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GIBONACCI EXTENSIONS OF A CATALAN DELIGHT WITH GRAPH-THEORETIC CONFIRMATIONS REVISITED

Abstract: We explore the extension of the Catalan-like identity $g_{n+k}g_{n-k} - g_n^2 = (-1)^{n+k+1}\mu f_k^2$ [6] and its ramifications to the Pell, Jacobsthal, Vieta, and Chebyshev families, and give graph-theoretic confirmations of the Gibonacci and Jacobsthal versions.

Keywords: Fibonacci Numbers, Pell Numbers, Jacobsthal Vieta, and Chebyshev Families, Graph-Theoretic Confirmations.

Mathematical Subject Classification (2010) No.: 05A19, 11B37, 11B39, 11Cxx.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_n(x) = a(x)z_{n-1}(x) + b(x)z_{n-2}(x)$, where x is a complex variable; $a(x)$, $b(x)$, $z_0(x)$ and $z_1(x)$ are arbitrary complex polynomials; and $n \geq 2$.

Fibonacci, Lucas, Pell-Lucas, Vieta, Vieta-Lucas, and Chebyshev polynomials belong to the family $\{z_n(x)\}$; they are denoted by $f_n(x)$, $l_n(x)$, $p_n(x)$, $q_n(x)$, $J_n(x)$, $V_n(x)$, $v_n(x)$, $T_n(x)$, and $U_n(x)$, respectively [7, 8]. The nth Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, and Jacobsthal Lucas numbers are denoted by F_n, L_n, P_n, Q_n, J_n, and j_n, respectively; they are given by

$F_n = f_n(1)$, $L_n = l_n(1)$, $P_n = p_n(1) = f_n(2)$, $2Q_n = q_n(1) = l_n(1)$, $J_n = J_n(2)$, and $j_n = j_n(2)$, [7, 8].

These subfamilies are linked by the following relationships [1, 7, 8]:

$$J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x}) \qquad j_n(x) = x^{n/2} l_n(1/\sqrt{x})$$

$$V_n(x) = i^{n-1} f_n(-ix) \qquad v_n(x) = i^n l_n(-ix)$$

$$V_n(x) = U_{n-1}(x/2) \qquad v_n(x) = 2T_n(x/2),$$

where $i = \sqrt{-1}$.

In the interest of brevity, clarity, and convenience, we *omit* the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$; we also let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $c_n = J_n(x)$ or $j_n(x)$, and correspondingly, $G_n = F_n$ or L_n, $B_n = P_n$ or Q_n, and $C_n = J_n$ or j_n.

2. Gibonacci Extensions of a Catalan Delight

The charming identity [4]

$$F_{n+k}F_{n-k} - F_n^2 = (-1)^{n-k+1} F_k^2$$

has a gibonacci extension [6, 7], where $n \geq k$:

$$g_{n+k}g_{n-k} - g_n^2 = (-1)^{n-k+1} \mu f_k^2 \ ,$$

where

$$\mu = \mu(x) = \begin{cases} 1 & \text{if} \quad g_n = f_n \\ -(x^2+4) & \text{if} \quad g_n = l_n. \end{cases}$$

This identity has a delightful extension [6]:

$$g_{m+k}g_{n-k} - g_m g_n = (-1)^{n-k+1} \mu f_k f_{m-n+k} \, . \tag{1}$$

We can establish this using the Binet-like formulas

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad l_n = \alpha^n + \beta^n,$$

where $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are the solutions of the equation $t^2 - xt - 1 = 0$.

For example,

$$\begin{aligned} l_m l_n + (-1)^{n-k}(x^2+4) f_k f_{m-n+k} &= (\alpha^m + \beta^m)(\alpha^n + \beta^n) \\ &\quad + (\alpha\beta)^{n-k}(\alpha^k - \beta^k)(\alpha^{m-n+k} - \beta^{m-n+k}) \\ &= \alpha^{m+n} + \alpha^{m+k}\beta^{n-k} + \alpha^{n-k}\beta^{m+k} + \beta^{m+n} \\ &= (\alpha^{m+k} + \beta^{m+k})(\alpha^{n-k} + \beta^{n-k}) \\ &= l_{m+k} l_{n-k}. \end{aligned}$$

This gives identity (1) when $g_n = l_n$. Its Fibonacci counterpart follows similarly.

It follows from identity (1) that

$$\begin{aligned} G_{m+k}G_{n-k} - G_m G_n &= (-1)^{n-k+1}\mu(1) F_k F_{m-n+k}; \qquad (2) \\ b_{m+k}b_{n-k} - b_m b_n &= (-1)^{n-k+1}\mu(2x) p_k p_{m-n+k}; \\ B_{m+k}B_{n-k} - B_m B_n &= (-1)^{n-k+1}\mu(2) P_k P_{m-n+k}. \end{aligned}$$

Identity (2) with $G_n = F_n$ is the d'Ocagne identity [4, 6].

Next we explore the consequences of identity (1) to the Jacobsthal family.

2.1 Jacobsthal Implications: Replacing x with $1/\sqrt{x}$ in identity (1), we get

$$g_{m+k}g_{n-k} - g_m g_n = (-1)^{n-k+1}\mu(1/\sqrt{x}) f_k f_{m-n+k}, \qquad (3)$$

where $g_n = g_n(1/\sqrt{x})$.

Suppose $g_n = f_n$. Multiplying the resulting equation with $x^{(m+n-2)/2}$, we get

$$J_{m+k}(x)J_{n-k}(x) - J_m(x)J_n(x) = -(-x)^{n-k} J_k(x)J_{m-n+k}(x).$$

Likewise, when $g_n = l_n$, multiplying the corresponding equation with $x^{(m+n)/2}$ yields

$$j_{m+k}(x)j_{n-k}(x) - j_m(x)j_n(x) = (-x)^{n-k}(4x+1)J_k(x)J_{m-n+k}(x).$$

Combining the two cases, we get

$$c_{m+k}c_{n-k} - c_m c_n = -(-x)^{n-k}\nu(x)J_k J_{m-n+k}, \tag{4}$$

where

$$\nu(x) = \begin{cases} 1 & \text{if} \quad c_n = J_n(x) \\ -(4x+1) & \text{if} \quad c_n = j_n(x). \end{cases}$$

This can be confirmed independently using the Binet-like formulas for $J_n(x)$ and $j_n(x)$.

