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SECOND ORDER FINITE DIFFERENCE SCHEME FOR ONE DIMENSIONAL SHALLOW WATER EQUATIONS

Abstract: In this paper, second order finite difference scheme for one dimensional shallow water equations is presented. Derivatives and functions appearing in one dimensional shallow water equations are approximated by finite differences evaluated at $t = t_{n+\frac{1}{2}}$. Finite difference method is proved to be consistent and is of second order in both space and time variables. Stability of the method is discussed. The numerical solutions obtained by proposed method are listed to demonstrate the reliability of method.

Keywords: Finite Difference Method, Shallow Water Equations.

Mathematical Subject Classification (2020) No.: 65M06.

1. Introduction

The shallow water equations are one of the simplest form of equations of motion that can be used to describe the horizontal structure of an atmosphere and ocean that model the propagation of disturbances in fluids. They are widely used to model the free surface water flows such as periodic flows (tidal), transient wave phenomena (tsunamis, flood waves, and dam break problems etc.).

Several explicit and implicit finite difference methods have been used to solve the shallow water equations. Shallow water equations are used to model tsunami wave propagation near coastal line and that model correctly predicts behavior of tsunami water wave. In [1], a modified two-four finite difference scheme is developed to solve shallow water equations for simulating dam break flows over wet and dry bed. Stability is discussed using CFL condition. For large value of time *t*, it is observed that Mohapatra and Choudhari scheme is unstable even though CFL condition is satisfied. In [2], two Predictor-Corrector methods are developed for one dimensional shallow water equations. Numerical methods used for this model are Lax Wendroff finite difference method and MacCormack finite difference method. Both methods are stable under CFL condition and are of second order. Numerical solutions developed by these methods seem to be capable of describing the propagation of flood wave after the failure of a reinforced concrete dam with an open channel downstream. In [3], the numerical schemes such as Mac cormack method, method of characteristics, Leap Frog and Lax Wendroff methods are used to simulate dam break flow and results are compared with analytic solution. Some of the methods discussed in [3] are stable and has best accuracy while some of them generate most unstable results. In [4], explicit finite difference method is developed to solve one dimensional shallow water equations. In [5], finite difference schemes, adoption of Roe's approximate Riemann solver and Q schemes of Bermudez and Vazquez are considered to obtain approximate solution of the Shallow water equations with good accuracy.

In this paper, we proposed a finite difference scheme for homogeneous one dimensional shallow water equations,

$$h_t + (hu)_x = 0; \tag{1.1}$$

$$(hu)_t + (hu^2 + \tfrac{1}{2} gh^2)_x = 0 \tag{1.2}$$

with initial conditions, $u(x,0) = g_1(x)$, $h(x,o) = g_2(x)$ and boundary conditions $u(x_a,t) = f_1(x)$, $u(x_b,t) = f_2(x)$, $h(x_a,t) = f_3(x)$, $h(x_b,t) = f_4(x)$. In equation (1.1) and (1.2), $h(x,t)$ and $u(x,t)$ represents the wavelength and the horizontal velocity of fluid (water) respectively. On eliminating h_t from equation (1.2), equations (1.1) and (1.2) get reduced to

$$h_t + hu_x + h_x u = 0; \tag{1.3}$$

$$u_t + uu_x + gh_x = 0 \tag{1.4}$$

Second order implicit numerical scheme is developed for equations (1.3) and (1.4) by approximating functions h, u and the partial derivatives at $t = (n + \frac{1}{2})\Delta t$. The paper is arranged as follows:

In section II, the difference scheme is developed for equations (1.3) and (1.4). It is proved that the method is consistent and is of second order in both space and time variables. In section III, stability of the proposed method is discussed. In section IV, numerical solutions of certain test problems are obtained by using proposed method. The results obtained by this method are compared with exact solutions and solutions obtained by different numerical schemes available in the literature.

2. Second Order Finite Difference Method for 1D SWEs

Consider equations (1.3) and (1.4) with initial conditions $u(x,0) = g_1(x)$, $h(x,o) = g_2(x)$ and boundary conditions $u(x_a,t) = f_1(x)$, $u(x_b,t) = f_2(x)$, $h(x_a,t) = f_3(x)$, $h(x_b,t) = f_4(x)$.

Let $x_a = x_0 < x_1 < x_2 < x_3 < \ldots < x_N = x_b$ be a uniform partition of $[x_a, x_b]$, where $x_i = x_0 + i\Delta x$, $\Delta x = \frac{x_b - x_a}{N}$ and $t_{n+1} = t_n + \Delta t$, where Δt is some increment and $t_0 = 0$. The numerical value of u at $t = t_n$ and $x = x_i$ is denoted by u_i^n whereas numerical value of h at $t = t_n$ and $x = x_i$ is denoted by h_i^n.

Implicit finite difference scheme for (1.3) and (1.4) is modeled at $t = t_{n+\frac{1}{2}}$. The time derivatives h_t and u_t are approximated by central difference of h and u at $t = t_{n+\frac{1}{2}}$. The function h and u at $t = t_{n+\frac{1}{2}}$ are approximated by the average value of h and u at t_n and t_{n+1} whereas u_x and h_x are approximated by average value of central difference at t_n and t_{n+1}.

Thus, finite difference scheme for (1.3) and (1.4) becomes

$$\frac{h_i^{n+1} - h_i^u}{\Delta t} + \left(\frac{h_i^{n+1} + h_i^u}{2}\right)\left(\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{4\Delta x} + \frac{u_{i+1}^n - u_{i-1}^n}{4\Delta x}\right)$$
$$+ \left(\frac{u_i^{n+1} + u_i^n}{2}\right)\left(\frac{h_{i+1}^{n+1} - h_{i-1}^{n+1}}{4\Delta x} + \frac{h_{i+1}^n - h_{i-1}^n}{4\Delta x}\right) = 0 \qquad (2.1)$$

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+\left(\frac{u_i^{n+1}+u_i^n}{2}\right)\left(\frac{u_{i+1}^{n+1}-u_{i-1}^{n+1}}{4\Delta x}+\frac{u_{i+1}^{n}-u_{i-1}^{n}}{4\Delta x}\right)$$
$$+g\left(\frac{h_{i+1}^{n+1}-h_{i-1}^{n+1}}{4\Delta x}+\frac{h_{i+1}^{n}-h_{i-1}^{n}}{4\Delta x}\right)=0 \qquad (2.2)$$

On rearranging equations (2.1) and (2.2), we have

$$-rA_i^n h_{i-1}^{n+1}+h_i^{n+1}+rA_i^n h_{i+1}^{n+1}-rB_i^n u_{i-1}^{n+1}+rB_i^n u_{i+1}^{n+1}$$
$$=rA_i^n h_{i-1}^{n}+h_i^{n}-rA_i^n h_{i+1}^{n}+rB_i^n u_{i-1}^{n}-rB_i^n u_{i+1}^{n} \qquad (2.3)$$

$$-rgh_{i-1}^{n+1}+rgh_{i+1}^{n+1}-rA_i^n u_{i-1}^{n+1}+u_i^{n+1}+rA_i^n u_{i+1}^{n+1}$$
$$=rgh_{i-1}^{n}-rgh_{i+1}^{n}+rA_i^n u_{i-1}^{n}+u_i^{n}-rA_i^n u_{i+1}^{n} \qquad (2.4)$$

where $A_i^n=\frac{u_i^{n+1}+u_i^n}{2}$ $B_i^n=\frac{h_i^{n+1}+h_i^n}{2}$ and $r=\frac{\Delta t}{4\Delta x}$.

Substituting expansions of h and u in Taylor series at $x=x_i$ and $t=t_n$ in equations (2.1) and (2.2) and rearranging them we get following equations

$$h_t+hu_x+h_xu+\frac{\Delta t}{2}(h_t+hu_x+h_xu)_t$$
$$+\frac{(\Delta t)^2}{4}\left(\frac{2}{3}h_{ttt}+hu_{xtt}+h_tu_{xt}+h_{tt}u_x+h_{xtt}u+h_{xt}u_t+h_xu_{tt}\right)$$
$$+\frac{(\Delta x)^2}{3!}(hu_{xxx}+h_{xxx}u)+o((\Delta t)^3,(\Delta x)^2(\Delta t))=0 \qquad (2.5)$$

$$ut+uu_x+gh_x+\frac{\Delta t}{2}(ut+uu_x+gh_x)_t+\frac{(\Delta t)^2}{4}\left(\frac{2}{3}u_{ttt}+uu_{xtt}+u_xu_{tt}+u_tu_{xt}+gh_{xtt}\right)$$
$$+\frac{(\Delta x)^2}{3!}(uu_{xxx}+gh_{xxx})+o((\Delta t)^3,(\Delta x)^2(\Delta t))=0 \qquad (2.6)$$

In the view of equation (1.3), (2.5) and equations (1.4), (2.6) it is observed that the finite difference scheme (2.3), (2.4) is consistent and of order 2 in both space and time variables.

3. Stability of Finite difference Scheme

To analyze stability of numerical scheme (2.3) and (2.4), consider linearized form of equations (2.3) and (2.4) as follows

$$-rAh_{i-1}^{n+1}+h_i^{n+1}+rAh_{i+1}^{n+1}-rBu_{i-1}^{n+1}+rBu_{i+1}^{n+1}$$
$$=rAh_{i-1}^{n}+h_i^{n}-rAh_{i+1}^{n}+rBu_{i-1}^{n}-rBu_{i+1}^{n} \quad (3.1)$$

$$-rgh_{i-1}^{n+1}+rgh_{i+1}^{n+1}-rAu_{i-1}^{n+1}+u_i^{n+1}+rAu_{i+1}^{n+1}$$
$$=rgh_{i-1}^{n}-rgh_{i+1}^{n}+rAu_{i-1}^{n}+u_i^{n}-rAu_{i+1}^{n} \quad (3.2)$$

where $A=A_i^n$ and $B=B_i^n$

The error equations corresponding to (3.1) and (3.2) gives system of equations

$$\epsilon \qquad M\begin{bmatrix}\epsilon_1^{n+1}\\ \epsilon_2^{n+1}\end{bmatrix}=N\begin{bmatrix}\epsilon_1^{n}\\ \epsilon_2^{n}\end{bmatrix} \quad (3.3)$$

where $\varepsilon_1^n=\begin{bmatrix}\epsilon_{11}^n\\ \epsilon_{21}^n\\ \vdots\\ \epsilon_{(N-1)1}^n\end{bmatrix}$ and $\varepsilon_2^n=\begin{bmatrix}\epsilon_{12}^n\\ \epsilon_{22}^n\\ \vdots\\ \epsilon_{(N-1)2}^n\end{bmatrix}$

$\epsilon_{k1}^n=H_k^n-h_k^n,\ \epsilon_{k2}^n=U_k^n-u_k^n,$

$H_k^n,\ U_k^n$ are exact solutions of (3.1), (3.2) at $t=t_n$, $x=x_k$.

The matrices M, N are block matrices

$$M=\begin{bmatrix}P & Q\\ R & P\end{bmatrix} \text{ and } N=\begin{bmatrix}P^T & Q^T\\ R^T & P^T\end{bmatrix}$$

where P Q and R are tridiagonal matrices given by

$$P = \begin{bmatrix} 1 & rA & 0 & \dots & 0 & 0 \\ -rA & 1 & rA & \dots & 0 & 0 \\ 0 & -rA & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -rA & 1 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0 & rB & 0 & \dots & 0 & 0 \\ -rB & 1 & rB & \dots & 0 & 0 \\ 0 & -rB & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -rB & 0 \end{bmatrix},$$

$$R = \begin{bmatrix} 0 & rg & 0 & \dots & 0 & 0 \\ -rg & 0 & rg & \dots & 0 & 0 \\ 0 & -rg & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -rg & 0 \end{bmatrix}.$$

Now matrix equation (3.3) can be written as

$$\begin{bmatrix} \varepsilon_1^{n+1} \\ \varepsilon_2^{n+1} \end{bmatrix} = M^{-1} N \begin{bmatrix} \varepsilon_1^{n} \\ \varepsilon_2^{n} \end{bmatrix} \tag{3.4}$$

Matrix M can be written as $M = \begin{bmatrix} P & Q \\ R & P \end{bmatrix} = \begin{bmatrix} I+K & Q \\ R & I+K \end{bmatrix}$

Eigenvalues of M are given by roots of characteristic equation,

$$det(M - \lambda I) = 0$$

i.e.

$$\begin{vmatrix} I + K - \lambda I & Q \\ R & I + K - \lambda I \end{vmatrix} = 0 .$$

Since, $(I + K - \lambda I)R = R(I + K - \lambda I)$, $QR = RQ$, from [6] the characteristic equation of matrix M becomes,

$$det\left[(\lambda - 1)^2 I + 2(\lambda - 1)K + K^2 - QR\right] = 0$$

Simillarly, eigenvalues of N are given by roots of characteristic equation,

$$det\left[(\lambda - 1)^2 I + 2(\lambda - 1)K + K^2 - QR\right] = 0$$

Since characteristic polynomials of matrices M and N are identical, M and N have same eigenvalues.

By Brauer's Theorem [7], all the eigenvalues of M and N denoted by λ lie in the region,

$$\left|\lambda - 1\right| \leq 2r \max \{\left|A\right| + \left|B\right|, \left|A\right| + g\}$$

Therefore *Minimum eigenvalue of* $M \geq 1 - 2r \max \{\left|A\right| + \left|B\right|, \left|A\right| + g\}$ and *Maximum eigenvalue of* $N \leq 1 + 2r \max \{\left|A\right| + \left|B\right|, \left|A\right| + g\}$.

Thus, the spectral radius,

$$\rho(M^{-1}N) \leq \frac{1 + 2r \max \{\left|A\right| + \left|B\right|, \left|A\right| + g\}}{1 - 2r \max \{\left|A\right| + \left|B\right|, \left|A\right| + g\}}$$

$$\approx 1 + 4r \max \{\left|A\right| + \left|B\right|, \left|A\right| + g\}$$

Therefore $\rho(M^{-1}N) \geq 1$ and the proposed method is unconditionally unstable. Though the method is unstable, $\rho(M^{-1}N)$ lies in neighbourhood of 1 since value of r is very small and the numerical scheme given by equation (2.3) and (2.4) will give reliable solutions.

