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PRODUCTS OF EXTENDED GIBONACCI POLYNOMIAL EXPRESSIONS

Abstract: We explore finite and infinite products of extended gibonacci polynomial expressions, and their numeric counterparts.

Keywords: Fibonacci Polynomial, Pell-Lucas Polynomials, Gibonacci Polynomials, Jacobsthal-Lucas Polynomial.

Mathematical Subject Classification (2010) No.: Primary 05A19, 11B37, 11B39, 11Cxx.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary complex variable; $a(x),\ b(x),\ z_0(x)$ and $z_1(x)$ are arbitrary complex polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*.

They can also be defined by the *Binet-like formulas*

$$f_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} \quad \text{and} \quad l_n(x) = \alpha^n(x) + \beta^n(x)$$

where $$\alpha(x) = \frac{x + \sqrt{x^2 + 4}}{2} \quad \text{and} \quad \beta(x) = \frac{x - \sqrt{x^2 + 4}}{2}.$$

Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 5, 6].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. They also can be defined by the *Binet-like formulas*

$$p_n(x) = \frac{\gamma^n(x) - \delta^n(x)}{\gamma(x) - \delta(x)} \quad \text{and} \quad q_n(x) = \gamma^n(x) + \delta^n(x),$$

where $\gamma(x) = x + \sqrt{x^2 + 1}$ and $\delta(x) = x - \sqrt{x^2 + 1}$. In particular, the *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [6].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. They can also be defined by the *Binet-like formulas*

$$J_n(x) = \frac{u^n(x) - v^n(x)}{u(x) - v(x)} \quad \text{and} \quad j_n(x) = u^n(x) + v^n(x),$$

where $u(x) = \frac{1 + \sqrt{4x + 1}}{2}$ and $v(x) = \frac{1 - \sqrt{4x + 1}}{2}$. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$.

Fibonacci and Jacobsthal polynomials, and Lucas and Jacobsthal-Lucas polynomials are closely related by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [4, 5, 6, 7].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $\alpha = \alpha(1)$, $\beta = \beta(1)$, $\gamma = \gamma(1)$, $\delta = \delta(1)$, $\Delta = \sqrt{x^2 + 4}$, $D = \sqrt{4x + 1}$, and omit a lot of basic algebra.

2. Products of Fibonacci Polynomial Expressions

Our discourse hinges on the Cassini-like identity [6, 8]

$$g_{n+k} g_{n-k} - g_n^2 = (-1)^{n-k} \mu \, f_k^2, \tag{1}$$

where

$$\mu = \begin{cases} -1 & \text{if } gn = fn \\ \Delta^2 & \text{otherwise} \end{cases}.$$

With this background, we begin our explorations.

Theorem 1:

$$\prod_{n=3}^{m} \frac{g_{n-2} g_{n-1} g_{n+1} g_{n+2}}{g_n^4} = \frac{g_1 g_2^2}{g_3^2 g_4} \cdot \frac{g_{m+1}^2 g_{m+2}}{g_{m-1} g_m^2}. \tag{2}$$

Proof: We will establish the formula using recursion [2, 6]. Let A_m = LHS and $B_m = \text{RHS}$. Then

$$\begin{aligned} \frac{B_m}{B_{m-1}} &= \frac{g_{m+1}^2 g_{m+2}}{g_{m-1} g_m^2} \cdot \frac{g_{m-2} g_{m-1}^2}{g_m^2 g_{m+1}} \\ &= \frac{g_{n-2} g_{n-1} g_{m+1} g_{m+2}}{g_m^4} \\ &= \frac{A_m}{A_{m-1}}. \end{aligned}$$

Thus, $\frac{A_m}{B_m} = \frac{A_{m-1}}{B_{m-1}} = \ldots = \frac{A_3}{B_3} = 1$. Consequently, $A_m = B_m$, as desired. □

In particular, we have

$$\prod_{n=3}^{m} \frac{f_{n-2}f_{n-1}f_{n+1}f_{n+2}}{f_n^4} = \frac{x^2}{(x^2+1)^2(x^3+2x)} \cdot \frac{f_{m+1}^2 f_{m+2}}{f_{m-1}f_m^2}; \tag{3}$$

$$\prod_{n=3}^{m} \frac{l_{n-2}l_{n-1}l_{n+1}l_{n+2}}{l_n^4} = \frac{x(x^2+2)^2}{(x^3+3x)^2(x^4+4x^2+2)} \cdot \frac{l_{m+1}^2 l_{m+2}}{l_{m-1}l_m^2}; \tag{4}$$

$$\prod_{n=3}^{m} \frac{F_{n-2}F_{n-1}F_{n+1}F_{n+2}}{F_n^4} = \frac{1}{12} \cdot \frac{F_{m+1}^2 F_{m+2}}{F_{m-1}F_m^2};$$

$$\prod_{n=3}^{m} \frac{L_{n-2}L_{n-1}L_{n+1}L_{n+2}}{L_n^4} = \frac{9}{112} \cdot \frac{L_{m+1}^2 L_{m+2}}{L_{m-1}L_m^2}.$$

By equation (4), we have

$$\begin{aligned}(f_{n+2}f_{n-2})(f_{n+1}f_{n-1}) &= \left[f_n^2 - (-1)^n x^2\right]\left[f_n^2 + (-1)^n\right] \\ &= f_n^4 - \left[(-1)^n(x^2-1)f_n^2 + x^2\right].\end{aligned}$$

Using Theorem 1, this yields

$$\begin{aligned}\prod_{n=3}^{m}\left[1 - \frac{(-1)^n(x^2-1)f_n^2 + x^2}{f_n^4}\right] &= \prod_{n=3}^{m} \frac{f_{n-2}f_{n-1}f_{n+1}f_{n+2}}{f_n^4} \\ &= \frac{x^2}{(x^2+1)^2(x^3+2x)} \cdot \frac{f_{m+1}^2 f_{m+2}}{f_{m-1}f_m^2}; \end{aligned} \tag{5}$$

$$\prod_{n=3}^{\infty}\left[1 - \frac{(-1)^n(x^2-1)f_n^2 + x^2}{f_n^4}\right] = \frac{x^2}{(x^2+1)^2(x^3+2x)}\alpha^5(x), \tag{6}$$

where $$\lim_{m\to\infty} \frac{f_{m+k}}{f_m} = \alpha^k(x).$$

Consequently,

$$\prod_{n=3}^{m}\left(1-\frac{1}{F_n^4}\right) = \frac{1}{12}\cdot\frac{F_{m+1}^2 F_{m+2}}{F_{m-1}F_m^2};$$

$$\prod_{n=3}^{\infty}\left(1-\frac{1}{F_n^4}\right) = \frac{\alpha^5}{12},$$

as in [3, 9], where $2\alpha = 1+\sqrt{5}$.

Next we extract the Pell consequences of equations (5) and (6).

2.1 Pell Implications: Since $p_n(x) = f_n(2x)$, it follows from equations (5) and (6) that

$$\prod_{n=3}^{m}\left[1-\frac{(-1)^n(4x^2-1)p_n^2+4x^2}{p_n^4}\right] = \frac{x}{(2x^2+1)(4x^2+1)^2}\cdot\frac{p_{m+1}^2 p_{m+2}}{p_{m-1}p_m^2};$$

$$\prod_{n=3}^{m}\left[1-\frac{3(-1)^n P_n^2+4}{P_n^4}\right] = \frac{1}{75}\cdot\frac{P_{m+1}^2 P_{m+2}}{P_{m-1}P_m^2};$$

$$\prod_{n=3}^{\infty}\left[1-\frac{(-1)^n(4x^2-1)p_n^2+4x^2}{p_n^4}\right] = \frac{x}{(2x^2+1)(4x^2+1)^2}\gamma^5(x);$$

$$\prod_{n=3}^{\infty}\left[1-\frac{3(-1)^n P_n^2+4}{P_n^4}\right] = \frac{\gamma^5}{75},$$

where $\gamma(x) = x+\sqrt{x^2+1}$ and $\gamma = 1+\sqrt{2}$.

Next we explore the Lucas counterparts of formulas (5) and (6).

3. Products of Lucas Polynomial Expressions

Using equation (1), we have

$$(l_{n+2}l_{n-2})(l_{n+1}l_{n-1}) = \left[l_n^2 + (-1)^n \Delta^2 x^2\right]\left[l_n^2 - (-1)^n \Delta^2\right]$$

$$= l_n^4 + (-1)^n (x^2 - 1)\Delta^2 l_n^2 - \Delta^4 x^2 .$$

By Theorem 1, we then have

$$\prod_{n=3}^{m}\left[1 + \frac{(-1)^n (x^2 - 1)\Delta^2 l_n^2 - \Delta^4 x^2}{l_n^4}\right] = \prod_{n=3}^{m} \frac{l_{n-2}l_{n-1}l_{n+1}l_{n+2}}{l_n^4}$$

$$= \frac{x(x^2 + 2)^2}{(x^3 + 3x)^2(x^4 + 4x^2 + 2)} \cdot \frac{l_{m+1}^2 l_{m+2}}{l_{m-1}l_m^2} . \qquad (7)$$

Since $\lim_{m\to\infty} \frac{l_{m+k}}{l_m} = \alpha^k(x)$, this implies

$$\prod_{n=3}^{\infty}\left[1 + \frac{(-1)^n (x^2 - 1)\Delta^2 l_n^2 - \Delta^4 x^2}{l_n^4}\right] = \frac{x(x^2 + 2)^2}{(x^3 + 3x)^2(x^4 + 4x^2 + 2)}\alpha^5(x) . \qquad (8)$$

It follows by equations (7) and (8) that

$$\prod_{n=3}^{m}\left(1 - \frac{25}{L_n^4}\right) = \frac{9}{112} \cdot \frac{L_{m+1}^2 L_{m+2}}{L_{m-1}L_m^2};$$

$$\prod_{n=3}^{\infty}\left(1 - \frac{25}{L_n^4}\right) = \frac{9\alpha^5}{112},$$

respectively, where $2\alpha = 1 + \sqrt{5}$.

Next we find the Pell-Lucas consequences of formulas (7) and (8).

3.1 Pell-Lucas Implications: Since $q_n(x) = l_n(2x)$ and $\gamma(x) = \alpha(2x)$, it follows from equations (7) and (8) that

$$\prod_{n=3}^{m}\left[1 + \frac{4(-1)^n(x^2+1)(4x^2-1)q_n^2 - 64x^2(x^2+1)^2}{q_n^4}\right] = \frac{(2x^2+1)^2}{x(4x^2+3)^2(8x^4+8x^2+1)} \cdot \frac{q_{m+1}^2 q_{m+2}}{q_{m-1}q_m^2};$$

$$\prod_{n=3}^{\infty}\left[1 + \frac{4(-1)^n(x^2+1)(4x^2-1)q_n^2 - 64x^2(x^2+1)^2}{q_n^4}\right] = \frac{(2x^2+1)^2}{x(4x^2+3)^2(8x^4+8x^2+1)}\gamma^5(x),$$

where $\gamma(x) = x + \sqrt{x^2+1}$.

Since $q_n(1) = 2Q_n$, we then get

$$\prod_{n=3}^{m}\left[1 + \frac{6(-1)^n Q_n^2 - 16}{Q_n^4}\right] = \frac{9}{833} \cdot \frac{Q_{m+1}^2 Q_{m+2}}{Q_{m-1}Q_m^2};$$

$$\prod_{n=3}^{\infty}\left[1 + \frac{6(-1)^n Q_n^2 - 16}{Q_n^4}\right] = \frac{9}{833}\gamma^5,$$

where $\gamma = 1 + \sqrt{2}$.

4. Products of Jacobsthal Polynomial Expressions

Next we find the Jacobsthal counterparts of formulas (4) and (5) using the relationship $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$.

4.1 Jacobsthal Versions: Consider the rational expression $A = \dfrac{f_n^4 - \left[(-1)^n(x^2-1)f_n^2 + x^2\right]}{f_n^4}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator of the resulting expression with x^{2n-3}, we get

$$A = \frac{\left[x^{(n-1)/2} f_n\right]^4 + (-1)^n (x-1) x^{n-2}\left[x^{(n-1)/2} f_n\right]^2 - x^{2n-3}}{\left[x^{(n-1)/2} f_n\right]^4}$$

$$= \frac{J_n^4 + (x-1)(-x)^{n-2} J_n^2 - x^{2n-3}}{J_n^4}$$

$$= 1 + \frac{(x-1)(-x)^{n-2} J_n^2 - x^{2n-3}}{J_n^4};$$

$$\text{LHS} = \prod_{n=3}^{m}\left[1 + \frac{(x-1)(-x)^{n-2} J_n^2 - x^{2n-3}}{J_n^4}\right],$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Now let $B = \dfrac{x^2}{(x^2+1)^2(x^3+2x)} \cdot \dfrac{f_{m+1}^2 f_{m+2}}{f_{m-1} f_m^2}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator of the resulting expression with $x^{(3m+1)/2}$, we get

$$B = \frac{x^2\sqrt{x} f_{m+1}^2 f_{m+2}}{(x+1)^2(2x+1) f_{m-1} f_m^2}$$

$$= \frac{\left[x^{m/2} f_{m+1}\right]^2 \left[x^{(m+1)/2} f_{m+2}\right]}{(x+1)^2(2x+1)\left[x^{(m-2)/2} f_{m-1}\right]\left[x^{(m-1)/2} f_m\right]^2};$$

$$\text{RHS} = \frac{1}{(x+1)^2(2x+1)} \cdot \frac{J_{m+1}^2 J_{m+2}}{J_{m-1} J_m^2},$$

where $f_n = f_n(1/\sqrt{x})$ and $J_m = J_m(x)$.

Equating the two sides, we get

$$\prod_{n=3}^{m}\left[1 + \frac{(x-1)(-x)^{n-2} J_n^2 - x^{2n-3}}{J_n^4}\right] = \frac{1}{(x+1)^2(2x+1)} \cdot \frac{J_{m+1}^2 J_{m+2}}{J_{m-1} J_m^2}, \tag{9}$$

where $J_k = J_k(x)$.

It then follows that

$$\prod_{n=3}^{m}\left(1-\frac{1}{F_n^4}\right)=\frac{1}{12}\cdot\frac{F_{m+1}^2F_{m+2}}{F_{m-1}F_m^2};$$

$$\prod_{n=3}^{\infty}\left(1-\frac{1}{F_n^4}\right)=\frac{\alpha^5}{12},$$

as found earlier. In addition,

$$\prod_{n=3}^{m}\left[1+\frac{(-2)^{n-2}J_n^2-2^{2n-3}}{J_n^4}\right]=\frac{1}{45}\cdot\frac{J_{m+1}^2J_{m+2}}{J_{m-1}J_m^2};$$

$$\prod_{n=3}^{\infty}\left[1+\frac{(x-1)(-x)^{n-2}J_n^2-x^{2n-3}}{J_n^4}\right]=\frac{u^5(x)}{45}; \tag{10}$$

$$\prod_{n=3}^{\infty}\left[1+\frac{(-2)^{n-2}J_n^2-2^{2n-3}}{J_n^4}\right]=\frac{32}{45},$$

where $\lim_{m\to\infty}\frac{J^{m+k}}{Jm}=u^k(x)$, and $u(x)=\frac{1+D}{2}$.

Since

$$J_{n-2}J_{n-1}J_{n+1}J_{n+2}=J_n^4+(x-1)(-x)^{n-2}J_n^2-x^{2n-3},$$

by the *Cassini-like identity* $J_{n+k}J_{n-k}-J_n^2=-(-x)^{n-k}J_k^2$ [6], formulas (9) and (10) can be rewritten as

$$\prod_{n=3}^{m}\frac{J_{n-2}J_{n-1}J_{n+1}J_{n+2}}{J_n^4}=\frac{1}{(x+1)^2(2x+1)}\cdot\frac{J_{m+1}^2J_{m+2}}{J_{m-1}J_m^2};$$

$$\prod_{n=3}^{\infty} \frac{J_{n-2}J_{n-1}J_{n+1}J_{n+2}}{J_n^4} = \frac{u^5(x)}{45} ,$$

respectively.

Next we explore the Jacobsthal-Lucas versions of formulas (7) and (8) using the relationship $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$.

4.2 Jacobsthal-Lucas Versions: Consider the expression $A = \frac{l_4^n + (-1)^n (x^2 - 1)\Delta^2 l_n^2 - \Delta^4 x^2}{l_n^4}$. Replacing x with $1/\sqrt{x}$, then multiplying the numerator and denominator of the resulting expression with x^{2n-3}, we get

$$A = \frac{x^3 l_n^4 - (-1)^n x(x-1)D^2 l_n^2 - D^4}{x^3 l_n^4}$$

$$= \frac{(x^{n/2} l_n)^4 - (-1)^n x(x-1)D^2 x^{n-3} \ (x^{n/2} l_n)^2 - D^4 x^{2n-3}}{(x^{n/2} ln)^4}$$

$$= \frac{j_n^4 - (-1)^n (x-1)D^2 x^{n-2} j_n^2 - D^4 x^{2n-3}}{j_n^4} ;$$

$$\text{LHS} = \prod_{n=3}^{m} \left[1 - \frac{(-1)^n (x-1)D^2 x^{n-2} j_n^2 + D^4 x^{2n-3}}{j_n^4} \right] ,$$

where $l_n = l_n(1/\sqrt{x})$, $j_n = j_n(x)$, and $D^2 = 4x + 1$.

Now let

$$B = \frac{x(x^2 + 2)^2}{(x^3 + 3x)^2 (x^4 + 4x^2 + 2)} \cdot \frac{l_{m+1}^2 l_{m+2}}{l_{m-1} l_m^2} .$$

Replacing x with $1/\sqrt{x}$, then multiplying the numerator and denominator of the resulting expression with $x^{(3m+4)/2}$, we get

$$B = \frac{(2x+1)^2}{x^2(3x+1)^2(2x^2+4x+1)} \cdot \frac{\left[x^{(m+1)/2} l_{m+1}\right]^2 \left[x^{(m+2)/2} l_{m+2}\right]}{\left[x^{(m-1)/2} l_{m-1}\right] \left(x^{m/2} l_m\right)^2}$$

$$\text{RHS} = \frac{(2x+1)^2}{x^2(3x+1)^2(2x^2+4x+1)} \cdot \frac{j_{m+1}^2 j_{m+2}}{j_{m-1} j_m^2},$$

where $l_m = l_m(1/\sqrt{x})$ and $j_m = j_m(x)$.

Combining the two sides yields

$$\prod_{n=3}^{m}\left[1 - \frac{(-1)^n (x-1) D^2 x^{n-2} j_n^2 + D^4 x^{2n-3}}{j_n^4}\right] = \frac{(2x+1)^2}{(3x+1)^2(2x^2+4x+1)} \cdot \frac{j_{m+1}^2 j_{m+2}}{j_{m-1} j_m^2}, \tag{11}$$

where $j_m = j_m(x)$ and $D^2 = 4x+1$.

Since

$$\lim_{m\to\infty} \frac{j_{m+k}}{j_m} = u^k(x) \text{ and } u(x) = \frac{1+D}{2}, \text{ this yields}$$

$$\prod_{n=3}^{\infty}\left[1 - \frac{(-1)^n (x-1) D^2 x^{n-2} j_n^2 + D^4 x^{2n-3}}{j_n^4}\right] = \frac{(2x+1)^2}{(3x+1)^2(2x^2+4x+1)} u^5(x), \tag{12}$$

Since, $j_n(1) = L_n$, $j_n(2) = j_n$, and $u(2) = 2$, it follows from equations (11) and (12) that

$$\prod_{n=3}^{m}\left(1 - \frac{25}{L_n^4}\right) = \frac{9}{112} \cdot \frac{L_{m+1}^2 L_{m+2}}{L_{m-1} L_m^2} \;;$$

$$\prod_{n=3}^{\infty}\left(1-\frac{25}{L_n^4}\right)=\frac{9}{112}\alpha^5\,;$$

$$\prod_{n=3}^{m}\left[1-\frac{9(-2)^{n-2}j_n^2+81\cdot 2^{2n-3}}{j_n^4}\right]=\frac{25}{833}\cdot\frac{j_{m+1}^2 j_{m+2}}{j_{m-1}j_m^2}\,;$$

$$\prod_{n=3}^{\infty}\left[1-\frac{9(-2)^{n-2}j_n^2+81\cdot 2^{2n-3}}{j_n^4}\right]=\frac{800}{833}\,.$$

Since

$$j_{n-2}j_{n-1}j_{n+1}j_{n+2}=j_n^4-(x-1)(-x)^{n-2}D^2j_n^2-D^4x^{2n-3}\,,$$

by the Cassini-like identity $j_{n+k}j_{n-k}-j_n^2=(-x)^{n-k}D^2J_k^2$, we can rewrite formulas (11) and (12) as

$$\prod_{n=3}^{m}\frac{j_{n-2}j_{n-1}j_{n+1}j_{n+2}}{j_n^4}=\frac{(2x+1)^2}{(3x+1)^2(2x^2+4x+1)}\cdot\frac{j_{m+1}^2 j_{m+2}}{j_{m-1}j_m^2}\,;$$

$$\prod_{n=3}^{\infty}\frac{j_{n-2}j_{n-1}j_{n+1}j_{n+2}}{j_n^4}=\frac{(2x+1)^2}{(3x+1)^2(2x^2+4x+1)}\,u^5(x)\,,$$

respectively.

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PRODUCTS OF EXTENDED GIBONACCI POLYNOMIAL EXPRESSIONS REVISITED

Abstract: Using graph-theoretic tools, we confirm the products of extended gibonacci and Jacobsthal polynomial expressions, investigated in [8].

Keywords: Gibonacci Polynomials, Jacobsthal-Lucas Polynomial, Pell Polynomials.

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1. Introduction

Extended Gibonacci polynomials $z_n(x)$ are defined by the recurrence, $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$ where x is an arbitrary complex variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary complex polynominals; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*.

Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 5, 6].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* Pn and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [6].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$ [4, 6].

Fibonacci and Jacobsthal polynomials, and Lucas and Jacobsthal-Lucas polynomials are closely related by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [4, 5, 6, 7].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we omit a lot of basic algebra.

1.1 Products of gibonacci and Jacobsthal Polynomial Expressions: In [8], we studied the following products of gibonacci and Jacobsthal polynomial expressions:

$$\prod_{n=3}^{m} \frac{f_{n-2}f_{n-1}f_{n+1}f_{n+2}}{f_n^4} = \frac{x^2}{(x^2+1)^2(x^3+2x)} \cdot \frac{f_{m+1}^2 f_{m+2}}{f_{m-1}f_m^2}; \tag{1}$$

$$\prod_{n=3}^{m} \frac{l_{n-2}l_{n-1}l_{n+1}l_{n+2}}{l_n^4} = \frac{x(x^2+2)^2}{(x^3+3x)^2(x^4+4x^2+2)} \cdot \frac{l_{m+1}^2 l_{m+2}}{l_{m-1}l_m^2}; \tag{2}$$

$$\prod_{n=3}^{m} \frac{J_{n-2}J_{n-1}J_{n+1}J_{n+2}}{J_n^4} = \frac{1}{(x+1)^2(2x+1)} \cdot \frac{J_{m+1}^2 J_{m+2}}{J_{m-1}J_m^2}; \tag{3}$$

$$\prod_{n=3}^{m} \frac{j_{n-2}j_{n-1}j_{n+1}j_{n+2}}{j_n^4} = \frac{(2x+1)^2}{(3x+1)^2(2x^2+4x+1)} \cdot \frac{j_{m+1}^2 j_{m+2}}{j_{m-1} j_m^2}. \quad (4)$$

Our goal is to confirm these formulas using graph-theoretic techniques. To this end, we first present the needed graph-theoretic tools.

2. Graph-Theoretic Tools

Consider the Fibonacci digraph D_1 in Figure 1 with vertices v_1 and v_2, where a $weight$ is assigned to each edge [6, 7]. It follows by induction from its $weighted\ adjacency\ matrix$ $Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$, that

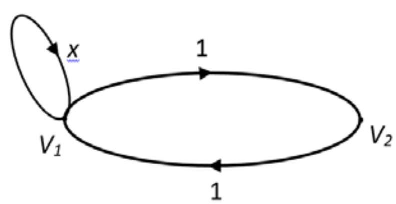

Figure 1: Weighted Fibonacci Digraph D_1

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $n \geq 1$ [6, 7].

A $walk$ from vertex v_i to vertex v_j is a sequence $v_i - e_i - v_{i+1} - \cdots - v_{j-1} - e_{j-1} - v_j$ of vertices v_k and edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The walk is $closed$ if $v_i = v_j$; otherwise, it is $open$. The $length$ of a walk is the number of edges in the walk. The $weight$ of a walk is the product of the weights of the edges along the walk.

We can employ the matrix Q^n to compute the weight of a walk of length n from any vertex v_i to any vertex v_j, as the following theorem shows [6, 7].

Theorem 1: *Let* M *be the weighted adjacency matrix of a weighted, connected digraph with vertices* $v_1, v_2, \ldots, vk$. *Then the* ij*th entry of the matrix* M^n *gives the sum of the weights of all walks of length* n *from* v_i *to* v_j, *where* $n \geq 1$. □

The next result follows from this theorem.

Corollary 1: *The* ij*th entry of* Q^n *gives the sum of the weights of all walks of length* n *from* v_i *to* v_j *in the weighted digraph* D_1, *where* $1 \leq i,\ j \leq n$. □

It follows by this corollary that the sum of the weights of closed walks of length n originating at v_1 in the digraph is f_{n+1} and that of those originating at v_2 is f_{n-1}. Consequently, the sum of the weights of all closed walks of length n in the digraph is $f_{n+1} + f_{n-1} = l_n$. These facts play a major role in the graph-theoretic proofs of the gibonacci formulas (1) and (2).

Let A and B denote sets of walks of varying lengths originating at a vertex v. Then the sum of the weights of the elements $(a,\ b)$ in the product set $A \times B$ is *defined* as the product of the sums of weights from each component. This definition can be extended to any finite number of components [7].

With these tools at our disposal, we are now ready for the graph-theoretic proofs.

3. Graph-Theoretic Confirmations

3.1 Confirmation of Formula (1): Proof: Let W_n denote the sum of the weights of the elements in the set C_n of closed walks of length n originating at v_1. By Corollary 1, $W_n = f_{n+1}$. Then W_n^4 gives the sum of the weights of the elements in the product set $C_n \times C_n \times C_n \times C_n$, and $W_{n-3}W_{n-2}W_nWn+1$ that of those in the set $C_{n-3} \times C_{n-2} \times C_n \times C_{n+1}$, where $n \geq 3$.