Identity (4) implies

$$C_{m+k}C_{n-k} - C_m C_n = -(-2)^{n-k}\nu J_k J_{m-n+k},$$

where

$$\nu = \begin{cases} 1 & \text{if} \quad c_n = J_n \\ -9 & \text{if} \quad c_n = j_n. \end{cases}$$

2.2 Vieta and Chebyshev Implications: Identity (1) has Vieta and Chebyshev consequences as well. In the interest of brevity, we omit the details.

$$d_{m+k}d_{n-k} - d_m d_n = \delta(x)V_k V_{m-n+k},$$

where

$$\delta(x) = \begin{cases} -1 & \text{if} \quad d_n = V_n \\ x^2 - 4 & \text{if} \quad d_n = v_n. \end{cases}$$

Since $U_n(x) = V_{n+1}(2x)$ and $2T_n(x) = v_n(2x)$, it then follows that

$$e_{m+k}e_{n-k} - e_m e_n = (x)U_{k-1}U_{m-n+k-1},$$

where

$$e(x) = \begin{cases} -1 & \text{if} \quad e_n = U_n \\ x^2 - 1 & \text{if} \quad e_n = T_n. \end{cases}$$

Next we confirm identities (1) and (4) using graph-theoretic tools.

3. Graph-theoretic Confirmations

In order to confirm identity (1), first we present some basic facts.

Consider the *weighted digraph* D_1 with vertices v_1 and v_2 in Figure 1. A *weight* is assigned to each edge.

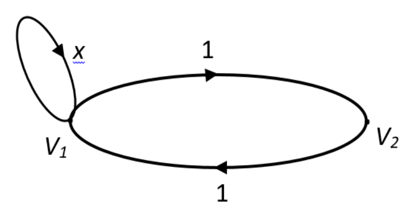

Figure 1: Weighted Digraph D_1

Its *weighted adjacency matrix* is the Q-matrix

$$Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix},$$

where $Q = Q(x)$. It then follows by induction that

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $n \geq 1$ [3].

A *walk* from vertex v_i to vertex v_j is a sequence $v_i - e_i - v_{i+1} - \cdots - v_{j-1} - e_{j-1} - v_j$ of vertices v_k and edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The walk is *closed* if $v_i = v_j$; otherwise, it is *open*. The *length* of a walk is the number of edges in the walk. The *weight* of a walk is the product of the weights of the edges along the walk.

The following theorem provides a powerful tool for computing the weight of a walk of length n from v_i to v_j [2, 3].

Theorem 1: *Let A be the weighted adjacency matrix of a weighted and connected digraph with vertices $v_1, v_2, \ldots, v_k$. Then the ijth entry of the matrix A^n gives the sum of the weights of all walks of length n from v_i to v_j, where $n \geq 1$.* □

The next result follows from this theorem.

Corollary 1: *The ijth entry of Q^n gives the sum of the weights of all walks of length n from v_i to v_j in the weighted digraph D_1, where $1 \leq i, j \leq n$.* □

Consequently, the sum of the weights of all closed walks of length n originating at v_1 is f_{n+1}, and that of walks of length n originating at v_2 is f_{n-1}. So the sum of the weights of all closed walks of length n is $f_{n+1} + f_{n-1} = l_n$. These facts play a pivotal role in our graph-theoretic proofs.

3.1 Proof of Identity (1): Let A, B, C, and D be the sets of closed walks of lengths $m+k-1$, $n-k-1$, $k-1$ and $m-n+k-1$ from v_1 to v_1, respectively. The sum S_1 of the weights of pairs (v, w) in $A \times B$ is given by $S_1 = f_{m+k} f_{n-k}$, and the sum S_2 of the weights of elements (v, w) in $C \times D$ is given by

$S_2 = f_k f_{m-n+k}$. So $S_1 + (-1)^{n-k+1} S_2 = f_{m+k} f_{n-k} + (-1)^{n-k+1} f_k f_{m-n+k}$.

We will now compute this sum in a different way. Let (v, w) be an arbitrary element of $A \times B$. If both v and w begin with a loop, the sum of the weights of such pairs is $(xf_{m+k-1})(xf_{n-k-1}) = x^2 f_{m+k-1} f_{n-k-1}$; if v begins with a loop and w does not, the corresponding sum is $(xf_{m+k-1})(1 \cdot 1 \cdot f_{n-k-2}) = xf_{m+k-1}f_{n-k-2}$; if v does not and w does, the corresponding sum is $(1 \cdot 1 \cdot f_{m+k-2})(xf_{n-k-1}) = xf_{m+k-2}f_{n-k-1}$ and if neither does, the resulting sum is $(1 \cdot 1 \cdot f_{m+k-2})(1 \cdot 1 \cdot f_{n-k-2}) = f_{m+k-2}f_{n-k-2}$.