4. Numerical Experiments

Problem 4.1: Consider one dimensional shallow water equations (1.3) and

(1.4) defined on the domain $D = \{x \,/\, -1 \le x \le 1\}$ with $g = 1$ satisfying initial conditions

$$u(x,0) = \frac{2}{3}(x \ -1), \qquad h(x,0) = \frac{1}{4} + \frac{4}{9}(x-1)^2$$

and boundary conditions

$$u(-1,t) = \frac{-4}{3(t+1)}, \ \ u(1,t) = 0,$$

$$h(-1,t) = \frac{1}{4}\frac{1}{(t+1)^{2/3}} + \frac{16}{9(t+1)^2}, \ \ h(1,t) = \frac{1}{4}\frac{1}{(t+1)^{2/3}}$$

Exact solutions u and h denoted by U and H of equations (1.3) and (1.4) with above intial and boundary conditions are obtained from [8]

$$U(x,t) = \frac{2(x-1)}{3(t+1)}, \quad H(x,t) = \frac{1}{4}\frac{1}{(t+1)^{2/3}} + \frac{4}{9}\frac{(x-1)^2}{(t+1)^2} \tag{4.1}$$

The numerical solution for this problem is obtained from the finite difference scheme (2.3) and (2.4) with $\Delta t = 0.01$, 0.001 and $\Delta x = 0.01$, $0.05, 0.1$. These numerical solutions h, hu are compared with exact solutions H, HU. The comparision of the solutions obtained by difference scheme (2.3) and (2.4) and exact solutions at different time t are shown in Table (1) to Table (5). From Table (1) to Table (5), it is observed that the solutions obtained from the scheme (2.3) - (2.4) are correct upto three decimal places.

Table 1: Comparison of numerical and exact solution for
$\Delta t = 0.01$, $\Delta x = 0.01$, at $t = 0.5$

x	h	H	hu	HU
-1.	0.388317	0.388317	-0.34517	-0.34517
-0.8	0.350737	0.350786	-0.280631	-0.280629
-0.6	0.317147	0.317205	-0.22555	-0.225568
-0.4	0.287485	0.287576	-0.178875	-0.178936

x	h	H	hu	HU
-0.2	0.261766	0.261897	-0.139574	-0.139678
0.	0.240066	0.240168	-0.10667	-0.106742
0.2	0.222311	0.222391	-0.0790246	-0.0790722
0.4	0.208502	0.208563	-0.055587	-0.0556169
0.6	0.198639	0.198687	-0.0353049	-0.0353221
0.8	0.192727	0.192761	-0.017129	-0.0171343
1.	0.190786	0.190786	0.	0.

Table 2: Comparison of numerical and exact solution for
$\Delta t = 0.01$, $\Delta x = 0.01$, at $t = 1$

x	h	H	hu	HU
-1.	0.268601	0.268601	-0.179067	-0.179067
-0.8	0.247485	0.24749	-0.148523	-0.148494
-0.6	0.228595	0.228601	-0.121964	-0.121921
-0.4	0.211942	0.211935	-0.0989507	-0.0989028
-0.2	0.197498	0.19749	-0.0790361	-0.0789961
0.	0.185248	0.185268	-0.0617678	-0.061756
0.2	0.175191	0.175268	-0.0467069	-0.0467381
0.4	0.167426	0.16749	-0.0334769	-0.033498
0.6	0.16189	0.161935	-0.0215834	-0.0215913
0.8	0.158586	0.158601	-0.0105703	-0.0105734
1.	0.15749	0.15749	0.	0.

Table 3: Comparison of numerical and exact solution for
$\Delta t = 0.01$, $\Delta x = 0.05$, at $t = 0.5$

x	h	H	hu	HU
-1.	0.388317	0.388317	-0.34517	-0.34517
-0.8	0.350742	0.350786	-0.280634	-0.280629
-0.6	0.31715	0.317205	-0.225554	-0.225568
-0.4	0.287472	0.287576	-0.17886	-0.178936
-0.2	0.26177	0.261897	-0.139578	-0.139678
0.	0.240066	0.240168	-0.10667	-0.106742
0.2	0.222311	0.222391	-0.0790246	-0.0790722
0.4	0.208502	0.208563	-0.055587	-0.0556169
0.6	0.198639	0.198687	-0.035305	-0.0353221
0.8	0.192728	0.192761	-0.0171282	-0.0171343
1.	0.190786	0.190786	0.	0.

Table 4: Comparison of numerical and exact solution for
$\Delta t = 0.01$, $\Delta x = 0.05$, at $t = 1$

x	h	H	hu	HU
-1.	0.268601	0.268601	-0.179067	-0.179067
-0.8	0.24748	0.24749	-0.148523	-0.148494
-0.6	0.228597	0.228601	-0.121966	-0.121921
-0.4	0.211945	0.211935	-0.0989532	-0.0989028
-0.2	0.197496	0.19749	-0.0790344	-0.0789961
0.	0.185245	0.185268	-0.0617652	-0.061756
0.2	0.175197	0.175268	-0.0467112	-0.0467381
0.4	0.167428	0.16749	-0.0334776	-0.033498
0.6	0.16189	0.161935	-0.0215824	-0.0215913
0.8	0.158587	0.158601	-0.0105703	-0.0105734
1.	0.15749	0.15749	0.	0.

Table 5: Comparison of numerical and exact solution for
$\Delta t = 0.001$, $\Delta x = 0.1$, at $t = 0.5$

x	h	H	hu	HU
-1.	0.388317	0.388317	-0.34517	-0.34517
-0.8	0.350781	0.350786	-0.280629	-0.280629
-0.6	0.3172	0.317205	-0.225567	-0.225568
-0.4	0.287565	0.287576	-0.178928	-0.178936
-0.2	0.261884	0.261897	-0.139669	-0.139678
0.	0.240158	0.240168	-0.106734	-0.106742
0.2	0.222382	0.222391	-0.0790674	-0.0790722
0.4	0.208557	0.208563	-0.0556139	-0.0556169
0.6	0.198682	0.198687	-0.0353204	-0.0353221
0.8	0.192758	0.192761	-0.0171337	-0.0171343
1.	0.190786	0.190786	0.	0.

Problem 4.2: Consider one dimensional shallow water equations (1.3) and (1.4) defined on the domain $D = \{x \,/ -5 \le x \le 5\}$ with $g = 1$ satisfying initial conditions

$$u(x,0) = 0,\;\; h(x,0) = 1 + \frac{2}{5}e^{(-5)x^2}$$

and boundary conditions

$$u(-5,t) = 0,\;\; u(5,t) = 0,\;\; h(-5,t) = 1,\;\; h(5,t) = 1.$$

The example models dam break problem. Solution graphs are shown in Figure (1) to Figure (10).

In Figures (1) to (10), figures with odd number are from [9] whereas figures with even number are constructed from numerical scheme (2.3–2.4). From all figures it is seen that the solutions obtained by the proposed numerical scheme are reliable. It is observed that as time passes the initial condition produces two waves one moving in each direction. Each of the wave shows the same behavior at all time in future. Initially the momentum is zero since the velocity in x direction is zero. As time passes momentum curve also shows two waves one moving in each direction but mirror image about X axis.

Since the numerical solutions of problem (4.2) are not available in literature the numerical solutions of (4.2) by using finite difference scheme at different time levels are listed in Table (6) and (7).

Table 6: Numerical solution for $\Delta t = 0.1$, $\Delta x = 0.1$, at $t = 0.5$

x	h	u	hu
-5	1	0	0
-4	1.	-1:65115 × 10-15	-1:65115 × 10-15
-3	1.	-4:10305 × 10-10	-4:10305 × 10-10
-2	1.00003	-0.0000294016	-0.0000294024
-1	1.06608	-0.0651121	-0.0694147
0	1.1128	8:48731 × 10-19	9:444655951555686 × 10-19
1	1.06608	0.0651121	0.0694147
2	1.00003	0.0000294016	0.0000294024
3	1.	4:10305 × 10-10	4:10305 × 10-10
4	1.	1:21870 × 10-15	1:21870 × 10-15
5	1	0	0

Table 7: Numerical solution for $\Delta t = 0.1$, $\Delta x = 0.1$, at $t = 1$

x	h	u	hu
-5	1	0	0
-4	1.	-2:25827 × 10-11	-2:25827 × 10-11
-3	1.	-8:30577 × 10-7	-8:30578 × 10-7
-2	1.00399	-0.00398586	-0.00400176

x	h	u	hu
-1	1.1798	-0.173156	-0.20429
0	0.996458	2:35858 × 10-16	2:35023 × 10-16
1	1.1798	0.173156	0.20429
2	1.00399	0.00398586	0.00400176
3	1.	8:30577 × 10-7	8:30578 × 10-7
4	1.	2:25824 × 10-11	2:25824 × 10-11
5	1	0	0

Finite difference Model Results

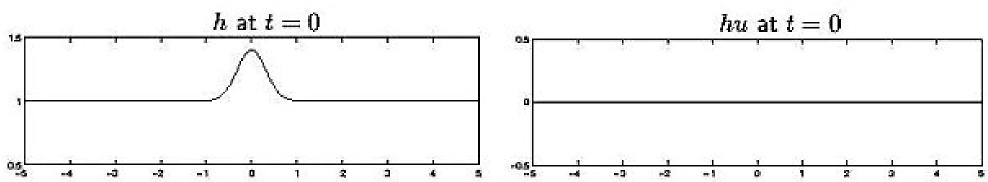

Figure 1: Graph of h and hu from [9] at $t = 0$

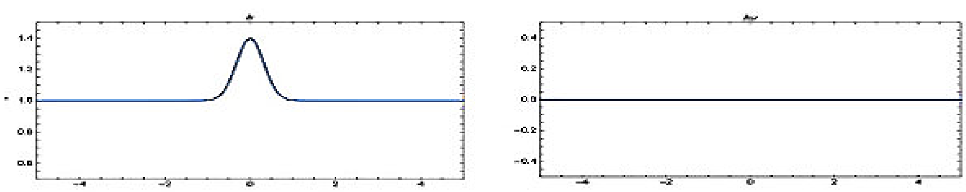

Figure 2: Graph of h and hu from proposed finite difference scheme at $t = 0$

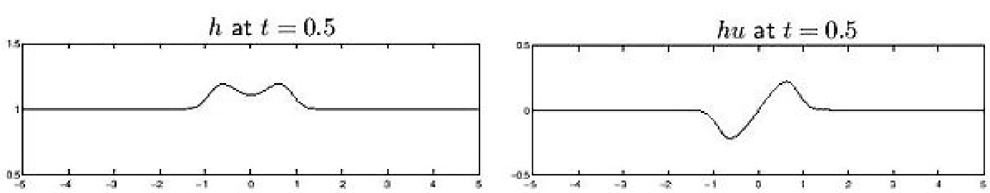

Figure 3: Graph of h and hu from [9] at $t = 0.5$

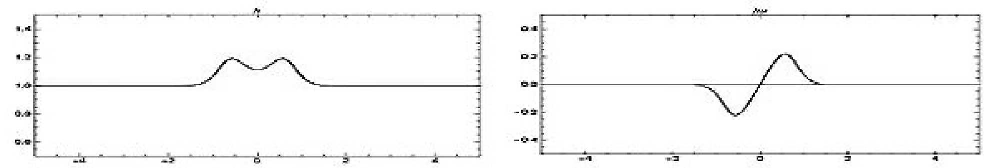

Figure 4: Graph of h and hu from proposed finite difference scheme at $t = 0.5$

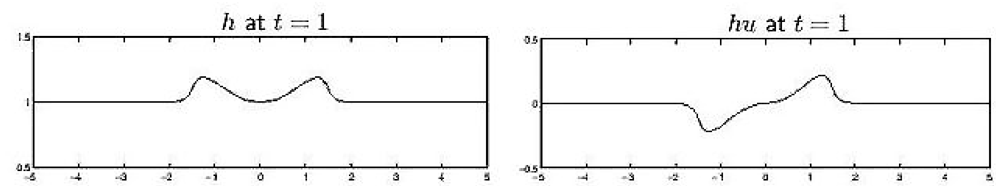

Figure 5: Graph of h and hu from [9] at $t = 1$

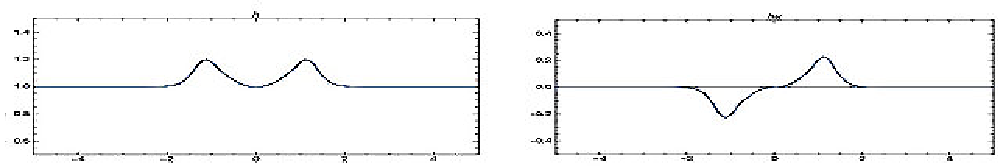

Figure 6: Graph of h and hu from proposed finite difference scheme at $t = 1$

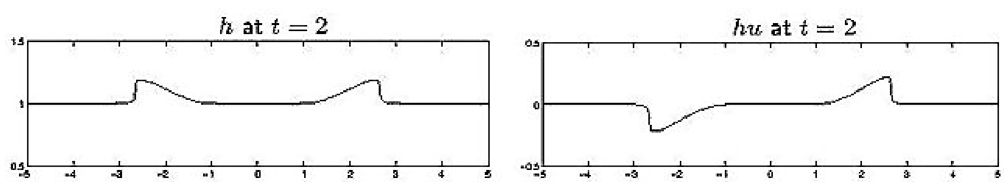

Figure 7: Graph of h and hu from [9] at $t = 2$

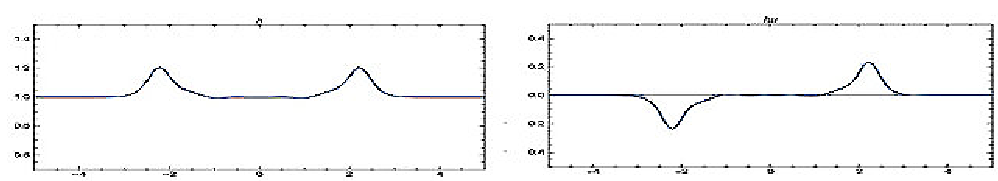

Figure 8: Graph of h and hu from proposed finite difference scheme at $t = 2$

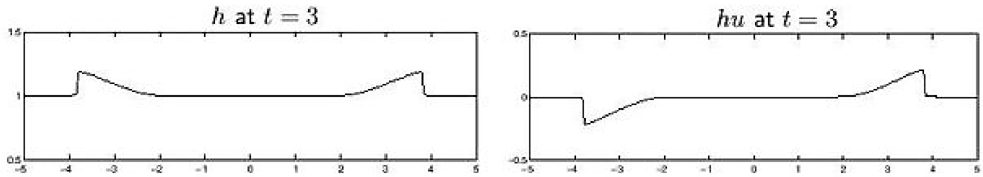

Figure 9: Graph of h and hu from [9] at $t = 3$

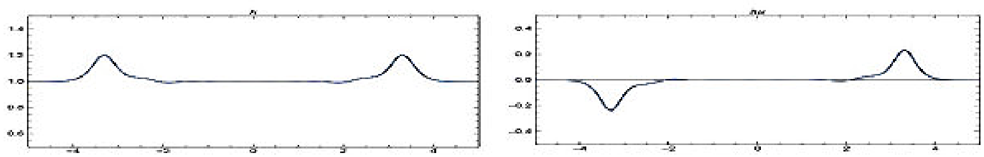

Figure 10: Graph of h and hu from proposed finite difference scheme at $t = 3$

5. Conclusion

The finite difference scheme is presented for one dimensional Shallow Water Equations. The method is proved to be consistent and is of order two in both space and time variables. Though method is unstable, it gives reliable solutions. The reliability of solutions is assured by comparing numerical solutions with exact solutions.