Let

$$Am = \prod_{n=3}^{m} \frac{W_{n-3}W_{n-2}W_nW_{n+1}}{W_{n-1}^4}$$

$$= \prod_{n=3}^{m} \frac{f_{n-2}f_{n-1}f_{n+1}f_{n+2}}{f_n^4}. \tag{5}$$

Suppose w denotes the weight of the loop at v_1 in the digraph and

$$Bm = \frac{w^2}{(w^2+1)^2(w^3+2w)} \cdot \frac{W_m^2W_{m+1}}{W_{m-2}W_{m-1}^2}$$

$$= \frac{x^2 \cdot}{(x^2+1)^2(x^3+2x)} \cdot \frac{f_{m+1}^2f_{m+2}}{f_{m-1}f_m^2}. \tag{6}$$

Using recursion [2, 8], we will now establish that $A_m = B_m$, where $m \geq 3$. We have

$$\frac{B_m}{B_{m-1}} = \frac{x^2f_{m+1}^2f_{m+2}}{(x^2+1)^2(x^3+2x)f_{m-1}f_m^2} \cdot \frac{(x^2+1)^2(x^3+2x)f_{m-2}f_{m-1}^2}{x^2f_m^2f_{m+1}}$$

$$= \frac{f_{m-2}f_{m-1}f_{m+1}f_{m+2}}{f_m^4}$$

$$= \frac{A_m}{A_{m-1}}.$$

So $\frac{A_m}{B_m} = \frac{A_{m-1}}{B_{m-1}} = \ldots = \frac{A_3}{B_3} = \frac{f_1f_2f_4f_5}{f_1f_2f_4f_5} = 1$. Consequently, $A_m = B_m$.

It then follows by equations (5) and (6) that

$$\prod_{n=3}^{m} \frac{f_{n-2}f_{n-1}f_{n+1}f_{n+2}}{f_n^4} = \frac{x^2}{(x^2+1)^2(x^3+2x)} \cdot \frac{f_{m+1}^2f_{m+2}}{f_{m-1}f_m^2},$$

as desired. □

Since

$$f_{n-2}f_{n-1}f_{n+1}f_{n+2} = f_n^4 - \left[(-1)^n(x^2-1)f_n^2 + x^2\right],$$

it then follows that

$$\prod_{n=3}^{m}\left[1 - \frac{(-1)^n(x^2-1)f_n^2 + x^2}{f_n^4}\right] = \frac{x^2}{(x^2+1)^2(x^3+2x)} \cdot \frac{f_{m+1}^2 f_{m+2}}{f_{m-1}f_m^2}$$

$$\prod_{n=3}^{\infty}\left[1 - \frac{(-1)^n(x^2-1)f_n^2 + x^2}{f_n^4}\right] = \frac{x^2}{(x^2+1)^2(x^3+2x)}\alpha^5(x)$$

$$\prod_{m}^{n=3}\left(1 - \frac{1}{F_n^4}\right) = \frac{1}{2}\ 1 \cdot \frac{F_{m+1}^2 F_{m+2}}{F_{m-1}F_m^2};$$

$$\prod_{n=3}^{\infty}\left(1 - \frac{1}{F_n^4}\right) = \frac{\alpha^5}{12},$$

as in [3, 8, 10], where $\lim_{m\to\infty}\frac{f_{m+k}}{f_m} = \alpha^k(x)$, $2\alpha(x) = x + \sqrt{x^2+4}$, and $2\alpha = 1 + \sqrt{5}$.

Next we explore the graph-theoretic proof of formula (2).

3.2 Confirmation of Formula (2): Proof: Let W_n denote the sum of the weights of elements in the set C_n of all closed walks of length n in the digraph. By Corollary 1, $W_n = f_{n+1} + f_{n-1} = l_n$. Then W_n^4 gives the sum of the weights of the elements in the product set $C_n \times C_n \times C_n \times C_n$, and $W_{n-2}W_{n-1}W_{n+1}W_{n+2}$ the sum of the weights of the elements in the set $C_{n-2} \times C_{n-1} \times C_{n+1} \times C_{n+2}$, where $n \geq 3$.

Let

$$A_m = \prod_{n=3}^{m}\frac{W_{n-2}W_{n-1}W_{n+1}W_{n+2}}{W_n^4}$$

$$= \prod_{n=3}^{m}\frac{l_{n-2}l_{n-1}l_{n+1}l_{n+2}}{l_n^4}. \tag{7}$$

Suppose w denotes the weight of the loop at v_1 and

$$B_m = \frac{w(w^2+2)^2}{(w^3+3w)^2(w^4+4w^2+2)} \cdot \frac{W_{m+1}^2 W_{m+2}}{W_{m-1}W_m^2}$$

$$= \frac{x(x^2+2)^2}{(x^3+3x)^2(x^4+4x^2+2)} \cdot \frac{l_{m+1}^2 l_{m+2}}{l_{m-1}l_m^2}.$$

As before, we will now confirm that $A_m = B_m$ using recursion [2, 8]. We have

$$\frac{B_m}{B_{m-1}} = \frac{x(x^2+2)^2 l_{m+1}^2 l_{m+2}}{(x^3+3x)(x^4+4x^2+2)l_{m-1}l_m^2} \cdot \frac{(x^3+3x)(x^4+4x^2+2)l_{m-2}l_{m-1}^2}{x(x^2+2)^2 l_m^2 l_{m+1}}$$

$$= \frac{l_{m-2}l_{m-1}l_{m+1}l_{m+2}l}{l_m^4}$$

$$= \frac{A_m}{A_{m-1}}.$$

This implies, $\frac{A_m}{B_m} = \frac{A_{m-1}}{B_{m-1}} = \ldots = \frac{A_3}{B_3} = \frac{l_1 l_2 l_4 l_5}{l_1 l_2 l_4 l_5} = 1.$

So $A_m = B_m.$

This yields

$$\prod_{n=3}^{m} \frac{l_{n-2}l_{n-1}l_{n+1}l_{n+2}}{l_n^4} = \frac{x(x^2+2)^2}{(x^3+3x)^2(x^4+4x^2+2)} \cdot \frac{l_{m+1}^2 l_{m+2}}{l_{m-1}l_m^2},$$

as desired. □

Since $(l_{n+2}l_{n-2})(l_{n+1}l_{n-1}) = l_n^4 + (-1)^n(x^2-1)\Delta^2 l_n^2 - \Delta^4 x^2$ [8], this yields

$$\prod_{n=3}^{m} \left[1 + \frac{(-1)^n(x^2-1)\Delta^2 l_n^2 - \Delta^4 x^2}{l_n^4}\right] = \frac{x(x^2+2)^2}{(x^3+3x)^2(x^4+4x^2+2)} \cdot \frac{l_{m+1}^2 l_{m+2}}{l_{m-1}l_m^2};$$

$$\prod_{n=3}^{\infty}\left[1+\frac{(-1)^n(x^2-1)\Delta^2 l_n^2-\Delta^4 x^2}{l_n^4}\right]=\frac{x(x^2+2)^2}{(x^3+3x)^2(x^4+4x^2+2)}\alpha^5(x);$$

$$\prod_{n=3}^{m}\left(1-\frac{25}{L_n^4}\right)=\frac{9}{112}\cdot\frac{L_{m+1}^2 L_{m+2}}{L_{m-1}L_m^2};$$

$$\prod_{n=3}^{\infty}\left(1-\frac{25}{L_n^4}\right)=\frac{9}{112}\alpha^5,$$

as in [8], where $\lim_{m\to\infty}\frac{l_{m+k}}{l_m}=\alpha^k(x)$.

Since $b_n(x)=g_n(2x)$, the graph-theoretic confirmations of the Pell versions [8] of the gibonacci formulas (1) and (2) follow from these two proofs, where $g_n=f_n$ or l_n and $b_n=p_n$ or q_n.

4. Graph-Theoretic Tools Revisited

To confirm the Jacobsthal results (3) and (4), consider the *weighted Jacobsthal digraph* D_2 in Figure 2 with vertices v_1 and v_2 [6, 7]. It follows from its *weighted adjacency matrix* $M=\begin{bmatrix}1 & x\\ 1 & 0\end{bmatrix}$, that, $M^n=\begin{bmatrix}J_{n+1} & xJ_n\\ J_n & xJ_{n-1}\end{bmatrix}$,

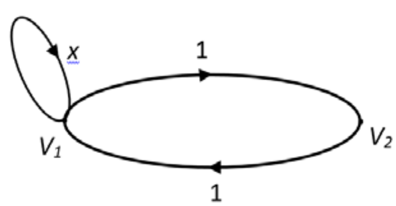

Figure 2: Weighted Jacobsthal Digraph D_2

where $J_n=J_n(x)$ and $n\geq 1$.

It then follows that the sum of the weights of closed walks of length n originating at v_1 is J_{n+1}, and that of those originating at v_2 is xJ_{n-1}. So the sum of the weights of all closed walks of length n in the digraph is $J_{n+1} + xJ_{n-1} = j_n$. These facts play a major role in the graph-theoretic proofs.

As before, let A and B denote sets of walks of varying lengths originating at a vertex v. Then the sum of the weights of the elements (a, b) in the product set $A \times B$ is $\mathit{defined}$ as the product of the sums of weights from each component. This definition can be extended to any finite number of components [7].

With these tools, we now explore the graph-theoretic proofs of (3) and (4).

4.1 Confirmation of Formula (3): Proof: Let W_n denote the sum of the weights of elements in the set C_n of closed walks of length n originating at v_1 in the digraph D_2. Then $W_n = J_{n+1}$. Clearly, W_n^4 denotes the sum of the weights of all elements in the product set $C_n \times C_n \times C_n \times C_n$, and $W_{n-3}W_{n-2}W_nW_{n+1}$ the sum of those in the set $C_{n-3} \times C_{n-2} \times C_n \times C_{n+1}$, where $n \geq 3$.

Let

$$Am = \prod_{n=3}^{m} \frac{W_{n-3}W_{n-2}W_nW_{n+1}}{W_{n-1}^4}$$

$$= \prod_{n=3}^{m} \frac{J_{n-2}J_{n-1}J_{n+1}J_{n+2}}{J_n^4}. \tag{8}$$

Let w denote the weight of the edge v_1v_2 and

$$B_m = \frac{1}{(w+1)^2(2w+1)} \cdot \frac{W_m^2W_{m+1}}{W_{m-2}W_{m-1}^2}$$

$$= \frac{1}{(x+1)^2(2x+1)} \cdot J_{m+1}^2J_{m+2}J_{m-1}J_m^2. \tag{9}$$

We will now establish that $A_m = B_m$ using recursion [2, 8]. We have

$$\frac{B_m}{B_{m-1}} = \frac{J_{m+1}^2 J_{m+2}}{J_{m-1}J_m^2} \cdot \frac{J_{m-2}J_{m-1}^2}{J_m^2 J_{m+1}}$$

$$= \frac{J_{m-2}J_{m-1}J_{m+1}J_{m+2}}{J_m^4}$$

$$= \frac{A_m}{A_{m-1}}.$$

This yields, $\frac{A_m}{B_m} = \frac{A_{m-1}}{B_{m-1}} = \ldots = \frac{A_3}{B_3} = \frac{J_1J_2J_4J_5}{J_1J_2J_4J_5} = 1$. So $A_m = B_m$.

Consequently,

$$\prod_{n=3}^{m} \frac{J_{n-2}J_{n-1}J_{n+1}J_{n+2}}{J_n^4} = \frac{1}{(x+1)^2(2x+1)} \cdot \frac{J_{m+1}^2 J_{m+2}}{J_{m-1}J_m^2},$$

as in formula (3) [8].

Since $J_{n-2}J_{n-1}J_{n+1}J_{n+2} = J_n^4 + (x-1)(-x)^{n-2}J_n^2 - x^{2n-3}$ [8], we can rewrite this formula as

$$\prod_{n=3}^{m} \left[1 + \frac{(x-1)(-x)^{n-2}J_n^2 - x^{2n-3}}{J_n^4}\right] = \frac{1}{(x+1)^2(2x+1)} \cdot \frac{J_{m+1}^2 J_{m+2}}{J_{m-1}J_m^2},$$

where $J_k = J_k(x)$ [8].

We then have
$$\prod_{n=3}^{\infty} \frac{J_{n-2}J_{n-1}J_{n+1}J_{n+2}}{J_n^4} = \frac{u^5(x)}{(x+1)^2(2x+1)}$$

$$\prod_{n=3}^{m}\left[1 + \frac{(x-1)(-x)^{n-2}J_n^2 - x^{2n-3}}{J_n^4}\right] = \frac{u^5(x)}{(x+1)^2(2x+1)} ,$$

where $\lim_{m\to\infty} \frac{J_{m+k}}{J_m} = u^k(x)$, and $2^u(x) = 1 + \sqrt{4x+1}$ [8]. □

Finally, pursue the confirmation of formula (4).

4.2 Confirmation of Formula (4): Proof: Let W_n denote the sum of the weights of elements in the set C_n of all closed walks of length n originating in the digraph D_2. Then $W_n = J_{n+1} + xJ_{n-1} = j_n$. Clearly, $W_n^4 = j_n^4$ gives the sum of the weights of all elements in the product set $C_n \times C_n \times C_n \times C_n$, and $W_{n-2}W_{n-1}W_{n+1}W_{n+2}$ that of those in the set $C_{n-2} \times C_{n-1} \times C_{n+1} \times C_{n+2}$, where $n \geq 3$.

Let

$$A_m = \prod_{n=3}^{m} \frac{W_{n-2}W_{n-1}W_{n+1}W_{n+2}}{W_{n-1}^4}$$

$$= \prod_{n=3}^{m} \frac{j_{n-2}j_{n-1}j_{n+1}j_{n+2}}{j_n^4}. \quad (10)$$

We will now compute A_m in a different way. To this end, we let w be the weight of the edge v_1v_2 and

$$B_m = \frac{(2w+1)^2}{(3w+1)^2(2w^2+4w+1)} \cdot \frac{W_{m+1}^2 W_{m+2}}{W_{m-1}W_m^2}$$

$$= \frac{(2x+1)^2}{(3x+1)^2(2x^2+4x+1)} \cdot \frac{j_{m+1}^2 j_{m+2}}{j_{m-1}j_m^2}. \quad (11)$$

Next we establish that $A_m = B_m$ using recursion [2, 8]. We have

$$\frac{B_m}{B_{m-1}} = \frac{j_{m+1}^2 j_{m+2}}{j_{m-1}j_m^2} \cdot \frac{j_{m-2}j_{m-1}^2}{j_m^2 j_{m+1}}$$

$$= \frac{j_{m-2}j_{m-1}j_{m+1}j_{m+2}}{j_m^4}$$

$$= \frac{A_m}{A_{m-1}}.$$

This implies, $\frac{A_m}{B_m} = \frac{A_{m-1}}{B_{m-1}} = \ldots = \frac{A_3}{B_3} = \frac{j_1j_2j_4j_5}{j_1j_2j_4j_5} = 1$. So $A_m = B_m$.

Consequently, we have

$$\prod_{n=3}^{m} \frac{j_{n-2}j_{n-1}j_{n+1}j_{n+2}}{j_n^4} = \frac{(2x+1)^2}{(3x+1)^2(2x^2+4x+1)} \cdot \frac{j_{m+1}^2 j_{m+2}}{j_{m-1}j_m^2},$$

as in formula (4) [8]. □

Since

$$j_{n-2}j_{n-1}j_{n+1}j_{n+2} = j_n^4 - (x-1)(-x)^{n-2}D^2 j_n^2 - D^4 x^{2n-3},$$

we can rewrite this equation as [8]

$$\prod_{n=3}^{m} \left[1 - \frac{(-1)^n(x-1)D^2x^{n-2}j_n^2 + D^4x^{2n-3}}{j_n^4}\right] = \frac{(2x+1)^2}{(3x+1)^2(2x^2+4x+1)} \cdot \frac{j_{m+1}^2 j_{m+2}}{j_{m-1}j_m^2}.$$

Since $\lim_{m\to\infty} \frac{j_{m+k}}{j_m} = u^k(x)$, we then get

$$\prod_{n=3}^{\infty} \frac{j_{n-2}j_{n-1}j_{n+1}j_{n+2}}{j_n^4} = \frac{(2x+1)^2}{(3x+1)^2(2x^2+4x+1)} u^5(x);$$

$$\prod_{n=3}^{\infty} \left[1 - \frac{(-1)^n(x-1)D^2x^{n-2}j_n^2 + D^4x^{2n-3}}{j_n^4}\right] = \frac{(2x+1)^2}{(3x+1)^2(2x^2+4x+1)} u^5(x),$$

as in [8].

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$\bar{H}$-FUNCTION AND GENERALIZED BESSEL FUNCTION INVOLVING THE GENERALIZED MELLIN-BARNES CONTOUR INTEGRALS

Abstract: This paper deal with a new approach to evaluate three theorems which are associated with the product of two hypergeometric functions, generalized Bessel function of first kind and the $\bar{H}$-function. These theorems are in most general nature which give many interesting particular cases. We give some new and known special cases of our main theorems.

Keywords: $\bar{H}$-Function, Generalized Bessel function of first kind and Hypergeometric functions.

Mathematical Subject Classification (2010) No.: 26A33, 33C05, 33C10, 33C60.

1. Introduction

In recent years, a large number of integral formulae involving different types of special functions have been developed by many authors i.e. Srivastava [20]. Garg and Mittal [9], Saxena *et al*. [17, 18] and others several researchers, obtained an interesting unified integral involving Fox H-function. Inayat Hussain [12] has pointed the usefulness of Feynman integrals in the statistical mechanics. The $\bar{H}$-function which is a new generalization of the familiar Fox`s function [6]. Using This $\bar{H}$-function and the following formulae, we establish our main theorems.

We have the three integrals ([10], also see [13], p. 77 Eqs. (3.1), (3.2) and (3.3)]:

(i) $$\int_0^\infty \left[\left(\alpha x+\frac{\beta}{x}\right)^2+\gamma\right]^{-\rho-1} dx = \frac{\sqrt{\pi}\,\Gamma(\rho+\frac{1}{2})}{2\alpha(4\alpha\beta+\gamma)^{\rho+\frac{1}{2}}\,\Gamma(\rho+1)} \tag{1.1}$$

$$(\alpha>0,\ \beta\geq 0,\ 4\alpha\beta+\gamma>0,\ \mathrm{Re}(\rho)+\frac{1}{2}>0).$$

(ii) $$\int_0^\infty \frac{1}{x^2}\left[\left(\alpha x+\frac{\beta}{x}\right)^2+\gamma\right]^{-\rho-1} dx = \frac{\sqrt{\pi}\,\Gamma(\rho+\frac{1}{2})}{2\beta(4\alpha\beta+\gamma)^{\rho+\frac{1}{2}}\,\Gamma(\rho+1)} \tag{1.2}$$

$$(\alpha\geq 0,\ \beta>0,\ 4\alpha\beta+\gamma>0,\ \mathrm{Re}(\rho)+\frac{1}{2}>0).$$

(iii) $$\int_0^\infty \left(\alpha+\frac{\beta}{x^2}\right)\left[\left(\alpha x+\frac{\beta}{x}\right)^2+\gamma\right]^{-\rho-1} dx = \frac{\sqrt{\pi}\,\Gamma(\rho+\frac{1}{2})}{(4\alpha\beta+\gamma)^{\rho+\frac{1}{2}}\,\Gamma(\rho+1)} \tag{1.3}$$

$$(\alpha>0,\ \beta>0,\ 4\alpha\beta+\gamma>0,\ \mathrm{Re}(\rho)+\frac{1}{2}>0).$$

(iv) Inayat Hussian [12, 13] defined and introduced the $\bar{H}$ function and represented by Bushman and Srivastava [5] in the following form:

$$\bar{H}^{M,N}_{P,Q}[z] = \bar{H}^{M,N}_{P,Q}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi i}\int_{-\omega\infty}^{\omega\infty}\bar{\phi}(\xi)\,z^{\xi}\,d\xi \tag{1.4}$$

where

$$\bar{\phi}(\xi) = \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \tag{1.5}$$

and $\omega=\sqrt{-1}$. Here a_j (j = 1,…,P) and b_j (j = 1,…,Q) are complex parameters, $\alpha_j\geq 0\,(j=1,...,P)$ and $\beta_j\geq 0\,(j=1,...,Q)$ and the exponents A_j (j = 1,…,N) and B_j (j = N + 1,…,Q) can take any non-integer values.

The contour in (1.4) is imaginary axis Re(ξ) = 0. It is suitably indented in order to avoid the singularities of the gamma functions and to keep those singularities on appropriate sides. The poles of the gamma functions of numerator in (1.5) are converted to branch points. However, as long as there is no coincidence of poles from any $\Gamma(b_j - \beta_j \xi)$ (j = 1, . . ., M) and $\Gamma(1 - a_j + \alpha_j \xi)$ (j = 1, . . ., N) pair.

Buschman and Srivastava [5] has proved that the integral represented by Eq. (1.4) is absolutely convergent when

$$\Omega = \sum_{j=1}^{M} \beta_j + \sum_{j=1}^{N} A_j \alpha_j - \sum_{j=M+1}^{Q} B_j \beta_j - \sum_{j=N+1}^{P} \alpha_j > 0 \tag{1.6}$$

and

$$| \arg z | < 1/2\, \pi\, \Omega \tag{1.7}$$

When all the exponents A_j and B_j takes the value unity, the $\bar{H}$-function reduces to the well-known Fox's $\bar{H}$-function [8] (see also [21]).

The following two particular cases of the $\bar{H}$-function which are not special cases of the $\bar{H}$-function.

(a) The function connected with certain class of Feynman integrals

$$g[\gamma,\eta,\tau,s;z] = \frac{K_{d-1}\Gamma(s+1)\Gamma\left(\frac{1}{2}+\frac{\tau}{2}\right)}{2^{s+2}\sqrt{\pi}(-1)^{s}\,\Gamma(\gamma)\Gamma\left(\gamma-\frac{\tau}{2}\right)}$$

$$\bar{H}^{1,3}_{3,3}\left[-z \left| \begin{matrix} (1-\gamma,1;1), \left(1-\gamma+\frac{\tau}{2},1;1\right), (1-\eta,1;1+p) \\ (0,1), \left(-\frac{\tau}{2},1;1\right), (-\eta,1;p+1) \end{matrix} \right.\right] \tag{1.8}$$

where

$$K_d = 2^{1-d}\,\pi^{-\frac{d}{2}} / \Gamma\left(\frac{d}{2}\right) \text{ [13, eq. p. 4121].}$$

(b) The polylogarithm function of order s introduced by Erdelyi *et al*. [7] is

$$F\left[z\ ,s\right]=\bar{H}^{1,2}_{2,2}\left[-z\ \middle|\ \begin{matrix}(1,1;1)\ ,\ (1\ ,1\ ;\ s)\\ (1,\ 1)\ ,\ (0\ ,1\ ;\ s)\end{matrix}\right] \tag{1.10}$$

(v) The function ${}_2F_1[z]$ was defined and introduced by Gauss and known as Gauss hypergeometric function.

$${}_2F_1[\alpha,\beta;\ \gamma;\ z]=\sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{r!(\gamma)_r}z^r \tag{1.11}$$

Where α, β, γ and z may be real or complex. The series is convergent for all values of z, real or complex such that $|z|<1$ and divergent for all values of z real or complex, such that $|z|>1$ (see also [19, eq.(1.1.1.4), p. 3].

We also use the following results related with hypergeometric function ${}_2F_1[z]$ ([19], p. 75, Th. 1)

$$(1-y)^{\alpha+\beta-\gamma}\ {}_2F_1[2\alpha,\ 2\beta;\ 2\gamma;\ z]=\sum_{r=1}^{\infty}a_r y^r \tag{1.12}$$

Then

$${}_2F_1[\alpha,\beta;\ \gamma+1/2;\ z]\ {}_2F_1[\gamma-\alpha,\gamma-\beta;\ \gamma+1/2;\ z]=\sum_{r=o}^{\infty}\frac{(\gamma)_r}{(\gamma+\frac{1}{2})}z^r \tag{1.13}$$

(vi) Generalized Bessel function is defined by the following series (see, e.g., [2, eq. (1.15), pp. 10]); for a recent work, see also [1, 3 and 4], [15, eq. (2.2), pp. 182], and [14, eq. (8), pp. 2]:

$$W_\nu(z)=\sum_{k=0}^{\infty}\frac{(-1)^k\ c^k\ (z/2)^{\nu+2k}}{k!\Gamma(\nu+k+(1+b)/2)} \tag{1.14}$$

($z \in C/\{0\}$ and b, c, $\nu \in C$ with Re(ν) > - 1.)

Where C denotes the set of complex numbers and $\Gamma(z)$ is the familiar Gamma function (see [22, sec. 1.1]).

2. Main Results

In this section, we obtain three theorems in the form of integrals will be evaluated. The integrals are associated with the product of two Gauss hypergeometric function, generalized Bessel function and the $\bar{H}$ -function.

Theorem 1: Suppose that

(i) $\alpha > 0,\ \beta \geq 0,\ ,\ 4\alpha\beta + \gamma > 0,\ \mathrm{Re}(\rho) + \frac{1}{2} > 0\ --\frac{1}{2} < \alpha - \beta - \gamma < \frac{1}{2}$,

(ii) $z \in C/\{0\}$ and $b, c, \nu \in C$ with $\mathrm{Re}(\nu) > -1$, $\eta \geq 0$, $\mu \geq 0$

(iii) $\mathrm{Re}(\rho + \eta\nu) + \mu \min_{1 \leq j \leq M} \mathrm{Re}(\frac{b_j}{\beta_j}) > 0$ for (j = 1, . . ., M) and

(iv) The $\bar{H}$ -function occurring in the theorem (2.1) satisfying the conditions corresponding appropriately to those given by (1.5) and (1.6).