Thus, by identity (1), we have

$$\begin{aligned} S_1 &= x^2 f_{m+k-1} f_{n-k-1} + xf_{m+k-1}f_{n-k-2} + xf_{m+k-2}f_{n-k-1} + f_{m+k-2}f_{n-k-2} \\ &= (xf_{m+k-1} + f_{m+k-2})(xf_{n-k-1} + f_{n-k-2}) \\ &= f_{m+k}f_{n-k} \\ &= f_m f_n + (-1)^{n-k} f_k f_{m-n+k}. \end{aligned}$$

To re-compute S_2, let (v, w) be an arbitrary element of $C \times D$. If both v and w begin with a loop, the sum of the weights of such pairs is $(xf_{k-1})(xf_{m-n+k-1}) = x^2 f_{k-1} f_{m-n+k-1}$; if v does and w does not, the corresponding sum is $(xf_{k-1})(1 \cdot 1 \cdot f_{m-n+k-2}) = xf_{k-1}f_{m-n+k-2}$; if v does not and w does, then the sum is $(1 \cdot 1 \cdot f_{k-2})(xf_{m-n+k-1}) = xf_{k-2}f_{m-n+k-1}$; and if neither does, then the sum is $(1 \cdot 1 \cdot f_{k-2})(1 \cdot 1 \cdot f_{m-n+k-2}) = f_{k-2}f_{m-n+k-2}$. So

$$\begin{aligned} S_2 &= x^2 f_{k-1} f_{m-n+k-1} + xf_{k-1}f_{m-n+k-2} + xf_{k-2}f_{m-n+k-1} + f_{k-2}f_{m-n+k-2} \\ &= (xf_{k-1} + f_{k-2})(xf_{m-n+k-1} + f_{m-n+k-2}) \\ &= f_k f_{m-n+k}. \end{aligned}$$

Thus,

$$S_1 + (-1)^{n-k+1} S_2 = [f_m f_n + (-1)^{n-k} f_k f_{m-n+k}] + (-1)^{n-k+1} f_k f_{m-n+k} = f_m f_n.$$

Equating the two sums yields identity (1) when $g_n = f_n$, as desired.

Now let $g_n = l_n$. Let A and B be the sets closed walks of length $m+k$ originating at v_1 and v_2, respectively; and C and D the sets closed walks of length $n-k$ originating at v_1 and v_2, respectively. Then $A \cup B$ and $C \cup D$ denote the sets of closed walks of lengths $m+k$ and $n-k$ in the digraph, respectively. The sum S_1 of the weights of the pairs (v, w) of walks in $(A \cup B) \times (C \cup D)$ is given by $S_1 = l_{m+k} l_{n-k}$. Let R and S be the sets of closed walks of lengths $k-1$ and $m-n+k-1$ from v_1 to v_1, respectively. The sum S_2 of the weights of elements (v, w) in $R \times S$ is given by $S_2 = f_k f_{m-n+k}$. Then

$$S_1 - (-1)^{n-k}(x^2+4)S_2 = l_{m+k} l_{n-k} - (-1)^{n-k}(x^2+4) f_k f_{m-n+k}.$$

We will now compute this sum in a different way. To this end, let (v, w) be an arbitrary element of $(A \cup B) \times (C \cup D)$.

Case 1: Suppose $v \in A$ and $w \in C$. If both v and w begin with a loop, the sum of the weights of such pairs (v, w) is $(x f_{m+k})(x f_{n-k}) = x^2 f_{m+k} f_{n-k}$; if v does and w does not, the corresponding sum is $(x f_{m+k})(1 \cdot 1 \cdot f_{n-k-1}) = x f_{m+k} f_{n-k-1}$; if v does not and w does, the resulting sum is $(1 \cdot 1 \cdot f_{m+k-1})(x f_{n-k}) = x f_{m+k-1} f_{n-k}$; and if neither does, the corresponding sum is $(1 \cdot 1 \cdot f_{m+k-1})(1 \cdot 1 \cdot f_{n-k-1}) = f_{m+k-1} f_{n-k-1}$.

Case 2: Suppose $v \in A$ and $w \in D$. If v begins with a loop, the sum of the weights of such pairs is $(x f_{m+k}) f_{n-k-1} = x f_{m+k} f_{n-k-1}$; and if v does not, the corresponding sum is $(1 \cdot 1 \cdot f_{m+k-1}) f_{n-k-1} = f_{m+k-1} f_{n-k-1}$.

Case 3: Suppose $v \in B$ and $w \in C$. If w begins with a loop, the sum of the weights of such pairs is $f_{m+k-1}(x f_{n-k}) = x f_{m+k-1} f_{n-k}$; and if w does not, the corresponding sum is $f_{m+k-1}(1 \cdot 1 \cdot f_{n-k-1}) = f_{m+k-1} f_{n-k-1}$.

Case 4: Suppose $v \in B$ and $w \in D$. The sum of the weights of such pairs is $f_{m+k-1} f_{n-k-1}$.

Combining the four cases, by identity (1) we have

$$
\begin{aligned}
S_1 &= (xf_{m+k} + f_{m+k-1})(xf_{n-k} + f_{n-k-1}) + (xf_{m+k} + f_{m+k-1})f_{n-k-1} \\
&\qquad + f_{m+k-1}(xf_{n-k} + f_{n-k-1}) + f_{m+k-1}f_{n-k-1} \\
&= (f_{m+k+1} + f_{m+k-1})(f_{n-k+1} + f_{n-k-1}) \\
&= l_{m+k} l_{n-k} \\
&= l_m l_n + (-1)^{n-k}(x^2+4) f_k f_{m-n+k} .
\end{aligned}
$$

With the sets R and S above, we have $S_2 = f_k f_{m+n-k}$. Thus,

$$
\begin{aligned}
S_1 - (-1)^{n-k}(x^2+4)S_2 &= [l_m l_n + (-1)^{n-k}(x^2+4) f_k f_{m-n+k}] \\
&\qquad - (-1)^{n-k}(x^2+4) f_k f_{m-n+k} = l_m l_n .
\end{aligned}
$$

This, coupled with the earlier sum, gives the desired result.

Next we confirm the Jacobsthal delight in (4) using graph-theoretic tools.

4. Graph-theoretic Confirmation of Identity (4)

Consider the weighted digraph D_2 in Figure 2 with vertices v_1 and v_2. Its weighted adjacency matrix is given by

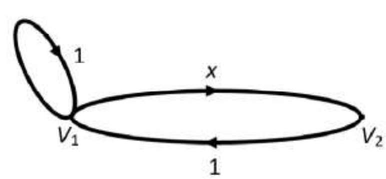

Figure 2: Weighted Digraph D_2

$$
M = \begin{bmatrix} 1 & x \\ 1 & 0 \end{bmatrix}.
$$

Since,

$$M^n = \begin{bmatrix} J_{n+1}(x) & xJ_n(x) \\ J_{n(x)} & xJ_{n-1(x)} \end{bmatrix},$$

by induction [5], it follows that the sum of the closed walks of length n from v_1 to itself is $J_{n+1}(x)$, and that from v_2 to itself is $xJ_{n-1}(x)$. Consequently, the sum of the weights of all closed walks of length n is $J_{n+1}(x) + xJ_{n-1}(x) = j_n(x)$.