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GRAPH-THEORETIC CONFIRMATIONS OF THREE SUMS OF GIBONACCI POLYNOMIAL PRODUCTS OF ORDER 3

Abstract: Using graph-theoretic tools, we confirm three identities involving sums of gibonacci polynomial products of order 3, investigated in [5].

Keywords: Fibonacci polynomial, Lucas.

Mathematical Subject Classification (2110) No.: Primary 05A19, 11B37, 11B39, 11Cxx.

1. Introduction

Gibonacci polynomials $z_n(x)$ are defined by the recurrence, $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$ where x is an arbitrary complex variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary complex polynominals; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*.

Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2, 3].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. We let $g_n = f_n$ or l_n and $\Delta^2 = x^2 + 4$. We also omit a lot of basic algebra.

It is well known [3] that $f_{n+2} + f_{n-2} = (x^2+2)f_n$, $f_{n+2} - f_{n-2} = xl_n$, $f_{n+1} + f_{n-1} = l_n$, $f_n = (x^2+1)f_{n-2} + xf_{n-3}$, $f_{2n} = f_n l_n$, $f_{2n+1} = f_{2+1}^n + f_n^2$, and the gibonacci addition formula

$$g_{a+b} = f_{a+1}g_b + f_a g_{b-1}.$$

A *gibonacci polynomial product of order m* is a product of gibonacci polynomials g_{n+k} of the form $\prod_{k\in\mathcal{Z}} g_{n+k}^{sj}$, where $\sum_{s_j\geq 1} s_j = m$ [4, 6].

1.1 Sums of Gibonacci Polynomial Products of Order 3: In [5], we explored the following sums of gibonacci polynomial products of order 3:

$$x^2 f_{3n} = 3f_{n+2}^2 f_n - (2x^2+5)f_{n+2}f_n^2 + (x^2+1)f_n^3 + (x^2+1)f_n^2 f_{n-2}. \tag{1}$$

$$x^2 l_{3n+1} = f_{n+2}^3 + 3f_{n+2}^2 f_n - (2x^2+7)f_{n+2}f_n^2 + (x^2+1)f_n^3 + (x^2+2)f_n^2 f_{n-2}. \tag{2}$$

$$\begin{aligned}\Delta^2 x^2 f_{3n+2} = (x^2+4)f_{n+2}^3 + 2x^2 f_{n+2}^2 f_n - 2(x^4+3x^2+4)f_{n+2}f_n^2 + x^2 f_{n+2}f_n f_{n-2} \\ + (x^4+3x^2+4)f_n^2 f_{n-2} - x^2 f_n f_{n-2}^2.\end{aligned} \tag{3}$$

2. Some Graph-theoretic Tools

To confirm these three polynomial identities using graph-theoretic techniques, we now develop the needed tools. To this end, consider the *Fibonacci digraph* D_1 in Figure 1 with vertices v_1 and v_2, where a *weight* is assigned to each edge [3, 4]. It follows by induction from its *weighted adjacency matrix* $Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$, that

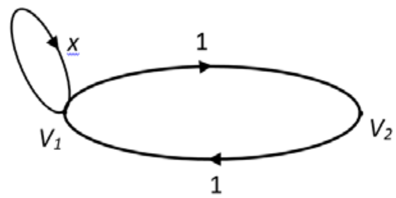

Figure 1: Weighted Fibonacci Digraph D_1

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $n \geq 1$ [3, 4].

A *walk* from vertex v_i to vertex v_j is a sequence $v_i - e_i - v_{i+1} - \cdots - v_{j-1} - e_{j-1} - v_j$ of vertices v_k and edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The walk is *closed* if $v_i = v_j$; otherwise, it is *open*. The *length* of a walk is the number of edges in the walk. The *weight* of a walk is the product of the weights of the edges along the walk.

We can employ the matrix Q^n to compute the weight of a walk of length n from any vertex v_i to any vertex v_j, as the following theorem shows [3, 4].

Theorem 1: *Let M be the weighted adjacency matrix of a weighted, connected digraph with vertices $v_1, v_2, \ldots, v_k$. Then the ijth entry of the matrix M^n gives the sum of the weights of all walks of length n from v_i to v_j, where $n \geq 1$.* □

The next result follows from this theorem.

Corollary 1: *The ijth entry of Q^n gives the sum of the weights of all walks of length n from v_i to v_j in the weighted digraph D_1, where $1 \leq i$, $j \leq n$.* □

It follows by this corollary that the sum of the weights of closed walks of length n originating at v_1 in the digraph is f_{n+1} and that of those originating at v_2 is

f_{n-1}. Consequently, the sum of the weights of all closed walks of length n in the digraph is $f_{n+1} + f_{n-1} = l_n$. These facts play a crucial role in the graph-theoretic proofs.

Let A, B, and C denote the sets of walks of lengths r, s, and t all originating at a vertex v, respectively. Then the sum of the weights of the elements (a, b, c) in the product set $A \times B \times C$ is *defined* as the product of the sums of weights from each component [4].

We are now ready for the proofs.

3. Graph-theoretic Proofs

3.1 Proof of Identity (1): Let S denote the sum of the weights of closed walks of length $3n-1$ in the digraph from v_1 to v_2. Then $S = f_{3n}$, and hence, $x^2 S = x^2 f_{3n}$.

We will now compute the sum $x^2 S$ in a different way. To this end, let w be an arbitrary closed walk of length $3n-1$ from v_1 to v_1. It can land at v_1 or v_2 at the nth and $2n$th steps: $w = \underbrace{v_1 - \ldots - x}_{\text{subwalk of length } n} \; \underbrace{x - \ldots - x}_{\text{subwalk of length } n} \; \underbrace{x - \ldots - v_1}_{\text{subwalk of length } n-1}$,

where $x = v_1$ or v_2.

Table 1: Sum of the Weights of Closed Walks Originating at v_1

w lands at v_1 at the nth step?	w lands at v_1 at the $2n$th step?	w lands at v_1 at the $(3n-1)$ st step?	sum of the weights of walks w
yes	yes	yes	$f_{n+1}^2 f_n$
yes	no	yes	$f_{n+1} f_n f_{n-1}$
no	yes	yes	f_n^3
no	no	yes	$f_n f_{n-1}^2$

Table 1 shows the possible cases and the sums of weights of the corresponding walks. It follows from the table that the sum S of the weights of all closed walks originating at v_1 is given by

$$S = f_{n+1}^2 f_n + f_{n+1} f_n f_{n-1} + f_n^3 + f_n f_{n-1}^2 .$$

Then

$$\begin{aligned} x^2 S &= (x f_{n+1})^2 f_n + (x f_{n+1}) f_n (x f_{n-1}) + x^2 f_n^3 + f_n (x f_{n-1})^2 \\ &= (f_{n+2} - f_n)^2 f_n + (f_{n+2} - f_n) f_n (f_n - f_{n-2}) + x^2 f_n^3 + f_n (f_n - f_{n-2})^2 \\ &= f_{n+2}^2 f_n - f_{n+2} f_n^2 - f_{n+2} f_n f_{n-2} + (x^2 + 1) f_n^3 - f_n^2 f_{n-2} + f_n f_{n-2}^2 \\ &= 3 f_{n+2}^2 f_n - f_{n+2} f_n^2 + (x^2 + 1) f_n^3 + A , \end{aligned}$$

where

$$\begin{aligned} A &= -2 f_{n+2}^2 f_n - f_{n+2} f_n f_{n-2} + f_n f_{n-2}^2 - f_n^2 f_{n-2} \\ &= -f_{n+2} f_n (f_{n+2} + f_{n-2}) - f_n (f_{n+2}^2 - f_{n-2}^2) - f_n^2 f_{n-2} \\ &= -f_n (f_{n+2} + f_{n-2})(2 f_{n+2} - f_{n-2}) - f_n^2 f_{n-2} \\ &= -f_n \left[(x^2 + 2) f_n \right] (2 f_{n+2} - f_{n-2}) - f_n^2 f_{n-2} \\ &= -2(x^2 + 2) f_{n+2} f_n^2 + (x^2 + 1) f_n^2 f_{n-2}. \end{aligned}$$

Consequently,

$$x^2 S = 3 f_{n+2}^2 f_n - (2x^2 + 5) f_{n+2} f_n^2 + (x^2 + 1) f_n^3 + (x^2 + 1) f_n^2 f_{n-2}.$$

Equating this value of $x^2 S$ with the earlier one yields the desired result. □

We now turn to identity (2).

3.2 Proof of Identity (2): Let S' denote the sum of the weights of all closed walks of length $3n + 1$ in the digraph. Then $S' = l_{3n+1}$; so $x^2 S' = x^2 l_{3n+1}$.

We will now compute x^2S' in a different way. Let w be an arbitrary closed walk of length $3n+1$.

Case 1: Suppose w originates (and ends) at v_1. It can then land at v_1 or v_2 at the $(n+1)$st and $(2n+1)$st steps:

$$w = \underbrace{v_1 - \ldots - x}_{\text{subwalk of length } n+1} \quad \underbrace{x - \ldots - x}_{\text{subwalk of length } n} \quad \underbrace{x - \ldots - v_1}_{\text{subwalk of length } n},$$

where $x = v_1$ or v_2.

Table 2: Sums of the Weights of Closed Walks Originating at v_1

w lands at v_1 at the $(n+1)$th step?	w lands at v_1 at the $(2n+1)$th step?	w lands at v_1 at the $(3n+1)$st step?	sum of the weights of walks w
yes	yes	yes	$f_{n+2}f_{n+1}^2$
yes	no	yes	$f_{n+2}f_n^2$
no	yes	yes	$f_{n+1}^2f_n$
no	no	yes	$f_{n+1}f_nf_{n-1}$

Table 2 shows the possible cases and the sums of weights of the respective walks. It follows from the table that the sum S_1' of the weights of such walks w is given by

$$\begin{aligned} S_1' &= f_{n+2}f_{n+1}^2 + f_{n+2}f_n^2 + f_{n+1}^2f_n + f_{n+1}f_nf_{n-1} \\ &= f_{n+2}(f_{n+1}^2 + f_n^2) + f_{n+1}f_n(f_{n+1} + f_{n-1}) \\ &= f_{n+2}(f_{n+1}^2 + f_n^2) + f_{n+1}f_nl_n; \end{aligned}$$

$$\begin{aligned} x^2S_1' &= f_{n+2}\left[(xf_{n+1})^2 + (xf_n)^2\right] + (xf_{n+1})f_n(xl_n) \\ &= f_{n+2}\left[(f_{n+2} - f_n)^2 + x^2f_n^2\right] + f_n(f_{n+2} - f_{n-2})(f_{n+2} - f_n) \end{aligned}$$

$$= f_{n+2}^3 - f_{n+2}f_n\left[(f_{n+2} + f_{n-2}) - x^2 f_n\right] + f_n^2 f_{n-2}$$

$$= f_{n+2}^3 - f_{n+2}f_n \left[(x^2 + 2)f_n - x^2 f_n\right] + f_n^2 f_{n-2}$$

$$= f_{n+2}^3 - 2f_{n+2}f_n^2 + f_n^2 f_{n-2}.$$

Case 2: Suppose w originates (and ends) at v_2. Then w can land at v_1 or v_2 at the $(n+1)$st and $(2n+1)$st steps:

$$w = \underbrace{v_2 - \ldots - x}_{\text{subwalk of length } n+1} \underbrace{x - \ldots - x}_{\text{subwalk of length } n} \underbrace{x - \ldots - v_2}_{\text{subwalk of length } n},$$

where $x = v_1$ or v_2.

Table 3: Sums of the Weights of Closed Walks Originating at v_2

w lands at v_1 at the $(n+1)$th step?	w lands at v_1 at the $(2n+1)$th step?	w lands at v_2 at the $(3n+1)$st step?	sum of the weights of walks w
yes	yes	yes	$f_{n+1}^2 f_n$
yes	no	yes	$f_{n+1} f_n f_{n-1}$
no	yes	yes	f_n^3
no	no	yes	$f_n f_{n-1}^2$

Table 3 shows the possible cases and the corresponding sums of weights of the walks. Clearly, the sum S_2' of the weights of all such walks w is given by

$$S_2' = f_{n+1}^2 f_n + f_{n+1} f_n f_{n-1} + f_n^3 + f_n f_{n-1}^2.$$

By the algebraic proof in Section 2.1, we then have

$$x^2 S_2' = 3f_{n+2}^2 f_n - (2x^2 + 5)f_{n+2}f_n^2 + (x^2 + 1)f_n^3 + (x^2 + 1)f_n^2 f_{n-2};$$

$$x^2 S' = x^2 S_1' + x^2 S_2'$$

$$= (f_{n+2}^3 - 2f_{n+2}f_n^2 + f_n^2 f_{n-2})$$

$$+\left[3f_{n+2}^2 f_n - (2x^2+5)f_{n+2}f_n^2 + (x^2+1)f_n^3 + (x^2+1)f_n^2 f_{n-2}\right]$$

$$= f_{n+2}^3 + 3f_{n+2}^2 f_n - (2x^2+7)f_{n+2}f_n^2 + (x^2+1)f_n^3 + (x^2+2)f_n^2 f_{n-2}.$$

This value of x^2S', coupled with its original value, yields the desired result, as expected. □

Finally, we present the proof of identity (3).

3.3 Proof of Identity (3): Let S^* denote the sum of the weights of closed walks of length $3n+1$ originating at v_1 in the digraph. Clearly, $S^* = f_{3n+2}$, and hence, $\Delta^2 x^2 S^* = \Delta^2 x^2 f_{3n+2}$.

To compute $\Delta^2 x^2 S^*$ in a different way, we first let w be an arbitrary closed walk of length $3n+1$ originating at v_1.