Then the following integral formula holds:

$$\int_0^\infty R^{-\rho-1}\ {}_2F_1[\alpha, \beta; \gamma + 1/2; uR]\ {}_2F_1[\gamma - \alpha, \gamma - \beta; \gamma + 1/2; uR]$$

$$W_\nu[z`R^{-\eta})]\ \bar{H}^{M,N}_{P,Q}\left[A' z R^{-\mu} \left| \begin{matrix} (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right. \right] dx$$

$$= \frac{\sqrt{\pi}}{2\alpha(4\alpha\beta+\gamma)^{\rho+1/2}} \sum_{k,r=0}^{\infty} \frac{(-1)^k c^k (\gamma)_r a_r (z`/2)^{\nu+2k} u^r}{k!(\gamma+1/2)_r \Gamma(\nu + k + (1+b)/2)(4\alpha\beta+\gamma)^{\eta(\nu+2k)-r}}$$

$$\bar{H}^{M,\ N+1}_{P+1,\ Q+1}\left[\frac{z}{(4\alpha\beta+\gamma)^{\mu}} \left| \begin{matrix} (r-\rho-\eta(\nu+2k)+1/2,\ \mu; 1), (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q}, (r-\rho-\eta(\nu+2k),\ \mu;\ 1) \end{matrix} \right. \right] \quad (2.1)$$

Where $R = \{ (\alpha x + \frac{\beta}{x})^2 + \gamma \}$, (2.2)

Theorem 2: Suppose that

(a) $\alpha \geq 0,\ \beta > 0,\ 4\alpha\beta + \gamma > 0,\ \mathrm{Re}(\rho) + \frac{1}{2} > 0,\ -\frac{1}{2} < \alpha - \beta - \gamma < \frac{1}{2},$

(b) $z \in C/\{0\}$ and $b, c, \nu \in C$ with $\mathrm{Re}(\nu) > -1,\ \eta \geq 0,\ \mu \geq 0$

(c) $\mathrm{Re}(\rho + \eta\nu) + \mu \min_{1 \leq j \leq M} \mathrm{Re}(\frac{b_j}{\beta_j}) > 0$ for (j = 1, . . ., M) and

(d) The $\bar{H}$-function occurring in the theorem (2.1) satisfying the conditions corresponding appropriately to those given by (1.5) and (1.6). Then the following formula holds:

$$\int_0^\infty \frac{1}{x^2} R^{-\rho-1}\ {}_2F_1[\alpha, \beta; \gamma + 1/2; uR]\ {}_2F_1[\gamma - \alpha, \gamma - \beta; \gamma + 1/2; uR]$$

$$W_\nu[z`R^{-\eta}]]\ \bar{H}^{M,N}_{P,Q}\left[A' zR^{-\mu} \left| \begin{matrix} (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right. \right] dx =$$

$$\frac{\sqrt{\pi}}{2\beta(4\alpha\beta + \gamma)^{\rho+1/2}} \sum_{k,r=0}^{\infty} \frac{(-1)^k c^k (\gamma)_r a_r (z`/2)^{\nu+2k} u^r}{k!(\gamma + 1/2)_r \Gamma(\nu + k + (1+b)/2)(4\alpha\beta + \gamma)^{\eta(\nu+2k)-r}}$$

$$\bar{H}^{M,N+1}_{P+1,Q+1}\left[\frac{z}{(4\alpha\beta + \gamma)^\mu} \left| \begin{matrix} (r - \rho - \eta(\nu + 2k) + 1/2, \mu; 1), (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q}, (r - \rho - \eta(\nu + 2k), \mu; 1) \end{matrix} \right. \right] \tag{2.3}$$

Where R is defined by (2.2)

Theorem 3: Suppose that

(a) $\alpha > 0, \beta > 0, 4\alpha\beta + \gamma > 0,\ \mathrm{Re}(\rho) + \frac{1}{2} > 0,\ -\frac{1}{2} < \alpha - \beta - \gamma < \frac{1}{2}$

(b) z ϵ C/{0} and b, c, ν ϵ C with Re(ν) > - 1, $\eta \geq 0$, $\mu \geq 0$

(c) $\mathrm{Re}(\rho+\eta\nu)+\mu \min_{1\leq j\leq M} \mathrm{Re}(\frac{b_j}{\beta_j})>0$ for (j = 1, . . ., M) and

(d) The $\bar{H}$ -function occurring in the theorem (2.1) satisfying the conditions corresponding appropriately to those given by (1.5) and (1.6).

$$\int_0^\infty (\alpha+\frac{\beta}{x^2})R^{-\rho-1}\ {}_2F_1[\alpha,\beta;\ \gamma+1/2;\ uR]\ {}_2F_1[\gamma-\alpha,\gamma-\beta;\ \gamma+1/2;\ uR]$$

$$W_\nu[z`R^{-\eta})]\ \bar{H}^{M,N}_{P,Q}\left[A'zR^{-\mu}\ \middle|\ \begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right]dx$$

$$=\frac{\sqrt{\pi}}{(4\alpha\beta+\gamma)^{\rho+1/2}}\sum_{k,r=0}^{\infty}\frac{(-1)^k c^k\ (\gamma)_r\, a_r\, (z`/2)^{\nu+2k} u^r}{k!(\gamma+1/2)_r\,\Gamma(\nu+k+(1+b)/2)\,(4\alpha\beta+\gamma)^{\eta(\nu+2k)-r}}$$

$$\bar{H}^{M,\ N+1}_{P+1,\ Q+1}\left[\frac{A'z}{(4\alpha\beta+\gamma)^{\mu}}\ \middle|\ \begin{matrix}(r-\rho-\eta(\nu+2k)+1/2,\ \mu;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(r-\rho-\eta(\nu+2k),\ \mu;\ 1)\end{matrix}\right]\tag{2.4}$$

Proof: To establish theorem (2.1), initially, using the property given by the equations (1.12) and (1.13), we obtain the following form

$$\Delta=\int_0^\infty\sum_{r=o}^{\infty}\frac{(\gamma)_r\ u^r}{(\gamma+\frac{1}{2})_r}R^{r-\rho-1}W_\nu[z`R^{-\eta})]\bar{H}^{M,N}_{P,Q}\left[A'zR^{-\mu}\ \middle|\ \begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right]dx$$

Again, using the series representation (1.14) for $W_\nu(z`)$ and Mellin Barnes contour integral (1.4) for $\bar{H}^{M,\ N}_{P,\ Q}[z]$ in (2.5), and collecting the powers of R defined by (2.2), we can obtain the following form

$$= \int_0^\infty \{ \sum_{k,r=o}^{\infty} \frac{(-1)^k c^k (z`/2)^{\nu+2k} (\gamma)_r u^r}{k!\Gamma(\nu+k+(1+b)/2)(\gamma+\frac{1}{2})_r}$$

$$(\frac{1}{2\pi i} \int_{-\omega\infty}^{\omega\infty} \bar{\phi}(\xi) A'^{\xi} z^{\xi} d\xi) R^{r-\rho-\mu\xi-\eta(\nu+2k)-1} \} dx$$

In the above resulting expression, the order of integration and summation is interchanged (which is permissible under the conditions stated with (2.1)), we evaluate the intermost integral with the help of the result given by (1.1) and the interpreting the resulting Mellin–Barnes contour integral as an $\bar{H}$-function, we arrive at the right hand side of (2.1) after a little simplification.

Two theorems (2.3) and (2.4) can also be evaluated similarly by using the results given by (1.2) and (1.3) respectively. However, we omit the details here of these results.

3. Special Cases

Each of our integral formulae (2.1), (2.3) and (2.4) are unified in nature and possesses manifold generality. On suitably specializing the parameters of the $\bar{H}$-function, the generalized Bessel function of first kind in our main theorems, a large number of new integrals can be obtained as their special cases. one of them are discussed below.

In our main theorems, if we take A` = 1, M = 1, N = P = Q = 2 and reduce the $\bar{H}$-function to F(-z, s) function to the free energy model as given by Eq(1.10) following result are obtained.

Corollary 3.1: If $\alpha > 0$, $\beta \geq 0$, $4\alpha\beta + \gamma > 0$, $\mathrm{Re}(\rho) + \frac{1}{2} > 0$, $\mu > 0$, $\mu \geq 0$, $-\frac{1}{2} < \alpha - \beta - \gamma < \frac{1}{2}$, z ϵ C/{0} and b, c, ν ϵ C with Re(ν) > - 1, $\eta \geq 0$ and R = $\{ (\alpha x + \frac{\beta}{x})^2 + \gamma \}$ then there holds

$$\int_0^\infty R^{-\rho-1} {}_2F_1[\alpha,\beta;\gamma+1/2;uR]\, {}_2F_1[\gamma-\alpha,\gamma-\beta;\gamma+1/2;uR]$$

$$W_\nu[z`R^{-\eta})] \mathrm{F}[-zR^{-\mu}]\, dx$$

$$= \frac{\sqrt{\pi}}{2\alpha(4\alpha\beta+\gamma)^{\rho+1/2}} \sum_{k,r=0}^{\infty} \frac{(-1)^k c^k (\gamma)_r a_r (z`/2)^{\nu+2k} u^r}{k!(\gamma+1/2)_r \Gamma(\nu+k+(1+b)/2)(4\alpha\beta+\gamma)^{\eta(\nu+2k)-r}}$$

$$\bar{H}_{3,3}^{1,3}\left[\frac{-z}{(4\alpha\beta+\gamma)^{\mu}}\left|\begin{matrix}(r-\rho-\eta(\nu+2k)+1/2,\ \mu;1),(1,1;2),\ (1/2,1;\ d)\\(1,1),\ (0,1;\ 1+d),\ (r-\rho-\eta(\nu+2k),\ \mu;\ 1)\end{matrix}\right.\right] \quad (3.1)$$

Corollary 3.2: If $\alpha \geq 0, \alpha > 0,\ \beta \geq 0,\ 4\alpha\beta+\gamma > 0,\ \mathrm{Re}(\rho)+\frac{1}{2} > 0,\ \mu > 0,$ $\mu \geq 0,$

$-\frac{1}{2} < \alpha - \beta - \gamma < \frac{1}{2}$, z ϵ C/{0} and b, c, ν ϵ C with Re(ν) > - 1,

$\eta \geq 0$ and R = $\{(\alpha x+\frac{\beta}{x})^2+\gamma\}$ then there holds

$$\int_0^{\infty} \frac{1}{x^2} R^{-\rho-1}\ {}_2F_1[\alpha,\beta;\ \gamma+1/2;\ uR]\ {}_2F_1[\gamma-\alpha,\gamma-\beta;\ \gamma+1/2;\ uR]$$

$$W_{\nu}[z`R^{-\eta})]\,F[-zR^{-\mu}]\,dx$$

$$= \frac{\sqrt{\pi}}{2\beta(4\alpha\beta+\gamma)^{\rho+1/2}} \sum_{k,r=0}^{\infty} \frac{(-1)^k c^k (\gamma)_r a_r (z`/2)^{\nu+2k} u^r}{k!(\gamma+1/2)_r \Gamma(\nu+k+(1+b)/2)(4\alpha\beta+\gamma)^{\eta(\nu+2k)-r}}$$

$$\bar{H}_{3,3}^{1,3}\left[\frac{-z}{(4\alpha\beta+\gamma)^{\mu}}\left|\begin{matrix}(r-\rho-\eta(\nu+2k)+1/2,\ \mu;1),(1,1;2),\ (1/2,1;\ d)\\(1,1),\ (0,1;\ 1+d),\ (r-\rho-\eta(\nu+2k),\ \mu;\ 1)\end{matrix}\right.\right] \quad (3.2)$$

Corollary 3.3: If $\alpha > 0, \beta > 0$, $\mu > 0$, $\mu \geq 0$,

$4\alpha\beta+\gamma > 0,\ \mathrm{Re}(\rho)+\frac{1}{2} > 0,$

$-\frac{1}{2} < \alpha - \beta - \gamma < \frac{1}{2}$, z ϵ C/{0} and b, c, ν ϵ C with Re(ν) > - 1,

$\eta \geq 0$ and R = $\{(\alpha x+\frac{\beta}{x})^2+\gamma\}$ then there holds

$$\int_0^\infty (\alpha+\frac{\beta}{x^2})R^{-\rho-1}\ {}_2F_1[\alpha,\beta;\gamma+1/2;uR]\ {}_2F_1[\gamma-\alpha,\gamma-\beta;\gamma+1/2;uR]$$

$$W_\nu[z`R^{-\eta})]\,\mathrm{F}[-zR^{-\mu}]\ \mathrm{dx}$$

$$=\frac{\sqrt{\pi}}{(4\alpha\beta+\gamma)^{\rho+1/2}}\sum_{k,r=0}^{\infty}\frac{(-1)^k c^k\ (\gamma)_r\, a_r\,(z`/2)^{\nu+2k}u^r}{k!(\gamma+1/2)_r\,\Gamma(\nu+k+(1+b)/2)\,(4\alpha\beta+\gamma)^{\eta(\nu+2k)-r}}$$

$$\bar{\mathrm{H}}_{3,3}^{1,3}\left[\frac{-z}{(4\alpha\beta+\gamma)^{\mu}}\left|\begin{matrix}(r-\rho-\eta(\nu+2k)+1/2,\ \mu;1),(1,1;2),\ (1/2,1;\ \mathrm{d})\\ (1,\ 1),\ (0,1;\ 1+\mathrm{d}),\ (r-\rho-\eta(\nu+2k),\ \mu;\ 1)\end{matrix}\right.\right] \qquad (3.3)$$

The importance of our main theorems of present work lies in the many fold generality. Several other interesting special cases of the theorems involving numerous simpler special functions for example defined by (1.8) and a many variety of generalized Bessel function as a special functions can also be worked out We give only two of such cases. Thus, if we reduce the $\bar{H}$-function occurring on the left-hand side of our main findings to the Fox`s $\bar{H}$-function and generalized Bessel function occurring therein to unity, we get a known integrals [6, pp. 1463, Remark (2.7)] and if we reduce the $\bar{H}$-function and generalized Bessel function occurring on the left-hand side of our main findings, to unity, we arrive at a known basic formula [16, eqns.(3.1), (3.2 and (3.3) , pp. 75-76].

4. Conclusion

In this paper, we investigate the generalized fractional integration involving the definite integrals Gradshteyn-Ryzhik of the $\bar{H}$-function and generalized Bessel function of first kind. Therefore, we conclude this paper with remark that, the main findings obtained above are significant and can lead to yield numerous other fractional integrals involving various functions which are not special cases of H-function, Bessel functions and trigonometric functions by the suitable specializations of arbitrary parameters in the theorems.

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SUMS INVOLVING RECIPROCALS OF GIBONACCI POLYNOMIALS

Abstract: We explore sums involving reciprocals of gibonacci polynomials, and their Pell and numeric versions.

Keywords: Fibonacci Polynomial, Lucas Polynomial, Pell Polynomial, Binet-Like Formula, Pell- Lucas Polynomial.

Mathematical Subject Classification (2010) No.: Primary 05A19, 11B39, 11Cxx.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an positive integer variable; $a(x),\ b(x),\ z_0(x)$ and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*.

They can also be defined by *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth *Lucas number* [1, 3, 4].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* P_n and *Pell-Lucas* numbers Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n. It follows by their *Binet-like formulas* [4] that $\lim_{m\to\infty} \frac{g_{m+k}}{g_m} = \alpha^k(x)$, $\lim_{m\to\infty} \frac{f_{m+k}}{l_m} = \frac{\alpha^k(x)}{\Delta}$, $\lim_{m\to\infty} \frac{l_{m+k}}{f_m} = \alpha^k(x)\Delta$, where $x > 0$, $\Delta = \sqrt{x^2+4}$ and $2\alpha(x) = x + \Delta$. We also let $\alpha = \alpha(1)$, and omit a lot of basic algebra.

1.1 Fundamental Identities: Gibonacci polynomials satisfy the following fundamental properties [4]:

(a) $f_{2n} = f_n l_n$;

(b) $l_n = f_{n+1} + f_{n-1}$;

(c) $l_n l_{n+1} = l_{2n+1} + (-1)^n x$;

(d) $l_n^2 - \Delta^2 f_n^2 = 4(-1)^n$;

(e) $f_{n+k} f_{n-k} - f_n^2 = (-1)^{n-k+1} f_k^2$;

(f) $l_{n+k} l_{n-k} - l_n^2 = (-1)^{n-k} \Delta^2 f_k^2$;

(g) $f_{m-n} = (-1)^n (f_m f_{n+1} - f_{m+1} f_n)$;

(h) $l_{m-n} = (-1)^n (f_{m+1} l_n - f_m l_{n+1})$.

Properties (e) and (f) are the well known *Cassini-like identities*, and properties (g) and (h) are the *addition formulas*. They play a major role in our discourse.

2. Fibonacci Sums

With this background, we begin our explorations with a sum involving the reciprocals of odd numbered Fibonacci polynomials.

Theorem 1: *Let* $m \geq 2$. *Then*

$$\sum_{n=2}^{m} \frac{x}{f_{2n-1}^2 - 1} = \frac{f_{2m-2}}{xf_{2m}}. \tag{1}$$

Proof: We will establish this formula using a recursive technique [2, 4]. To this end, let A_n denote the LHS of equation (1) and B_n its RHS. Using the addition formula and the Cassini-like identity, we have

$$\begin{aligned} B_m - B_{m-1} &= \frac{f_{2m-2}}{xf_{2m}} - \frac{f_{2m-4}}{xf_{2m-2}} \\ &= \frac{f_{2m-2}^2 - f_{2m}f_{2m-4}}{xf_{2m}f_{2m-2}} \\ &= \frac{x}{f_{2m-1}^2 - 1} \\ &= A_m - A_{m-1}. \end{aligned}$$

Thus, $A_m - B_m = A_{m-1} - B_{m-1} = \ldots = A_2 - B_2 = \dfrac{x}{f_3^2 - 1} - \dfrac{f_2}{xf_4} = 0$.

This implies, $A_m = B_m$, as desired. □

Using identity (e), we can rewrite formula (1) in a different form:

$$\sum_{n=2}^{m} \frac{x}{f_{2n}f_{2n-2} - 1} = \frac{f_{2m-2}}{xf_{2m}}.$$

It follows from equation (1) that

$$\sum_{n=2}^{m} \frac{1}{F_{2n-1}^2 - 1} = \frac{F_{2m-2}}{F_{2m}};$$

$$\sum_{n=2}^{\infty} \frac{1}{f_{2n-1}^2 - 1} = \frac{1}{x^2\alpha^2(x)};$$

$$\sum_{n=2}^{\infty} \frac{1}{F_{2n-1}^2 - 1} = \frac{3-\sqrt{5}}{2}. \tag{2}$$

Theorem 1 has an interesting byproduct. Using identity (d), it yields

$$\sum_{n=2}^{m} \frac{\Delta^2 x}{\Delta^2 f_{2n-1}^2 - \Delta^2} = \frac{f_{2m-2}}{x f_{2m}};$$

$$\sum_{n=2}^{m} \frac{\Delta^2 x}{l_{2n-1}^2 - x^2} = \frac{f_{2m-2}}{x f_{2m}}. \tag{3}$$

This implies that

$$\sum_{n=2}^{m} \frac{5}{L_{2n-1}^2 - 1} = \frac{F_{2m-2}}{F_{2m}};$$

$$\sum_{n=2}^{\infty} \frac{\Delta^2}{l_{2n-1}^2 - x^2} = \frac{1}{x^2 \alpha^2(x)};$$

$$\sum_{n=2}^{\infty} \frac{1}{L_{2n-1}^2 - 1} = \frac{3-\sqrt{5}}{10}.$$

Next we establish a corresponding result for even-numbered Fibonacci polynomials.

Theorem 2: *Let* $m \geq 2$ *and* $\kappa(x) = \dfrac{f_{10}}{f_4 f_6} + \dfrac{f4}{x f_6}$. *Then*

$$\sum_{n=2}^{m} \frac{x^3 + 2}{x f_{2n}^2 - x^2} = \kappa(x) - \frac{f_{4m+2}}{f_{2m+2} f_{2m}}. \tag{4}$$

Proof: Again, we will confirm this formula using recursion. We let $A_n = \text{LHS}$ and $B_n = \text{RHS}$.

Using identities (a), (e), and (g), we get

$$
\begin{aligned}
B_m - B_{m-1} &= \frac{f_{4m-2}}{f_{2m}f_{2m-2}} - \frac{f_{4m+2}}{f_{2m+2}f_{2m}} \\
&= -\frac{f_{4m+2}f_{2m-2} - f_{4m-2}f_{2m+2}}{f_{2m+2}f_{2m}f_{2m-2}} \\
&= -\frac{\left[(x^2+1)f_{4m} + xf_{4m-1}\right](f_{2m} - xf_{2m-1})}{f_{2m+2}f_{2m}f_{2m-2}} \\
&\qquad + \frac{(f_{4m} - xf_{4m-1})\left[(x^2+1)f_{2m} + xf_{2m-1}\right]}{f_{2m+2}f_{2m}f_{2m-2}} \\
&= -\frac{(x^3+2x)(f_{4m-1}f_{2m} - f_{4m}f_{2m-1})}{f_{2m+2}f_{2m}f_{2m-2}} \\
&= \frac{(x^3+2x)f_{4m-2m}}{f_{2m+2}f_{2m}f_{2m-2}} \\
&= \frac{x^3+2x}{f_{2m+2}f_{2m-2}} \\
&= \frac{x^3+2x}{f_{2m}^2 - x^2} \\
&= A_m - A_{m-1}.
\end{aligned}
$$

Consequently, $A_m - B_m = A_{m-1} - B_{m-1} = \ldots = A^2 - B^2 = \dfrac{f_4}{xf_3l_3} - \dfrac{f_4}{xf_6} = 0$.

Thus, $A_m = B_m$, as expected. □

With identity (e), we can rewrite formula (4) as

$$
\sum_{n=2}^{m} \frac{x^3+2x}{f_{2n+2}f_{2n-2} - x^2} = \kappa(x) - \frac{f_{4m+2}}{f_{2m+2}f_{2m}}.
$$

It follows from equation (4) that

$$\sum_{n=2}^{m} \frac{3}{F_{2n}^2 - 1} = \frac{8}{3} - \frac{F_{4m+2}}{F_{2m+2}F_{2m}};$$

$$\sum_{n=2}^{\infty} \frac{x^3 + 2x}{f_{2n}^2 - x^2} = \kappa - \Delta;$$

$$\sum_{n=2}^{\infty} \frac{3}{F_{2n}^2 - 1} = \frac{8}{3} - \sqrt{5}\ . \tag{5}$$

An interesting byproduct: Using equations (2) and (5), we get

$$\sum_{n=3}^{\infty} \frac{1}{F_n^2 - 1} = \frac{3 - \sqrt{5}}{2} + \frac{8 - 3\sqrt{5}}{9}$$

$$= \frac{43}{18} - \frac{5\sqrt{5}}{6},$$

as in [5, 6].

A Lucas implication: Using identity (d), equation (5) yields

$$\sum_{n=2}^{m} \frac{(x^3 + 2x)\Delta^2}{\Delta^2 f_{2n}^2 - \Delta^2 x^2} = \kappa - \frac{f_{4m+2}}{f_{2m+2}f_{2m}};$$

$$\sum_{n=2}^{m} \frac{(x^3 + 2x)\Delta^2}{l_{2n}^2 - \Delta^2 x^2 - 4} = \kappa - \frac{f_{4m+2}}{f_{2m+2}f_{2m}}\ . \tag{6}$$

Consequently, we have

$$\sum_{n=2}^{m} \frac{15}{L_{2n}^2 - 9} = \frac{8}{3} - \frac{F_{4m+2}}{F_{2m+2}F_{2m}};$$

$$\sum_{n=2}^{\infty} \frac{(x^3 + 2x)\Delta^2}{l_{2n}^2 - \Delta^2 x^2 - 4} = \kappa - \Delta;$$

$$\sum_{n=2}^{\infty} \frac{1}{L_{2n}^2 - 9} = \frac{8}{3} - \sqrt{5}\ .$$

Next we explore sums involving reciprocals of Lucas polynomials.

3. Lucas Sums

We begin our exploration with sums involving odd-numbered Lucas polynomials.

The next sum involves squares of odd-numbered Lucas polynomials. It is a delightful application of the identity $l_n^2 - \Delta^2 f_n^2 = 4(-1)^n$ and the Cassini-like identity (e).

Theorem 3: *Let* $m \geq 1$. *Then*

$$\sum_{n=1}^{m} \frac{1}{l_{2n+1}^2 - x^2} = \frac{1}{\Delta^2 x^2} \cdot \frac{f_{2m}}{f_{2m+2}}\ . \tag{7}$$

Proof: Let $A_m = \text{LHS}$ and $B_m = \text{RHS}$. Then

$$\begin{aligned}
B_m - B_{m-1} &= \frac{1}{\Delta^2 x^2}\left(\frac{f_{2m}}{f_{2m+2}} - \frac{f_{2m-2}}{f_{2m}}\right) \\
&= \frac{-\left(f_{2m+2} f_{2m-2} - f_{2m}^2\right)}{\Delta^2 x^2 f_{2m+2} f_{2m}} \\
&= \frac{1}{\Delta^2 f_{2m+1}^2 - \Delta^2} \\
&= \frac{1}{l_{2m+1}^2 - 4(-1)^{2m+1} - (x^2 + 4)} \\
&= \frac{1}{l_{2m+1}^2 - x^2} \\
&= A_m - A_{m-1}\ .
\end{aligned}$$

This implies

$$A_m - B_m = A_{m-1} - B_{m-1} = \dots = A_1 - B_1 = \frac{f_2}{\Delta^2 x^2 f_4} - \frac{f_2}{\Delta^2 x^2 f_4} = 0\,.$$

Thus, $A_m = B_m$, as desired. □

It follows from formula (7) that

$$\sum_{n=1}^{m} \frac{1}{L_{2n+1}^2 - 1} = \frac{F_{2m}}{5F_{2m+2}};$$

$$\sum_{n=1}^{\infty} \frac{1}{l_{2n+1}^2 - x^2} = \frac{1}{\alpha^2(x)\Delta^2 x^2};$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n+1}^2} - 1 = \frac{3-\sqrt{5}}{10}\,. \tag{8}$$

The next sum also involves squares of odd-numbered Lucas polynomials.

Theorem 4: *Let* $m \ge 0$. *Then*

$$\sum_{n=0}^{m} \frac{\Delta^2 x}{l_{2n+1}^2 + \Delta^2} = \frac{l_{2m+3}}{l_{2m+2}} - \frac{x}{2}\,. \tag{9}$$

Proof: Let $A_m = \text{LHS}$ and $B_m = \text{RHS}$, as usual. By the gibonacci recurrence and identity (f), we then have

$$B_m - B_{m-1} = \frac{l_{2m+3}}{l_{2m+2}} - \frac{l_{2m+1}}{l_{2m}}$$

$$= \frac{l_{2m+3} l_{2m} - l_{2m+2} l_{2m+1}}{l_{2m+2} l_{2m}}$$

$$= \frac{(x l_{2m+2} + l_{2m+1})(l_{2m+2} - x l_{2m+1}) - l_{2m+2} l_{2m+1}}{l_{2m+2} l_{2m}}$$

$$= \frac{xl_{2m+2}(l_{2m+2} - xl_{2m+1}) - xl_{2m+1}^2}{l_{2m+1}^2 + \Delta^2}$$

$$= \frac{x(l_{2m+2}l_{2m} - l_{2m+1}^2)}{l_{2m+1}^2 + \Delta^2}$$

$$= \frac{\Delta^2}{xl_{2m+1}^2 + \Delta^2}$$

$$= A_m - A_{m-1}.$$

Consequently,

$$A_m - B_m = A_{m-1} - B_{m-1} = \ldots = A_0 - B_0 = \frac{\Delta^2 x}{x^2 + \Delta^2} - \left(\frac{l_3}{l_2} - \frac{x}{2}\right) = 0.$$

Thus, $A_m = B_m$, as desired. □

It follows from formula (9) that

$$\sum_{n=0}^{m} \frac{5}{L_{2n+1}^2 + 5} = \frac{L_{2m+3}}{L_{2m+2}} - \frac{1}{2};$$

$$\sum_{n=0}^{\infty} \frac{\Delta^2 x}{l_{2n+1}^2 + \Delta^2} = \frac{\Delta^2}{2};$$

$$\sum_{n=0}^{\infty} \frac{5}{L_{2n+1}^2 + 5} = \frac{\sqrt{5}}{2}.$$

Finally, we explore a sum involving squares of even-numbered Lucas polynomials.