We are now ready to confirm identity (4). We begin with $c_n = J_n(x)$. Let A, B, C, and D be the sets of closed walks of lengths $m+k-1$, $n-k-1$, $k-1$, and $m-n+k-1$ from v_1 to v_1, respectively. The sum S_1 of the weights of pairs of walks in $A \times B$ is $S_1 = J_{m+k}(x)J_{n-k}(x)$; and the sum S_2 of the weights of pairs of walks in $C \times D$ is $S_2 = J_k(x)J_{m-n+k}(x)$. Then

$$S_1 + (-x)^{n-k} S_2 = J_{m+k}(x)J_{n-k}(x) + (-x)^{n-k} J_k(x)J_{m-n+k}(x).$$

We will now compute this sum in a different way. To re-compute S_1, we let (v,w) an arbitrary element of $A \times B$. If both v and w begin with a loop, the sum of the weights of such pairs is $[1 \cdot J_{m+k-1}(x)][1 \cdot J_{n-k-1}(x)] = J_{m+k-1}(x)J_{n-k-1}(x)$; if v does and w does not, the corresponding sum is $[1 \cdot J_{m+k-1}(x)][x \cdot 1 \cdot J_{n-k-2}(x)] = xJ_{m+k-1}(x)J_{n-k-2}(x)$; if v does not and w does, the corresponding sum is $[x \cdot 1 \cdot J_{m+k-2}(x)][1 \cdot J_{n-k-1}(x)] = xJ_{m+k-2}(x)J_{n-k-1}(x)$ and if neither does, the resulting sum is $[x \cdot 1 \cdot J_{m+k-2}(x)][x \cdot 1 \cdot J_{n-k-2}(x)] = x^2 J_{m+k-2}(x)J_{n-k-2}(x)$. Then, by identity (4), we have

$$\begin{aligned} S_1 &= [J_{m+k-1}(x) + xJ_{m+k-2}(x)][J_{n-k-1}(x) + xJ_{n-k-2}(x)] \\ &= J_{m+k}(x)J_{m-k}(x) \\ &= J_m(x)J_n(x) - (-x)^{n-k} J_k(x)J_{m-n+k}(x). \end{aligned}$$

We will now re-compute S_2. Let (v,w) be an arbitrary element of $C \times D$. If both v and w begin with a loop, the sum of the weights of such pairs is $[1 \cdot J_{k-1}(x)][1 \cdot J_{m-n+k-1}(x)] = J_{k-1}(x)J_{m-n+k-1}(x)$; if v does and w does not, the corresponding sum is $[1 \cdot J_{k-1}(x)][x \cdot 1 \cdot J_{m-n+k-2}(x)] = xJ_{k-1}(x)J_{m-n+k-2}(x)$; if v does not and w does, the corresponding sum is $[x \cdot 1 \cdot J_{k-2}(x)][1 \cdot J_{m-n+k-1}(x)] = xJ_{k-2}(x)J_{m-n+k-1}(x)$; and if neither does, the resulting sum is $[x \cdot 1 \cdot J_{k-2}(x)][x \cdot 1 \cdot J_{m-n+k-2}(x)] = x^2 J_{k-2}(x)J_{m-n+k-2}(x)$. Then

$$\begin{aligned} S_2 &= [J_{k-1}(x) + xJ_{k-2}(x)][J_{m-n+k-1}(x) + xJ_{m-n+k-2}(x)] \\ &= J_k(x)J_{m-n+k}(x) \ . \end{aligned}$$

Thus,

$$\begin{aligned} S_1 + (-x)^{n-k} S_2 &= [J_m(x)J_n(x) - (-x)^{n-k} J_k(x)J_{m-n+k}(x)] \\ &\qquad + (-x)^{n-k} J_k(x)J_{m-n+k}(x) \\ &= J_m(x)J_n(x) \, . \end{aligned}$$

Equating this with the earlier sum yields identity (4) with $c_n = J_n(x)$.

Finally, consider the case $c_n = j_n(x)$. Let A and B be the sets of closed walks of length $m+k$ originating at v_1 and v_2, respectively; and R and S the sets closed walks of length $n-k$ originating at v_1 and v_2, respectively. Let C and D denote the sets of closed walks of lengths $k-1$ and $m-n+k-1$, originating at v_1, respectively. The sum S_1 of the weights of pairs of walks in $(A \cup B) \times (R \cup S)$ is given by $S_1 = j_{m+k}(x)j_{n-k}(x)$; and the sum S_2 of the weights of pairs of walks in $C \times D$ is $S_2 = J_k(x)J_{m-n+k}(x)$. So

$$S_1 - (4x+1)(-x)^{n-k} S_2 = j_{m+k}(x)j_{n-k}(x) - (4x+1)(-x)^{n-k} J_k(x)J_{m-n+k}(x) \, .$$

We will now compute this sum in a different way. Let (v,w) be an arbitrary element of $(A \cup B) \times (R \cup S)$.

