M\to\infty} \lim\sup_m [M^{-1}(\sum_k \left|\Delta(b(n,k,m) - \alpha_k)\right| + \lim_k \left|b(n,k,m) - \alpha k\right|)]^{pm} = 0$.

where

$$b(n,k,m) = \frac{1}{m+1} \sum_{j=0}^{m} a(n+j,k).$$

**Proof:** Taking the mapping $\sigma(n) = n + 1$ instead of mapping $\sigma$ as the translation mapping, the space $V_\sigma(p)$ of Theorem 1 reduces to $f(p)$. Hence it is proved.

**Corollary 4:** Let $p \in \ell_\infty$. Then $A \in (cs, f_0(p))$ if and only if

(i) $\left|b(n,k,m)\right|^{pm} \to 0$ as $m \to \infty$, uniformly in $n$, for each $k$,

(ii) $\lim_M \lim\sup_m [M^{-1}(\sum_k \left|\Delta b(n,k,m)\right| + \lim_k \left|b(n,k,m)\right|)]^{pm} = 0$.

**Proof:** Taking the mapping $\sigma(n) = n + 1$ in Theorem 2.

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