Theorem 5: Let $m \geq 1$. Then

$$\sum_{n=1}^{m} \frac{x^2}{l_{2n}^2 - \Delta^2} = \frac{f_{2m}}{l_{2m+1}}. \tag{10}$$

Proof: As usual, we let $A_m = \text{LHS}$ and $B_m = \text{RHS}$. Then, by the gibonacci recurrence and identity (h), we have

$$\begin{aligned}
B_m - B_{m-1} &= \frac{f_{2m}}{l_{2m+1}} - \frac{l_{2m-2}}{f_{2m-1}} \\
&= \frac{xf_{2m}l_{2m-1} - xf_{2m-2}l_{2m+1}}{xl_{2m+1}l_{2m-1}} \\
&= \frac{(f_{2m+1} - f_{2m-1})l_{2m-1} - x(f_{2m} - xf_{2m-1})(xl_{2m} + xl_{2m-1})}{xl_{2m+1}l_{2m-1}} \\
&= \frac{(f_{2m+1} - xf_{2m})l_{2m-1} + (x^2 - 1)f_{2m-1}l_{2m-1} - x^2(f_{2m} - xf_{2m-1})l_{2m}}{xl_{2m+1}l_{2m-1}} \\
&= \frac{x^2(f_{2m-1}l_{2m-1} - f_{2m-2}l_{2m})}{xl_{2m+1}l_{2m-1}} \\
&= \frac{-x^2 l_{-1}}{xl_{2m+1}l_{2m-1}} \\
&= \frac{x^2}{l_{2m}^2 - \Delta^2} \\
&= A_m - A_{m-1}.
\end{aligned}$$

This implies

$$Am - Bm = A_{m-1} - B_{m-1} = \ldots = A_1 - B_1 = \frac{1}{x^2 + 3} - \frac{x}{x^3 + 3x} = 0.$$

Consequently, $A_m = B_m$, as desired. □

Formula (10) yields

$$\sum_{n=1}^{m} \frac{1}{L_{2n}^2 - 5} = \frac{F_{2m}}{L_{2m+1}};$$

$$\sum_{n=1}^{\infty} \frac{x^2}{l_{2n}^2 - \Delta^2} = \frac{1}{\alpha(x)\Delta};$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n}^2 - 5} = \frac{1}{2} - \frac{\sqrt{5}}{10}.$$

Using the identity $l_n^2 - \Delta^2 f_n^2 = 4(-1)^n$, formula (10) yields

$$\sum_{n=1}^{m} \frac{x^2}{\Delta^2 f_{2n}^2 - x^2} = \frac{f_{2m}}{l_{2m+1}}; \tag{11}$$

$$\sum_{n=1}^{\infty} \frac{x^2}{\Delta^2 f_{2n}^2 - x^2} = \frac{1}{\alpha(x)\Delta};$$

$$\sum_{n=1}^{\infty} \frac{1}{5F_{2n}^2 - 1} = \frac{1}{2} - \frac{\sqrt{5}}{10}. \tag{12}$$

An interesting byproduct: Using equations (8) and (12), we get

$$\sum_{n=1}^{\infty} \frac{1}{5F_{2n+1}^2 - 1} = \frac{3 - \sqrt{5}}{10};$$

$$\sum_{n=3}^{\infty} \frac{1}{5F_n^2 - 1} = \sum_{n=1}^{\infty} \frac{1}{5F_{2n+1}^2 - 1} + \sum_{n=1}^{\infty} \frac{1}{5F_{2n}^2 - 1}$$

$$= \frac{4}{5} - \frac{\sqrt{5}}{5}.$$

Theorem 6: *Let* $m \geq 0$. *Then*

$$\sum_{n=0}^{m} \frac{(-1)^n \Delta^2}{l_{2n+1} + (-1)^n x} = \frac{l_{m+2}}{l_{m+1}} - \frac{x}{2}. \tag{13}$$

Proof: Again, we let $A_m = \text{LHS}$ and $B_m = \text{RHS}$. Using identities (c) and (f), we then have

$$B_m - B_{m-1} = \frac{l_{m+2}}{l_{m+1}} - \frac{l_{m+1}}{l_m}$$

$$= \frac{l_{m+2} l_m - l_{m2+1}}{l_{m+1} l_m}$$

$$= \frac{(-1)^m \Delta^2}{l_{2m+1} + (-1)^n x}$$

$$= A_m - A_{m-1}.$$

This yields

$$A_m - B_m = A_{m-1} - B_{m-1} = \ldots = A_0 - B_0 = \frac{\Delta^2}{2x} - \frac{\Delta^2}{2x} = 0.$$

So, $A_m = B_m$, as expected. □

Formula (13) implies

$$\sum_{n=0}^{m} \frac{5(-1)^n}{L_{2n+1} + (-1)^n} = \frac{L_{m+2}}{L_{m+1}} - \frac{1}{2};$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \Delta^2}{l_{2n+1} + (-1)^n x} = \frac{\Delta}{2};$$

$$\sum_{n=0}^{\infty} \frac{5(-1)^n}{L_{2n+1} + (-1)^n} = \frac{\sqrt{5}}{2}.$$

4. Pell and Pell-Lucas Implications

Since $b_n(x) = g_n(2x)$, equations (1), (3), (4), (6), (7), (9), (10), (11), and (13), yield the following Pell versions, respectively:

$$\sum_{n=2}^{m} \frac{2x}{p_{2n-1}^2 - 1} = \frac{p_{2m-2}}{2xp_{2m}}$$

$$\sum_{n=2}^{m} \frac{8x(x^2+1)}{q_{2n-1}^2 - 4x^2} = \frac{p_{2m-2}}{2xp_{2m}};$$

$$\sum_{n=2}^{m} \frac{4x(2x^2+1)}{p_{2n}^2 - 4x^2} = \kappa(2x) - \frac{p_{4m+2}}{p_{2m+2}p_{2m}};$$

$$\sum_{n=2}^{m} \frac{16x(x^2+1)(2x^2+1)}{q_{2n-1}^2 6x^2(x^2+1) - 4} = \kappa(2x) - \frac{p_{4m+2}}{p_{2m+2}p_{2m}};$$

$$\sum_{n=1}^{m} \frac{1}{q_{2n+1}^2 - 4x^2} = \frac{1}{16x^2(x^2+1)} \cdot \frac{p_{2m}}{p_{2m+2}};$$

$$\sum_{n=0}^{m} \frac{8x(x^2+1)}{q_{2n+1}^2 + 4(x^2+1)} = \frac{q_{2m+3}}{q_{2m+2}} - x;$$

$$\sum_{n=1}^{m} \frac{4x^2}{q_{2n}^2 - 4(x^2+1)} = \frac{p_{2m}}{q_{2m+1}};$$

$$\sum_{n=1}^{m} \frac{x^2}{(x^2+1)p_{2n}^2 - x^2} = \frac{p_{2m}}{q_{2m+1}};$$

$$\sum_{n=0}^{m} \frac{4(-1)^n(x^2+1)}{q_{2n+1} + 2(-1)^n x} = \frac{q_{m+2}}{q_{m+1}} - x.$$

Their numeric versions follow by letting $x = 1$. For brevity, we omit them.

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SUMS INVOLVING RECIPROCALS OF JACOBSTHAL POLYNOMIALS

Abstract: We explore the Jacobsthal versions of the gibonacci sums investigated in [6].

Keywords: Fibonacci Polynomial, Jacobsthal-Lucas Polynomial, Gibonacci Polynomials.

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1. Introduction

Extended Gibonacci polynomials $z_n(x)$ are defined by the recurrence, $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$ where x is an arbitrary complex variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary complex polynominals; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*.

Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 4, 5].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$.

Fibonacci and Jacobsthal polynomials, and Lucas and Jacobsthal-Lucas polynomials are closely related by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [2, 3].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. We let $\Delta = \sqrt{x^2+4}$, $D = \sqrt{4x+1}$, $2\alpha(x) = x + \Delta$, $2u(x) = 1 + D$, $2v(x) = 1 - D$ and $\alpha = \alpha(1)$. Using the *Binet-like* formulas [5], it follows that $\lim_{m\to\infty} \frac{c_{m+k}}{c_m} = u^k(x)$, $\lim_{m\to\infty} \frac{J_{m+k}}{j_m} = \frac{\alpha^\kappa(x)}{\Delta}$, $\lim_{m\to\infty} \frac{J_{m+k}}{j_m} = \alpha^\kappa(x)\Delta$. We also omit a lot of basic algebra.

2. Sums Involving Reciprocals of Gibonacci Polynomials

We investigated the following gibonacci sums in [6]:

$$\sum_{n=2}^{m} \frac{x}{f_{2n-1}^2 - 1} = \frac{f_{2m-2}}{x f_{2m}}; \tag{1}$$

$$\sum_{n=2}^{m} \frac{x^3 + 2x}{f_{2n}^2 - x^2} = \kappa(x) - \frac{f_{4m+2}}{f_{2m+2} f_{2m}}; \tag{2}$$

$$\sum_{n=1}^{m} \frac{1}{l_{2n+1}^2 - x^2} = \frac{1}{\Delta^2 x^2} \cdot \frac{f_{2m}}{f_{2m+2}}; \tag{3}$$

$$\sum_{n=0}^{m} \frac{\Delta^2 x}{l_{2n+1}^2 + \Delta^2} = \frac{l_{2m+3}}{l_{2m+2}} - \frac{x}{2}; \tag{4}$$

$$\sum_{n=1}^{m} \frac{x^2}{l_{2n}^2 - \Delta^2} = \frac{f_{2m}}{l_{2m+1}}; \tag{5}$$

$$\sum_{n=0}^{m} \frac{(-1)^n \Delta^2}{l_{n+1}^2 + (-1)^n x} = \frac{l_{m+2}}{l_{m+1}} - \frac{x}{2}, \tag{6}$$

where $\kappa(x) = \dfrac{f_{10}}{f_4 f_6} + \dfrac{f_4}{x f_6}$.

Our objective is to explore their Jacobsthal counterparts. The investigations hinge on the gibonacci-Jacobsthal relationships cited above and the Jacobsthal identity $j_n^2 - D^2 J_n^2 = 4(-x)^n$ [5].

3. Sums Involving Reciprocals of Jacobsthal Polynomials

We begin our explorations with the first gibonacci sum.

3.1 Jacobsthal Version of Identity (1): Let $A = \dfrac{x}{f_{2n-1}^2 - 1}$ and $B = \dfrac{f_{2m-2}}{x f_{2m}}$.

Replacing x with $1/\sqrt{x}$ in A, and then multiplying the numerator and denominator of the resulting expression with x^{2n-2}, we get

$$\text{LHS} = \sum_{n=2}^{m} \frac{x^{(4n-5)/2}}{\left[x^{(2n-2)/2} f_{2n-1}\right]^2 - x^{2n-2}}$$

$$= \sum_{n=2}^{m} \frac{x^{(4n-5)/2}}{J_{2n-1}^2 - x^{2n-2}},$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Now replace x with $1/\sqrt{x}$ in B, and then multiply the numerator and denominator of the resulting expression with $x^{(2m-1)/2}$. This yields

$$\begin{aligned}\text{RHS} &= \frac{\sqrt{x} f_{2m-2}}{x f_{2m}} \\ &= \frac{x\sqrt{x}\left[x^{(2m-3)/2} f_{2m-2}\right]}{x^{(2m-1)}/2 f_{2m}} \\ &= \frac{x\sqrt{x} J_{2m-2}}{J_{2m}},\end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Equating the two sides, we get

$$\sum_{n=2}^{m} \frac{x^{2n-4}}{J_{2n-1}^2 - x^{2n-2}} = \frac{J_{2m-2}}{J_{2m}}. \tag{7}$$

It follows from equation (7) that

$$\sum_{n=2}^{m} \frac{1}{F_{2n-1}^2 - 1} = \frac{F_{2m-2}}{F_{2m}};$$

$$\sum_{n=2}^{\infty} \frac{x^{2n-4}}{J_{2n-1}^2 - x^{2n-2}} = \frac{1}{u^2(x)};$$

$$\sum_{n=2}^{\infty} \frac{1}{F_{2n-1}^2 - 1} = \frac{3-\sqrt{5}}{2};$$

$$\sum_{n=2}^{\infty} \frac{2^{2n-4}}{J_{2n-1}^2 - 2^{2n-2}} = \frac{1}{4};$$

see [6, 7, 8].

Using the identity $j_n^2 - D^2 J_n^2 = 4(-x)^n$, it also implies

$$\sum_{n=2}^{m} \frac{D^2 x^{2n-4}}{j_{2n-1}^2 - x^{2n-2}} = \frac{J_{2m-2}}{J_{2m}}.$$

This yields

$$\sum_{n=2}^{\infty} \frac{D^2 x^{2n-4}}{j_{2n-1}^2 - x^{2n-2}} = \frac{1}{u^2(x)};$$

$$\sum_{n=2}^{\infty} \frac{5}{L_{2n-1}^2 - 1} = \frac{3-\sqrt{5}}{2};$$

$$\sum_{n=2}^{\infty} \frac{9 \cdot 2^{2n-4}}{j_{2n-1}^2 - 2^{2n-2}} = \frac{1}{4}.$$

Next we turn to identity (2)

3.2 Jacobsthal Version of Identity (2): Let $A = \dfrac{x^3 + 2x}{f_{2n}^2 - x^2}$, and $B = \dfrac{f_{10}}{f_4 f_6} + \dfrac{f_4}{x f_6} - \dfrac{f_{4m+2}}{f_{2m+2} f_{2m}}$. Replacing x with with $1/\sqrt{x}$ in A, and then multiplying the numerator and denominator of the resulting expression with x^{2n-2}, we get

$$\text{LHS} = \sum_{n=2}^{m} \frac{2x+1}{\sqrt{x}(x f_{2n-1}^2)}$$

$$= \sum_{n=2}^{m} \frac{(2x+1)x^{2n-2}}{\sqrt{x}\left\{\left[x^{(2n-1)/2} f_{2n}\right]^2 - x^{2n-2}\right\}}$$

$$= \sum_{n=2}^{m} \frac{(2x+1)x^{2n-2}}{\sqrt{x}(J_{2n}^2 - x^{2n-2})},$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Now replace x with with $1/\sqrt{x}$ in B, and then multiply each numerator and each denominator of the resulting expression with $x^{(4m+1)/2}$. We then get

$$\text{RHS} = \frac{x^{9/2}f_{10}}{\sqrt{x}(x^{5/2}f_6)(x^{3/2}f_4)} + \frac{x^3\sqrt{x}(x^{3/2}f_4)}{x^2(x^{5/2}f_6)} - \frac{x^{(4m+1)/2}f_{4m+2}}{\sqrt{x}[x^{(2m+1)/2}f_{2m+2}][x^{(2m-1)/2}f_{2m}]}$$

$$= \frac{J_{10}}{\sqrt{x}J_6J_4} + \frac{x\sqrt{x}J_4}{xJ_6} - \frac{J_{4m+2}}{\sqrt{x}J_{2m+2}J_{2m}},$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Equating the two sides then yields

$$\sum_{n=2}^{m} \frac{(2x+1)x^{2n-2}}{J_{2n}^2 - x^{2n-2}} = \kappa^*(x) - \frac{J_{4m+2}}{J_{2m+2}J_{2m}}, \qquad (8)$$

where $\kappa^*(x) = \dfrac{J_{10}}{J_6J_4} + \dfrac{x^2J_4}{J_6}$.

This implies

$$\sum_{n=2}^{m} \frac{3}{F_{2n}^2 - 1} = \frac{8}{3} - \frac{F_{4m+2}}{F_{2m+2}F_{2m}};$$

$$\sum_{n=2}^{\infty} \frac{(2x+1)x^{2n-2}}{J_{2n}^2 - x^{2n-2}} = \kappa^*(x) - D;$$

$$\sum_{n=2}^{\infty} \frac{3}{F_{2n}^2 - 1} = \frac{8}{3} - \sqrt{5};$$

$$\sum_{n=2}^{\infty} \frac{5 \cdot 2^{2n-2}}{J_{2n}^2 - 2^{2n-2}} = \frac{6}{5}.$$

Using the identity $j_n^2 - D^2 J_n^2 = 4(-x)^n$, it also yields

$$\sum_{n=2}^{m} \frac{(2x+1)D^2 x^{2n-2}}{j_{2n}^2 - (2x+1)^2 x^{2n-2}} = \kappa^*(x) - \frac{J_{4m+2}}{J_{2m+2}J_{2m}};$$

$$\sum_{n=2}^{\infty} \frac{(2x+1)D^2 x^{2n-2}}{j_{2n}^2 - (2x+1)^2 x^{2n-2}} = \kappa^*(x) - D;$$

$$\sum_{n=2}^{\infty} \frac{15}{L_{2n}^2 - 9} = \frac{8}{3} - \sqrt{5};$$

$$\sum_{n=2}^{\infty} \frac{45 \cdot 2^{2n-2}}{j_{2n}^2 - 25 \cdot 2^{2n-2}} = \frac{6}{5}.$$

Next we explore the Jacobsthal counterpart of identity (3).

3.3 Jacobsthal Version of Identity (3): Let $A = \dfrac{1}{l_{2n+1}^2 - x^2}$, and $B = \dfrac{1}{\Delta^2 x^2} \cdot \dfrac{f_{2m}}{f_{2m+2}}$. Replacing x with with $1/\sqrt{x}$ in A, and then multiplying the numerator and denominator of the resulting expression with x^{2n-1}, yields

$$\text{LHS} = \sum_{n=2}^{m} \frac{x}{x l_{2n+1}^2 - 1}$$

$$= \sum_{n=2}^{m} \frac{x^{2n}}{\left[x^{(2n+1)/2} l_{2n+1}\right]^2 - x^{2n-1}}$$

$$= \sum_{n=2}^{m} \frac{x^{2n}}{j_{2n+1}^2 - x^{2n-1}},$$

where $l_n = l_n(1/\sqrt{x})$ and $j_n = j_n(x)$.

Now replace x with with $1/\sqrt{x}$ in B, and then multiply the numerator and denominator of the resulting expression with $x^{(2m+1)/2}$. This yields

$$\text{RHS} = \frac{x^2}{D^2} \cdot \frac{f_{2m}}{f_{2m+2}};$$

$$= \frac{x^2}{D^2} \cdot \frac{x\left[x^{(2m-1)/2} f_{2m}\right]}{x^{(2m+1)/2} f_{2m+2}};$$

$$= \frac{x^3}{D^2} \cdot \frac{J_{2m}}{J_{2m+2}},$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Equating the two sides, we then get

$$\sum_{n=1}^{m} \frac{x^{2n}}{j_{2n+1}^2 - x^{2n-1}} = \frac{x^3}{D_2} \cdot \frac{J_{2m}}{J_{2m+2}}. \tag{9}$$

It follows from equation (9) that

$$\sum_{n=1}^{m} \frac{1}{L_{2n+1}^2 - 1} = \frac{F_{2m}}{5F_{2m+2}};$$

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{j_{2n+1}^2 - x^{2n-1}} = \frac{x^3}{D^2 u^2(x)};$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n+1}^2 - 1} = \frac{3-\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{{}^{2}_{2n+1} - 2^{2n-1}} j = \frac{2}{9}.$$

Using the identity $j_n^2 - D^2 J_n^2 = 4(-x)^n$, it also follows that

$$\sum_{n=1}^{m} \frac{x^{2n}}{D^2 J_{2n+1}^2 - (4x^2+1)x^{2n-1}} = \frac{x^3}{D^2} \cdot \frac{J_{2m}}{J_{2m+2}};$$

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{D^2 J_{2n+1}^2 - (4x^2+1)x^{2n-1}} = \frac{x^3}{D^2 u^2(x)};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n+1}^2 - 1} = \frac{3-\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{9J_{2n+1}^2 - 17 \cdot 2^{2n-1}} = \frac{2}{9}.$$

Next we investigate the Jacobsthal counterpart of identity (4)

3.4 Jacobsthal Version of Identity (4): Let $A = \dfrac{\Delta^2 x}{l_{2n+1}^2 + \Delta^2}$, and $B = \dfrac{l_{2m+3}}{l_{2m+2}} - \dfrac{x}{2}$. Replacing x with $1/\sqrt{x}$ in A, and then multiplying the numerator and denominator of the resulting expression with x^{2n}, yields

$$\text{LHS} = \sum_{n=0}^{m} \frac{D^2}{\sqrt{x}(xl_{2n+1}^2 + D^2)}$$

$$= \sum_{n=0}^{m} \frac{D^2 x^{2n}}{\sqrt{x}\left\{\left[x^{(2n+1)/2} l_{n+1}^2\right]^2 + D^2 x^{2n}\right\}}$$

$$= \sum_{n=0}^{m} \frac{D^2 x^{2n}}{\sqrt{x}(j_{2n+1}^2 + D^2 x^{2n})},$$

where $f_n = f_n(1/\sqrt{x})$ and $j_n = j_n(x)$.

Now replace x with $1/\sqrt{x}$ in B, and then multiplying the numerator and denominator of the resulting expression with $x^{(2m+3)/2}$. This gives

$$\mathrm{RHS} = \frac{x^{(2m+3)/2} l_{2m+3}}{\sqrt{x}\left[x^{(2m+2)/2} l_{2m+2}\right]} - \frac{x^{(2m+3)/2}}{2\sqrt{x}\cdot x^{(2m+3)/2}}$$

$$= \frac{j_{2m+3}}{\sqrt{x}\, j_{2m+2}} - \frac{1}{2\sqrt{x}},$$

where $l_n = l_n(1/\sqrt{x})$ and $j_n = j_n(x)$.

Equating the two sides yields

$$\sum_{n=0}^{m} \frac{D^2 x^{2n}}{j_{2n+1}^2 + D^2 x^{2n}} = \frac{j_{2m+3}}{j_{2m+2}} - \frac{1}{2}. \tag{10}$$

This yields

$$\sum_{n=0}^{m} \frac{5}{L_{2n+1}^2 + 5} = \frac{L_{2m+3}}{L_{2m+2}} - \frac{1}{2};$$

$$\sum_{n=0}^{\infty} \frac{D^2 x^{2n}}{j_{2n+1}^2 + D^2 x^{2n}} = u(x) - \frac{1}{2};$$

$$\sum_{n=0}^{\infty} \frac{5}{L_{2n+1}^2 + 5} = \frac{\sqrt{5}}{2};$$

$$\sum_{n=0}^{\infty} \frac{9\cdot 2^{2n}}{j_{2n+1}^2 + 9\cdot 2^{2n}} = \frac{3}{2}.$$

Using the identity $j_n^2 - D^2 J_n^2 = 4(-x)^n$, it also implies that

$$\sum_{n=0}^{m} \frac{D^2 x^{2n}}{D^2 J_{2n+1}^2 + x^{2n}} = \frac{j_{2m+3}}{j_{2m+2}} - \frac{1}{2};$$

$$\sum_{n=0}^{\infty} \frac{D^2 x^{2n}}{D^2 J_{2n+1}^2 + x^{2n}} = u(x) - \frac{1}{2};$$

$$\sum_{n=0}^{\infty} \frac{5}{5F_{2n+1}^2 + 1} = \frac{\sqrt{5}}{2};$$

$$\sum_{n=0}^{\infty} \frac{9 \cdot 2^{2n}}{9J_{2n+1}^2 + 2^{2n}} = \frac{3}{2}.$$

Next we investigate the Jacobsthal counterpart of identity (5)

3.5 Jacobsthal Version of Identity (5): Let $A = \dfrac{x^2}{l_{2n}^2 - \Delta^2}$, and $B = \dfrac{f_{2m}}{l_{2m+1}}$.

Replacing x with $1/\sqrt{x}$ in A, and then multiplying the numerator and denominator of the resulting expression with x^{2n-1}, we get

$$\text{LHS} = \sum_{n=1}^{m} \frac{1}{x l_{2n}^2 - D^2}$$

$$= \sum_{n=1}^{m} \frac{x^{2n-1}}{\left[x^{2n/2} l_{2n}\right]^2 - D^2 x^{2n-1}}$$

$$= \sum_{n=1}^{m} \frac{x^{2n-1}}{j_{2n}^2 - D^2 x^{2n-1}},$$

where $l_n = l_n(1/\sqrt{x})$ and $j_n = j_n(x)$.

Replacing x with $1/\sqrt{x}$ in B, and then multiplying the numerator and denominator of the resulting expression with $x^{(2m+1)/2}$, yields

$$\text{RHS} = \frac{x\left[x^{(2m-1)/2}\right]f_{2m}}{x^{(2m+1)/2}l_{2m+1}}$$

$$= \frac{xJ_{2m}}{j_{2m+1}},$$

where $l_n = l_n(1/\sqrt{x})$ and $j_n = j_n(x)$

Equating the two sides, we then get

$$\sum_{n=1}^{m} \frac{x^{2n-1}}{j_{2n}^2 - D^2x^{2n-1}} = \frac{xJ_{2m}}{j_{2m+1}}. \tag{11}$$

It then follows that

$$\sum_{n=1}^{m} \frac{1}{L_{2n}^2 - 5} = \frac{F_{2m}}{L_{2m+1}};$$

$$\sum_{n=1}^{\infty} \frac{x^{2n-1}}{j_{2n}^2 - D^2x^{2n-1}} = \frac{x}{Du(x)};$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n}^2 - 5} = \frac{1}{2} - \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{2^{2n-1}}{j_{2n}^2 - 9 \cdot 2^{2n-1}} = \frac{1}{9}.$$

Using the identity $j_n^2 - D^2J_n^2 = 4(-x)^n$, it also implies that

$$\sum_{n=1}^{m} \frac{x^{2n-1}}{D^2J_{2n}^2 - x^{2n-1}} = \frac{xJ_{2m}}{j_{2m+1}};$$

$$\sum_{n=1}^{\infty} \frac{x^{2n-1}}{D^2 J_{2n}^2 - x^{2n-1}} = \frac{x}{Du(x)};$$

$$\sum_{n=1}^{\infty} \frac{1}{5F_{2n}^2 - 1} = \frac{1}{2} - \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{2^{2n-1}}{9J_{2n}^2 - 2^{2n-1}} = \frac{1}{9}.$$

Finally, we explore the Jacobsthal counterpart of identity (6)

3.6 Jacobsthal Version of Identity (6): The desired counterpart of identity (6) is

$$\sum_{n=0}^{m} \frac{D^2(-x)^n}{j_{2n+1} + (-x)^n} = \frac{j_{m+2}}{j_{m+1}} - \frac{1}{2}. \tag{12}$$

Consequently,

$$\sum_{n=0}^{\infty} \frac{D^2(-x)^n}{j_{2n+1} + (-x)^n} = u(x) - \frac{1}{2}.$$

In the interest of brevity, we omit the basic algebra and the consequences of formula (12).

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COMMON FIXED POINT THEOREM FOR OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS SATISFYING INTEGRAL TYPE INEQUALITY IN SYMMETRIC SPACES

Abstract: In this paper, a common fixed point theorem for three pair of self and occasionally weakly compatible mappings by using contractive condition of integral type in symmetric space is established which generalize and improves several known results.

Keywords: Fixed Point, Symmetric Spaces, Occasionally Weakly Compatible Mappings

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1. Introduction

Sessa [11], initiated the tradition of improving commutativity conditions in metrical common fixed point theorems. While doing so Sessa [11] introduced the notion of weak commutativity. Motivated by Sessa [11], Jungck [6] defined the concept of compatibility of two mappings, which includes weakly commuting mappings as a proper subclass. After this definition there is a multitude of compatibility like conditions. Jungck and Rhoades [7] introduced the notion of weakly compatible (coincidentally commuting) mappings and showed that compatible mappings are weakly compatible but not conversely. Many interesting fixed point theorems for weakly compatible maps satisfying contractive type conditions have been obtained by various authors. Recently, Jungck and Rhoades [8] introduced occasionally weakly compatible mappings, which is more general among the commutativity concepts. Several authors have obtained several common fixed point theorems by using the idea of occasionally weakly compatible mappings.