Case 1: Suppose w begins with a loop. It can then land at v_1 or v_2 at the $(n+1)$st and $(2n+1)$st steps:

$$w = \underbrace{v_1 - \ldots - x}_{\text{subwalk of length } n+1} \quad \underbrace{x - \ldots - x}_{\text{subwalk of length } n} \quad \underbrace{x - \ldots - v_1}_{\text{subwalk of length } n},$$

where $x = v_1$ or v_2.

Table 4: Sums of the Weights of Closed Walks Beginning with a Loop

w lands at v_1 at the $(n+1)$th step?	w lands at v_1 at the $(2n+1)$th step?	w lands at v_1 at the $(3n+1)$st step?	sum of the weights of walks w
yes	yes	yes	xf_{n+1}^3
yes	no	yes	$xf_{n+1}f_n^2$
no	yes	yes	$xf_{n+1}f_n^2$
no	no	yes	$xf_n^2 f_{n-1}f_n$

Table 4 shows the various possible cases and the respective sums of weights of such walks. It then follows that the sum S_1^* of the weights of walks w is given by

$$\begin{aligned} S_1^* &= xf_{n+1}^3 + 2xf_{n+1}f_n^2 + xf_n^2 f_{n-1}f_n \\ &= xf_{n+1}(f_{n+1}^2 + f_n^2) + xf_n^2(f_{n+1} + f_{n-1}) \\ &= xf_{n+1}(f_{n+1}^2 + f_n^2) + xf_n^2 ln \\ &= xf_{n+1}f_{2n+1} + xf_n f_{2n} \\ &= xf_{3n+1}. \end{aligned}$$

Case 2: Suppose w does not begin with a loop. Then also it can land at v_1 or v_2 at the $(n+1)$st and $(2n+1)$st steps:

$$w = \underbrace{v_1 - \ldots - x}_{\text{subwalk of length } n+1} \quad \underbrace{x - \ldots - x}_{\text{subwalk of length } n} \quad \underbrace{x - \ldots - v_1}_{\text{subwalk of length } n},$$

where $x = v_1$ or v_2.

Table 5: Sums of the Weights of Closed Walks not Beginning a Loop

w lands at v_1 at the $(n+1)$th step?	w lands at v_1 at the $(2n+1)$th step?	w lands at v_1 at the $(3n+1)$st step?	sum of the weights of walks w
yes	yes	yes	$f_{n+1}^2 f_n$
yes	no	yes	f_n^3
no	yes	yes	$f_{n+1}f_n f_{n-1}$
no	no	yes	$f_n f_{n-1}^2$

Table 5 shows the possible cases and the corresponding sums of weights of such walks. It follows from the table that the sum S_2^* of the weights of such walks w is given by

$$
\begin{aligned}
S_2^* &= f_{n+1}^2 f_n + f_{n+1} f_n f_{n-1} + f_n^3 + f_n f_{n-1}^2 \\
&= f_n (f_{n+1}^2 + f_n^2) + f_n f_{n-1}(f_{n+1} + f_{n-1}) \\
&= f_{2n+1} f_n + f_{2n} f_{n-1} \\
&= f_{3n}.
\end{aligned}
$$

By the identities [5]

$$
\begin{aligned}
\Delta^2 x^2 f_{3n} &= (x^2+12) f_{n+2}^2 f_n - (9x^2+20) f_{n+2} f_n^2 - x^2 f_{n+2} f_n f_{n-2} \\
&\quad + (x^2+1)(x^2+4) f_n^3 + (3x^2+4) f_n^2 f_{n-2} + x^2 f_n f_{n-2}^2; \\
\Delta^2 x^3 f_{3n+1} &= (x^2+4) f_{n+2}^3 + (x^2-12) f_{n+2}^2 f_n - (2x^4 - 3x^2 - 12) f_{n+2} f_n^2 \\
&\quad + 2x^2 f_{n+2} f_n f_{n-2} - (x^2+1)(x^2+4) f_n^3 + x^4 f_n^2 f_{n-2} - 2x^2 f_n f_{n-2}^2,
\end{aligned}
$$

we then get

$$
\begin{aligned}
\Delta^2 x^2 S^* &= \Delta^2 x^2 (S_1^* + S_2^*) \\
&= \Delta^2 x^3 f_{3n+1} + \Delta^2 x^2 f_{3n} \\
&= (x^2+4) f_{n+2}^3 + 2x^2 f_{n+2}^2 f_{n-2} (x^4 + 3x^2 + 4) f_{n+2} f_n^2 + x^2 f_{n+2} f_n f_{n-2} \\
&\quad + (x^4 + 3x^2 + 4) f_n^2 f_{n-2} - x^2 f_n f_{n-2}^2.
\end{aligned}
$$

Equating this value of $\Delta^2 x^2 S^*$ with its earlier version yields the desired result, as expected. □

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GRAPH-THEORETIC CONFIRMATIONS OF THREE SUMS OF JACOBSTHAL POLYNOMIAL PRODUCTS OF ORDER 3

Abstract: Using graph-theoretic techniques, we confirm three identities involving sums of Jacobsthal polynomial products of order 3, investigated in [5].

Keywords: Fibonnaci Polymals, Lucas Polymals, Jacobsthal Polynomial.

Mathematical Subject Classification (2010) No.: Primary 05A19, 11B37, 11B39, 11Cxx.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary complex variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary complex polynomials; and $n \geq 0$ [1, 2, 5, 6].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial* [1, 2]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$ and $j_n(1) = L_n$.

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. We also let $D^2 = 4x+1$ and omit a lot of basic algebra.

It is well known [3] that $J_{n+1} + xJ_{n-1} = j_n$, $J_{2n} = J_n j_n$, $J_{n+1}^2 = J_{n+1}^2 + xJ_n^2$, and the *Jacobsthal addition formula* $J_{m+n} = J_{m+1}J_n + xJ_m J_{n-1}$.

An *extended gibonacci polynomial product of order m* is a product of polynomials z_{n+k} of the form $\prod_{k \in \mathcal{Z}} z_{n+k}^{s_j}$, where $\sum_{sj \geq 1} sj = m$ [4, 7].

1.1 Sums of Jacobsthal Polynomial Products of Order 3: In [5], we investigated the following sums of gibonacci polynomial products of order 3:

$$\begin{aligned} xJ_{3n-1} &= J_{n+2}^3 - (3x+1)J_{n+2}^2 J_n + 3(x^2+x)J_{n+2}J_n^2 + x^2 J_{n+2}J_n J_{n-2} \\ &\quad - (x^3 + 2x^2 + x)J_n^3 - x^4 J_n J_{n-2}^2. \end{aligned} \tag{1}$$

$$\begin{aligned} j_{3n} &= 2J_{n+2}^3 - (6x+1)J_{n+2}^2 J_n + (6x^2+5x)J_{n+2}J_n^2 + x^2 J_{n+2}J_n J_{n-2} \\ &\quad - (2x^3 + 3x^2 + x)J_n^3 - x^3 J_n^2 J_{n-2} - x^4 J_n J_{n-2}^2; \end{aligned} \tag{2}$$

$$\begin{aligned} D^2 J_{3n+1} &= D^2 J_{n+2}^3 - (12x^2 - x)J_{n+2}^2 J_n + (12x^3 + 3x^2 - 2x)J_{n+2}J_n^2 \\ &\quad + 2x^3 J_{n+2}J_n J_{n-2} - (4x^4 + 5x^3 + x^2)J_n^3 \\ &\quad + x^3 J_n^2 J_{n-2} - 2x^5 J_n J_{n-2}^2. \end{aligned} \tag{3}$$

Our goal is to confirm these Jacobsthal identities using graph-theoretic tools.

2. Some Graph-theoretic Tools

To confirm these Jacobsthal results, consider the *weighted Jacobsthal digraph* D_2 in Figure 1 with vertices v_1 and v_2 [3, 4]. It follows from its *weighted adjacency matrix* $Q = \begin{bmatrix} 1 & x \\ 1 & 0 \end{bmatrix}$, that

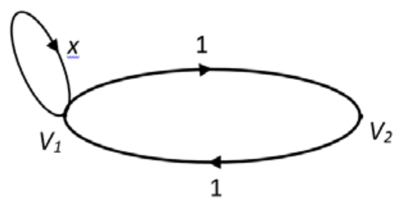

Figure 1: Weighted Digraph D_2

$$M^n = \begin{bmatrix} J_{n+1} & xJ_n \\ J_n & xJ_{n-1} \end{bmatrix},$$

where $J_n = J_n(x)$ and $n \geq 1$.

It then follows that the sum of the weights of closed walks of length n originating at v_1 is J_{n+1}, and that of those originating at v_2 is xJ_{n-1}. So the sum of the weights of all closed walks of length n in the digraph is $J_{n+1} + xJ_{n-1} = j_n$. These facts play a major role in the graph-theoretic proofs.

Let A, B, and C denote the sets of closed walks of lengths a, b, and c originating at vertex v, respectively. Then the sum of the weights of the elements in the product set $A \times B \times C$ is *defined* as the product the sums of the walks in each component [4].

With these tools at our fingertips, we are now ready for the graph-theoretic proofs.

3. Graph-theoretic Proofs

3.1 Proof of Identity (1): Let S denote the sum of the weights of closed walks of length $3n-2$ originating at v_1. Clearly, $S = J_{3n-1}$ and hence, $xS = xJ_{3n-1}$.

We will now compute the sum xS in a different way. To this end, let w be an arbitrary closed walk of length $3n-2$ originating at v_1.

Case 1: Suppose w begins with a loop. It can land at v_1 or v_2 at the $(n+1)$st and $(2n+1)$st steps:

$$w = \underbrace{v_1 - \ldots - x}_{\text{subwalk of length } n+1} \underbrace{x - \ldots - x}_{\text{subwalk of length } n} \underbrace{x - \ldots - v_1}_{\text{subwalk of length } n-3},$$

where $x = v_1$ or v_2.

Table 1: Sum of the Weights of Closed Walks

w lands at v_1 at the $(n+1)$st step?	w lands at v_1 at the $(2n+1)$st step?	w lands at v_1 at the $(3n-2)$ nd step?	sums of the weights of walks w
yes	yes	yes	$J_{n+1}^2 J_{n-2}$
yes	no	yes	$xJ_{n+1}J_nJ_{n-3}$
no	yes	yes	$xJ_n^2 J_{n-2}$
no	no	yes	$x^2 J_n J_{n-1} J_{n-3}$

Table 1 shows the possible cases and the sums of weights of the corresponding walks. It follows from the table that the sum S_1 of the weights of such closed walks is given by

$$\begin{aligned} S_1 &= J_{n+1}^2 J_{n-2} + xJ_{n+1}J_nJ_{n-3} + xJ_n^2 J_{n-2} + x^2 J_n J_{n-1} J_{n-3} \\ &= (J_{n+1}^2 + xJ_n^2)J_{n-2} + xJ_nJ_{n-3}(J_{n+1} + xJ_{n-1}) \\ &= J_{2n+1}J_{n-2} + xJ_{2n}J_{n-3} \\ &= J_{3n-2}. \end{aligned}$$

Case 2: Suppose w does not begin with a loop. Once again, it can land at v_1 or v_2 at the $(n+1)$st and $(2n+1)$st steps:

$$w = \underbrace{v_1 - \ldots - x}_{\text{subwalk of length } n+1} \underbrace{x - \ldots - x}_{\text{subwalk of length } n} \underbrace{x - \ldots - v_1}_{\text{subwalk of length } n-3},$$

where $x = v_1$ or v_2.

Table 2: Sum of the Weights of Closed Walks

w lands at v_1 at the $(n+1)$st step?	w lands at v_1 at the $(2n+1)$st step?	w lands at v_1 at the $(3n-2)$ nd step?	sums of the weights of walks w
yes	yes	yes	$xJ_{n+1}J_nJ_{n-2}$
yes	no	yes	$x^2J_n^2J_{n-3}$
no	yes	yes	$x^2J_nJ_{n-1}J_{n-2}$
no	no	yes	$x^3J_{n-1}^2J_{n-3}$

It follows from Table 2 that the sum S_2 of the weights of all such closed walks w is given by

$$\begin{aligned} S_2 &= xJ_{n+1}J_nJ_{n-2} + x^2J_n^2J_{n-3} + x^2J_nJ_{n-1}J_{n-2} + x^3J_{n-1}^2J_{n-3} \\ &= xJ_nJ_{n-2}(J_{n+1} + xJ_{n-1}) + x^2J_{n-3}(J_n^2 + xJ_{n-1}^2) \\ &= x(J_n^2J_{n-2} + xJ_{2n-1}J_{n-3}) \\ &= xJ_{3n-3}\,. \end{aligned}$$

Combining the two cases and using the Jacobsthal identity [6]

$$\begin{aligned} xJ_{3n-1} = J_{n+2}^3 - (3x+1)J_{n+2}^2J_n + 3(x^2+x)J_{n+2}J_n^2 + x^2J_{n+2}J_nJ_{n-2} \\ - (x^3+2x^2+x)J_n^3 - x^4J_nJ_{n-2}^2\,, \end{aligned}$$

we then get

$$\begin{aligned} xS &= x(S_1 + S_2) \\ &= xJ_{3n-1} \\ &= J_{n+2}^3 - (3x+1)J_{n+2}^2J_n + 3(x^2+x)J_{n+2}J_n^2 + x^2J_{n+2}J_nJ_{n-2} \\ &\qquad - (x^3+2x^2+x)J_n^3 - x^4J_nJ_{n-2}^2\,. \end{aligned}$$

This value of xS, coupled with its earlier value, yields the desired result. □

Next we explore the graph-theoretic proof of identity (2).

3.2 Proof of Identity (2): Let S' denote the sum of the weights of all closed walks of length $3n$ in the digraph. Then $S' = j_{3n}$.

To compute S' in a different way, let w be an arbitrary closed walk of length $3n$.

Case 1: Suppose w originates (and ends) at v_1. It can then land at v_1 or v_2 at the nth and $2n$th steps:

$$w = \underbrace{v_1 - \dots - x}_{\text{subwalk of length } n} \quad \underbrace{x - \dots - x}_{\text{subwalk of length } n} \quad \underbrace{x - \dots - v_1}_{\text{subwalk of length } n},$$

where $x = v_1$ or v_2.