Case 1: Suppose $v \in A$ and $w \in R$. If both v and w begin with a loop, the sum of the weights of such pairs (v, w) is $[1 \cdot J_{m+k}(x)][1 \cdot J_{n-k}(x)] = J_{m+k}(x)J_{n-k}(x)$; if v does and w does not, the corresponding sum is $[1 \cdot J_{m+k}(x)][x \cdot 1 \cdot J_{n-k-1}(x)] = xJ_{m+k}(x)J_{n-k-1}(x)$; if v does not and w does, the resulting sum is $[x \cdot 1 \cdot J_{m+k-1}(x)][1 \cdot J_{n-k}(x)] = xJ_{m+k-1}(x)J_{n-k}(x)$; and if neither does, the corresponding sum is $[x \cdot 1 \cdot J_{m+k-1}(x)][x \cdot 1 \cdot J_{n-k-1}(x)] = x^2 J_{m+k-1}(x)J_{n-k-1}(x)$. Their sum is

$$[J_{m+k}(x) + xJ_{m+k-1}(x)][J_{n-k}(x) + xJ_{n-k-1}(x)] = J_{m+k+1}(x)J_{n-k+1}(x)\,.$$

Case 2: Suppose $v \in A$ and $w \in S$. If v begins with a loop, the sum of the weights of such pairs is $[1 \cdot J_{m+k}(x)][xJ_{n-k-1}(x)] = xJ_{m+k}(x)J_{n-k-1}(x)$; and if v does not, the corresponding sum is $[x \cdot 1 \cdot J_{m+k-1}(x)][xJ_{n-k-1}(x)] = x^2 J_{m+k-1}(x)J_{n-k-1}(x)$. Their sum is

$$[J_{m+k}(x) + xJ_{m+k-1}(x)][xJ_{n-k-1}(x)] = xJ_{m+k+1}(x)J_{n-k+1}(x)\,.$$

Case 3: Suppose $v \in B$ and $w \in R$. If w begins with a loop, the sum of the weights of such pairs is $[xJ_{m+k-1}(x)][1 \cdot J_{n-k}(x)] = xJ_{m+k-1}(x)J_{n-k}(x)$; and if w does not, the corresponding sum is $[xJ_{m+k-1}(x)][x \cdot 1 \cdot J_{n-k-1}(x)] = x^2 J_{m+k-1}(x)J_{n-k-1}(x)$. Their sum is

$$xJ_{m+k-1}(x)[J_{n-k}(x) + xJ_{n-k-1}(x)] = xJ_{m+k+1}(x)J_{n-k+1}(x)\ .$$

Case 4: Suppose $v \in B$ and $w \in S$. The sum of the weights of such pairs is

$$[xJ_{m+k-1}(x)][xJ_{n-k-1}(x)] = x^2 J_{m+k-1}(x)J_{n-k-1}(x)\,.$$

Combining the four cases, by identity (4) we have

$$
\begin{aligned}
S_1 &= [J_{m+k+1}(x) + xJ_{m+k-1}(x)][J_{n-k+1}(x) + xJ_{n-k-1}(x)] \\
&= j_{m+k}(x)j_{n-k}(x) \\
&= j_m(x)j_n(x) + (4x+1)(-x)^{n-k} J_k(x)J_{m-n+k}(x)\,.
\end{aligned}
$$

Earlier, we found that $S_2 = J_k(x)J_{m-n+k}(x)$.

Consequently,

$$
\begin{aligned}
S_1 - (4x+1)(-x)^{n-k} S2 &= [j_m(x)j_n(x) + (4x+1)(-x)^{n-k} J_k(x)J_{m-n+k}(x)] \\
&\quad - (4x+1)(-x)^{n-k} J_k(x)J_{m-n+k}(x) \\
&= j_m(x)j_n(x)\,.
\end{aligned}
$$

Equating the two sums yields the desired result.

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A FAMILY OF SUMS OF GIBONACCI POLYNOMIAL PRODUCTS OF ORDER 3

Abstract: We explore twelve sums of gibonacci polynomial products of order 3, involving g_{3n-1}, g_{3n}, g_{3n+1}, and g_{3n+2}, where g_n denotes the nth gibonacci polynomial.

Keywords: Gibonacci Polynomial, Fibonacci Polynomial.

Mathematical Subject Classification (2010) No.: 05A19, 11B37, 11B39, 11CXX.

1. Introduction

Gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary complex variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary complex polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*.

Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 6, 8].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. We let $g_n = f_n$ or l_n, and omit a lot of basic algebra.

A *gibonacci polynomial product of order* m *is* a product of gibonacci polynomials g_{n+k} of the form $\prod_{k\in\mathcal{Z}} g_{n+k}^{s_j}$, where $\sum_{s_j\geq 1} s_j = m$ [9, 10].

1.1 Sums of Gibonacci Polynomial Products of Orders 2 and 3: Table 1 shows some well known gibonacci polynomials involving sums of products of orders 2 and 3, where $\Delta^2 = x^2 + 4$ [5, 6, 8].

Table 1
Sums of Gibonacci Polynomial Products of Orders 2 and 3

$f_{n+1}^2 + f_n^2 = f_{2n+1}$	$l_{n+1}^2 + l_n^2 = \Delta^2 f_{2n+1}$
$f_{n+2}^2 - f_{n-2}^2 = (x^3 + 2x)f_{2n}$	$l_{n+2}^2 - l_{n-2}^2 = (x^3 + 2x)\Delta^2 f_{2n}$
$f_{n+1}^3 + xf_n^3 - f_{n-1}^3 = xf_{3n}$	$l_{n+1}^3 + xl_n^3 - l_{n-1}^3 = x\Delta^2 l_{3n}$
$f_{n+2}^3 - (x^2+2)f_n^3 + f_{n-2}^3 = x^2(x^2+2)f_{3n}$	$l_{n+2}^3 - (x^2+2)l_n^3 + l_{n-2}^3 = x^2(x^2+2)\Delta^2 l_{3n}$

With this background, we now investigate a family of similar formulas for g_{3n}, g_{3n-1}, g_{3n+1}, and g_{3n+2} as sums of gibonacci polynomial products of order 3.

2. A Family of Sums of Gibonacci Polynomial Products of Order 3

The development of the desired formulas hinges on gibonacci recurrence, identities $f_{n+2} + f_{n-2} = (x^2 + 2)f_n$, $f_{n+2} - f_{n-2} = xl_n$, $f_{n+1} + f_{n-1} = l_n$, $f_n = (x^2 + 1)f_{n-2} + xf_{n-3}$, $f_{2n} = f_n l_n$, $f_{2n+1} = f_{n+1}^2 + f_n^2$, and the gibonacci addition formula [8]

$$g_{a+b} = f_{a+1}g_b + f_a g_{b-1}.$$

We begin our exploration with $x^2 f_{3n+2}$.