Fixed point theory in symmetric (semi-metric) space is one of the emerging areas of interdisciplinary mathematical research. Menger [9] introduced symmetric (semi-metric) space as a generalization of metric space. Cicchese [2] introduced the notion of a contractive mapping in symmetric (semi-metric) space and proved the first fixed point theorem for this class of spaces. Hicks and Rhoades [4] generalized Banach contraction principle in symmetric (semi-metric) space. Jha *et al*. [5] proved a common fixed point theorem for three pairs of self-mappings using occasionally weakly compatible mappings.

Branciari [1] introduced the notion of contraction of integral type and proved first fixed point theorem for this class of mapping. Goyal and Jaiswal [3] proved some common fixed point theorems for compatible mappings in symmetric space satisfying a general contractive condition of integral type. In this paper, we prove a fixed point common theorem using the idea of occasionally weakly compatible mappings in symmetric space satisfying a general contractive condition of integral type.

2. Basic Definition

We recall that a symmetric on a set X is a non negative real valued function d on $X \times X$ such that

(i) $d(x, y) = 0$ if and only if $x = y$, for $x, y \in X$

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$

Let d be a symmetric on a set X and for $r > 0$ and any $x \in X$, let $B(x, r) = \{y \in X : d(x, y) < r\}$. A topology $t(d)$ on X is given by $U \in t(d)$ if and only if for each $x \in U$, $B(x, r) \subset U$ for some $r > 0$. A symmetric d is a symmetric if for each $x \in X$ and each $r > 0$, $B(x, r)$ is a neighbourhood of x in the topology $t(d)$. Note that $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology $t(d)$.

The following two axioms were given by Wilson [12]. Let (X, d) be a symmetric space.

(W3) Given $\{x_n\}$, x and y in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ implies $x = y$.

(W4) Given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ implies that $\lim_{n\to\infty} d(y_n, x) = 0$.

It is easy to see that for a symmetric d, if $t(d)$ is a Hausdorff, then (W3) holds.

In the sequel, we need a function $F^* = \{\varphi : R_+ \to R_+\}$ such that φ is a Lebesgue integrable mapping which is summable, non-negative and satisfy $\int_0^{\varepsilon} \varphi(t)dt > 0$ for all $\varepsilon > 0$ and ϕ will be a function defined by, $\phi : R_+ \to R_+$ such that $0 < \phi(t) < t$ for all $t > 0$.

Definition 2.1: Let S and T be two self mappings of a symmetric space (X, d). S and T are said to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} d(Sx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = 0$ for some $t \in X$.

Definition 2.2: Let X be a non-empty set and $S, T: X \to X$ be an arbitrary mapping. If $w = Sx = Tx$ for some x in X, then x is called a coincidence point of S and T and w is called a point of coincidence of S and T.

Definition 2.3: Two self mappings S and T of a symmetric space (X, d) are said to be weakly compatible if they commute at their coincidence points.

Definition 2.4: Let S and T be two self mappings of a symmetric space (X, d).Then S and T are said to be occasionally weakly compatible (owc) if there is a point $x \in X$ which is coincidence point of S and T at which S and T commute. It is important to note that weakly compatible mappings are occasionally weakly compatible mappings but not the converse.

Example 2.5: Let $X = [0, \infty)$ with the usual metric. Define $S, T: X \to X$ by $S(x) = 2$ and $T(x) = x^2$ for all $x \in X$.

Then,

$$C(S,T) = \{x \in X: S(x) = T(x)\} = \{0,2\},$$

$$S(T(0) = T\big(S(0)\big) \text{ and } S(T(2) \neq T\big(S(2)\big).$$

Thus, (S, T) is a occasionally weakly compatible pair but not weakly compatible.

Example 2.6: Consider $x = [2, 20]$ with the symmetric space (X, d) defined by $d(x, y) = (x - y)^2$.

Let $S, T: X \to X$ be maps defined by

$S(2) = 2$ at $x = 2$ and $S(x) = 6$ for $x > 2$

$T(2) = 2$ at $x = 2$ and $T(x) = 12$ for $2 < x \leq 5$ and $T(x) = x - 3$ for $x > 5$.

Again, $S(9) = T(9) = 6$, therefore, $x = 9$ is another coincidence point of S and T except $x = 2$.

$ST(2) = TS(2)$ but $ST(9) = 6, TS(9) = 3, ST(9) \neq TS(9)$

Hence, S and T are occasionally weakly compatible but not weakly compatible. Therefore, weakly compatible mappings are occasionally weakly compatible but converse is not true.

Lemma 2.7: Let (X, d) be a symmetric space. If the self mappings S and T on *X* have a unique point of coincidence $w = Sx = Tx$, then *w* is the unique common fixed point of *S* and *T*.

Definition 2.8: A point $x \in X$ is called a commuting pair of S and T if

$$STx = TSx.$$

3. Main Result

Theorem 3.1: Let (X, d) be a symmetric space. Let A, B, S, T, I and J be self mappings of X such that

(i) (AB, I) and (ST, J) are occasionally weakly compatible (OWC) (1)

(ii) $\int_0^{d(ABx,STy)} \emptyset(t)dt \leq F(max\{\int_0^{m_1(x,y)} \emptyset(t)dt, \int_0^{m_2(x,y)} \emptyset(t)dt,$
$\int_0^{m_3(x,y)} \emptyset(t)dt\})$ (2)

for all $(x, y) \in X \times X$, where

$$m_1(x, y) = d(Ix, Jy),$$

$$m_2(x,y) = \frac{1}{2}[d(ABx, Ix) + d(STy, Jy)],$$

$$m_3(x,y) = \frac{1}{2}[d(ABx, Jy) + d(STy, Ix)]$$

$\emptyset: R^+ \to R^+$ is a Lebesgue integrable mapping which is summable, non-negative and such that $\int_0^{\epsilon} \emptyset(t)dt > 0$ for all $\epsilon > 0$.

$F: R^+ \to R^+$ be a function such that

(a) F is non-decreasing on R^+

(b) $0 < F(t) < t$ for all $t \in (0, \infty)$

(c) $F(0) = 0$

Then AB, ST, I and J have a unique common fixed point.

Furthermore, if the pairs (A, B) and (S, T) are commuting pair of mappings then A, B, S, T, I and J have a unique common fixed point in X.

Proof: By (1), (AB,I) and (ST, J) are occasionally weakly compatible, then there exists $x, y \in X$ which is a coincidence point of (AB, I) and (ST, J) at which (AB, I) and (ST, J) commute i.e. ABx = Ix and Sty = Jy.

We show that ABx = STy.

Using condition (2), we get

$$\int_0^{d(ABx,STy)} \emptyset(t)dt \le F(max\{\int_0^{m_1(x,y)} \emptyset(t)dt, \int_0^{m_2(x,y)} \emptyset(t)dt, \int_0^{m_3(x,y)} \emptyset(t)dt\})$$

where,

$$m_1(x,y) = d(Ix, Jy)$$

$$m_2(x,y) = \frac{1}{2}[d(ABx, Ix) + d(STy, Jy)]$$

$$m_3(x,y) = \frac{1}{2}[d(ABx, Jy) + d(STy, Ix)]$$

Since, $ABx = Ix$ and $STy = Jy$, therefore,

$$m_1(x,y) = d(ABx, STy)$$

$$m_2(x,y) = \frac{1}{2}[d(ABx,ABx) + d(STy,STy)] = 0$$

$$m_3(x,y) = \frac{1}{2}[d(ABx,STy) + d(ABx,STy)]$$

$$\therefore \int_0^{d(ABx,STy)} \emptyset(t)dt \ \le F\left(max\left\{\int_0^{d(ABx,STy)} \emptyset(t)dt\, , 0, \int_0^{d(ABx,STy)} \emptyset(t)dt\right\}\right)$$

$$= F\left(\int_0^{d(ABx,STy)} \emptyset(t)\, dt\right)$$

or, $\int_0^{d(ABx,STy)} \emptyset(t)\, dt < \int_0^{d(ABx,STy)} \emptyset(t)\, dt$, which is a contradiction.

Hence, $\int_0^{d(ABx,STy)} \emptyset(t)\, dt = 0$ [by def. of '$\emptyset$']

or, $d(ABx,STy) = 0$

or, $ABx = STy$

Therefore, $ABx = Ix = STy = Jy$ (3)

Again, if there is another point of coincidence z such that $ABz = Iz$, then on using condition (2), we get,

$$ABz = Iz = STy = Jy \tag{4}$$

Now, from (3) and (4), it follows that $ABz = ABx$. This implies that $z = x$.

Therefore, $u = ABx = Ix$, for $u \in X$ is the unique point of coincidence of *AB* and *I*.

By using Lemma (2.7), *u* is a unique common fixed point of *AB* and *I*.

Hence, $ABu = Iu = u$.

Similarly, there is a unique common fixed point $w \in X$ such that $w = STw = Jw$.

Suppose that $u \neq w$ then on using condition (2), we get

$$\int_0^{d(u,w)} \emptyset(t)dt = \int_0^{d(ABu,STw)} \emptyset(t)dt$$

$$\le F\left(max\left\{\int_0^{m_1(u,w)} \emptyset(t)dt\, , \int_0^{m_2(u,w)} \emptyset(t)dt \int_0^{m_3(u,w)} \emptyset(t)dt\right\}\right)$$

where

$$m_1(u,w) = d(Iu,Jw) = d(u,w)$$

$$m_2(u,w) = \frac{1}{2}[d(ABu,Iu) + d(STw,Jw)]$$

$$= \frac{1}{2}[d(u,u) + d(w,w)] = 0$$

$$m_3(u,w) = \frac{1}{2}[d(ABu,Jw) + d(STw,Iu)] = d(u,w)$$

$$\therefore \quad \int_0^{d(u,w)} \emptyset(t)dt \ \le F\left(max\left\{\int_0^{d(u,w)} \emptyset(t)dt\,, 0, \int_0^{d(u,w)} \emptyset(t)dt\right\}\right)$$

$$= F(\int_0^{d(u,w)} \emptyset(t)dt)$$

or, $\int_0^{d(u,w)} \emptyset(t)dt < \int_0^{d(u,w)} \emptyset(t)dt$, which is a contradiction.

Hence, we get $\int_0^{d(u,w)} \emptyset(t)dt = 0$ [by def. of $'\emptyset'$]

or, $$d(u,w) = 0$$

or, $$u = w.$$

Therefore, u is the unique common fixed point of AB, ST, I and J.

It remains only to show that u is only the common fixed point of mappings A, B, S, T, I and J. If the pairs (A, B) and (S, T) are commuting pairs then, we get

$$Au = A(ABu) = A(BAu) = AB(Au)$$

This gives $Au = u.$

Again, $Bu = B(ABu) = BA(Bu) = AB(Bu).$

This gives $Bu = u.$

Similarly, we get $Su = u$ and $Tu = u.$

Consequently, $Au = Bu = Iu = Ju = Su = Tu = u.$ Hence, A, B, S, T, I and J have a unique common fixed point.

It should be noted that our theorem (3.1) generalized the result of Jha *et al*. [5], Jungck and Rhoades [8] and Pant and Chauhan [10].

On the basis of theorem (3.1), we get the following result.

Corollary 3.2: Let (X, d) be a symmetric space. Let A, B, S, T, I and J be self mappings of X such that

(i) $\{AB, I\}$ and $\{ST, J\}$ are occasionally weakly compatible (OWC).

(ii) $$\int_0^{d(ABx,STy)} \emptyset(t)dt \leq F\left(max\left\{\int_0^{m_1(x,y)} \emptyset(t)dt, \int_0^{m_2(x,y)} \emptyset(t)dt, \int_0^{m_3(x,y)} \emptyset(t)dt, \int_0^{m_4(x,y)} \emptyset(t)dt\right\}\right)$$

where $$m_1(x,y) = d(Ix, Jy)$$

$$m_2(x,y) = d(ABx, Jy)$$

$$m_3(x,y) = d(STy, Ix)$$

and $$m_4(x,y) = \frac{1}{2}[d(ABx, Ix) + d(STy, Jy)]$$

for all $(x, y) \in X \times X$, then AB,ST, I and J have a unique common fixed point.

Furthermore, if the pairs (A, B) and (S, T) are commuting pair of mappings then A, B, S, T, I and J have a unique common fixed point in X.

The proof of the above corollary (3.2) is same as that of theorem (3.1).

In the above theorem (3.1), if we take $AB = A$ and $ST = S$, then we have the following corollary which generalizes the results of Jungck and Rhoades [8] to integral type inequality.

Corollary 3.3: Let (X, d) be a symmetric space. Let A, S, I and J be self mappings of X such that

(i) $\{A, I\}$ and $\{S, J\}$ are occasionally weakly compatible (OWC).

(ii) $\int_0^{d(Ax,Sy)} \emptyset(t)dt \leq F(max\{\int_0^{m_1(x,y)} \emptyset(t)dt, \int_0^{m_2(x,y)} \emptyset(t)dt, \int_0^{m_3(x,y)} \emptyset(t)dt\})$

for all $(x, y) \in X \times X$ where

$m_1(x,y) = d(Ix, Jy)$

$m_2(x,y) = \frac{1}{2}[d(Ax, Ix) + d(Sy, Jy)]$

$m_3(x,y) = \frac{1}{2}[d(Ax, Jy) + d(Sy, Ix)]$

then A, S, I and J have a unique common fixed point in *X*.

Our next corollary is obtained by putting $AB = J = A$ and $ST = I = S$ in theorem (3.1).

Corollary 3.4: Let (X, d) be a symmetric space. Let A and S be self mappings of X such that

(i) *A* and S are occasionally weakly compatible (OWC).

(ii) $\int_0^{d(Ax,Sy)} \emptyset(t)dt \leq F(max\{\int_0^{m_1(x,y)} \emptyset(t)dt, \int_0^{m_2(x,y)} \emptyset(t)dt, \int_0^{m_3(x,y)} \emptyset(t)dt\})$

(iii) where,

$m_1(x,y) = d(Sy, Ay)$

$m_2(x,y) = \frac{1}{2}[d(Ax, Sx) + d(Sy, Ay)]$

$m_3(x,y) = \frac{1}{2}[d(Ax, Ay) + d(Sy, Sx)]$

for all $(x, y) \in X \times X$.

then A and S have a unique common fixed point in *X*.

Now we give an example to demonstrate the validity of our theorem (3.1) and corollary (3.2).

Example 3.5: Let $X = [0, 1]$ with symmetric space (X, d) defined by $d(x, y) = (x - y)^2$. Define self mappings A, B, S, T, I and J by

$$Ax = \frac{x+1}{2}, Bx = \frac{2+3x}{5}, Sx = \frac{2x+1}{3}, Tx = \frac{x+3}{4}, Ix = \frac{3x+1}{4}, Jx = \frac{2x+3}{5}$$

Then the mappings satisfy all the conditions of corollary (3.2) and theorem (3.1) and hence have a unique common fixed point at $x = 1$.

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RELATIVE ASYMPTOTIC REGULARITY AND COMMON FIXED POINT IN 2-METRIC SPACES

Abstract: Many authors have extended Banach fixed point theorem by introducing more general contractive conditions, which implies the existence of a fixed point. Almost all of the conditions imply the asymptotic regularity of the mappings under consideration. So, the investigation of asymptotically regular maps plays an important role in fixed point theory. In this paper, we obtain a common fixed point theorem in complete 2-metric space using the concept of relative asymptotic regularity at a point for two pairs of weakly commuting mappings. Our work generalizes, extend and improve several results.

Keywords: Fixed Point, Complete 2-Metric Spaces, Weakly Commuting Mappings, Relative Asymptotic Regularity.

Mathematical Subject Classification (2010) No.: 54H25, 47H10.

1. Introduction

The concept of **2-metric spaces** has been investigated initially by Gahler [4]. This concept was subsequently enhanced by Gahler ([5], [6]), White [20] and several others. On the other hand Guay and Singh [10], Sharma and Yuel [16], Ciric [3] and a number of other authors have studied the aspects of fixed point theory in the setting of 2-metric space.

Rhoades *et al*. [14] introduced the concept of relative asymptotic regularity for a pair of mapping on a metric space and Jungck [11] proposed the concept of compatible mappings and weakly commuting mappings. Sessa [15] and others used both cited concepts and gave many interesting results.

Singh and Virendra [19] have proved a common fixed point theorem for three weakly commuting mappings by using the concept of relative asymptotic regularity of a sequence in 2-metric spaces. Nesic [12] gave a general result about fixed points for asymptotic regular mappings on complete metric spaces. Gosain and Goyal [7], Singh *et al*. [17], Baskaran and Rajesh [1] and Rajesh and Baskaran [13] proved some results of fixed point theorems for asymptotically regular mappings on complete 2-metric spaces with some Nesic [12] type contractive condition. Goyal ([8], [9]), obtained some common fixed point theorems in complete 2-metric spaces by using the concept of relative asymptotically regularity at a point for weakly commuting and compatible mappings. They have been motivated by various concepts already known for metric spaces and have thus introduced analogous of various concepts in the frame-work of the 2-metric space.

In this paper, we establish a common fixed point theorem in complete 2-metric spaces using the concept of relative asymptotical regularity at a point for weakly commuting mappings.

2. Basic Definitions

Following Gahler ([4], [5]) and White [20], we have the following definitions:

Definition 2.1: Let X be a set consisting of atleast three points. A **2-metric** on X is a real valued function $d : X \times X \times X \to R^+$, which satisfies the following conditions:

(i) To each pair of distinct points x, y in X, there exists a point z in X such that $d(x, y, z) \neq 0$,

(ii) $d(x, y, z) = 0$, when at least two of x, y, z are equal;

(iii) $d(x, y, z) = d(y, z, x) = d(x, z, y)$ for all x, y, z in X,

(iv) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$, for all x, y, z, w in X.

The pair (X, d) is called a **2-metric space**.

Definition 2.2: A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be **convergent** with limit x in X if $\mathrm{Lim}_{n\to\infty} d(x_n, x, a) = 0$ for all a in X.

Definition 2.3: A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be a **Cauchy sequence** if

$$\text{Lim}_{m,n\to\infty} d(x_m, x_n, a) = 0 \text{ for all a in } X.$$

Definition 2.4: A 2-metric space (X, d) is said to be **complete** if every Cauchy sequence in X is convergent.

Remark 2.5: In a complete 2-metric space a convergent sequence need not be a Cauchy sequence. It has been illustrated by the following example:

Example 2.6: Let $X = \left\{0,1,\frac{1}{2},\frac{1}{3},\dots\right\}$

Define $d : X \times X \times X \to [0,\infty)$ as

$$d(x,y,z) = \begin{cases} 1, & \text{if } x,y,z \text{ are distinct and } \left\{\frac{1}{n},\frac{1}{n+1}\right\} \subset \{x,y,z\} \text{for some positie integer } n \\ 0, & \text{otherwise} \end{cases}$$

Then it can be shown that (X, d) is a complete 2-metric space, the sequence $\left\{\frac{1}{n}\right\}$ converges to zero. But $\left\{\frac{1}{n}\right\}$ is not Cauchy sequence.

Motivated from Browder and Petryshyn [2], we have

Definition 2.7: Let (X, d) be a 2 metric space and T be a self mapping on X. Then T is said to be **asymptotically regular** at a point $x \in X$ if

$$\text{Lim}_{n\to\infty}\ d\left(T^n x, T^{n+1}x, a\right) = 0 \text{ for all } a \in X .$$

Sessa [15] introduced the notion of weakly commuting mapping as follows:

Definition 2.8: Let (X, d) be a 2-metric space, and S, T be self mappings of X. Then (S, T) is said to be **weakly commuting** pair if $d(STx, TSx, a) \le d(Sx, Tx, a)$ for all x, a in X.

Remark 2.9: A commuting pair of self maps on a 2-metric space is weakly commutative. The following example shows that the converse need not be true.

Example 2.10: Let $X=\{1,2,3,4\}$ be a set. Define $d: X\times X\times X\to R^{+}$ as

$$d(x,y,z)=\begin{cases}0, & \text{if } x=y \text{ or } y=z \text{ or } z=x \text{ or } \{x,y,z\}=\{1,2,3\}\\ \frac{1}{2}, & \text{otherwise}\end{cases}$$

Define $S,T: X\to X$ as $S1=S2=S3=S4=2$ and $T1=T2=T3=T4=3$

Thus, (X, d) is a 2-metric space and S, T commute weakly on X but they do not commute on X.

Rhoades *et al*. [14] introduced the concept of asymptotic regularity for pair of mapping as follows:

Definition 2.11: Let S and T be self mappings of a 2-metric space (X, d). Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called **asymptotically regular** with respect to pair (S, T) if

$$\mathrm{Lim}_{n\to\infty} d\left(Sx_n, Tx_n, a\right)=0 \text{ for all } a \text{ in } X.$$

Definition 2.12: Let B and T be two self mappings of a 2-metric space (X, d) then the sequence $\{x_n\}$of X is said to be **asymptotic T-regular** with respect to B if

$$\mathrm{Lim}_{n\to\infty} d\left(Bx_n, Tx_n, a\right)=0$$

Example 2.13: Let $X = [0, 2]$ define $d(x, y, z) = \mathrm{Min}\ \{|x-y|, |y-z|, |z-x|\}$

for all $x, y, z\in X$. Define self maps B and T on X.

$$Bx=\begin{cases}1, & \text{if } x\in[0,1)\\ 2, & \text{if } x=1\\ \frac{x+3}{5} & \text{if } x\in(1,2]\end{cases} \quad\text{and}\quad Tx=\begin{cases}2, & \text{if } x\in[0,1]\\ \frac{x}{2} & \text{if } x\in(1,2]\end{cases}$$

Take $x_n=2-\frac{1}{2n}$, then we have

$$B(1) = T(1) = 2$$

$$B(2) = T(2) = 1$$

Hence, $Tx_n \to 1$ and $Bx_n \to 1$

$$\mathrm{Lim}_{n\to\infty} d\left(Bx_n, Tx_n, a\right) = d(1,1,2) = 0.$$

Therefore, the sequence $\{x_n\}$ is T-regular with respect to B.

3. Main Results

Let R^+ be the set of non-negative reals and let $F_i: R^+ \to R^+$ be functions such that $F_i(0) = 0$ and F_i is continuous at $0 (i = 1, 2)$.

Now, we prove our main result which is motivated by the contractive condition studied by Ciric [3].

Theorem 3.1: Let A, B, S, T be self mappings of a complete 2-metric space (X, d) satisfying

(i) $$\begin{aligned} d(ABx, ABy, a) \le\ & a_1 d(Sx, ABx, a) + a_2 d(Tx, ABx, a) + a_3 d(Sy, ABy, a) \\ & + a_4 d(Ty, ABy, a) \\ & + F_1[min\{d(Sx, ABx, a). d(Sy, ABy, a), \\ & \qquad d(Tx, ABx, a). d(Ty, ABy, a)\}] \\ & + F_2[d(Sx, ABx, a). d(Sy, ABy, a) \\ & + d(Tx, ABx, a). d(Ty, ABy, a)] \end{aligned} \tag{1}$$

For all x, y, a in X, where a_1 is bounded and a_2, a_3, a_4 are non-negative numbers such that $a_2 + a_3 < 1, a_3 + a_4 < 1$.

(ii) (AB, S) and (AB, T) are weakly commuting pairs (2)

(iii) There exists a sequence $\{x_n\}$ which is asymptotically regular with respect to $(AB, S), (AB, T)$ and (S, T); (3)

(iv) S and T are continuous. (4)

If d is continuous the AB, S and T have a unique common fixed point.

Furthermore, if the pairs (A, B), (A, S), (B, S), (A, T) and (B, T) are commuting mappings then A, B, S and T have a unique common fixed point.

Proof: Let $\{x_n\}$ be a sequence in X which satisfy conditions (3) then on using (1), we have

$$d(ABx_n, ABx_m, a) \leq a_1 d(Sx_n, ABx_n, a) + a_2 d(Tx_n, ABx_n, a)$$
$$+a_3 d(Sx_m, ABx_m, a) + a_4 d(Tx_m, ABx_m, a)$$
$$+F_1[min\{d(Sx_n, ABx_n, a). d(Sx_m, ABx_m, a),$$
$$d(Tx_n, ABx_n, a). d(Tx_m, ABx_m, a)\}]$$
$$+F_2[d(Sx_n, ABx_n, a). d(Sx_m, ABx_m, a)$$
$$+d(Tx_n, ABx_n, a). d(Tx_m, ABx_m, a)]$$

Letting $m, n \to \infty$ and using condition (3), we get

$$\lim_{m,n\to\infty} d(ABx_n, ABx_m, a) = 0 \text{ for all } a \text{ in } x.$$

Therefore $\{ABx_n\}$ is a Cauchy sequence and converges to some point z in X (as X is complete 2-metric space.

Again,

$$d(Sx_n, z, a) \leq d(Sx_n, ABx_n, a) + d(ABx_n, z, a) + d(Sx_n, ABx_n, z)$$
[using def. of 2-metric space]

Letting $n \to \infty$ and using (3), we have

$$\lim_{n\to\infty} d(Sx_n, z, a) \to 0$$

which shows that $Sx_n \to z$. Similarly, we can show that $Tx_n \to z$

Since, S and T are continuous, we get

$SABx_n \to Sz$, $S^2x_n = SSx_n \to Sz$, $STx_n \to Sz$,

$TABx_n \to Tz$, $T^2x_n = TTx_n \to Tz$, $TABx_n \to Tz$

Again, the pair (AB, S) are weakly commuting and the sequence $\{x_n\}$ is asymptotically regular with respect to (AB, S), then on using definition of 2-metric space, we have

$$d(ABSx_n, Sz, a) \leq d(ABSx_n, SABx_n, a) + d(SABx_n, Sz, a)$$
$$+d(ABSx_n, SABx_n, Sz)$$
$$\leq d(Sx_n, ABx_n, a) + d(SABx_n, Sz, a)$$
$$+d(Sx_n, ABx_n, Sz)$$
$$\to 0 \text{ as } n \to \infty$$

which shows that $ABSx_n \to Sz$.

Similarly, $ABTx_n \to Tz$.