Table 3: Sums of the Weights of Closed Walks Originating at v_1

w lands at v_1 at the nth step?	w lands at v_1 at the $2n$th step?	w lands at v_1 at the $3n$th step?	sums of the weights of walks w
yes	yes	yes	J_{n+1}^3
yes	no	yes	$xJ_{n+1}J_n^2$
no	yes	yes	$xJ_{n+1}J_n^2$
no	no	yes	$x^2J_n^2J_{n-1}$

Using the identity [6]

$$J_{3n+1} = J_{n+2}^3 - 3xJ_{n+2}^2J_n + (3x^2 + 2x)J_{n+2}J_n^2 - (x^3 + x^2)J_n^3 - x^3J_n^2J_{n-2},$$

it then follows by Table 3 that the sum S_1' of the weights of such walks w is given by

$$\begin{aligned} S_1' &= J_{n+1}^3 + 2x_{n+1}J_n^2 + x^2J_n^2J_{n-1} \\ &= J_{n+1}(J_{n+1}^2 + xJ_n^2) + xJ_n^2(J_{n+1} + xJ_{n-1}) \\ &= J_{n+1}J_{2n+1} + xJ_nJ_n^2 \\ &= J_{3n+1} \\ &= J_{n+2}^3 - 3xJ_{n+2}^2J_n + (3x^2 + 2x)J_{n+2}J_n^2 - (x^3 + x^2)J_n^3 - x^3J_n^2J_{n-2}. \end{aligned}$$

Case 2: Suppose w originates (and ends) at v_2. Then w can land at v_1 or v_2 at the nth and $2n$th steps:

$$w = \underbrace{v_1 - \ldots - x}_{\text{subwalk of length } n} \quad \underbrace{x - \ldots - x}_{\text{subwalk of length } n} \quad \underbrace{x - \ldots - v_1}_{\text{subwalk of length } n},$$

where $x = v_1$ or v_2.

Table 4: Sums of the Weights of Closed Walks Originating at v_2

w lands at v_1 at the nth step?	w lands at v_1 at the $2n$th step?	w lands at v_1 at the $3n$th step?	sums of the weights of walks w
yes	yes	yes	$xJ_{n+1}J_n^2$
yes	no	yes	$x^2J_n^2J_{n-1}$
no	yes	yes	$x^2J_n^2J_{n-1}$
no	no	yes	$x^3J_{n-1}^3$

Using the identity [6]

$$\begin{aligned} xJ_{3n-1} &= J_{n+2}^3 - (3x+1)J_{n+2}^2J_n + 3(x^2+x)J_{n+2}J_n^2 + x^2J_{n+2}J_nJ_{n-2} \\ &\quad - (x^3+2x^2+x)J_n^3 - x^4J_nJ_{n-2}^2, \end{aligned}$$

it follows by Table 4 that the sum S_2' of the weights of all such walks w is given by

$$\begin{aligned} S_2' &= xJ_{n+1}J_n^2 + 2x^2J_n^2J_{n-1} + x^3J_{n-1}^3 \\ &= x^2J_n^2(J_{n+1} + xJ_{n-1}) + x^2J_{n-1}(J_n^2 + xJ_{n-1}^2) \\ &= x(J_n^2J_n + xJ_{2n-1}J_{n-1}) \\ &= xJ_{3n-1} \\ &= J_{n+2}^3 - (3x+1)J_{n+2}^2J_n + 3(x^2+x)J_{n+2}J_n^2 + x^2J_{n+2}J_nJ_{n-2} \\ &\qquad - (x^3+2x^2+x)J_n^3 - x^4J_nJ_{n-2}^2. \end{aligned}$$

Thus

$$S' = S_1' + S_2'$$
$$= 2J_{n+2}^3 - (6x+1)J_{n+2}^2 J_n + (6x^2+5x)J_{n+2}J_n^2 + x^2 J_{n+2}J_n J_{n-2}$$
$$- (2x^3+3x^2+x)J_n^3 - x^3 J_n^2 J_{n-2} - x^4 J_n J_{n-2}^2 .$$

Equating the two values of S' yields the desired result, as expected. □

Finally, we explore the graph-theoretic proof of identity (3).

3.3 Proof of Identity (3): Let S^* denote the sum of the weights of closed walks of length $3n$ originating at v_1. Clearly, $S^* = J_{3n+1}$ and hence, $D^2 S^* = D^2 J_{3n+1}$.

To compute $D^2 S^*$ in a different way, suppose that w is an arbitrary closed walk of length $3n$ originating at v_1.

Case 1: Suppose w begins with a loop. It can then land at v_1 or v_2 at the nth and $2n$th steps:

$$w = \underbrace{v_1 - \ldots - x}_{\text{subwalk of length } n} \quad \underbrace{x - \ldots - x}_{\text{subwalk of length } n} \quad \underbrace{x - \ldots - v_1}_{\text{subwalk of length } n},$$

where $x = v_1$ or v_2.

Table 5: Sums of the Weights of Closed Walks Beginning with a Loop

w lands at v_1 at the nth step?	w lands at v_1 at the $2n$th step?	w lands at v_1 at the $3n$th step?	sums of the weights of walks w
yes	yes	yes	$J_{n+1}^2 J_n$
yes	no	yes	xJ_n^3
no	yes	yes	$xJ_{n+1}J_n J_{n-1}$
no	no	yes	$x^2 J_n J_{n-1}^2$

Using the identity [6]

$$D^2 J_{3n} = (12x+1)J_{n+2}^2 J_n - (20x^2+9x)J_{n+2}J_n^2 - x^2 J_{n+2}J_n J_{n-2} + (4x^3+5x^2+x)J_n^3 + (4x^4+3x^3)J_n^2 J_{n-2} + x^4 J_n J_{n-2}^2 ,$$

it follows by Table 5 that the sum S_1^* of the weights of walks w is given by

$$\begin{aligned} S_1^* &= J_{n+1}^2 J_n + xJ_n^3 + xJ_{n+1}J_n J_{n-1} + x^2 J_n J_{n-1}^2 \\ &= J_{n+1}J_n(J_{n+1} + xJ_{n-1}) + xJ_n(J_n^2 + xJ_{n-1}^2) \\ &= J_{n+1}J_{2n} + xJ_n J_{n-1}^2 \\ &= J_{3n} \end{aligned}$$

$$D^2 S_1^* = (12x+1)J_{n+2}^2 J_n - (20x^2+9x)J_{n+2}J_n^2 - x^2 J_{n+2}J_n J_{n-2} + (4x^3+5x^2+x)J_n^3 + (4x^4+3x^3)J_n^2 J_{n-2} + x^4 J_n J_{n-2}^2 .$$

Case 2: Suppose w does not begin with a loop. Then also it can land at v_1 or v_2 at the nth and $2n$th steps:

$$w = \underbrace{v_2 - \ldots - x}_{\text{subwalk of length } n} \; \underbrace{x - \ldots - x}_{\text{subwalk of length } n} \; \underbrace{x - \ldots - v_2}_{\text{subwalk of length } n} ,$$

where $x = v_1$ or v_2.

Table 6: Sums of the Weights of Closed Walks not Beginning a Loop

w lands at v_1 at the nth step?	w lands at v_1 at the $2n$th step?	w lands at v_1 at the $3n$th step?	sums of the weights of walks w
yes	yes	yes	$xJ_{n+1}^2 J_{n-1}$
yes	no	yes	$x^2 J_n^2 J_{n-1}$
no	yes	yes	$x^2 J_{n+1}J_n J_{n-2}$
no	no	yes	$x^2 J_n J_{n-1}J_{n-2}$

Using the identity [6]

$$D^2 xJ_{3n-1} = D^2 J_{n+2}^3 - (12x^2 + 11x + 1)J_{n+2}^2 J_n + (12x^3 + 23x^2 + 7x)J_{n+2}J_n^2 + (2x^3 + x^2)J_{n+2}J_n J_{n-2} - (x+1)^2(4x^2 + x)J_n^3,$$

it follows by Table 6 that the sum S_2^* of the weights of such walks w is given by

$$\begin{aligned} S_2^* &= xJ_{n+1}^2 J_{n-1} + x^2 J_n^2 J_{n-1} + x^2 J_{n+1}J_n J_{n-1} + x^3 J_n J_{n-1}J_{n-2} \\ &= xJ_{n-1}J_{n+1}^2 + xJ_n^2 + x^2 J_n J_{n-2}(J_{n+1} + xJ_{n-1}) \\ &= x(J_{n+1}^2 J_{n-1} + xJ_{2n}J_{n-2}) \\ &= xJ_{3n-1} \end{aligned}$$

$$D^2 S_2^* = D^2 J_{n+2}^3 - (12x^2 + 11x + 1)J_{n+2}^2 J_n + (12x^3 + 23x^2 + x)J_{n+2}J_n^2 + (2x^3 + x^2)J_{n+2}J_n J_{n-2} - (x+1)^2(4x^2 + x)J_n^3.$$

Combining the two cases, we then get

$$\begin{aligned} D^2 S^* &= D^2 S_1^* + D^2 S_2^* \\ &= D^2 J_{n+2}^3 - (12x^2 - x)J_{n+2}^2 J_n + (12x^3 + 3x^2 - 2x)J_{n+2}J_n^2 \\ &\quad + 2x^3 J_{n+2}J_n J_{n-2} - (4x^4 + 5x^3 + x^2)J_n^3 + x^3 J_n^2 J_{n-2} - 2x^5 J_n J_{n-2}^2. \end{aligned}$$

This value of $D^2 S^*$, together with its earlier value, yields the desired result, as expected. □

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EXTENSION OF GENERATING FUNCTIONS INVOLVING MODIFIED LAGUERRE & MODIFIED BESSEL POLYNOMIALS

Abstract: In the present paper, we discuss the generating functions involving the product of modified Laguerre polynomials $L_n^{(\alpha-n)}(x)$, modified Bessel polynomials $Y_m^{(\alpha+m)}[q]$ and the confluent hypergeometric functions ${}_1F_1[.]$ and then obtain some more generating functions by group-theoretic approach and discuss their applications. Earlier Chandel, Kumar and Senger [1] introduce the generating functions involving the product of modified Bessel polynomials $Y_n^{(\alpha+n)}[x]$ and the confluent hypergeometric functions ${}_1F_1[.]$.

Keywords: Generating Functions, Modified Laguerre Polynomials, Modified Bessel Polynomials, Confluent Hypergeometric Functions.

Mathematical Subject Classification No.: 33C45, 33C99, 22E30.

1. Introduction

The modified Laguerre polynomials $L_n^{(\alpha-n)}(x)$ and modified Bessel polynomials $Y_m^{(\alpha+m)}(u)$ are defined by Srivastava and Manocha [6] as:

$$L_n^{(\alpha-n)}(x) = \frac{\Gamma(1+\alpha)}{\Gamma(1+n)\Gamma(1+\alpha-n)}\ {}_1F_1[-n; 1+\alpha-n; x] \tag{1.1}$$

$$Y_m^{(\alpha+n)}(u) = {}_2F_0\left[-m, m+n+\alpha-1\,; -\,;\, -\frac{u}{\beta}\right] \tag{1.2}$$

The confluent hypergeometric functions ${}_1F_1[.]$ can be replaced by many special functions. Srivastava and Manocha [6] defined and studied various bilinear, bilateral and multilinear generating functions.

In this paper, we introduce the following new general class of generating functions:

$$G(x,u,q,w) = \sum_{n=0}^{\infty} a_n \, L_n^{(\alpha-n)}(x) \, Y_m^{(\alpha+m)}(u) \, {}_1F_1\left[-n; m+1; q\right] w^n \quad (1.3)$$

where a_n is any arbitrary sequence independent of x, u, q and w.

Again in (1.3) setting various values of a_n, we may find several results on generating functions involving different special functions, hence (1.3) is a general class of generating functions.

In this paper, we evaluate some more general class of generating functions and finally discuss their applications.

2. Group-Theoretic Operators: In our investigations, we use the following group-theoretic operators:

The operators R_1 due to Majumdar [4] is given by

$$R_1 = xyz \frac{\partial}{\partial x} - y^2 z \frac{\partial}{\partial y} - (x-\alpha) yz \quad (2.1)$$

Such that

$$R_1 \left[L_n^{(\alpha-n)}(x) y^n z^\alpha \right] = (n+1) \, L_{n+1}^{(\alpha-n-1)}(x) y^{n+1} z^{\alpha+1} \quad (2.2)$$

The operators R_2 due to Chongdar [2] is given by

$$R_2 = u^2 t^{-1} v \frac{\partial}{\partial u} + uv \frac{\partial}{\partial t} + u t^{-1} v^2 \frac{\partial}{\partial v} + t^{-1} v \, (\beta - u) \quad (2.3)$$

Such that

$$R_2 \left[Y_m^{(\alpha+n)}(u) \, t^n v^m \right] = \beta \, Y_{m+1}^{(\alpha+n-1)}(u) \, t^{n-1} v^{m+1} \quad (2.4)$$

The operator R_3 due to Miller Jr. [5] is given by

$$R_3 = r \frac{\partial}{\partial p} + rqp^{-1} \frac{\partial}{\partial q} - rq \, p^{-1} \quad (2.5)$$

Such that

$$R_3[{}_1F_1[-n;m+1;q]r^n p^m] = m\ {}_1F_1\left[-n-1;m;q\right]r^{n+1}\,p^{m-1} \qquad (2.6)$$

The actions of R_1, R_2 and R_3 on function f are obtained as follows:

$$e^{wR_1}f(x,y,z) = (1+wyz)^{\alpha}\ exp(-wxyz)\ f\left[x(1+wyz),\frac{y}{1+wyz},z\right] \qquad (2.7)$$

(cf. Majumdar [4])

$$e^{wR_2}F(u,t,v) = \ (1-wut^{-1}v)\ exp(\beta wt^{-1}v)\ F\left[\frac{u}{1-wut^{-1}v},\frac{t}{1-wut^{-1}v},\frac{v}{1-wut^{-1}v}\right] \qquad (2.8)$$

(cf. Chongdar [2])

And

$$e^{wR_3}f(r,p,q) = \exp\left(\frac{-qrw}{p}\right) f\left[\ r,p+wr,\ \ q\left(1+\frac{wr}{p}\right)\right]$$

(2.9)(cf. Miller Jr. [5])

3. Some More General Class of Generating Functions

In this sections, making an use of the general class of generating function (1.3) and group-theoretic operators R_1 R_2 and R_3 with their actions given in the section 2, we obtain some more general class of generating functions through following theorem:

Theorem: If there exists a general class of generating functions involving the triple product of modified Laguerre polynomials $L_n^{(\alpha-n)}(x)$, modified Bessel polynomials $Y_m^{(\alpha+m)}(u)$ and the confluent hypergeometric functions ${}_1F_1[-n;\ m+1;q]$ given by

$$G(x,u,q,w) = \ \ \sum_{n=0}^{\infty} a_n\ L_n^{(\alpha-n)}(x)\,Y_m^{(\alpha+m)}(u)\ {}_1F_1\left[-n;m+1;q\right]w^n \qquad (3.1)$$

Then the following more general class of generating functions holds:

$$(1+w)^{\alpha+m}(1-\mathrm{wut}^{-1}v)^{1-\mathrm{m}}\ \cdot \exp[-w(x-\beta\mathrm{t}^{-1}v+q)].$$

$$G\left[x(1+w),\frac{u}{1-wu},q(1+w),\frac{wyv}{1+w}\right] = \sum_{n,i,j,k=0}^{\infty}\ \frac{a_n\ (n+1)_i}{i!\ \ j!\ \ k!}.$$

$$L_{n+i}^{(\alpha-n-i)}(x)\,Y_{m+j}^{(\alpha+n-j)}(u)\,{}_1F_1[-n-k;\ m-k+\ 1;\ q]\ w^i\ (w\beta t^{-1}v)^j(mw)^k(wyr)^n \tag{3.2}$$

Proof of the Theorem: In the general class of generating functions (3.1), replacing w by $wytr$ and then multiplying by $z^\alpha\ v^m\ p^m$ on both sides, we get

$$G(x,q,u,wytr)\ z^\alpha\ v^m\ p^m\ =\ \sum_{n=0}^{\infty} a_n\ L_n^{(\alpha-n)}(x)\ P_m^{(\alpha,\ \beta-m)}(q)\ .$$

$${}_1F_1[-n;\ m+1;u]\ y^n t^n\ r^n.z^\alpha\ v^m\ p^m\ .w^n \tag{3.3}$$

Now, operating both the sides of (3.3) with $e^{wR_1}e^{wR_2}e^{wR_3}$, we obtain

$$e^{wR_1}e^{wR_2}e^{wR_3}[\ G(x,q,u,wytr)\ z^\alpha\ v^m\ p^m]$$

$$= e^{wR_1}e^{wR_2}e^{wR_3}\sum_{n=0}^{\infty} a_n\ L_n^{(\alpha-n)}(x)y^n z^\alpha.$$

$$Y_m^{(\alpha+n)}(u)\ t^n v^m.\ {}_1F_1\big[-n;m+\ 1;q\big]r^n p^m.\,w^n \tag{3.4}$$

The left hand side of (3.4) becomes

$$z^\alpha\ (1+wyz)^\alpha\ (1-wut^{-1}v)\left(\frac{v}{1-wut^{-1}v}\right)^m (p+wr)^m\ \exp\left(-wxyz+\beta wt^{-1}v-\frac{qrw}{p}\right) G\left[x(1+wyz)\,,\frac{u}{1-wut^{-1}v}\,,q\left(1+\frac{wr}{p}\right),\frac{wyv}{1+wyz}\right] \tag{3.5}$$

And the right hand side of (3.4) becomes

$$\sum_{n,i,j,k=0}^{\infty} \frac{a_n\ (n+1)_i\,\beta^j\ m^k w^{n+i+j+k}}{i!\ j!\ k!}\ \ L_{n+i}^{(\alpha-n-i)}(x)y^{n+i}\ \ z^{\alpha+i}\,Y_{m+j}^{(\alpha+n-j)}(u)t^{n-j}v^{m+j}\ \ {}_1F_1\big[-n-1;m;q\big]\ r^{n+k}\ p^{m-k} \tag{3.6}$$

Now equating (3.5) and (3.6), and setting $\mathrm{r}=\mathrm{p}$ and $\mathrm{yz}=1$,

$$(1+\mathrm{w})^{\alpha+\mathrm{m}}\ (1-\mathrm{wut}^{-1}v)^{1-\mathrm{m}}\ \ \exp(-\mathrm{w}(\mathrm{x}-\beta\mathrm{t}^{-1}v+\ \mathrm{q})$$

$$.\,\mathrm{G}\left[\mathrm{x}(1+\mathrm{w})+\frac{\mathrm{u}}{1-\mathrm{wut}^{-1}v}\,,\mathrm{q}(1+\mathrm{w}),\frac{\mathrm{wy}v}{1+\mathrm{w}}\right]=$$

$$\sum_{n,i,j,k=0}^{\infty} \frac{a_n\ (n+1)_i}{i!\ \ j!\ \ k!}\ \ L_{n+i}^{(\alpha-n-i)}(x)\ \ Y_{m+j}^{(\alpha+n-j)}(u).$$

$${}_1F_1\big[-n-k;m-k+1;q\big]\,.\,w^i(w\beta t^{-1}v)^j\ (mw)^k(wytp)^n \tag{3.7}$$

which is the required result.

4. Special Case

Taking $u = 0, q = 0$ in given theorem and proceeding as the proof of the main theorem, we get

$$exp(-wx)\ G\left[x(1+w), \frac{wyv}{1+w}\right] = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \frac{a_n\ (n+1)_i\ w^{n+i}}{i!}\ L_{n+i}^{(\alpha-n-i)}(x)\ y^{n+i}\ z^i$$

$$\sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{a_{n-i}\ (n-i+1)_i\ w^n}{i!}\ L_n^{(\alpha-n)}(x)\ y^n\ z^i$$

$$= \sum_{n=0}^{\infty} \sigma_n(x,z).(wy)^n \tag{4.1}$$

where

$$\sigma_n(x,z) = \sum_{i=0}^{n} \frac{a_{n-i}\ (n-i+1)_i}{p!}\ L_n^{(\alpha-n)}(x)\ z^i \tag{4.2}$$

which is given by Majumdar [4].

(ii) If we set $x = 0\ , q = 0\ \&\ t = v$ in given theorem and proceeding as the proof of main theorem with operator R_2, we get

$$(1-wu)^{1-m}\ exp(w\beta)\ G\left[\frac{u}{1-wu}, \frac{wyv}{1+w}\right] = \sum_{n,q=0}^{\infty} \frac{a_n\ \beta^j\ w^{n+j}}{j!}\ Y_{m+j}^{(\alpha+n-j)}(u) y^n \tag{4.3}$$

which is a known result and as parallel to Kar [3].

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REVIEW STUDY ON DIFFERENTIAL TRANSFORM FOR LAPLACE EQUATION

Abstract: In this article, the analytical solution for Laplace equation using differential transform method has been presented. To represent this we have obtained the corresponding exact solutions by considering four models with two well-known boundary conditions known as Dirichlet and Neumann. The achieved outcome shows the easiness of the method and substantial decrease in successive iterations in comparison to the other well-known iterative methods. We can say that very less number of iterations gives the desired output for the problem nearer to the expansions of series for the identified functions.

Keywords: Laplace equation, boundary Conditions of Dirichlet and Neumann, Finite Difference.

Mathematical Subject Classification: 35K20.

Introduction

It is very difficult to solve analytically the problems which are related to engineering and physics where the governing equations are in the form of standard boundary value problems such as Laplace, heat and wave equations in one and two dimensions. The exact solution of the governing differential equation corresponding to the problem can be obtained after difficult calculations. To reduce this difficulty, different approximations and direct methods, such as Adomian decomposition [1], Homotopy analysis [2], Variational iteration [3], New iterative [4, 5] and Differential transform [6] are invented.

The different methods have been applied by investigators for obtaining the required outcomes of the problems [7–12].

The solution of the Laplace equation $u_{xx} + u_{yy} = 0$ with Dirichlet and Neumann boundary conditions using differential transform scheme (two dimensional) with less number of iterations is the main purpose of this article. Different examples with different boundary conditions have been solved. Applied scheme builds one analytic output without need of liberalization or discretization. The computational iterations have been reduced and desired output in the series form with fast convergence has been obtained.

Methodology

If $u(x, y) = f(x)g(y)$ is a function of variables x and y, where the functions $f(x)$ and $g(y)$ are two different functions of variable x and y respectively. Now by considering the differential transform property of one dimension we can write $u(x, y)$ as

$$u(x,y) = \sum_{m=0}^{\infty} F(m)x^m \sum_{n=0}^{\infty} G(n)\, y^n = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} U(m,n)x^m y^n \qquad (1)$$
1

here $U(m,n) = F(m)\, G(n)$ is known as spectrum of the function $u(x,y)$, given as follows:

$$U(m,n) = \frac{1}{m!n!}\left[\frac{\partial^{m+n}u(x,y)}{\partial x^m \partial y^n}_{x=x_0, y=y_0}\right] \qquad (2)$$

and the inverse of differential transform of $U(m,n)$ is of the form:

$$u(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} U(m,n)(x-x_0)^m(y-y_0)^n \qquad (3)$$

On combining equations (2) and (3), we have

$$u(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!}\left[\frac{\partial^{m+n}u(x,y)}{\partial x^m \partial y^n}_{x=x_0, y=y_0}\right](x-x_0)^m(y-y_0)^n \qquad (4)$$

Now suppose that $U(m,n), V(m,n)$ and $W(m,n)$ represents the transformations of the functions $u(x,y), v(x,y)$ and $w(x,y)$ at $(0,0)$ respectively then:

(a) If $u(x,y) = v(x,y) \pm w(x,y)$, then $U(m,n) = V(m,n) \pm W(m,n)$

(b) If $u(x,y) = av(x,y)$ then $U(m,n) = aV(m,n)$

(c) If $u(x,y) = v(x,y)w(x,y)$, then $U(m,n) = \sum_{k=0}^{m}\sum_{l=0}^{n} V(k,n-l)W(m-k,l)$

(d) If $u(x,y) = \frac{\partial^{r+s}v(x,y)}{\partial x^r \partial y^s}$, then $U(m,n) = \frac{(m+r)!}{m!}\frac{(n+s)!}{n!}V(m+r,n+s)$

(e) If $u(x,y) = e^{av(x,y)}$, then

$$U(m,n) = \begin{cases} e^{av(0,0)}, & m = n = 0 \\ a\sum_{k=0}^{m-1}\sum_{l=0}^{n}\frac{m-k}{m}V(m-k,l)U(k,n-l), & m \geq 1 \\ a\sum_{k=0}^{m}\sum_{l=0}^{n-1}\frac{n-l}{n}V(k,n-l)U(m-k,n), & n \geq 1. \end{cases}$$

(f) If $u(x,y) = x^k y^h$, then

$$U(m,n) = \begin{cases} \partial(m-k,n-h), m = k, n = h \\ \quad 0, \qquad \text{otherwise} \end{cases}$$

(g) If $u(x,y) = x^k e^{ay}$, then $U(m,n) = \partial(m-k)\frac{a^n}{n!}$.

Different Examples

A. Solve: $u_{xx} + u_{yy} = 0, \quad 0 < x, y < \pi$ (5)

With Dirichlet boundary conditions:

$$u(x,0) = \sin hx, \quad U(x,\pi) = -\sin hx,$$

$$u(0,y) = 0, \quad u(\pi,y) = \sin h(\pi)\cos y. \tag{6}$$

Now taking the differential transform of (5) we have

$$(m+1)(m+2)U(m+2,n) + (n+1)(n+2)U(m,n+2) = 0 \tag{7}$$

from (6) and (3) it implies that

$$u(x,0) = \sum_{m=0}^{\infty} U(m,0)\, x^m = \sin hx = \sum_{m=1,3,5\ldots\ldots}^{\infty}\frac{x^m}{m!}, \tag{8}$$

Now on comparing the both sides, we have

$$U(m,0) = \begin{cases} \frac{1}{m!} & where\ m\ is\ odd \\ 0 & otherwise \end{cases} \tag{9}$$

Also from (6) and (3) we have

$$u(0,y) = \sum_{n=0}^{\infty} U(0,n)y^n = 0 \tag{10}$$

which gives

$$U(0,n) = 0 \tag{11}$$

Substituting (9) and (11) into (7) and after some calculations, we reach at

$$U(m,n) = \begin{cases} \frac{(-1)^{\frac{n}{2}}}{m!n!}, & \text{when } m \text{ is odd and } n \text{ is even} \\ 0, & \text{otherwise} \end{cases} \tag{12}$$

Similarly by substituting (12) into (3), we have

$$u(x,y) = \sum_{m=1,3,5\dots}^{\infty} \sum_{n=0,2,4\dots}^{\infty} \frac{(-1)^{\frac{n}{2}}}{m!n!} x^m y^n,$$

$$= \left(\sum_{m=1,3,5\dots}^{\infty} \frac{x^m}{m!}\right)\left(\sum_{n=0,2,4\dots}^{\infty} \frac{(-1)^{\frac{n}{2}}}{n!} y^n\right),$$

$$= \sin hx \cos y \tag{13}$$

B. Solve: $$u_{xx} + u_{yy} = 0, \quad 0 < x, \quad y < \pi \tag{14}$$

With Dirichlet boundary conditions:

$$u(x,0) = 0, \ \ u(x,\pi) = 0,$$

$$u(0,y) = \sin y, \ \ u(\pi,y) = \cos h(\pi) \sin y \tag{15}$$

Now taking the differential transform of (14) we have

$$(m+1)(m+2)U(m+2,n) + (n+1)(n+2)U(m,n+2) = 0 \tag{16}$$

From conditions (15) and (3), we obtain

$$u(x,0) = \sum_{m=0}^{\infty} U(m,0)x^m = 0, \tag{17}$$

Now by comparison both sides we have

$$U(m,0) = 0. \tag{18}$$

Also, (3) and (15), imply that

$$u(0,y) = \sum_{n=0}^{\infty} U(0,n)y^n = \sin y = \sum_{0}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n!} y^n \tag{19}$$

Now by comparison of both sides we have

$$U(0,n) = \begin{cases} \frac{(-1)^{\frac{n-1}{2}}}{n!}, & \text{when } n \text{ is odd} \\ 0, & \text{otherwise} \end{cases} \tag{20}$$

from (18), (20) and (16), we have

$$U(m,n) = \begin{cases} \frac{(-1)^{\frac{n-1}{2}}}{m!n!}, & \text{when } m \text{ is even and } n \text{ is odd} \\ 0, & \text{otherwise} \end{cases} \tag{21}$$