2.1 A Gibonacci Sum for $x^2 f_{3n+2}$: By the Fibonacci addition formula, we have

$$\begin{aligned}
x^2 f_{3n+2} &= x^2 f_{2n+1} f_{n+2} + f_n(xl_n)(xf_{n+1}) \\
&= f_{n+2}\ [(xf_{n+1})2 + x^2 f_n^2] + f_n(f_{n+2} - f_{n-2})(f_{n+2} - f_n) \\
&= f_{n+2}[(f_{n+2} - f_n)^2 + x^2 f_n^2] + f_n(f_{n+2} - f_{n-2})(f_{n+2} - f_n) \\
&= f_{n+2}^3 + (-f_{n+2}^2 f_n + x^2 f_{n+2} f_n^2 - f_{n+2} f_n f_{n-2}) + f_n^2 f_{n-2} \\
&= f_{n+2}^3 - f_{n+2} f_n (f_{n+2} + f_{n-2} - x^2 f_n) + f_n^2 f_{n-2} \\
&= f_{n+2}^3 - f_{n+2} f_n [(x^2 + 2) f_n - x^2 f_n] + f_n^2 f_{n-2} \\
&= f_{n+2}^3 - 2 f_{n+2} f_n^2 + f_n^2 f_{n-2}\,. \qquad (1)
\end{aligned}$$

It then follows that

$$F_{3n+2} = F_{n+2}^3 - 2F_{n+2}F_n^2 + F_n^2 F_{n-2}\,; \qquad (2)$$

see [2, 7] for a graph-theoretic proof of this using path graphs.

Next we find a similar formula for $x^3 f_{3n+1}$.

2.2 A Gibonacci Sum for $x^3 f_{3n+1}$: Again by the addition formula, we get

$$\begin{aligned}
x^3 f_{3n+1} &= x^3 f_{2n+1} f_{n+1} + f_{2n} f_n \\
&= [(xf_{n+1})^2 + x^2 f_n^2]\ (xf_{n+1}) + x^2 f_n^2 (xl_n) \\
&= [(f_{n+2} - f_n)^2 + x^2 f_n^2 [(f_{n+2} - f_n) + x^2 f_n^2 (f_{n+2} - f_{n-2}) \\
&= f_{n+2}^3 - 3 f_{n+2}^2 f_n + (2x^2 + 3) f_{n+2} f_n^2 - (x^2 + 1) f_n^3 - x^2 f_n^2 f_{n-2}\,. \qquad (3)
\end{aligned}$$

This yields

$$F_{3n+1} = F_{n+2}^3 - 3F_{n+2}^2 Fn + 5F_{n+2}F_n^2 - 2F_n^3 - F_n^2 F_{n-2}\,. \tag{4}$$

Next we explore a formula for $x^3 f_{3n-1}$.

2.3 A Gibonacci Sum for $x^3 f_{3n-1}$: Since $f_{3n-1} = f_{2n+1}f_{n-1} + f_{2n}f_{n-2}$, we have

$$\begin{aligned} x^3 f_{3n-1} &= [(xf_{n+1})^2 + x^2 f_n^2[(xf_{n-1}) + x^2 f_n f_{n-2}(xl_n) \\ &= [(f_{n+2} - f_n)^2 + x^2 f_n^2][f_{n+2} - (x^2+1)f_n] \\ &\qquad + x^2 f_n f_{n-2}(f_{n+2} - f_{n-2}) \\ &= f_{n+2}^3 - (x^2+3)f_{n+2}^2 f_n + 3(x^2+1)f_{n+2}f_n^2 + x^2 f_{n+2}f_n f_{n-2} \\ &\qquad - (x^2+1)^2 f_n^3 - x^2 f_n f_{n-2}^2\,. \end{aligned} \tag{5}$$

Consequently,

$$F_{3n-1} = F_{n+2}^3 - 4F_{n+2}^2 F_n + 6F_{n+2}F_n^2 + F_{n+2}F_nF_{n-2} - 4F_n^3 - F_nF_{n-2}^2\,. \tag{6}$$

Identity (1) can be used to find a gibonacci sum for $x^5 f_{3n-1}$.

2.4 A Gibonacci Sum for $x^5 f_{3n-1}$: It follows from identity (1) that

$$x^2 f_{3n-1} = f_{n+1}^3 - 2f_{n+1}f_{n-1}^2 + f_{n-1}^2 f_{n-3}$$

$$x^5 f_{3n-1} = A + B\,,$$

where

$$\begin{aligned} A &= (xf_{n+1})^3 \\ &= (f_{n+2} - f_n)^3 \\ &= f_{n+2}^3 - 3f_{n+2}^2 f_n + 3f_{n+2}f_n^2 - f_n^3\,; \end{aligned}$$

$$\begin{aligned}B &= (xf_{n-1})^2[-2(xf_{n+1}) + (xf_{n-3})]\\ &= (f_n - f_{n-2})^2\{-2(f_{n+2} - f_n) + [f_n - (x^2+1)f_{n-2}\ ]\}\\ &= (f_n^2 - 2f_nf_{n-2} + f_{n-2}^2)[-2f_{n+2} + 3f_n - (x^2+1)f_{n-2}]\\ &= -2f_{n+2}f_n^2 + 4f_{n+2}f_nf_{n-2} - 2f_{n+2}f_{n-2}^2 + 3f_n^2 - (x^2+7)f_n^2f_{n-2}\\ &\qquad + (2x^2+5)f_nf_{n-2}^2 - (x^2+1)f_{n-2}^3\,.\end{aligned}$$