On using (1), we have

$$d(ABSx_n, ABTx_n, a) \leq a_1 d(S^2x_n, ABSx_n, a) + a_2 d(TSx_n, ABSx_n, a)$$
$$+a_3 d(STx_n, ABTx_n, a) + a_4 d(T^2x_n, ABTx_n, a)$$
$$+F_1[min\{d(S^2x_n, ABSx_n, a).\, d(STx_n, ABTx_n, a),$$
$$d(TSx_n, ABSx_n, a).\, d(T^2x_n, ABTx_n, a)\}]$$
$$+F_2[d(S^2x_n, ABSx_n, a).\, d(STx_n, ABTx_n, a)$$
$$+d(TSx_n, ABSx_n, a).\, d(T^2x_n, ABTx_n, a)] \quad (5)$$

Again,

$$d(STx_n, TSx_n, a) \leq d(STx_n, ABSx_n, a) + d(STx_n, TSx_n, ABSx_n)$$
$$+d(ABSx_n, TSx_n, a)$$
$$\leq d(STx_n, ABSx_n, a) + d(STx_n, TSx_n, ABSx_n)$$
$$+d(ABSx_n, ABTx_n, a) + d(ABTx_n, TSx_n, a)$$
$$+d(ABSx_n, TSx_n, ABTx_n) \quad (6)$$

On using (5) in (6), we get

$$d(STx_n, TSx_n, a) \leq d(STx_n, ABSx_n, a) + d(STx_n, TSx_n, ABSx_n)$$
$$+a_1 d(S^2x_n, ABSx_n, a) + a_2 d(TSx_n, ABSx_n, a)$$
$$+a_3 d(STx_n, ABTx_n, a) + a_4 d(T^2x_n, ABTx_n, a)$$

$$+F_1[min\{d(S^2x_n, ABSx_n, a). d(STx_n, ABTx_n, a),$$

$$d(TSx_n, ABSx_n, a). d(T^2x_n, ABTx_n, a)\}]$$

$$+F_2[d(S^2x_n, ABSx_n, a). d(STx_n, ABTx_n, a),$$

$$d(TSx_n, ABSx_n, a). d(T^2x_n, ABTx_n, a)]$$

$$+d(ABTx_n, TSx_n, a) + d(ABSx_n, TSx_n, ABTx_n)$$

Letting $n \to \infty$, we get

$$d(Sz, Tz, a) \leq d(Sz, Sz, a) + d(Sz, Tz, Sz) + a_1 d(Sz, Sz, a)$$

$$+a_2 d(Tz, Sz, a) + a_3 d(Sz, Tz, a) + a_4 d(Tz, Tz, a)$$

$$+F_1[min\{d(Sz, Sz, a). d(Sz, Tz, a),$$

$$d(Tz, Sz, a). d(Tz, Tz, a)\}]$$

$$+F_2[d(Sz, Sz, a). d(Sz, Tz, a)$$

$$+d(Tz, Sz, a). d(Tz, Tz, a)]$$

$$+d(Tz, Tz, a) + d(Sz, Tz, Tz)$$

or, $$d(Sz, Tz, a) \leq (a_2 + a_3 + a_5) d(Sz, Tz, a)$$

which is a contradiction.

As $(a_2 + a_3) < 1$, it follows that $d(Sz, Tz, a) = 0$, this means that $Sz = Tz$ (a being arbitrary).

Again from (1), we get

$$d(ABTx_n, ABz, a) \leq a_1 d(STx_n, ABTx_n, a) + a_2 d(T^2x_n, ABTx_n, a)$$

$$+a_3 d(Sz, ABz, a) + a_4 d(Tz, ABz, a)$$

$$+F_1[min\{d(STx_n, ABTx_n, a). d(Sz, ABz, a),$$

$$d(T^2x_n, ABTx_n, a). d(Tz, ABz, a)\}]$$

$$+F_2[d(STx_n, ABTx_n, a). d(Sz, ABz, a)$$

$$+d(T^2x_n, ABTx_n, a). d(Tz, ABz, a)]$$

Letting $n \to \infty$, we get

$$d(Tz, ABz, a) \le a_1 d(Sz, Tz, a) + a_2 d(Tz, Tz, a) + a_3 d(Sz, ABz, a)$$
$$+a_4 d(Tz, ABz, a) + a_5 d(Sz, Tz, a)$$
$$+F_1[min\{d(Sz, Tz, a). d(Sz, ABz, a),$$
$$d(Tz, Tz, a). d(Tz, ABz, a)\}]$$
$$+F_2[d(Sz, Tz, a). d(Sz, ABz, a)$$
$$+d(Tz, Tz, a). d(Tz, ABz, a)]$$

which gives,

$$d(Tz, ABz, a) \le (a_3 + a_4) d(Tz, ABz, a) \quad [\because Sz = Tz]$$

which is a contradiction as $(a_3 + a_4) < 1$, yielding thereby $ABz = Tz$.

This gives $Sz = Tz = ABz$.

Again, from (1), we have

$$d(ABABz, ABz, a) \le a_1 d(SABz, ABABz, a) + a_2 d(TABz, ABABz, a)$$
$$+a_3 d(Sz, ABz, a) + a_4 d(Tz, ABz, a)$$
$$+F_1[min\{d(SABz, ABABz, a). d(Sz, ABz, a),$$
$$d(TABz, ABABz, a). d(Tz, ABz, a)\}]$$
$$+F_2[d(SABz, ABABz, a). d(Sz, ABz, a)$$
$$+d(TABz, ABABz, a). d(Tz, ABz, a)]$$
$$= a_1 d(SABz, ABABz, a) + a_2 d(TABz, ABABz, a)$$
$$\le a_1 d(Sz, ABz, a) + a_2 d(Tz, ABz, a)$$

[The pairs (AB, S) and (AB, T) are weakly commuting]

$$= 0 \quad \text{[From (2)]}$$

Hence, $ABABz = ABz$ i.e. ABz is a fixed point of AB.

Put $ABz = t$, then

$$ABSz = ABABz = ABz = t \text{ and}$$

$$d(St, t, a) = d(SABz, t, a)$$

$$\leq d(SABz, ABSz, a) + d(SABz, ABSz, t) + d(ABSz, t, a)$$

[By def. of 2-metric space]

$$= 0 \quad \text{[From (2)]}$$

Hence, $St = t$. Similarly, $Tt = t$.

So, $St = ABt = Tt = t$ i.e, t is a common fixed point of S, AB and T.

To prove the uniqueness of t, let u be another common fixed point of AB, S and T then from (1), we get

$$d(ABt, ABu, a) \leq a_1 d(St, ABt, a) + a_2 d(Tt, ABt, a)$$

$$+a_3 d(Su, ABu, a) + a_4 d(Tu, ABu, a)$$

$$+F_1(min\{d(St, ABt, a).\, d(Su, ABu, a),$$

$$d(Tt, ABt, a).\, d(Tu, ABu, a)\})$$

$$+F_2[d(St, ABt, a).\, d(Su, ABu, a)$$

$$+d(Tt, ABt, a).\, d(Tu, ABu, a)]$$

$$= 0$$

Hence, $t = u$.

Finally, we have to prove that t is also a common fixed point of A, B, S and T.

Now,

$$At = A(ABt) = A(BAt) = AB(At) \quad [\because AB = BA]$$

$$At = A(St) = S(At) \quad [\because \text{AS} = \text{SA}]$$

$$Bt = B(ABt) = B(A(Bt)) = BA(Bt) = AB(Bt) \quad [\because \text{AB} = \text{BA}]$$

$$Bt = B(St) = S(Bt) \quad [\because BS = SB]$$

which shows that (AB, S) has a common fixed points which are At and Bt giving thereby $At = t = Bt = St = ABt$, in the view of uniqueness of common fixed point of the pair (AB, S).

Similarly, using the commutativity of (A, B), (A, T) and (B, T), we get

$$At = A(ABt) = A(BAt) = AB(At)$$

$$At = A(Tt) = T(At)$$

$$Bt = B(ABt) = B\big(A(Bt)\big) = BA(Bt) = AB(Bt)$$

$$Bt = B(Tt) = T(Bt)$$

which shows that (AB, T) has a common fixed points which are At and Bt giving thereby $At = t = Bt = Tt = ABt$ in view of uniqueness of common fixed point of the pair (AB, T).

Consequently, $At = Bt = St = Tt = t$ and t is a unique common fixed point of A, B, S and T.

Our theorem (3.1) generalize the result of Goyal [8]

Corollary 3.2: Let A, B, S, T be self mappings of a complete 2-metric space (X, d) satisfying

(i) $d(ABx, ABy, a)$

$$\leq a_1 d(Sx, ABx, a) + a_2 d(Tx, ABx, a) + a_3 d(Sy, ABy, a)$$
$$+ a_4 d(Ty, ABy, a)$$
$$+ F_1[min\{d(Sx, ABx, a).\, d(Sy, ABy, a), d(Tx, ABx, a).\, d(Ty, ABy, a)\}]$$
$$+ F_2[d(Sx, ABx, a).\, d(Sy, ABy, a)$$
$$+ d(Tx, ABx, a).\, d(Ty, ABy, a)]$$

For all x, y, a in X, where a_1 is bounded and a_2, a_3, a_4 are non-negative numbers such that $a_2 + a_3 < 1, a_3 + a_4 < 1$.

(ii) (AB, S) and (AB, T) are weakly commuting pairs

(iii) There exists a sequence $\{x_n\}$ which is asymptotically regular with respect to $(AB, S), (AB, T)$ and (S, T);

(iv) S and T are continuous

If d is continuous the AB, S and T have a unique common fixed point.

Furthermore, if the pairs $(A, B), (A, S), (B, S), (A, T)$ and (B, T) are commuting mappings then, A, B, S and T have a unique common fixed point.

If we take $AB = A, S = B, F_1 = F$ and $F_2(t) = 0$ for $t \in R^+$ in corollary (3.2), then we get the following result of Goyal [8].

Corollary 3.3: Let A, B and T be self mappings of a complete 2-metric space (X, d) satisfying

(i) $$d(Ax, Ay, a) \leq a_1 d(Bx, Ax, a) + a_2 d(Tx, Ax, a) + a_3 d(By, Ay, a)$$
$$+a_4 d(Ty, Ay, a)$$
$$+F(min\{d(Bx, Ax, a). d(By, Ay, a), d(Tx, Ax, a). d(Ty, Ay, a)\})$$

for all x, y, a in X, where a_1, a_2, a_3 and a_4 are non-negative numbers such that $a_2 + a_3 < 1,\ a_3 + a_4 < 1$;

(ii) the pairs (A, B) and (A, T) are weakly commuting pairs;

(iii) there exists a sequence $\{x_n\}$ which is asymptotically regular with respect to (A, B) and (A, T);

(iv) B and T are continuous.

If d is continuous then, A, B and T have a unique common fixed point.

Our next corollary is obtained by putting $AB = A,\ S = B$ and $F_i(t) = 0$ for all $t \in R^+$ in corollary (3.5), which is a result of Singh and Virendra [18]

Corollary 3.7: Let A, B and T be self mappings of a complete 2-metric space (X, d) satisfying

(i) $$d(Ax, Ay, a) \leq a_1 d(Bx, Ax, a) + a_2 d(Tx, Ax, a) + a_3 d(By, Ty, a) + a_4 d(Ty, Ay, a)$$

for all x, y, a in X, where a_1, a_2, a_3 and a_4 are non-negative numbers such that $a_2 + a_3 < 1,\ a_3 + a_4 < 1$;

(ii) the pairs (A, B) and (A, T) are weakly commuting pairs;

(iii) there exists a sequence $\{x_n\}$ which is asymptotically regular with respect to (A, B) and (A, T) both;

(iv) B and T are continuous.

If d is continuous then, A, B and T have a unique common fixed point.

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A FAMILY OF SUMS OF GIBONACCI POLYNOMIAL PRODUCTS OF ORDER 6

Abstract: We explore twelve sums of gibonacci polynomial products of order 6, involving g6n+k, where gn denotes the nth gibonacci polynomial and 0 ≤ k ≤ 5.

Keywords: Extended Gibonacci Polynomials, Fibonacci Number, Lucas Number.

Mathematical Subject Classification (2020) No.: Primary 05C20, 05C22, 11B39, 11B83, 11C08.

1. Introduction

Extended Gibonacci polynomials $z_n(x)$ are defined by the recurrence, $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$ where x is an arbitrary complex variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary complex polynominals; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*.

Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2, 3].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, and omit a lot of basic algebra.

A *gibonacci polynomial product of order* m is a product of gibonacci polynomials g_{n+k} of the form $\prod_{k\in\mathcal{Z}} g_{n+k}^{sj}$, where $\sum_{s_j\geq 1} s_j = m$ [4, 6].

Again, in the interest of clarity, conciseness, and convenience, we let

$$\begin{array}{lll}
A = f_{n+2}^6 & B = f_{n+2}^5 f_n & C = f_{n+2}^4 f_n^2 \\
D = f_{n+2}^4 f_n f_{n-2} & E = f_{n+2}^3 f_n^3 & F = f_{n+2}^3 f_n^2 f_{n-2} \\
G = f_{n+2}^3 f_n f_{n-2}^2 & I = f_{n+2}^2 f_n^3 f_{n-2} & J = f_{n+2}^2 f_n^2 f_{n-2}^2 \\
K = f_{n+2}^2 f_n f_{n-2}^3 & L = f_{n+2} f_n^5 & M = f_{n+2} f_n^4 f_{n-2} \\
N = f_{n+2} f_n^3 f_{n-2}^2 & O = f_{n+2} f_n^2 f_{n-2}^3 & P = f_n^6 \\
Q = f_n^5 f_{n-2} & R = f_n^4 f_{n-2}^2 & S = f_n^3 f_{n-2}^3 \\
T = f_n^2 f_{n-2}^4\,; & &
\end{array}$$

and a through t denote their numeric Fibonacci counterparts, respectively.

1.1 Sums of Fibonacci Polynomial Products of Order 4: Our discourse hinges on gibonacci recurrence, identities $f_{n+1} + f_{n-1} = l_n$, $f_{2n} = f_n l_n$, $f_{n+2} - f_{n-2} = x l_n$, $f_{n+2} + f_{n-2} = (x^2 + 2) f_n$, $f_{2n+1} = f_{n+1}^2 + f_n^2$, and the addition formula $f_{m+n} = f_{m+1} f_n + f_m f_{n-1}$ [3] and the following sums of Fibonacci polynomial products of order 4 [5]:

$$\begin{aligned}
x^4 f_{4n-1} = f_{n+2}^4 - 4(x^2+1) f_{n+2}^3 f_n + (4x^4 + 13x^2 + 6) f_{n+2}^2 f_n^2 \\
- (x^6 + 7x^4 + 10x^2 + 4) f_{n+2} f_n^3 - 2(x^4 + 2x^2) f_{n+2} f_n^2 f_{n-2} \\
+ (x^6 + 3x^4 + 2x^2 + 1) f_n^4 + (x^6 + 3x^4 + 2x^2) f_n^3 f_{n-2} + x^2 f_n^2 f_{n-2}^2 . \quad (1)
\end{aligned}$$

$$x^3 f_{4n} = f_{n+2}^3 f_n - 2f_{n+2}^2 f_n^2 - f_{n+2}^2 f_n f_{n-2} + 2(x^2+1) f_{n+2} f_n^3 + f_{n+2} f_n f_{n-2}^2$$
$$- 2(x^2+1) f_n^3 f_{n-2} + 2f_n^2 f_{n-2}^2 - f_n f_{n-2}^3 . \qquad (2)$$

$$x^4 f_{4n+1} = f_{n+2}^4 - 4f_{n+2}^3 f_n + 2(2x^2+3) f_{n+2}^2 f_n^2 - (x^4+6x^2+4) f_{n+2} f_n^3$$
$$- 2x^2 f_{n+2} f_n^2 f_{n-2} + (x^2+1)^2 f_n^4 + (x^4+2x^2) f_n^3 f_{n-2} . \qquad (3)$$

2. Sums of Fibonacci Polynomial Products of Order 6

With this background, we begin our explorations of sums of Fibonacci polynomial products of order 6 with $x^5 f_{6n}$.

2.1 A Fibonacci Sum for $x^5 f_{6n}$: By the Fibonacci addition formula, and identities (1) and (2), we have

$$\begin{aligned} f_{6n} &= f_{2n+1} f_{4n} + f_{2n} f_{4n-1} \\ x^5 f_{6n} &= x^3 f_{4n}\left[(x f_{n+1})^2 + x^2 f_n^2\right] + x^4 f_{4n-1} f_n (x l_n) \\ &= x^3 f_{4n}\left[(f_{n+2} - f_n)^2 + x^2 f_n^2\right] + x^4 f_{4n-1} f_n (f_{n+2} - f_{n-2}) \\ &= V + W , \end{aligned}$$

where

$$\begin{aligned} V = \big[& f_{n+2}^3 f_n - 2f_{n+2}^2 f_n^2 - f_{n+2}^2 f_n f_{n-2} + 2(x^2+1) f_{n+2} f_n^3 \\ & + f_{n+2} f_n f_{n-2}^2 - 2(x^2+1) f_n^3 f_{n-2} \\ & + 2f_n^2 f_{n-2}^2 - f_n f_{n-2}^3 \big]\left[f_{n+2}^2 - 2f_{n+2} f_n + (x^2+1) f_n^2\right] \end{aligned}$$

$$\begin{aligned} = B - 4C - D + (3x^2+7)E + 2F + G - 6(x^2+1)H \\ - 3(x^2+1)I - K + 2(x^2+1)^2 L + 4(x^2+1)M \\ + (x^2-3)N + 2O - 2(x^2+1)^2 Q + 2(x^2+1)R - (x^2+1)S ; \end{aligned}$$

$$\begin{aligned}W = \big[& f_{n+2}^4 - 4(x^2+1)f_{n+2}^3 f_n + (4x^4+13x^2+6)f_{n+2}^2 f_n^2 \\ & - (x^6+7x^4+10x^2+4)f_{n+2}f_n^3 - 2(x^4+2x^2)f_{n+2}f_n^2 f_{n-2} \\ & + (x^6+3x^4+2x^2+1)f_n^4 + (x^6+3x^4+2x^2)f_n^3 f_{n-2} \\ & + x^2 f_n^2 f_{n-2}^2\big](f_{n+2}-f_{n-2})f_n\end{aligned}$$

$$\begin{aligned} = B &- 4(x^2+1)C - D + (4x^4+13x^2+6)E + 4(x^2+1)F \\ & -(x^6+7x^4+10x^2+4)H - (6x^4+17x^2+6)I \\ & +(x^6+3x^4+2x^2+1)L + 2(x^6+5x^4+6x^2+2)M \\ & +(2x^4+5x^2)N - (x^6+3x^4+2x^2+1)Q \\ & -(x^6+3x^4+2x^2)R - x^2 S \ .\end{aligned}$$

Thus,

$$\begin{aligned}x^5 f_{6n} = 2B &- 4(x^2+2)C - 2D + (4x^4+16x^2+13)E \\ & + 2(2x^2+3)F + G - (x^6+7x^4+16x^2+10)H \\ & -(6x^4+20x^2+9)I - K + (x^6+5x^4+6x^2+3)L \\ & + 2(x^6+5x^4+8x^2+4)M + (2x^4+6x^2-3)N \\ & + 2O - (x^6+5x^4+6x^2+3)Q \\ & -(x^6+3x^4-2)R - (2x^2+1)S \ . \qquad (4)\end{aligned}$$

Next we investigate a sum for $x^6 f_{6n+1}$.

2.2 A Fibonacci Sum for $x^6 f_{6n+1}$**:** Using the addition formula, and identities (2) and (3), we get

$$f_{6n+1} = f_{2n+1}f_{4n+1} + f_{2n}f_{4n}$$

$$x^6 f_{6n+1} = x^4 f_{4n+1}\left[(xf_{n+1})^2 + x^2 f_n^2\right] + x^3 f_{4n}(x^2 f_n)(xl_n)$$

$$= x^4 f_{4n+1}\left[(f_{n+2} - f_n)^2 + x^2 f_n^2\right] + x^3 f_{4n}(x^2 f_n)(f_{n+2} - f_{n-2})$$
$$= X + Y\,,$$

where

$$\begin{aligned} X &= \left[f_{n+2}^4 - 4f_{n+2}^3 f_n + 2(2x^2+3)f_{n+2}^2 f_n^2 - (x^4+6x^2+4)f_{n+2}f_n^3 \right.\\ &\qquad -2x^2 f_{n+2}f_n^2 f_{n-2} + (x^2+1)^2 f_n^4 \\ &\qquad \left. +(x^4+2x^2)f_n^3 f_{n-2}\right]\left[f_{n+2}^2 - 2f_{n+2}f_n + (x^2+1)f_n^2\right] \\ &= A - 6B + 5(x^2+3)C - (x^4+18x^2+20)E \\ &\qquad -2x^2F + (7x^4+24x^2+15)H + (x^4+6x^2)I \\ &\qquad -(x^6+9x^4+14x^2+6)L - 2(2x^4+3x^2)M \\ &\qquad +(x^2+1)^3P + (x^6+3x^4+2x^2)Q; \end{aligned}$$

$$\begin{aligned} Y &= \left[f_{n+2}^3 f_n - 2f_{n+2}^2 f_n^2 - f_n^2 + 2f_n f_{n-2} + 2(x^2+1)f_{n+2}f_n^3 + f_{n+2}f_n f_{n-2}^2\right.\\ &\qquad \left. -2(x^2+1)f_n^3 f_{n-2} + 2f_n^2 f_{n-2}^2 - f_n f_{n-2}^3\right](x^2 f_n)(f_{n+2} - f_{n-2}) \\ &= x^2C - 2x^2E - 2x^2F + 2(x^4+x^2)H + 2x^2I \\ &\qquad +2x^2J - 4(x^4+x^2)M + 2x^2N - 2x^2O \\ &\qquad +2(x^4+x^2)R - 2x^2S + x^2T\,. \end{aligned}$$

Consequently,

$$\begin{aligned} x^6 f_{6n+1} &= A - 6B + 3(2x^2+5)C - (x^4+20x^2+20)E - 4x^2F \\ &\qquad +(9x^4+26x^2+15)H + (x^4+8x^2)I + 2x^2J \\ &\qquad -(x^6+9x^4+14x^2+6)L - 2(4x^4+5x^2)M + 2x^2N \\ &\qquad -2x^2O + (x^2+1)^3P + (x^6+3x^4+2x^2)Q \\ &\qquad +2(x^4+x^2)R - 2x^2S + x^2T\,. \end{aligned} \tag{5}$$

Next we explore a sum for $x^5 f_{6n+2}$.

2.3 A Fibonacci Sum for $x^5 f_{6n+2}$: Using the gibonacci recurrence, and equations (2) and (3), we get

$$\begin{aligned}
x^5 f_{6n+2} &= x^6 f_{6n+1} + x^5 f_{6n} \\
&= \big[A - 6B + 3(2x^2+5)C - (x^4+20x^2+20)E - 4x^2F \\
&\qquad + (9x^4+26x^2+15)H + (x^4+8x^2)I + 2x^2J \\
&\qquad - (x^6+9x^4+14x^2+6)L - 2(4x^4+5x^2)M + 2x^2N \\
&\qquad - 2x^2O + (x^2+1)^3P + (x^6+3x^4+2x^2)Q \\
&\qquad + 2(x^4+x^2)R - 2x^2S + x^2T\big] + \big[2B - 4(x^2+2)C \\
&\qquad - 2D + (4x^4+16x^2+13)E + 2(2x^2+3)F + G \\
&\qquad - (x^6+7x^4+16x^2+10)H - (6x^4+20x^2+9)I \\
&\qquad - K + (x^6+5x^4+6x^2+3)L + 2(x^6+5x^4+8x^2+4)M \\
&\qquad + (2x^4+6x^2-3)N + 2O - (x^6+5x^4+6x^2+3)Q \\
&\qquad - (x^6+3x^4-2)R - (2x^2+1)S\big] \\
&= A - 4B + (2x^2+7)C - 2D + (3x^4-4x^2-7)E + 6F + G \\
&\qquad - (x^6-2x^4-10x^2-5)H - (5x^4+12x^2+9)I \\
&\qquad + 2x^2J - K - (4x^4+8x^2+3)L + 2(x^6+x^4+3x^2+4)M \\
&\qquad + (2x^4+8x^2-3)N - 2(x^2-1)O + (x^2+1)^3P \\
&\qquad - (2x^4+4x^2+3)Q - (x^6+x^4-2x^2-2)R \\
&\qquad - (4x^2+1)S + x^2T\,. \qquad (6)
\end{aligned}$$

Next we find a sum for $x^6 f_{6n+3}$.

2.4 A Fibonacci Sum for $x^6 f_{6n+3}$: Using the gibonacci recurrence, and identities (5) and (6), we get

$$
\begin{aligned}
x^6 f_{6n+3} &= x^2(x^5 f_{6n+2}) + x^5 f_{6n+1} \\
&= x^2[A - 4B + (2x^2+7)C - 2D + (3x^4 - 4x^2 - 7)E + 6F + G \\
&\qquad -(x^6 - 2x^4 - 10x^2 - 5)H - (5x^4 + 12x^2 + 9)I \\
&\qquad +2x^2 J - K - (4x^4 + 8x^2 + 3)L + 2(x^6 + x^4 + 3x^2 + 4)M \\
&\qquad +(2x^4 + 8x^2 - 3)N - 2(x^2 - 1)O + (x^2+1)^3 P \\
&\qquad -(2x^4 + 4x^2 + 3)Q - (x^6 + x^4 - 2x^2 - 2)R \\
&\qquad -(4x^2+1)S + x^2 T] + [A - 6B + 3(2x^2+5)C \\
&\qquad -(x^4 + 20x^2 + 20)E - 4x^2 F + (9x^4 + 26x^2 + 15)H \\
&\qquad +(x^4 + 8x^2)I + 2x^2 J - (x^6 + 9x^4 + 14x^2 + 6)L \\
&\qquad -2(4x^4 + 5x^2)M + 2x^2 N - 2x^2 O + (x^2+1)^3 P \\
&\qquad +(x^6 + 3x^4 + 2x^2)Q + 2(x^4 + x^2)R - 2x^2 S + x^2 T] \\
&= (x^2+1)A - 2(2x^2+3)B + (2x^4 + 13x^2 + 15)C - 2x^2 D \\
&\qquad +(3x^6 - 5x^4 - 27x^2 - 20)E + 2x^2 F + x^2 G \\
&\qquad -(x^8 - 2x^6 - 19x^4 - 31x^2 - 15)H - (5x^6 + 11x^4 + x^2)I \\
&\qquad +2(x^4 + x^2)J - x^2 K - (5x^6 + 17x^4 + 17x^2 + 6)L \\
&\qquad +2(x^8 + x^6 - x^4 - x^2)M + (2x^6 + 8x^4 - x^2)N \\
&\qquad -2x^4 O + (x^2+1)^4 P - (x^6 + x^4 + x^2)Q \\
&\qquad -(x^8 + x^6 - 4x^4 - 4x^2)R - (4x^4 + 3x^2)S \\
&\qquad +(x^4 + x^2)T. \qquad (7)
\end{aligned}
$$

Next we pursue a sum for $x^5 f_{6n+4}$.