Now making use of (21) in (3), we obtain

$$u(x,y) = \sum_{m=1,3,5\dots}^{\infty} \sum_{n=0,2,4\dots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{m!n!} x^m y^n,$$

$$= \left(\sum_{m=1,3,5\dots}^{\infty} \frac{x^m}{m!}\right) \left(\sum_{n=0,2,4\dots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n!} y^n\right),$$

$$= \cos hx \sin y \tag{22}$$

C. Solve: $u_{xx} + u_{yy} = 0, \quad 0 < x, \quad y < \pi$ (23)

With Neumann boundary conditions

$$u_y(x,0) = 0, \;\; u_y(x,\pi) = 2\cos 2x \sin 2\pi$$

$$u_x(0,y) = 0, \;\; u_x(\pi,y) = 0 \tag{24}$$

Now the differential transform of (23) gives

$$(m+1)(m+2)U(m+2,n)+(n+1)(n+2)U(m,n+2)=0 \tag{25}$$

From (3) and (24) we obtain

$$u_y(x,\pi)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} n\,\pi^{n-1}U(m,n)x^m,$$

$$=2\cos 2x\sin 2\pi,$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^{\frac{m}{2}}(2x)^m}{m!}\sum_{n=0}^{\infty}\frac{(2\pi)^n}{n!}. \tag{26}$$

On comparison after the changing the index $n$we have

$$U(m,n+1)=\frac{(-1)^{\frac{m}{2}}2^{m+n+1}}{(n+1)m!n!},$$

and $$U(m,n)=\begin{cases}\frac{(-1)^{\frac{m}{2}}2^{m+n}}{m!n!}, & \text{when } m \text{ and } n \text{ are even}\\ 0, & \text{otherwise}\end{cases} \tag{27}$$

as a result. Now putting (27) in (3), we obtain

$$u(x,y)=\sum_{m=0,2,4\ldots}^{\infty}\sum_{n=0,2,4\ldots}^{\infty}\frac{(-1)^{\frac{m}{2}}2^{m+n}}{m!n!}x^m y^n,$$

$$=\left(\sum_{m=0,2,4\ldots}^{\infty}\frac{(-1)^{\frac{m}{2}}(2x)^m}{m!}\right)\left(\sum_{n=0,2,4\ldots}^{\infty}\frac{(2y)^n}{n!}\right),$$

$$=\cos 2x\cos 2y, \tag{28}$$

D. Solve: $$u_{xx}+u_{yy}=0,\quad 0<x,\quad y<\pi \tag{29}$$

With Neumann boundary conditions

$$u_y(x,0)=\cos x,\quad u_y(x,\pi)=\cos h\pi\cos x,$$

$$u_x(0,y)=0,\ u_x(\pi,y)=0 \tag{30}$$

Taking the differential transform of (29) we have

$$(m+1)(m+2)U(m+2,n)+(n+1)(n+2)U(m,n+2)=0 \tag{31}$$

From (3) and (30), we obtain

$$u_y(x,0) = \sum_{m=0}^{\infty} U(m,1)x^m = \cos x = \sum_{m=0}^{\infty} \frac{(-1)^{\frac{m}{2}} x^m}{m!}, \tag{32}$$

Now by cosine series comparison, we obtain

$$U(m,1) = \begin{cases} \frac{(-1)^{\frac{m}{2}}}{m!}, & \text{when } m \text{ is even} \\ 0, & \text{otherwise.} \end{cases} \tag{33}$$

Also, from (3) and (30) we have

$$u_x(0,y) = \sum_{n=0}^{\infty} U(1,n)y^n = 0, \tag{34}$$

which gives

$$U(1,n) = 0 \tag{35}$$

Substituting (33) and (35) in (31), we have

$$U(m,1) = \begin{cases} \frac{(-1)^{\frac{m}{2}}}{m!n!}, & \text{when } m \text{ is even and } n \text{ is odd} \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

Making use of (36) in (3), we get

$$u(x,y) = \sum_{m=0,2,4\dots}^{\infty} \sum_{n=1,3,5\dots}^{\infty} \frac{(-1)^{\frac{m}{2}}}{m!n!} x^m y^n,$$

$$= \left(\sum_{m=0,2,4\dots}^{\infty} \frac{(-1)^{\frac{m}{2}}(x)^m}{m!}\right)\left(\sum_{n=1,3,5\dots}^{\infty} \frac{(y)^n}{n!}\right),$$

$$= \cos x \sin y \tag{37}$$

Conclusion

It has been observed in above examples with different conditions that differential transform scheme is very effective to achieve the exact solutions of Laplace equation and maintain the fast convergence rate with minimization of iterations. Also this method reduces the required calculation to achieve the desired output in comparisons to the other well-known available schemes. Hence we conclude that this method is very effective and perfect for the solution of different types of practical problems of the different fields of engineering and physics.

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A REVIEW STUDY OF MATHEMATICAL INDUCTION AND PIGEONHOLE PRINCIPLE WITH APPLICATIONS

Abstract: In this paper, we present a review study mathematical induction and pigeonhole principle with applications which have importance in the theory of automata.

Keywords: Mathematical Induction, Strong Induction, Pigeonhole Principle, Extended Pigeonhole Principle.

Mathematical Subject Classifications (2010) No.: 46A13, 03B48, 03D70, 05-XX.

1. Introduction

The very first use of mathematical induction was seen in the works of a sixteenth century mathematician named Francesco Maurolico (1494-1575). In his book Arithmeticorum Libri Duo, he presented various properties of integers and their proofs [1, 2].

Mathematical induction can be used to prove statements that assert that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function. A proof by mathematical induction has the following three parts,

Steps of Mathematical Induction

- Basis Step: Verifying the preposition $P(1)$ is true.

- Induction Hypothesis: Assuming it to be true for all positive integer $P(k)$.
- Inductive Step: Proving it is also true for $P(k+1)$.

1.1 Types of Induction [1, 2]

1.1.1 ***Strong Induction*****:** Strong mathematical induction assumes $P(1), P(2), \ldots P(k)$ are all true and uses them to show that $P(k+1)$ is also true.

Basis Step: We verify that the proposition $P(1)$ is true.

Inductive Step: We show that the conditional statement $\left[P(1) \wedge P(2) \wedge \ldots P(k)\right] \rightarrow P(k+1)$ is true for all positive integers k.

1.1.2 ***Recursive Induction*****:** We use two steps to define a function with the set of non-negative integers as its domain.

Basis step: Specify the value of the function at zero.

Recursive step: Give a rule for finding its value at an integer from its values at smaller integers.

1.1.3 ***Generalized Induction*****:** Under generalized mathematical induction we use a property i.e. lexicographic ordering where an ordered pair of non-negative integers $N \times N$ specify that (x_1, y_1) is less than or equal to (x_2, y_2) if either $x_1 < x_2$, or $x_1 = x_2$ and $y_1 < y_2$; has the property that every subset of $N \times N$ has a least element. This implies we can recursively define the terms $a_{m,n}$, with $m \in N$ and $n \in N$.

2. Pigeonhole Principle [1, 2]

If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

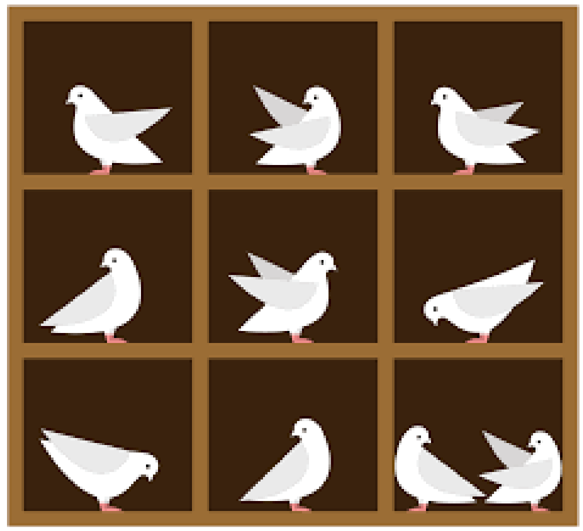

Figure 2: There are more pigeons than the pigeonholes

Generalized Pigeonhole Principle

The pigeonhole principle stated that there can be must be at least two objects in the same box when there are more objects than boxes. Generalized pigeonhole principles states that if N objects are placed into k boxes, then there is at least one box containing at least $\left\lceil N / k \right\rceil$ objects.

3. Applications of Induction [2, 3]

3.1 People Telling Secrets

Figure 3: People Telling Secrets

3.2 Dominoes

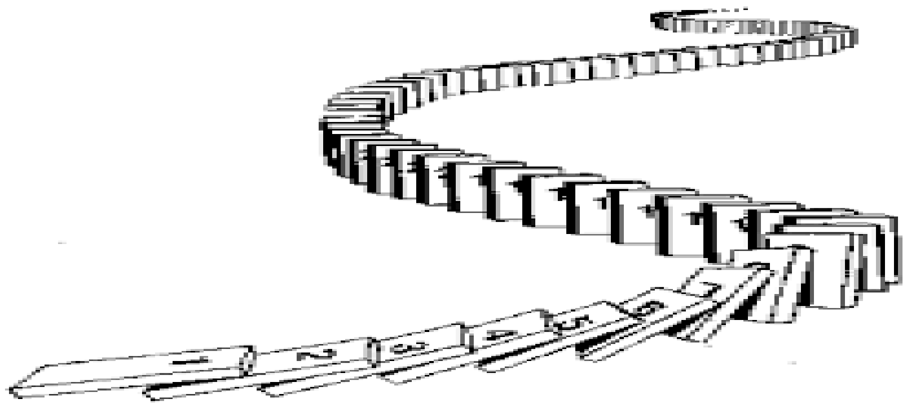

Figure 4: Illustrating How Mathematical Induction Works Using Dominoes

3.3 Checkerboard: Suppose n be a positive integer. Show that every $2^n \times 2^n$ checkerboard with one square removed can be titled using right triominoes, where these pieces cover three squares at a time, as shown in figure.

Figure 5: A Right Triominoes

Solution:

Figure 6: Tiling 2×2 Checkerboards with One Square Removed

3.4 Painting

Figure 7: A painting showing recursive induction

3.5 MatLab Program for Induction

3.5.1 ***Programs for Factorial*** **[4]**

I:
```
n=9;
%use iteration
f=1;
for i=1:n
    f=f*i;
end
disp('The factorial is:')
disp(f)
```

II:
```
function fout = fact_1 (n)
x = n;
for i=n-1:-1:1
x=x*i
end
fout=x
end
```

III:
```
function fout = fact_2(n)
%n=5;
f=n;
while n>1
    n=n-1;
    f=f*n
end
f
%disp(['n!='f])
end
```

IV:
```
function fout = fact(n)
x=1;
for i=1:n
    x=x*i
end
fout = x
end
```

3.5.2 *Program for Fibonacci Series* [5]

```
function fout = fact(n)
x=1;
for i=1:n
    x=x*i
end
fout = x
end
```

4. Applications of Pigeonhole Principle [1, 2]

- During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Using pigeonhole principle, there must be a period of some number of consecutive days which the team must play exactly 14 games.

- Using pigeonhole principle, among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

- The sequence $8, 11, 9, 1, 4, 6, 12, 10, 5, 7$ contains 10 terms. Note that $10 = 3^2 + 1$. There are four increasing subsequences of length four, namely, $1, 4, 6, 12; 1, 4, 6, 10$ and $1, 4, 5, 7$. Using pigeonhole principle, there is also a decreasing subsequence of length four, namely, $11, 9, 6, 5$.

4.1 MatLab Program for Pigeonhole Principle Birthday Paradox

```
function [A] = birthday(n)
A = ones(n,1);
p=1;
for i=1:n
    A(i) = 1-p;
    p = p * (365-i)/(365);
end
end
```

taking value of n = 100

we obtain the graph in Figure 8

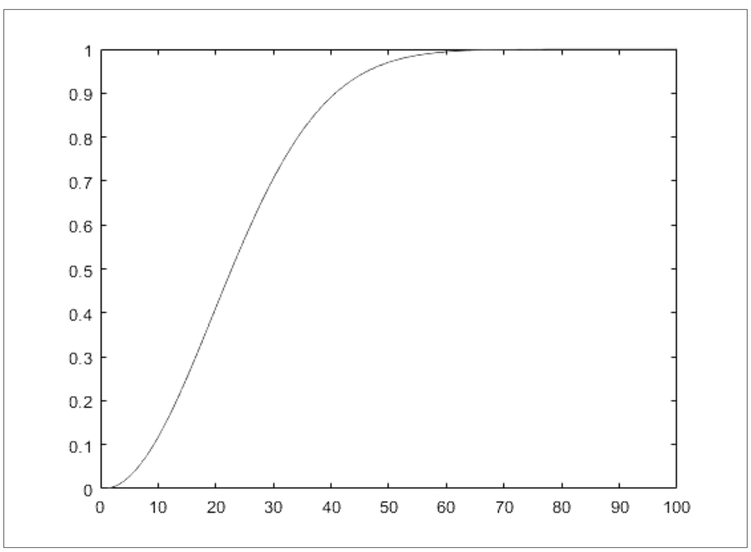

Figure: 8

5. Conclusion

In this paper, we demonstrated mathematical induction and pigeonhole principle with applications in the form of review study. The concept of proof by mathematical induction and pigeonhole principle are one of the most powerful tools for proving statements in discrete mathematics.

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A REVIEW STUDY OF ATTRIBUTES OF REAL NUMBERS WITH APPLICATIONS

Abstract: In this paper, we present a review study of attributes of real numbers with applications in signal processing.

Keywords: Natural Number, Integers, Rational and Irrational Numbers, Real Number, Signal.

Mathematical Subject Classifications (2010) No.: 46A30, 30C15, 26E70, 92F05, 05-XX.

1. Introduction

For defining the algebraic structure of the set of natural number N = {1, 2, 3....}, to each pair a, b of natural numbers, there corresponds a natural number denoted by $a + b$ the sum of a and b is called the Addition Composition and a natural number denoted by ab the product of a and b is called Multiplication Composition in the set of natural numbers. The fact of these existing in the set N of natural numbers these compositions is referred as possessing an algebraic structure [1, 7].

1.1. Basic Properties of the Two Compositions in N

1.1.1 Commutatively of addition and multiplication

$$a + b = b + a;\ ab = ba, \qquad \forall\, a, b \in N.$$

1.1.2 Associativity of addition and multiplication

$$a + (b + c) = (a + b) + c; \;\; a(bc) = (ab)c, \forall\, a, b, c \in N.$$

1.1.3 Cancellation laws

$$a + c = b + c \Rightarrow a = b; \quad ac = bc \Longrightarrow a = b.$$

1.1.4 Distributivity of addition with respect to multiplication

$$a(b + c) = ab + ac, \forall\, a, b, c \in N.$$

1.1.5 Multiplication property of 1

$$a.1 = a \;\forall\; a \in N.$$

Because of this property, 1 is called the Multiplicative Identity.