Thus,

$$\begin{aligned}x^5 f_{3n-1} &= f_{n+2}^3 - 3f_{n+2}^2f_n + f_{n+2}f_n^2 + 4f_{n+2}f_nf_{n-2} - 2f_{n+2}f_{n-2}^2 + 2f_n^3\\ &\qquad -(x^2+7)f_n^2f_{n-2} + (2x^2+5)f_nf_{n-2}^2 - (x^2+1)f_{n-2}^3\,. \qquad (7)\end{aligned}$$

This gives an alternate formula for F_{3n-1}:

$$\begin{aligned}F_{3n-1} &= F_{n+2}^3 - 3F_{n+2}^2F_n + F_n + 2F_n^2 + 4F_{n+2}F_nF_{n-2} - 2F_{n+2}F_{n-2}^2 + 2F_n^3\\ &\qquad - 8F_n^2F_{n-2} + 7F_nF_{n-2}^2 - 2F_{n-2}^3\,. \qquad (8)\end{aligned}$$

It follows from formulas (6) and (8) that

$$\begin{aligned}F_{n+2}^3 + 3F_{n+2}F_nF_{n-2} + 6F_n^3 + 8F_nF_{n-2}^2 &= 5F_{n+2}F_n^2\\ &+2F_{n+2}F_{n-2}^2 + 8F_n^2F_{n-2} + 2F_{n-2}^3;\end{aligned}$$

this can be confirmed independently.

2.5 A Gibonacci Sum for $x^2 f_{3n}$: Using identities (1) and (3), we can develop a sum of Fibonacci polynomial products of order 3 for

$$\begin{aligned}x^2 f_{3n} &= x^2 f_{3n+2} - x^3 f_{3n+1}\\ &= (f_{n+2}^3 - 2f_{n+2}f_n^2 + f_n^2f_{n-2})\\ &\qquad - [f_{n+2}^3 - 3f_{n+2}^2f_n + (2x^2+3)f_{n+2}f_n^2 - (x^2+1)f_n^3 - x^2f_n^2f_{n-2}]\\ &= 3f_{n+2}^2f_n - (2x^2+5)f_{n+2}f_n^2 + (x^2+1)f_n^3 + (x^2+1)f_n^2f_{n-2}. \qquad (9)\end{aligned}$$

An Alternate Method: By the addition formula, we have

$$\begin{aligned} f_n^3 &= f_{n+1}^2 f_n + f_{2n} f_{n-1} \\ x^2 f_{3n} &= [(xf_{n+1})^2 + x^2 f_n^2]\, f_n + f_n (xf_{n-1})(xl_n) \\ &= [(f_{n+2} - f_n)^2 + x^2 f_n^2]\, f_n + f_n [f_{n+2} - (x^2+1) f_n][f_{n+2} - f_{n-2}] \\ &= 2f_{n+2}^2 f_n - (x^2+3) f_{n+2} f_n^2 - f_{n+2} f_n f_{n-2} \\ &\qquad + (x^2+1) f_n^3 + (x^2+1) f_n^2 f_{n-2}\ . \end{aligned} \tag{10}$$

Since

$$\begin{aligned} 2f_{n+2}^2 f_n - (x^2+3) f_{n+2} f_n^2 - f_{n+2} f_n f_{n-2} &= 3f_{n+2}^2 f_n - f_{n+2} f_n\ [f_{n+2} + f_{n-2} + (x^2+3) f_n] \\ &= 3f_{n+2}^2 f_n - f_{n+2} f_n [(x^2+2) f_n + (x^2+3) f_n] \\ &= 3f_{n+2}^2 f_n - (2x^2+5) f_{n+2} f_n^2 , \end{aligned}$$

it follows that identities (9) and (10) are equivalent.

Identities (9) and (10) yield the following results:

$$F_{3n} = 3F_{n+2}^2 F_n - 7F_{n+2} F_n^2 + 2F_n^3 + 2F_n^2 F_{n-2}\,; \tag{11}$$

$$= 2F_{n+2}^2 F_n - 4F_{n+2} F_n^2 - F_{n+2} F_n F_{n-2} + 2F_n^3 + 2F_n^2 F_{n-2}\,. \tag{12}$$

Next we explore a gibonacci sum for $x^3 l_{3n}$.

2.6 A Gibonacci Sum for $x^3 l_{3n}$: Using identities (3) and (5), we get

$$\begin{aligned} x^3 l_{3n} &= x^3 f_{n+1}^3 + x^3 f_{n-1}^3 \\ &= [f_{n+2}^3 - 3f_{n+2}^2 f_n + (2x^2+3) f_{n+2} f_n^2 - (x^2+1) f_n^3 - x^2 f_n^2 f_{n-2}] \\ &\qquad + [f_{n+2}^3 - (x^2+3) f_{n+2}^2 f_n + 3(x^2+1) f_{n+2} f_n^2 + x^2 f_{n+2} f_n f_{n-2} \\ &\qquad - (x^2+1) 2f_n^3 - x^2 f_n f_{n-2}^2] \end{aligned}$$

$$= 2f_{n+2}^3 - (x^2+6)f_{n+2}^2 f_n + (5x^2+6)f_{n+2}f_n^2 + x^2 f_{n+2}f_n f_{n-2}$$
$$-(x^2+1)(x^2+2)f_n^3 - x^2 f_n^2 f_{n-2} - x^2 f_n f_{n-2}^2 . \tag{13}$$

An Alternate Formula: Since $xf_n + 2f_{n-1} = l_n$, it follows by formulas (5) and (10) that

$$\begin{aligned} x^3 l_{3n} &= x^2(x^2 f_{3n}) + 2(x^3 f_{3n-1}) \\ &= x^2[2f_{n+2}^2 f_n - (x^2+3)f_{n+2}f_n^2 - f_{n+2}f_n f_{n-2} + (x^2+1)f_n^3 + (x^2+1)f_n^2 f_{n-2}] \\ &\qquad +2[f_{n+2}^3 - (x^2+3)f_{n+2}^2 f_n + 3(x^2+1)f_{n+2}f_n^2 + x^2 f_{n+2}f_n f_{n-2} \\ &\qquad -(x^2+1)2f_n^3 - x^2 f_n f_{n-2}^2] \\ &= 2f_{n+2}^3 - 6f_{n+2}^2 f_n - (x^4 - 3x^2 - 6)f_{n+2}f_n^2 + x^2 f_{n+2}f_n f_{n-2} \\ &\qquad -(x^2+1)(x^2+2)f_n^3 + x^2(x^2+1)f_n^2 f_{n-2} - 2x^2 f_n f_{n-2}^2 . \end{aligned} \tag{14}$$