2.5 A Fibonacci Sum for $x^5 f_{6n+4}$: It follows by the gibonacci recurrence, and identities (6) and (7), we get

$$
\begin{aligned}
x^5 f_{6n+4} &= x^6 f_{6n+3} + x^5 f_{6n+2} \\
&= \big[A - 4B + (2x^2+7)C - 2D + (3x^4 - 4x^2 - 7)E + 6F + G \\
&\qquad -(x^6 - 2x^4 - 10x^2 - 5)H - (5x^4 + 12x^2 + 9)I \\
&\qquad +2x^2 J - K - (4x^4 + 8x^2 + 3)L + 2(x^6 + x^4 + 3x^2 + 4)M \\
&\qquad +(2x^4 + 8x^2 - 3)N - 2(x^2 - 1)O + (x^2+1)^3 P \\
&\qquad -(2x^4 + 4x^2 + 3)Q - (x^6 + x^4 - 2x^2 - 2)R \\
&\qquad -(4x^2+1)S + x^2 T\big] + \big[(x^2+1)A - 2(2x^2+3)B \\
&\qquad +(2x^4 + 13x^2 + 15)C - 2x^2 D + (3x^6 - 5x^4 - 27x^2 - 20)E \\
&\qquad +2x^2 + x^2 G - (x^8 - 2x^6 - 19x^4 - 31x^2 - 15)H \\
&\qquad -(5x^6 + 11x^4 + x^2)I + 2(x^4 + x^2)J - x^2 K \\
&\qquad -(5x^6 + 17x^4 + 17x^2 + 6)L + 2(x^8 + x^6 - x^4 - x^2)M \\
&\qquad +(2x^6 + 8x^4 - x^2)N - 2x^4 O + (x^2+1)^4 P \\
&\qquad -(x^6 + x^4 + x^2)Q - (x^8 + x^6 - 4x^4 - 4x^2)R \\
&\qquad -(4x^4 + 3x^2)S + (x^4 + x^2)T\big] \\
&= (x^2+2)A - 2(2x^2+5)B + (2x^4 + 15x^2 + 22)C - 2(x^2+1)D \\
&\qquad +(3x^6 - 2x^4 - 31x^2 - 27)E + 2(x^2+3)F + (x^2+1)G \\
&\qquad -(x^8 - x^6 - 21x^4 - 41x^2 - 20)H \\
&\qquad -(5x^6 + 16x^4 + 13x^2 + 9)I + 2(x^4 + 2x^2)J \\
&\qquad -(x^2+1)K - (5x^6 + 21x^4 + 25x^2 + 9)L \\
&\qquad +2(x^8 + 2x^6 + 2x^2 + 4)M + (2x^6 + 10x^4 + 7x^2 - 3)N \\
&\qquad -2(x^4 + x^2 - 1)O + (x^2+2)(x^2+1)^3 P \\
&\qquad -(x^6 + 3x^4 + 5x^2 + 3)Q - (x^8 + 2x^6 - 3x^4 - 6x^2 - 2)R \\
&\qquad -(4x^4 + 7x^2 + 1)S + (x^4 + 2x^2)T. \qquad (8)
\end{aligned}
$$

Next we investigate a sum for $x^6 f_{6n+5}$.

2.6 A Fibonacci Sum for $x^6 f_{6n+5}$: By the gibonacci recurrence, and identities (7) and (8), we have

$$\begin{aligned}
x^6 f_{6n+5} &= x^2(x^5 f_{6n+4}) + x^6 f_{6n+3} \\
&= x^2\big[(x^2+2)A - 2(2x^2+5)B + (2x^4+15x^2+22)C - 2(x^2+1)D \\
&\quad + (3x^6-2x^4-31x^2-27)E + 2(x^2+3)F + (x^2+1)G \\
&\quad - (x^8-x^6-21x^2-20)H - (5x^6+16x^4+13x^2+9)I \\
&\quad + 2(x^4+2x^2)J - (x^2+1)K - (5x^6+21x^4+25x^2+9)L \\
&\quad + 2(x^8+2x^6+2x^2+4)M + (2x^6 \;\; +10x^4+7x^2-3)N \\
&\quad - 2(x^4+x^2-1)O + (x^2+2)(x^2+1)^3 P \\
&\quad - (x^6+3x^4+5x^2+3)Q - (x^8+2x^6-3x^4-6x^2-2)R \\
&\quad - (4x^4+7x^2+1)S + (x^4+2x^2)T\big] + \big[(x^2+1)A \\
&\quad - 2(2x^2+3)B + (2x^4+13x^2+15)C - 2x^2 D \\
&\quad + (3x^6-5x^4-27x^2-20)E + 2x^2F + x^2G \\
&\quad - (x^8-2x^6-19x^4-31x^2-15)H - (5x^6+11x^4+x^2)I \\
&\quad + 2(x^4+x^2)J - x^2K - (5x^6+17x^4+17x^2+6)L \\
&\quad + 2(x^8+x^6-x^4-x^2)M + (2x^6+8x^4-x^2)N \\
&\quad - 2x^4O + (x^2+1)^4 P - (x^6+x^4+x^2)Q \\
&\quad - (x^8+x^6-4x^4-4x^2)R \;\; - (4x^4+3x^2)S + (x^4+x^2)T\big] \\
&= (x^4+3x^2+1)A - 2(2x^4+7x^2+3)B \\
&\quad + (2x^6+17x^4+35x^2) + 15C - 2(x^4+2x^2)D
\end{aligned}$$

$$+(3x^8+x^6-36x^4-54x^2-20)E+2(x^4+4x^2)F$$
$$+(x^4+2x^2)G-(x^{10}-23x^6-60x^4-51x^2-15)H$$
$$-(5x^8+21x^6+24x^4+10x^2)I+2(x^6+3x^4+x^2)J$$
$$-(x^4+2x^2)K-(5x^8+26x^6+42x^4+26x^2+6)L$$
$$+2(x^{10}+3x^8+x^6+x^4+3x^2)M$$
$$+(2x^8+12x^6+15x^4-4x^2)N-2(x^6+2x^4-x^2)O$$
$$+(x^4+3x^2+1)(x^2+1)^3P-(x^8+4x^6+6x^4+4x^2)Q$$
$$-(x^{10}+3x^8-2x^6-4x^4-10x^2-2)R$$
$$-(4x^6+11x^4+4x^2)S+(x^6+3x^4+x^2)T. \qquad (9)$$

Finally, we find a sum for $x^6 f_{6n-1}$. We will find it useful in the next subsection.

2.7 A Fibonacci Sum for $x^6 f_{6n-1}$: Since $x^6 f_{6n-1} = x^6 f_{6n+1} - x^2(x^5 f_{6n})$, it follows by equations (4) and (5) that

$$x^6 f_{6n-1} = A-2(x^2+3)B+(4x^4+14x^2+15)C+2x^2D$$
$$-(4x^6+17x^4+33x^2+20)E$$
$$-2(2x^4+5x^2)F-x^2G+(x^8+7x^6+25x^4+36x^2+15)H$$
$$+(6x^6+21x^4+17x^2)I+2x^2J+x^2K$$
$$-(x^8+6x^6+15x^4+17x^2+6)L$$
$$-2(x^8+5x^6+12x^4+9x^2)M-(2x^6+6x^4-5x^2)N$$
$$-4x^2O+(x^2+1)^3P+(x^8+6x^6+9x^4+5x^2)Q$$
$$+(x^8+3x^6+2x^4)R+(2x^4-x^2)S+x^2T. \qquad (10)$$

2.8 Numeric Fibonacci Byproducts: It follows from equations (4) through (10) that

$$F_{6n-1} = a - 8b + 33c + 2d - 74e - 14f - g + 84h + 44i$$
$$+2j + k - 45l - 54m - 3n - 4o + 8p + 21q + 6r + s + t$$

$$F_{6n} = 2b - 12c - 2d + 33e + 10f + g - 34h - 35i - k + 15l$$
$$+ 36m + 5n + 2o - 15q - 2r - 3s\,;$$

$$F_{6n+1} = a - 6b + 21c - 41e - 4f + 50h + 9i + 2j - 30l - 18m + 2n$$
$$-2o + 8p + 6q + 4r - 2s + t\,;$$

$$F_{6n+2} = a - 4b + +9c - 2d - 8e + 6f + g + 16h - 26i + 2j - k - 15l$$
$$+18m + 7n + 8p - 9q + 2r - 5s + t\,;$$

$$F_{6n+3} = 2a - 10b + 30c - 2d - 49e + 2f + g + 66h - 17i + 4j - k - 45l$$
$$+9n - 2o + 16p - 3q + 6r - 7s + 2t\,;$$

$$F_{6n+4} = 3a - 14b + 39c - 4d - 57e + 8f + 2g + 45h - 43i + 6j - 2k - 60l$$
$$+18m + 16n - 2o + 24p - 12q + 8r - 12s + 3t\,;$$

$$F_{6n+5} = 5a - 24b + 69c - 6d - 106e + 10f + 3g + 148h - 60i + 10j$$
$$-3k - 105l + 18m + 25n - 4o + 40p - 15q + 14r - 19s + 5t\,.$$

3. Sums of Lucas Polynomial Products of Order 6

Using gibonacci recurrence and the identity $l_n = f_{n+1} + f_{n-1}$, we now explore the counterparts for Lucas polynomials l_{6n+k}, where $0 \le k \le 5$.

3.1 A Fibonacci Sum for $x^6 l_{6n}$: Using equations (5) and (10), we have

$$x^6 l_{6n} = x^6 f_{6n+1} + x^6 f_{6n-1}$$

$$= \big[A - 6B + 3(2x^2 + 5)C - (x^4 + 20x^2 + 20)E$$

$$-4x^2F+(9x^4+26x^2+15)H+(x^4+8x^2)I+2x^2J$$

$$-(x^6+9x^4+14x^2+6)L-2(4x^4+5x^2)M+2x^2N$$

$$-2x^2O+(x^2+1)^3P+(x^6+3x^4+2x^2)Q$$

$$+2(x^4+x^2)R-2x^2S+x^2T\Big]+\Big[A-2(x^2+3)B$$

$$+(x^4+14x^2+15)C+2x^2D-(4x^6+17x^4+33x^2+20)E$$

$$-2(2x^4+5x^2)F-x^2G+(x^8+7x^6+25x^4+36x^2+15)H$$

$$+(6x^6+21x^4+17x^2)I+2x^2J+x^2K$$

$$-(x^8+6x^6+15x^4+17x^2+6)L$$

$$-2(x^8+5x^6+12x^4+9x^2)M-(2x^6+6x^4-5x^2)N$$

$$-4x^2O+(x^2+1)^3P+(x^8+6x^6+9x^4+5x^2)Q$$

$$+(x^8+3x^6+2x^4)R+(2x^4-x^2)S+x^2T\Big]$$

$$=2A-2(x^2+6)B+(4x^4+20x^2+30)C+2x^2D$$

$$-(4x^6+18x^4+53x^2+40)E-2(2x^4+7x^2)F$$

$$-x^2G+(x^8+7x^6+34x^4+62x^2+30)H$$

$$+(6x^6+22x^4+25x^2)I+4x^2J+x^2K$$

$$-(x^8+7x^6+24x^4+31x^2+12)L$$

$$-2(x^8+5x^6+16x^4+14x^2)M-(2x^6+6x^4-7x^2)N$$

$$-6x^2O+2(x^2+1)^3P+(x^8+7x^6+12x^4+7x^2)Q$$

$$+(x^8+3x^6+4x^4+2x^2)R+(2x^4-3x^2)S+2x^2T\,. \qquad (11)$$

Next we find a sum for x^5l_{6n+1}

3.2 A Fibonacci Sum for $x^5 l_{6n+1}$: Using equations (4) and (6), we get

$$\begin{aligned}
x^5 l_{6n+1} &= x^5 f_{6n+2} + x^5 f_{6n} \\
&= \big[A - 4B + (2x^2+7)C - 2D + (3x^4 - 4x^2 - 7)E + 6F + G \\
&\qquad -(x^6 - 2x^4 - 10x^2 - 5)H - (5x^4 + 12x^2 + 9)I \\
&\qquad + 2x^2 J - K - (4x^4 + 8x^2 + 3)L + 2(x^6 + x^4 + 3x^2 + 4)M \\
&\qquad + (2x^4 + 8x^2 - 3)N - 2(x^2 - 1)O + (x^2+1)^3 P \\
&\qquad - (2x^4 + 4x^2 + 3)Q - (x^6 + x^4 - 2x^2 - 2)R \\
&\qquad - (4x^2+1)S + x^2 T\big] + \big[2B - 4(x^2+2)C - 2D \\
&\qquad + (4x^4 + 16x^2 + 13)E + 2(2x^2+3)F + G \\
&\qquad - (x^6 + 7x^4 + 16x^2 + 10)H - (6x^4 + 20x^2 + 9)I \\
&\qquad - K + (x^6 + 5x^4 + 6x^2 + 3)L + 2(x^6 + 5x^4 + 8x^2 + 4)M \\
&\qquad + (2x^4 + 6x^2 - 3)N + 2O - (x^6 + 5x^4 + 6x^2 + 3)Q \\
&\qquad - (x^6 + 3x^4 - 2)R - (2x^2+1)S\big] \\
&= A - 2B - (2x^2+1)C - 4D + (7x^4 + 12x^2 + 6)E \\
&\qquad + 4(x^2+3)F + 2G - (2x^6 + 5x^4 + 6x^2 + 5)H \\
&\qquad - (11x^4 + 32x^2 + 18)I + 2x^2 J - 2K + (x^6 + x^4 - 2x^2)L \\
&\qquad + (4x^6 + 12x^4 + 22x^2 + 16)M \\
&\qquad + (4x^4 + 14x^2 - 6)N - 2(x^2 - 2)O + (x^2+1)^3 P \\
&\qquad - (x^6 + 7x^4 + 10x^2 + 6)Q - (2x^6 + 4x^4 - 2x^2 - 4)R \\
&\qquad - (6x^2+2)S + x^2 T. \qquad (12)
\end{aligned}$$

Next we investigate a sum for $x^6 l_{6n+2}$.

3.3 A Fibonacci Sum for $x^6 l_{6n+2}$**:** Using the gibonacci recurrence, and equations (11) and (12), we get

$$
\begin{aligned}
x^6 l_{6n+2} &= x^2(x^5 l_{6n+1}) + x^6 l_{6n} \\
&= x^2\big[A - 2B - (2x^2+1)C - 4D + (7x^4+12x^2+6)E \\
&\quad + 4(x^2+3)F + 2G - (2x^6+5x^4+6x^2+5)H \\
&\quad - (11x^4+32x^2+18)I + 2x^2J - 2K + (x^6+x^4-2x^2)L \\
&\quad + (4x^6+12x^4+22x^2+15)M + (4x^4+14x^2-6)N \\
&\quad - 2(x^2-2)O + (x^2+1)^3P - (x^6+7x^4+10x^2+6)Q \\
&\quad - (2x^6+4x^4-2x^2-4)R - (6x^2+2)S + x^2T\big] \\
&\quad + \big[2A - 2(x^2+6)B + (4x^4+20x^2+30)C + 2x^2 \\
&\quad - (4x^6+18x^4+53x^2+40)E - 2(2x^4+7x^2)F \\
&\quad - x^2G + (x^8+7x^6+34x^4+62x^2+30)H \\
&\quad + (6x^6+22x^4+25x^2)I + 4x^2J + x^2K \\
&\quad - (x^8+7x^6+24x^4+31x^2+12)L \\
&\quad - 2(x^8+5x^6+16x^4+14x^2)M - (2x^6+6x^4-7x^2)N \\
&\quad - 6x^2O + 2(x^2+1)^3P + (x^8+7x^6+12x^4+7x^2)Q \\
&\quad + (x^8+3x^6+4x^4+2x^2)R + (2x^4-3x^2)S + 2x^2T\big] \\
&= (x^2+2)A - 4(x^2+3)B + (2x^4+19x^2+30)C - 2x^2D \\
&\quad + (3x^6-6x^4-47x^2-40)E - 2x^2F + x^2G \\
&\quad - (x^8-2x^6-28x^4-57x^2-30)H - (5x^6+10x^4-7x^2)I \\
&\quad + 2(x^4+2x^2)J - x^2K - (6x^6+26x^4+31x^2+12)L \\
&\quad + 2(x^8+x^6-5x^4-6x^2)M + (2x^6+8x^4+x^2)N
\end{aligned}
$$

$$-2(x^4+x^2)O+(x^2+2)(x^2+1)^3P+(2x^4+x^2)Q$$
$$-(x^8+x^6-6x^4-6x^2)R-(4x^4+5x^2)S$$
$$+(x^4+2x^2)T\,. \tag{13}$$

Next we investigate a sum for $x^5 l_{6n+3}$.

3.4 A Fibonacci Sum for $x^5 l_{6n+3}$: Gibonacci recurrence, and equations (12) and (13) yield

$$\begin{aligned}
x^5 l_{6n+3} &= x^6 l_{6n+2} + x^5 l_{6n+1}\\
&= \big[(x^2+2)A-4(x^2+3)B+(2x^4+19x^2+30)C-2x^2D\\
&\quad +(3x^6-6x^4-47x^2-40)E-2x^2F+x^2G\\
&\quad -(x^8-2x^6-28x^4-57x^2-30)H-(5x^6+10x^4-7x^2)I\\
&\quad +2(x^4+2x^2)J-x^2K-(6x^6+26x^4+31x^2+12)L\\
&\quad +(2x^8+2x^6-10x^4-12x^2)M+(2x^6+8x^4+x^2)N\\
&\quad -2(x^4+x^2)O+(x^2+2)(x^2+1)^3P+(2x^4+x^2)Q\\
&\quad -(x^8+x^6-6x^4-6x^2)R-(4x^4+5x^2)S+(x^4+2x^2)T\big]\\
&\quad +\big[A-2B-(2x^2+1)C-4D+(7x^4+12x^2+6)E\\
&\quad +4(x^2+3)F+2G-(2x^6+5x^4+6x^2+5)H\\
&\quad -(11x^4+32x^2+18)I+2x^2J-2K+(x^6+x^4-2x^2)L\\
&\quad +(4x^6+12x^4+22x^2+15)M+(4x^4+14x^2-6)N\\
&\quad -2(x^2-2)O+(x^2+1)^3P-(x^6+7x^4+10x^2+6)Q\\
&\quad -(2x^6+4x^4-2x^2-4)R-2(3x^2+1)S+x^2T\big]\\
&= (x^2+3)A-2(2x^2+7)B+(2x^4+17x^2+29)C-2(x^2+2)D
\end{aligned}$$

$$+(3x^6+x^4-35x^2-34)E+2(x^2+6)F+(x^2+2)G$$
$$-(x^8-23x^4-51x^2-25)H-(5x^6+21x^4+25x^2+18)I$$
$$+2(x^4+3x^2)J-(x^2+2)K-(5x^6+25x^4+33x^2+12)L$$
$$+2(x^8+3x^6+x^4+5x^2+8)M$$
$$+(2x^6+12x^4+15x^2-6)N-2(x^4+2x^2-2)O$$
$$+(x^2+3)(x^2+1)^3P-(x^6+5x^4+9x^2+6)Q$$
$$-(x^8+3x^6-2x^4-8x^2-4)R-(4x^4+11x^2+2)S$$
$$+(x^4+3x^2)T. \tag{14}$$

Next we explore a sum for $x^6 l_{6n+4}$.

3.5 A Fibonacci Sum for $x^6 l_{6n+4}$: It follows by the gibonacci recurrence, and equations (13) and (14) that

$$\begin{aligned}
x^6 l_{6n+4} &= x^2(x^5 l_{6n+3}) + x^6 l_{6n+2} \\
&= x^2\big[(x^2+3)A-2(2x^2+7)B+(2x^4+17x^2+29)C-2(x^2+2)D \\
&\quad +(3x^6+x^4-35x^2-34)E+2(x^2+6)F+(x^2+2)G \\
&\quad -(x^8-23x^4-51x^2-25)H-(5x^6+21x^4+25x^2+18)I \\
&\quad +2(x^4+3x^2)J-(x^2+2)K-(5x^6+25x^4+33x^2+12)L \\
&\quad +2(x^8+6x^6+x^4+5x^2+8)M \\
&\quad +(2x^6+12x^4+15x^2-6)N-2(x^4+2x^2-2)O \\
&\quad +(x^2+3)(x^2+1)^3P-(x^6+5x^4+9x^2+6)Q \\
&\quad -(x^8+3x^6-2x^4-8x^2-4)R-(4x^4+11x^2+2)S \\
&\quad +(x^4+3x^2)T\big]+\big[(x^2+2)A-4(x^2+3)B
\end{aligned}$$

$$
\begin{aligned}
&+(2x^4+19x^2+30)C-2x^2D+(3x^6-6x^4-47x^2-40)E \\
&-2x^2F+x^2G-(x^8-2x^6-28x^4-57x^2-30)H \\
&-(5x^6+10x^4-7x^2)I+2(x^4+2x^2)J-x^2K \\
&-(6x^6+26x^4+31x^2+12)L+2(x^8+x^6-5x^4-6x^2)M \\
&+(2x^6+8x^4+x^2)N-(2x^4+2x^2)O+(x^2+2)(x^2+1)^3P \\
&+(2x^4+x^2)Q-(x^8+x^6-6x^4-6x^2)R \\
&-(4x^4+5x^2)S+(x^4+2x^2)T \\
=\ &(x^4+4x^2+2)A-2(2x^4+9x^2+6)B \\
&+(2x^6+19x^4+48x^2+30)C-2(x^4+3x^2)D \\
&+(3x^8+4x^6-41x^4-8x^2-40)E+2(x^4+5x^2)F \\
&+(x^4+3x^2)G-(x^{10}+x^8-25x^6-79x^4-82x^2-30)H \\
&-(5x^8+26x^6+35x^4+11x^2)I+2(x^6+4x^4+2x^2)J \\
&-(x^4+3x^2)K-(5x^8+31x^6+59x^4+43x^2+12)L \\
&+2(x^{10}+4x^8+2x^6+2x^2)M \\
&+(2x^8+14x^6+23x^4-5x^2)N-2(x^6+3x^4-x^2)O \\
&+(x^4+4x^2+2)(x^2+1)^3P-(x^8+5x^6+7x^4+5x^2)Q \\
&-(x^{10}+4x^8-x^6-14x^4-10x^2)R \\
&-(4x^6+15x^4+7x^2)S+(x^6+4x^4+2x^2)T\,. \qquad (15)
\end{aligned}
$$

We now find a sum for $x^5 l_{6n+5}$.

3.6 A Fibonacci Sum for $x^5 l_{6n+5}$: The gibonacci recurrence, coupled with equations (14) and (15) yield

$$\begin{aligned}
x^5 l_{6n+5} &= x^6 l_{6n+4} + x^5 l_{6n+3} \\
&= (x^4+5x^2+5)A - 2(2x^4+11x^2+13)B \\
&\quad + (2x^6+21x^4+65x^2+59)C - 2(x^4+4x^2+2)D \\
&\quad + (3x^8+7x^6-40x^4-116x^2-74)E + 2(x^4+6x^2+6)F \\
&\quad + (x^4+4x^2+2)G \\
&\quad - (x^{10}+2x^8-25x^6-102x^4-133x^2-55)H \\
&\quad - (5x^8+31x^6+56x^4+36x^2+18)I + 2(x^6+5x^4+5x^2)J \\
&\quad - (x^4+4x^2+2)K - (5x^8+36x^6+84x^4+76x^2+24)L \\
&\quad + 2(x^{10}+5x^8+5x^6+x^4+7x^2+8)M \\
&\quad + (2x^8+16x^6+35x^4+10x^2-6)N \\
&\quad - 2(x^6+4x^4+x^2-2)O + (x^4+5x^2+5)(x^2+1)^3 P \\
&\quad - (x^8+6x^6+12x^4+14x^2+6)Q \\
&\quad - (x^{10}+5x^8+2x^6-16x^4-18x^2-4)R \\
&\quad - (4x^6+19x^4+18x^2+2)S + (x^6+5x^4+5x^2)T. \qquad (16)
\end{aligned}$$

Finally, we extract the numeric counterparts of the Lucas sums.

3.7 Numeric Lucas Byproducts: Equations (11) through (16) yield the following numeric counterparts:

$$\begin{aligned}
L_{6n} &= 2a - 14b + 54c + 2d - 115e - 18f - g + 134h + 53i + 4j + k - 75l \\
&\quad - 72m - n - 6o + 16p + 27q + 10r - s + 2t;
\end{aligned}$$

$$\begin{aligned}
L_{6n+1} &= a - 2b - 3c - 4d + 25e + 16f + 2g - 18h - 61i + 2j - 2k \\
&\quad + 54m + 12n + 2o + 8p - 24q - 8s + t;
\end{aligned}$$

$$\begin{aligned}
L_{6n+2} &= 3a - 16b + 51c - 2d - 90e - 2f + g + 116h - 8i + 6j - k - 75l \\
&\quad - 18m + 11n - 4o + 24p + 3q + 10r - 9s + 3t \;;
\end{aligned}$$

$$L_{6n+3} = 4a - 18b + 48c - 6d - 65e + 14f + 3g + 98h - 69i + 8j - 3k - 75l + 36m + 23n - 2o + 32p - 21q + 10r - 17s + 4t\,;$$

$$L_{6n+4} = 7a - 34b + 99c - 8d - 155e + 12f + 4g + 214h - 77i + 14j - 4k - 150l + 18m + 34n - 6o + 56p - 18p - 18q + 20r - 26s + 7t\,;$$

$$L_{6n+5} = 11a - 52b + 147c - 14d - 220e + 26f + 7g + 312h - 146i + 22j - 7k - 225l + 54m + 37n - 8o + 88p - 39q + 30r - 43s + 11t\,.$$

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GRAPH-THEORETIC CONFIRMATIONS OF SUMS OF GIBONACCI POLYNOMIAL PRODUCTS OF ORDER 6

Abstract: Using graph-theoretic tools, we confirm four identities involving sums of gibonacci polynomial products of order 6, studied in [6].

Keywords: Gibonacci Polynomials, Pell-Lucas Polynomials, Lucas Numbers.

Mathematical Subject Classification (2010) No.: Primary 05A19, 11B39, 11Cxx.

1. Introduction

Gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the *nth Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the *nth Lucas polynomial*.

Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2, 3].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* Pn and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [3].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. We let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n,

$$\begin{array}{lll}
A = f_{n+2}^6 & B = f_{n+2}^5 f_n & C = f_{n+2}^4 f_n^2 \\
D = f_{n+2}^4 f_n f_{n-2} & E = f_{n+2}^3 f_n^3 & F = f_{n+2}^3 f_n^2 f_{n-2} \\
G = f_{n+2}^3 f_n f_{n-2}^2 & H = f_{n+2}^2 f_n^4 & I = f_{n+2}^2 f_n^3 f_{n-2} \\
J = f_{n+2}^2 f_n^2 f_{n-2}^2 & K = f_{n+2}^2 f_n f_{n-2}^3 & L = f_{n+2} f_n^5 \\
M = f_{n+2} f_n^4 f_{n-2} & N = f_{n+2} f_n^3 f_{n-2}^2 & O = f_{n+2} f_n^2 f_{n-2}^3 \\
P = f_n^6 & Q = f_n^5 f_{n-2} & R = f_n^4 f_{n-2}^2 \\
S = f_n^3 f_{n-2}^3 & T = f_n^2 f_{n-2}^4 \;; &
\end{array}$$

and also omit a lot of basic algebra.

It is well known that $f_{n+1} + f_{n-1} = l_n$, $f_n^2 = f_n l_n$, $f_{n+1}^2 = f_{n+1}^2 + f_n^2$, $f_{n+2} + f_{n-2} = (x^2 + 2) f_n$, $f_{n+2} - f_{n-2} = x l_n$, and the *gibonacci addition formula* $g_{a+b} = f_{a+1} g_b + f_a g_{b-1}$ [3].