1.2 Order Structure of the Set N of Natural Numbers: The relation for given any two different natural numbers a, b, we have, $a > b \Leftrightarrow b < a$, i.e., a is greater than $b \Leftrightarrow b$ is smaller than a.

The relation 'greater than' between different natural numbers is known as an 'Order relation' in the set of natural numbers and the presence of this relation in N is referred to as N having an order structure [2, 7].

1.2.1 *Properties of the Order Relation*: Transitivity of the relation as

$$[a > b] \wedge [b > c] \Rightarrow a > c.$$

This property is referred to as the transitivity of the order relation.

1.3 Compatibility of Algebraic Structure with Order Structure

1.3.1 Compositions separately.

1.3.2 Compatibility of the order relation with the addition composition

$$a > b \Longrightarrow a + c > b + c.$$

1.3.3 Compatibility of the order relation with the multiplication composition

$$a > b \Longrightarrow ac > bc.$$

As a result of the last two properties, we can say that the order structure in N is compatible with its algebraic structure or $vice - versa$.

1.3.4 Principle of finite induction

Let n ∈ N and let $P(n)$ denotes a statement pertaining to n. if

(a) $P(1)$ is true, i.e., the statement is true for $n = 1$ and

(b) $P(n)$ is true $\Longrightarrow P(n+1)$ is true , then P (n) is true for every natural number n.

We say that the set N of natural numbers satisfies the principle of finite induction [3, 7].

1.4. Inverse Operations and Corresponding Limitations (Subtraction and Division in *N*)

1.4.1 ***Subtraction*****:** Given two members a, c of N, does there exist $x \in N$ such that $a + x = c$?

It is easy to see that x, if it exists, is unique. This is a consequence of the cancellation principle in as much as $a + x = a + y \Longrightarrow x = y$.

Also x exist if and only if $c > a$.

For example, if $a = 5,\ c = 8$ so that $c = 8 > 5 = a,$ we have $x = 3$.

If, however, we take $a = 5, c = 3$ so that $c = 3 < 5 = a$, there exists no $x \in N$ such that $5 + x = 3$.

In case $c > a$, so that there exists x such that $a + x = c$, we denote this x by $c - a$.

The symbol $c - a$ denotes the natural number which when added to a gives c. This symbol is meaningful if and only if $c > a$.

1.4.2 ***Division*****:** Given two natural numbers a and c, does there exist a natural number x such that

$$ax = c?$$

The number x, if it exists, is unique. This is a consequence of the cancellation law which states that $ax = ay \Longrightarrow x = y$.

Also the number x exists if a is divisor of c.

For example, $if\ a = 3, c = 15$ so that 3 is a divisor of 15 , we have $x = 5$.

If, however, we take $a = 3, c = 14$ so that $a = 3$ is not a divisor of $c = 14$, there exists no natural number x such that $3x = 14$.

In case a is a divisor of c so that there exists x such that $ax = c$, we denote this x by $c \div a$.

From above, we see that if a, c be two given natural numbers, the symbol $c - a$ is meaningful if and only if c is greater than a and the symbol $c \div a$ is meaningful if and only if a is a divisor of c [4,7]

2. The Set I or Z of Integers

The set I or Z of integers consists of the number$\dots \dots \dots .., -3, -2, -1, 0, 1, 2, 3, \dots \dots$, so that we have $Z = I = \{0, -1, 1, -2, 2, -3, 3, -4, 4, \dots \dots \dots .\}$. In this section a, b, c etc., referred to arbitrary members of I , viz., arbitrary integer.

2.1 Algebraic Structure of the set I of integers

2.1.1 *Addition Composition in* I: Addition composition in I which associates to each pair of members a, b of I a number called their sum and denoted by $a + b$ has the following basic properties:

(1) $a + b = b + a\ \forall a, b \in I$. Commutativity.

(2) $(a + b) + c = a + (b + c) \forall a, b, c \in I$. Associativity

(3) The number $0 \in I$ is such that $a + 0 = a\ \forall\ a \in I$. The number ‘0’ because of this relation is referred to as the additive identity.

(4) To each $a \in I$ there corresponds another, viz., $-a \in I$ such that $a + (-a) = 0$.

The integer, $-a$, is said to be the negative of the integer a or the additive inverse of a.

Inverse of addition: The equation $\mathrm{a} + \mathrm{x} = \mathrm{b},\ \mathrm{a} \in \mathrm{I},\ \mathrm{b} \in \mathrm{I}$ admits of a unique solution x, viz., $b - a \in I$.

Subtraction in I: It will be seen that subtraction is always possible in I, i.e., given any two members, a, b of $I, a - b$ is again a member of I for all a, b so that here we have a property of I which does not hold for N[5,7].

Example: Deduce from the above properties of the addition composition in I, that the cancellation law holds for addition in I, viz., that $a + c = b + c \Longrightarrow a = b$.

2.1.2 *Multiplication composition in I*: Multiplication composition in I which associates to each pair of members a, b of I a member of I denoted by ab and called their product has the following basic properties:

(1) $ab = ba\ \forall\ a, b\ \in I.$ Commutativity

(2) $(ab)c = a(bc) \forall\ a, b, c\ \in I.$ Associativity

(3) The number '1' $\in I$ in such that $a.1 = a\ \forall a \in I$. Because of the property 3, the integer '1' is known as the multiplicative identity.

(4) $[ab = ac\ \wedge a \neq 0\ ]\ \Longrightarrow b = c.$ Cancellation law for multiplication.

The following law relates the two compositions:

$$a(b + c) = ab + ac\ \forall\ a, b, c\ \in I.\quad \text{Distributivity.}$$

We refer to this law by saying that Multiplication distributes addition in I .

Ex. Deduce from above the following basic properties of addition and multiplication in ***I***.

(1) $ab = 0 \Leftrightarrow a = 0\ \vee b = 0.$

(2) $a(-b) = -(ab), (-a)(b) = -(ab), (-a)(-b) = ab.$

2.2 Division in *I* (Factors and multiples): If a, b are two non-zero members of I, we say that a is a factor of b if there exists $c \in I$ such that $b = ac$. It will be seen that $b \div a$ is meaningful if and only if $a \neq 0$ and a is a factor of b or that a is a divisor of b.

2.3 Order Structure of I: Given any two different members $a, b \in I$, we have either $a > b$ or $b > a$.

The 'Greater than' relation is transitive in as much as $a > b \wedge b > c \implies a > c$.

Also $$a > b \implies a + c > b + c$$

and $$a > b, c > 0 \implies ac > bc.$$

Thus, the system I of integers has, what has already been referred to, an order structure compatible with its algebraic structure.

Ex. It is clear that $a > b \wedge c < 0 \implies ac < bc$.

3. The Set Q Of Rational Numbers

The rational numbers are of the form p/q where p, q are arbitrary integers with $q \neq 0$.

3.1 Algebraic structure of Q: As in I, the set Q of rational numbers admits of two compositions, viz., addition and multiplication. We give below the basic properties of these two compositions [6, 7].

Here a, b, c etc., denote arbitrary members of the set Q of rational numbers.

1. The addition composition is commutative, associative, admits of an additive identity, viz., 0 and each element a admits of an additive inverse, $viz., -a$.

2. The multiplication composition is commutative, associative, admits of a multiplicative identity, viz., 1 and each non-zero element p/q admits of multiplicative inverse, viz., q/p.

3. Multiplication distributes addition.

Let a, b be two given rational numbers. We write

$$\mathrm{a} - \mathrm{b} = \mathrm{a} + (-\mathrm{b}).$$

Thus, $a - b$ is obtained by adding to a the additive inverse $-b$, of b. Also, if $b \neq 0$, we write $a \div b = a\left(\frac{1}{b}\right)$ so that $a \div b$ is obtained on multiplying a with the multiplicative inverse $\frac{1}{b}$ of the non–zero b.

Ex. 1: It is clear that $a(b-c)=ab-ac$.

$$\mathrm{ab}=0 \Leftrightarrow \mathrm{a}=0 \vee \mathrm{b}=0.$$

$$a(-b)=-(ab),(-a)(b)=-(ab),(-a)(-b)=ab.$$

Ex. 2: It is clear that show that if $a \neq 0$, the equation $ax+b=0$ admits of a unique solution in Q; given that $a \in Q,\ b \in Q$.

It is also clear that if $a=0,\ b \neq 0$, the equation has no solution and if $a=0,\ b=0$, every member of Q is a root of the equation.

Ex. 3: It is clear that $ab=ac \wedge a \neq 0 \Rightarrow b=c$.

3.2 Order Structure of Q: Given any two different rational numbers a,b, we have either $a>b$ or $b>a$.

Moreover, the order relation is transitive and compatible with the addition and multiplication compositions, i.e., we have

1. $a>b \wedge b>c \Longrightarrow a>c$.

2. $a>b \Longrightarrow a+v>b+c$.

3. $a>b \wedge c>0 \Longrightarrow ac>bc$.

Ex. 1:(a) It is clear that $\mathrm{x}>y \wedge z<0 \Longrightarrow xz<yz$.

(b) It is clear that $x \geq 0\ \forall\ x \in Q$.

Ex. 2: Given two different rational numbers a,b such that $a<b$; it is clear that there exist an infinite number of rational numbers c such that $a<c<b$.

4. Attributes of Real Number

A set K of numbers containing at least two members is called a field, if it is such that when a,b are arbitrary members of K, then $a+b, ab, a-b$ are also members of K and if $b \neq 0$, then $a \div b$ is also a member of K. It will be seen that while the set Q of rational numbers is a field, the sets I and N are not fields [4.7].

Ex: It is clear that no proper sub–set of the field of rational numbers is a field.

Since the set Q of rational numbers, besides having a field structure, also has an order structure compatible with its field structure, we say that the set Q of rational numbers is an ordered field.

We now demonstrate the basic properties of the set of the set R of real numbers. These properties will be describes in three stages. The set of properties included in the first stage will describe the Field Structure of the set of real numbers. We shall then proceed to describe at the second stage the Order structure of the set of real numbers as an Ordered field. It will be seen that the set of rational numbers and the set of real numbers are both ordered fields. At the third stage, we shall describe a property of the ordered field of rational numbers. This property will be referred to by saying that the Field of real numbers is order-complete. On the basis of the properties of the set of real numbers enumerated in the three stages, we say that the set of real numbers is a complete ordered field. The set of rational numbers is an ordered field alright but not a complete ordered field. Every property of the set of real numbers can be derived as a consequence of the basic character of the set of real numbers as a complete ordered field. The character of the set of real numbers as a complete ordered field will now be described [7].

4.1 Field Structure (Addition Composition): To each ordered pair of real numbers, there corresponds a real number called their sum and denoted by $a + b$. This process of associating to each ordered pair of real numbers a real number called their sum is known as addition composition in the set. In the following $a, b, c,$ etc. denote real numbers. This addition composition has the following properties:

4.1 $a + b = b + a$ [$a, b \in \mathbb{R}$. [Commutativity]

4.1.2 $(a + b) + c = a + (b + c)$ [$a, b\ c \in \mathbb{R}$. [Associativity]

4.1.3 There exist a real number, viz., '0' such that [$a \in \mathbb{R}$.
$a + 0 = a$ [Existence of additive identity]

4.1.4 To each real number a there corresponds a real number, viz., $-a$, such that $a + (-a) = 0$ [Existence of additive inverse]

4.2 Field Structure (Multiplication Composition): To each ordered pair a, b of real numbers, there corresponds a real number called their product and denoted by ab. This process of associating to each ordered pair of real numbers a real number called their product is known as multiplication composition in the set. This multiplication composition has the following properties:

4.2.1 ab = ba [a , b $\in \mathbb{R}$. [Commutativity]

4.2.2 $(ab)c = a(bc)$ [$a, b, c \in \mathbb{R}$ [Associativity]

4.2.3 There exists a real numbers, viz., '1' such that [$a \in \mathbb{R}$.
$a.1 = a$ [Existence of multiplicative identity]

4.2.4 To each real number $a \neq 0$, there correspond another, viz., $1/a$ such that $a\left(\frac{1}{a}\right) = 1$ [Existence of multiplicative inverse]

There is also a law known as distributive law which relates the two compositions, viz.,

$$a(b + c) = ab + ac \ [a, b, c \in \mathbb{R}. \quad \text{[Distributive law]}$$

The fact of the set of real numbers admitting of two compositions satisfying the nine properties mentioned above is referred to as the set of real numbers having a field structure.

4.3 Applications: One of the most common ways of obtaining a discrete-time signal is by sampling a continuous-time signal. The discrete-time signal obtained by sampling the continuous-time signal can be denoted by $x(nT) = x(t)_{t=nT}$, where T is called the sampling period and n is an integer ranging from $-\infty\ to + \infty$ called the time index [8]. The instants at which the signal appears are called sampling instants. For convenience, we write $x(nT) = x(n), n = 0, \pm 1, \pm 2, \ldots$Thus, a discrete-time signal is represented by the sequence of numbers $\ldots x(-2), x(-1), x(0), x(1), x(2), \ldots$

We use real numbers for discrete-time signal as sequence:

$$x(n) = \left\{\ldots, x(-3), x(-2), x(-1), \underset{\uparrow}{x(0)}, x(1), x(2), x(3), \ldots\right\} \text{ as}$$

$$x(n) = \{\ldots, -2{,}1.3, \underset{\uparrow}{2}, -3{,}4, -1, 2, 1, \ldots\} \text{ means}$$

$$x(-2) = -2, x(-1) = 1{,}3, x(0) = 2, x(1) = -3.4, x(2) = -1, x(3) = 2, x(4) = 1.$$

FFT algorithms are based on the fundamental principle of decomposing the computation of discrete Fourier transform of a sequence of length N into successively smaller discrete Fourier transforms. There are basically two classes of FFT algorithms. They are decimation-in-time and decimation-in-frequency. In decimation-in-time, the sequence for which we need the DSFT is successively divided into smaller sequences and the DFTs of these subsequences are combined in a certain pattern to obtain the required DFT of the entire sequence. In the decimation-in-frequency approach, the frequency samples of the DFT are decomposed into smaller and smaller subsequences in a similar manner [8].

5. Conclusion

In this paper, we demonstrate in the form of review study that the property of the set R of real numbers, viz., which every non-empty sub-set of R which is bounded above admits of least upper bound, is referred to as its order-completeness property of real numbers. It will be also seen that the Archimedean property of R is a consequence of the order-completeness property of R. Attributes of real numbers are used in digital signal processing, to describe the different types of signals, fast Fourier transform algorithm, DIT-DIF FFT algorithm etc.

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