1.1 Sums of Gibonacci Polynomial Products of Order 4: Sums of gibonacci polynomial products of order 4 are studied in [5]; four of them play an important role in our discourse:

$$\begin{aligned} x^3 f_{4n} = {} & f_{n+2}^3 f_n - 2 f_{n+2}^2 f_n^2 - f_{n+2}^2 f_n f_{n-2} + 2(x^2+1) f_{n+2} f_n^3 + f_{n+2} f_n f_{n-2}^2 \\ & - 2(x^2+1) f_n^3 f_{n-2} + 2 f_n^2 f_{n-2}^2 - f_n f_{n-2}^3 ; \end{aligned} \tag{1}$$

$$x^4 f_{4n+1} = f_{n+2}^4 - 4f_{n+2}^3 f_n + 2(2x^2+3)f_{n+2}^2 f_n^2 - (x^4+6x^2+4)f_{n+2}f_n^3 - 2x^2 f_{n+2}f_n^2 f_{n-2} + (x^2+1)^2 f_n^4 + (x^4+2x^2)f_n^3 f_{n-2}; \quad (2)$$

$$x^3 f_{4n+2} = f_{n+2}^4 - 3f_{n+2}^2 f_n^2 + 2f_{n+2}f_n^2 f_{n-2} + f_n^4 - f_n^2 f_{n-2}^2. \quad (3)$$

$$x^4 f_{4n+3} = (x^2+1)f_{n+2}^4 - 4f_{n+2}^3 f_n + (x^2+6)f_{n+2}^2 f_n^2 - (x^4+6x^2+4)f_{n+2}f_n^3 + (x^4+3x^2+1)f_n^4 + (x^4+2x^2)f_n^3 f_{n-2} - x^2 f_n^2 f_{n-2}^2. \quad (4)$$

1.2 Sums of Gibonacci Polynomial Products of Order 6: In [6], we explored a family of sums of gibonacci polynomial products of order 6. Six of them are the following:

$$x^6 f_{6n+1} = A - 6B + 3(2x^2+5)C - (x^4+20x^2+20)E - 4x^2F + (9x^4+26x^2+15)H + (x^4+8x^2)I + 2x^2J - (x^6+9x^4+14x^2+6)L - 2(4x^4+5x^2)M + 2x^2N - 2x^2O + (x^2+1)^3P + (x^6+3x^4+2x^2)Q + 2(x^4+x^2)R - 2x^2S + x^2T. \quad (5)$$

$$x^5 f_{6n+2} = A - 4B + (2x^2+7)C - 2D + (3x^4-4x^2-7)E + 6F + G - (x^6-2x^4-10x^2-5)H - (5x^4+12x^2+9)I + 2x^2J - K - (4x^4+8x^2+3)L + 2(x^6+x^4+3x^2+4)M + (2x^4+8x^2-3)N - 2(x^2-1)O + (x^2+1)^3P - (2x^4+4x^2+3)Q - (x^6+x^4-2x^2-2)R - (4x^2+1)S + x^2T \quad . \quad (6)$$

$$x^6 f_{6n+3} = (x^2+1)A - 2(2x^2+3)B + (2x^4+13x^2+15)C - 2x^2D + (3x^6-5x^4-27x^2-20)E + 2x^2F + x^2G - (x^8-2x^6-19x^4-31x^2-15)H - (5x^6+11x^4+x^2)I + 2(x^4+x^2)J - x^2K - (5x^6+17x^4+17x^2+6)L$$

$$+ 2(x^8 + x^6 - x^4 - x^2)M + (2x^6 + 8x^4 - x^2)N - 2x^4 O + (x^2 + 1)^4 P$$
$$- (x^6 + x^4 + x^2)Q - (x^8 + x^6 - 4x^4 - 4x^2)R$$
$$- (4x^4 + 3x^2)S + (x^4 + x^2)T . \qquad (7)$$

$$x^6 f_{6n+5} = (x^4 + 3x^2 + 1)A - 2(2x^4 + 7x^2 + 3)B + (2x^6 + 17x^4 + 35x^2 + 15)C$$
$$- 2(x^4 + 2x^2)D + (3x^8 + x^6 - 36x^4 - 54x^2 - 20)E$$
$$+ 2(x^4 + 4x^2)F + (x^4 + 2x^2)G - (x^{10} - 23x^6 - 60x^4 - 51x^2 - 15)H$$
$$- (5x^8 + 21x^6 + 24x^4 + 10x^2)I + 2(x^6 + 3x^4 + x^2)J - (x^4 + 2x^2)K$$
$$- (5x^8 + 26x^6 + 42x^4 + 26x^2 + 6)L + 2(x^{10} + 3x^8 + x^6 + x^4 + 3x^2)M$$
$$+ (2x^8 + 12x^6 + 15x^4 - 4x^2)N - 2(x^6 + 2x^4 - x^2)O$$
$$+ (x^4 + 3x^2 + 1)(x^2 + 1)^3 P - (x^8 + 4x^6 + 6x^4 + 4x^2)Q$$
$$- (x^{10} + 3x^8 - 2x^6 - 4x^4 - 10x^2 - 2)R - (4x^6 + 11x^4 + 4x^2)S$$
$$+ (x^6 + 3x^4 + x^2)T . \qquad (8)$$

$$x^6 l_{6n+2} = (x^2 + 2)A - 4(x^2 + 3)B + (2x^4 + 19x^2 + 30)C - 2x^2 D$$
$$+ (3x^6 - 6x^4 - 47x^2 - 40)E - 2x^2 F + x^2 G$$
$$- (x^8 - 2x^6 - 28x^4 - 57x^2 - 30)H - (5x^6 + 10x^4 - 7x^2)I$$
$$+ 2(x^4 + 2x^2)J - x^2 K - (6x^6 + 26x^4 + 31x^2 + 12)L$$
$$+ 2(x^8 + x^6 - 5x^4 - 6x^2)M + (2x^6 + 8x^4 + x^2)N - 2(x^4 + x^2)O$$
$$+ (x^2 + 2)(x^2 + 1)^3 P + (2x^4 + x^2)Q - (x^8 + x^6 - 6x^4 - 6x^2)R$$
$$- (4x^4 + 5x^2)S + (x^4 + 2x^2)T . \qquad (9)$$

$$x^6 l_{6n+4} = (x^4 + 4x^2 + 2)A - 2(2x^4 + 9x^2 + 6)B + (2x^6 + 19x^4 + 48x^2 + 30)C$$
$$-2(x^4 + 3x^2)D + (3x^8 + 4x^6 - 41x^4 - 8x^2 - 40)E + 2(x^4 + 5x^2)F$$

$$+ (x^4 + 3x^2)G - (x10 + x^8 - 25x^6 - 79x^4 - 82x^2 - 30)H$$
$$- (5x^8 + 26x^6 + 35x^4 + 11x^2)I + 2(x^6 + 4x^4 + 2x^2)J - (x^4 + 3x^2)K$$
$$- (5x^8 + 31x^6 + 59x^4 + 43x^2 + 12)L + 2(x^{10} + 4x^8 + 2x^6 + 2x^2)M$$
$$+ (2x^8 + 14x^6 + 23x^4 - 5x^2)N - 2(x^6 + 3x^4 - x^2)O$$
$$+ (x^4 + 4x^2 + 2)(x^2 + 1)^3 P - (x^8 + 5x^6 + 7x^4 + 5x^2)Q$$
$$- (x^{10} + 4x^8 - x^6 - 14x^4 - 10x^2)R - (4x^6 + 15x^4 + 7x^2)S$$
$$+ (x^6 + 4x^4 + 2x^2)T . \quad (10)$$

2. Some Graph-theoretic Tools

Our goal is to confirm the polynomial identities (5), (7), (9), and (10) using graph-theoretic techniques. To this end, first we develop the needed tools. Consider the *Fibonacci digraph* D in Figure 1 with vertices v_1 and v_2, where a *weight* is assigned to each edge [3, 4]. It follows by induction

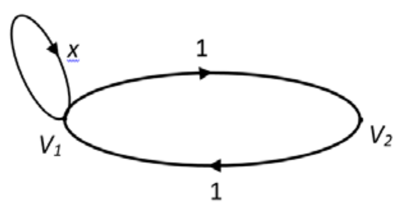

Figure 1: Weighted Fibonacci Digraph D

from its *weighted adjacency matrix* is the Q-matrix

$$Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}, \text{ that}$$

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $n \geq 1$ [3, 4].

A *walk* from vertex v_i to vertex v_j is a sequence $v_i - e_i - v_{i+1} - \cdots - v_{j-1} - e_{j-1} - v_j$ of vertices v_k and edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The walk is *closed* if $v_i = v_j$; otherwise, it is *open*. The *length* of a walk is the number of edges in the walk. The *weight* of a walk is the product of the weights of the edges along the walk.

The sum of the weights of closed walks of length n originating at v_1 in the digraph is f_{n+1} and that of those originating at v_2 is f_{n-1} [3, 4]. Consequently, the sum of the weights of all closed walks of length n in the digraph is $f_{n+1} + f_{n-1} = l_n$. These facts play a crucial role in the graph-theoretic proofs.

Let A, B, and C denote the sets of walks of varying lengths originating at a vertex v. Then the sum of the weights of the elements (a, b, c) in the product set $A \times B \times C$ is *defined* as the product of the sums of weights from each component [4].

With this background, we are on the way for the graph-theoretic confirmations.

3. Graph-Theoretic Confirmations

3.1 Confirmation of Identity (5): Proof: Let S denote the sum of the weights of closed walks of length $6n$ in the digraph D originating (and ending) at v_1. Then $S = f_{6n+1}$, and hence, $x^6 S = x^6 f_{6n+1}$.

We will now compute the sum $x^6 S$ in a different way. To this end, let w be an arbitrary closed walk of length $6n$ originating at v_1. Clearly, it can land at v_1 or v_2 at the $2n$th and $4n$th steps:

$$w = \underbrace{v_1 - \ldots - v}_{\text{subwalk of length } 2n} \quad \underbrace{v - \ldots - v}_{\text{subwalk of length } 2n} \quad \underbrace{v - \ldots - v_1}_{\text{subwalk of length } 2n},$$

where $v = v_1$ or v_2.

Table 1: Sum of the Weights of Closed Walks Originating at v_1

w lands at v_1 at the $2n$th step?	w lands at v_1 at the $4n$th step?	w lands at v_1 at the $6n$th step?	sum of the weights of walks w
yes	yes	yes	f_{2n+1}^3
yes	no	yes	$f_{2n+1}f_{2n}^2$
no	yes	yes	$f_{2n+1}f_{2n}^2$
no	no	yes	$f_{2n}^2 f_{2n-1}$

Table 1 shows the various possible cases and the corresponding sums of weights of walks w. Consequently, by equations (1) and (2), it follows from the table that the sum S of the weights of all such walks is given by

$$\begin{aligned} S &= f_{2n+1}^3 + 2f_{2n+1}f_{2n}^2 + f_{2n}^2 f_{2n-1} \\ &= f_{2n+1}\left(f_{2n+1}^2 + f_{2n}^2\right) + f_{2n}^2\left(f_{2n+1} + f_{2n-1}\right) \\ &= f_{2n+1}f_{4n+1} + f_{2n}f_{4n}\ ; \end{aligned}$$

$$\begin{aligned} x^6 S &= (x^4 f_{4n+1})\left[(xf_{n+1})^2 + x^2 f_n^2\right] + (x^3 f_{4n})(x^2 f_n)(xl_n) \\ &= (x^4 f_{4n+1})\left[(f_{n+2} - f_n)^2 + x^2 f_n^2\right] + (x^3 f_{4n})(x^2 f_n)(f_{n+2} - f_{n-2}) \\ &= (x^4 f_{4n+1})\left[f_{n+2}^2 - 2f_{n+2}f_n + (x^2+1)f_n^2\right] + (x^3 f_{4n})(x^2 f_n)(f_{n+2} - f_{n-2}) \\ &= \left[f_{n+2}^4 - 4f_{n+2}^3 f_n + 2(2x^2+3)f_{n+2}^2 f_n^2 - (x^4+6x^2+4)f_{n+2}f_n^3 - 2x^2 f_{n+2}f_n^2 f_{n-2}\right. \\ &\quad \left. + (x^2+1)^2 f_n^4 + (x^4+2x^2)f_n^3 f_{n-2}\right]\left[f_{n+2}^2 - 2f_{n+2}f_n + (x^2+1)f_n^2\right] \\ &\quad + \left[f_{n+2}^3 f_n - 2f_{n+2}^2 f_n^2 - f_{n+2}^2 f_n f_{n-2} + 2(x^2+1)f_{n+2}f_n^3 + f_{n+2}f_n f_{n-2}^2\right. \\ &\quad \left. - 2(x^2+1)f_n^3 f_{n-2} + 2f_n^2 f_{n-2}^2 - f_n f_{n-2}^3\right](x^2 f_n)(f_{n+2} - f_{n-2}) \\ &= A - 6B + 3(2x^2+5)C - (x^4+20x^2+20)E - 4x^2 F + (9x^4+26x^2+15)H \\ &\quad + (x^4+8x^2)I + 2x^2 J - (x^6+9x^4+14x^2+6)L - 2(4x^4+5x^2)M + 2x^2 N \\ &\quad - 2x^2 O + (x^2+1)^3 P + (x^6+3x^4+2x^2)Q + 2(x^4+x^2)R - 2x^2 S + x^2 T\ . \end{aligned}$$

This value of $x^6 S$, coupled with its initial value, yields identity (5), as desired. □

Next pursue the graph-theoretic proof of identity (7).

3.2 Confirmation of Identity (7): Proof: Let S' denote the sum of the weights of closed walks of length $6n+2$ originating at v_1 in the digraph. Then $S' = f_{6n+3}$; so $x^6 S' = x^6 f_{6n+3}$.

Table 2: Sums of the Weights of Closed Walks Originating at v_1

w lands at v_1 at the $2n$th step?	w lands at v_1 at the $(4n+1)$st step?	w lands at v_1 at the $(6n+2)$nd step?	sum of the weights of walks w
yes	yes	yes	$f_{2n+2}^2 f_{2n+1}$
yes	no	yes	f_{2n+1}^2
no	yes	yes	$f_{2n+2} f_{2n+1} f_{2n}$
no	no	yes	$f_{2n+1} f_{2n}^2$

We will now compute $x^6 S'$ in a different way. To achieve this, we let w be an arbitrary closed walk of length $6n+2$ originating at v_1. It can land at v_1 or v_2 at the $2n$th and $(4n+1)$st steps:

$$w = \underbrace{v_1 - \ldots - v}_{\text{subwalk of length } 2n} \quad \underbrace{v - \ldots - v}_{\text{subwalk of length } 2n+1} \quad \underbrace{v - \ldots - v_1}_{\text{subwalk of length } 2n+1},$$

where $v = v_1$ or v_2.

It follows by Table 2, and equations (3) and (4) that the sum S' of the weights of the corresponding walks w is given by

$$\begin{aligned} S' &= f_{2n+2}^2 f_{2n+1} + f_{2n+1}^3 + f_{2n+2} f_{2n+1} f_{2n} + f_{2n+1} f_{2n}^2 \\ &= f_{2n+1}\left(f_{2n+2}^2 + f_{2n+1}^2\right) + f_{2n+1} f_{2n}\left(f_{2n+2} + f_{2n}\right) \\ &= f_{2n+1} f_{4n+3} + f_{4n+2} f_{2n}; \end{aligned}$$

$$
\begin{aligned}
x^6 S' &= (x^4 f_{4n+3})\left[(xf_{n+1})^2 + x^2 f_n^2\right] + (x^3 f_{4n+2})(x^2 f_n)(xl_n)\\
&= (x^4 f_{4n+3})\left[(f_{n+2} - f_n)^2 + x^2 f_n^2\right] + (x^3 f_{4n+2})(x^2 f_n)(f_{n+2} - f_{n-2})\\
&= \left[(x^2+3)f_{n+2}^4 - 4f_{n+2}^3 f_n + x^2 f_{n+2}^2 f_n^2 - (x^4+6x^2+4)f_{n+2} f_n^3\right.\\
&\quad + 4f_{n+2} f_n^2 f_{n-2} + (x^4+3x^2+3)f_n^4 + (x^4+2x^2)f_n^3 f_{n-2}\\
&\quad \left. - (x^2+2)f_n^2 f_{n-2}^2\right]\left[f_{n+2}^2 - 2f_{n+2} f_n + (x^2+1)f_n^2\right]\\
&\quad + \left[(x^2+2)f_{n+2}^4 - 8f_{n+2}^3 f_n + (5x^2+12)f_{n+2}^2 f_n^2 - 2(x^4+6x^2+4)f_{n+2} f_n^3\right.\\
&\quad - 2x^2 f_{n+2} f_n^2 f_{n-2} + (2x^4+5x^2+2)f_n^4 + 2(x^4+2x^2)f_n^3 f_{n-2}\\
&\quad \left. - x^2 f_n^2 f_{n-2}^2\right](x^2 f_n)(f_{n+2} - f_{n-2})\\
&= (x^2+1)A - 2(2x^2+3)B + (2x^4+13x^2+15)C - 2x^2 D\\
&\quad + (3x^6 - 5x^4 - 27x^2 - 20)E + 2x^2 F + x^2 G - (x^8 - 2x^6 - 19x^4 - 31x^2 - 15)H\\
&\quad - (5x^6 + 11x^4 + x^2)I + 2(x^4 + x^2)J - x^2 K - (5x^6 + 17x^4 + 17x^2 + 6)L\\
&\quad + 2(x^8 + x^6 - x^4 - x^2)M + (2x^6 + 8x^4 - x^2)N - 2x^4 O + (x^2+1)^4 P\\
&\quad - (x^6 + x^4 + x^2)Q - (x^8 + x^6 - 4x^4 - 4x^2)R - (4x^4 + 3x^2)S + (x^4 + x^2)T\,.
\end{aligned}
$$

Equating this value of $x^6 S'$ with its earlier version yields identity (7), as expected.

3.3 Confirmation of Identity (9): Proof: Let S^* denote the sum of the weights of all closed walks of length $6n+2$ in the digraph.

Then $S^* = l_{6n+2}$ and hence, $x^6 S^* = x^6 l_{6n+2}$.

To compute $x^6 S^*$ in a different way, we let w be an arbitrary closed walk of length $6n+2$.

Case 1: Suppose w originates at v_1. It can land at v_1 or v_2 at the $2n$th and $4n$th steps:

$$w = \underbrace{v_1 - \ldots - v}_{\text{subwalk of length } 2n} \quad \underbrace{v - \ldots - v}_{\text{subwalk of length } 2n} \quad \underbrace{v - \ldots - v_1}_{\text{subwalk of length } 2n+2},$$

where $v = v_1$ or v_2.

Table 3: Sums of the Weights of Closed Walks Originating at v_1

w lands at v_1 at the $2n$th step?	w lands at v_1 at the $4n$th step?	w lands at v_1 at the $(6n+2)$nd step?	sum of the weights of walks w
yes	yes	yes	$f_{2n+3}f_{2n+1}^2$
yes	no	yes	$f_{2n+2}f_{2n+1}f_{2n}$
no	yes	yes	$f_{2n+3}f_{2n}^2$
no	no	yes	$f_{2n+2}f_{2n}f_{2n-1}$

It follows from Table 3 that the sum S_1^* of the weights of all such walks w is given by

$$\begin{aligned} S_1^* &= f_{2n+3}f_{2n+1}^2 + f_{2n+2}f_{2n+1}f_{2n} + f_{2n+3}f_{2n}^2 + f_{2n+2}f_{2n}f_{2n-1} \\ &= f_{2n+3}\left(f_{2n+1}^2 + f_{2n}^2\right) + f_{2n+2}f_{2n}\left(f_{2n+1} + f_{2n-1}\right) \\ &= f_{2n+3}f_{4n+1} + f_{2n+2}f_{4n} \\ &= f_{6n+3}. \end{aligned}$$

Case 2: Suppose w originates at v_2. Then also it can land at v_1 or v_2 at the $2n$th and $4n$th steps:

$$w = \underbrace{v_2 - \ldots - v}_{\text{subwalk of length } 2n} \quad \underbrace{v - \ldots - v}_{\text{subwalk of length } 2n} \quad \underbrace{v - \ldots - v_2}_{\text{subwalk of length } 2n+2},$$

where $v = v_1$ or v_2.

Table 4: Sums of the Weights of Closed Walks Originating at v_2

w lands at v_1 at the $2n$th step?	w lands at v_1 at the $4n$th step?	w lands at v_1 at the $(6n+2)$nd step?	sum of the weights of walks w
yes	yes	yes	$f_{2n+2}f_{2n+1}f_{2n}$
yes	no	yes	$f_{2n+1}f_{2n}^2$
no	yes	yes	$f_{2n+2}f_{2n}f_{2n-1}$
no	no	yes	$f_{2n+1}f_{2n-1}^2$

It follows from Table 4 that the sum S_2^* of the weights of all such walks w is given by

$$\begin{aligned} S_2^* &= f_{2n+2}f_{2n+1}f_{2n} + f_{2n+2}f_{2n}f_{2n-1} + f_{2n+1}f_{2n}^2 + f_{2n+1}f_{2n-1}^2 \\ &= f_{2n+1}f_{2n}(f_{2n+1} + f_{2n-1}) + f_{2n+1}\ (f_{2n}^2 + f_{2n-1}^2) \\ &= f_{2n+2}f_{4n} + f_{2n+1}f_{4n-1} \\ &= f_{6n+1}. \end{aligned}$$

Using equations (5) and (7), we then get

$$\begin{aligned} x^6S^* &= x^6S_1^* + x^6S_2^* \\ &= x^6f_{6n+3} + x^6f_{6n+1} \\ &= (x^2+2)A - 4(x^2+3)B + (2x^4+19x^2+30)C - 2x^2D \\ &\quad + (3x^6 - 6x^4 - 47x^2 - 40)E - 2x^2F + x^2G \\ &\quad - (x^8 - 2x^6 - 28x^4 - 57x^2 - 30)H - (5x^6 + 10x^4 - 7x^2)I \\ &\quad + 2(x^4 + 2x^2)J - x^2K - (6x^6 + 26x^4 + 31x^2 + 12)L \\ &\quad + 2(x^8 + x^6 - 5x^4 - 6x^2)M + (2x^6 + 8x^4 + x^2)N \\ &\quad - 2(x^4 + x^2)O + (x^2+2)(x^2+1)^3P + (2x^4 + x^2)Q \\ &\quad - (x^8 + x^6 - 6x^4 - 6x^2)R - (4x^4 + 5x^2)S + (x^4 + 2x^2)T. \end{aligned}$$

Equating the two values of x^6S^* yields identity (9), as desired. □

Finally, we present the graph-theoretic confirmation of identity (10).

3.4 Confirmation of Identity (10): Proof. Let S denote the sum of the weights of all closed walks of length $6n+4$ in the digraph.

Then $S = l_{6n+4}$; so $x^6S = x^6l_{6n+4}$.

We will now compute x^6S in a different way. To this end, we let w be an arbitrary walk of length $6n+4$.

Case 1: Suppose w originates at v_1. It can land at v_1 or v_2 at the $(2n+1)$st and $(4n+2)$nd steps:

$$w = \underbrace{v_1 - \ldots - v}_{\text{subwalk of length } 2n+1} \quad \underbrace{v - \ldots - v}_{\text{subwalk of length } 2n+1} \quad \underbrace{v - \ldots - v_1}_{\text{subwalk of length } 2n+2},$$

where $v = v_1$ or v_2.

Table 5: Sums of the Weights of Closed Walks Originating at v_1

w lands at v_1 at the $(2n+1)$st step?	w lands at v_1 at the $(4n+2)$nd step?	w lands at v_1 at the $(6n+4)$th step?	sum of the weights of walks w
yes	yes	yes	$f_{2n+3}f_{2n+2}^2$
yes	no	yes	$f_{2n+2}^2 f_{2n+1}$
no	yes	yes	$f_{2n+3}f_{2n+1}^2$
no	no	yes	$f_{2n+2}f_{2n+1}f_{2n}$

It follows from Table 5 that the sum S_1 of the weights of all such walks w is given by

$$\begin{aligned} S_1 &= f_{2n+3}f_{2n+2}^2 + f_{2n+2}^2 f_{2n+1} + f_{2n+3}f_{2n+1}^2 + f_{2n+2}f_{2n+1}f_{2n} \\ &= f_{2n+3}(f_{2n+2}^2 + f_{2n+1}^2) + f_{2n+2}f_{2n+1}(f_{2n+2} + f_{2n}) \\ &= f_{4n+3}f_{2n+3} + f_{4n+2}f_{2n+2} \\ &= f_{6n+5}. \end{aligned}$$

Case 2: Suppose w originates at v_2. It also can land at v_1 or v_2 at the $(2n+1)$st and $(4n+2)$nd steps:

$$w = \underbrace{v_2 - \ldots - v}_{\text{subwalk of length } 2n+1} \quad \underbrace{v - \ldots - v}_{\text{subwalk of length } 2n+1} \quad \underbrace{v - \ldots - v_2}_{\text{subwalk of length } 2n+2},$$

where $v = v_1$ or v_2.

Table 6: Sums of the Weights of Closed Walks Originating at v_2

w lands at v_1 at the $(2n+1)$st step?	w lands at v_1 at the $(4n+2)$nd step?	w lands at v_1 at the $(6n+4)$th step?	sum of the weights of walks w
yes	yes	yes	$f_{2n+2}^2 f_{2n+1}$
yes	no	yes	f_{2n+1}^3
no	yes	yes	$f_{2n+2} f_{2n+1} f_{2n}$
no	no	yes	$f_{2n+1} f_{2n}^2$

It follows from Table 6 that the sum S_2 of the weights of all such walks w is given by

$$\begin{aligned} S_2 &= f_{2n+2}^2 f_{2n+1} + f_{2n+1}^3 + f_{2n+2} f_{2n+1} f_{2n} + f_{2n+1} f_{2n}^2 \\ &= f_{2n+1}(f_{2n+2}^2 + f_{2n+1}^2) + f_{2n+1} f_{2n}(f_{2n+2} + f_{2n}) \\ &= f_{4n+3} f_{2n+1} + f_{4n+2} f_{2n} \\ &= f_{6n+3}\,. \end{aligned}$$

Thus, by equations (7) and (8), we get

$$\begin{aligned} x^6 S &= x^6 S_1 + x^6 S_2 \\ &= x^6 f_{6n+5} + x^6 f_{6n+3} \\ &= (x^4 + 4x^2 + 2)A - 2(2x^4 + 9x^2 + 6)B + (2x^6 + 19x^4 + 48x^2 + 30)C \\ &\quad -2(x^4 + 3x^2)D + (3x^8 + 4x^6 - 41x^4 - 8x^2 - 40)E + 2(x^4 + 5x^2)F \\ &\quad + (x^4 + 3x^2)G - (x^{10} + x^8 - 25x^6 - 79x^4 - 82x^2 - 30)H \\ &\quad - (5x^8 + 26x^6 + 35x^4 + 11x^2)I + 2(x^6 + 4x^4 + 2x^2)J \\ &\quad - (x^4 + 3x^2)K - (5x^8 + 31x^6 + 59x^4 + 43x^2 + 12)L \\ &\quad + 2(x^{10} + 4x^8 + 2x^6 + 2x^2)M + (2x^8 + 14x^6 + 23x^4 - 5x^2)N \\ &\quad - 2(x^6 + 3x^4 - x^2)O + (x^4 + 4x^2 + 2)(x^2 + 1)^3 P \end{aligned}$$

$$- (x^8 + 5x^6 + 7x^4 + 5x^2)Q - (x^{10} + 4x^8 - x^6 - 14x^4 - 10x^2)R$$
$$- (4x^6 + 15x^4 + 7x^2)S + (x^6 + 4x^4 + 2x^2)T .$$

This value of $x^6 S$, together with its earlier version, yields equation (10), as desired.

In conclusion, we add that the graph-theoretic confirmations of the Pell versions of the gibonacci identities (5), (7), (9), and (10), and hence the corresponding numeric versions follow from the above arguments.

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