Since

$$\begin{aligned} x^2 f_{n+2}^2 f_n &- (x^4+x^2)f_{n+2}f_n^2 + (x^4+x^2)f_n^2 f_{n-2} - x^2 f_n f_{n-2}^2 \\ &= x^2 f_{n+2}f_n[f_{n+2} - (x^2+2)f_n] + x^2 f_n f_{n-2}[(x^2+2)f_n - f_{n-2}] \\ &= x^2 f_{n+2}f_n[xf_{n+1} - (x^2+1)f_n] + x^2 f_n f_{n-2}[(x^2+1)f_n + xf_{n-1}] \\ &= x^2 f_{n+2}f_n(xf_{n-1} - f_n) + x^2 f_n f_{n-2}[x\left(xf_n + f_{n-1}\right) + f_n] \\ &= -x^2 f_{n+2}f_n f_{n-2} + x^2 f_n f_{n-2}(xf_{n+1} + f_n) \\ &= -x^2 f_{n+2}f_{n-2} + x^2 f_{n+2}f_n f_{n-2} \\ &= 0, \end{aligned}$$

it follows that identities (13) and (14) are indeed equivalent, as expected.

It then follows that

$$L_{3n} = 2F_{n+2}^3 - 7F_{n+2}^2 F_n + 11F_{n+2}F_n^2 + F_{n+2}F_n F_{n-2}$$
$$-6F_n^3 - F_n^2 F_{n-2} - F_n F_{n-2}^2 ; \tag{15}$$

$$= 2F_{n+2}^3 - 6F_{n+2}^2F_n + 8F_{n+2}F_n^2 + F_{n+2}F_nF_{n-2} - 6F_n^3 + 2F_n^2F_{n-2} - 2F_nF_{n-2}^2. \quad (16)$$

2.7 A Gibonacci Sum for x^2l_{3n+1}: Since $f_{n+1} + f_{n-1} = l_n$, it follows by identities (1) and (9) that

$$\begin{aligned} x^2l_{3n+1} &= x^2f_{3n+2} + x^2f_{3n} \\ &= (f_{n+2}^3 - 2f_{n+2}f_n^2 + f_n^2f_{n-2}) \\ &\quad + [3f_{n+2}^2f_n - (2x^2+5)f_{n+2}f_n^2 + (x^2+1)f_n^3 + (x^2+1)f_n^2f_{n-2}] \\ &= f_{n+2}^3 + 3f_{n+2}^2f_n - (2x^2+7)f_{n+2}f_n^2 + (x^2+1)f_n^3 + (x^2+2)f_n^2f_{n-2}. \quad (17) \end{aligned}$$

In particular,

$$L_{3n+1} = F_{n+2}^3 + 3F_{n+2}^2F_n - 9F_{n+2}F_n^2 + 2F_n^3 + 3F_n^2F_{n-2}. \quad (18)$$

2.8 A Gibonacci Sum for x^2l_{3n-1}: By identities (13) and (17), we have

$$\begin{aligned} x^2l_{3n-1} &= x^2l_{3n+1} - x^3l_{3n} \\ &= [f_{n+2}^3 + 3f_{n+2}^2f_n - (2x^2+7)f_{n+2}f_n^2 + (x^2+1)f_n^3 + (x^2+2)f_n^2f_{n-2}] \\ &= -[2f_{n+2}^3 - (x^2+6)f_{n+2}^2f_n + (5x^2+6)f_{n+2}f_n^2 + x^2f_{n+2}f_nf_{n-2} \\ &\quad - (x^2+1)(x^2+2)f_n^3 - x^2f_n^2f_{n-2} - x^2f_nf_{n-2}^2] \\ &= -f_{n+2}^3 + (x^2+9)f_{n+2}^2f_n - (7x^2+13)f_{n+2}f_n^2 - x^2f_{n+2}f_nf_{n-2} \\ &\quad + (x^2+1)(x^2+3)f_n^3 + 2(x^2+1)f_n^2f_{n-2} + x^2f_nf_{n-2}^2. \quad (19) \end{aligned}$$

It then follows that

$$\begin{aligned} L_{3n-1} &= -F_{n+2}^3 + 10F_{n+2}^2F_n - 20F_{n+2}F_n^2 - F_{n+2}F_nF_{n-2} \\ &\quad + 8F_n^3 + 4F_n^2F_{n-2} + F_nF_{n-2}^2. \quad (20) \end{aligned}$$

Next we express $x^3 l_{3n+2}$ as a sum of gibonacci polynomial products of order 3.

2.9 A Gibonacci Sum for $x^3 l_{3n+2}$: Using identities (13) and (17), we get

$$\begin{aligned}
x^3 l_{3n+2} &= x^4 l_{3n+1} + x^3 l_{3n} \\
&= [x^2 f_{n+2}^3 + 3x^2 f_{n+2}^2 f_n - (2x^4 + 7x^2) f_{n+2} f_n^2 + (x^4 + x^2) f_n^3 \\
&\qquad + (x^4 + 2x^2) f_n^2 f_{n-2}] + [2 f_{n+2}^3 - (x^2 + 6) f_{n+2}^2 f_n \\
&\qquad + (5x^2 + 6) f_{n+2} f_n^2 + x^2 f_{n+2} f_n f_{n-2} - (x^2 + 1)(x^2 + 2) f_n^3 \\
&\qquad - x^2 f_n^2 f_{n-2} - x^2 f_n f_{n-2}^2] \\
&= (x^2 + 2) f_{n+2}^3 + 2(x^2 - 3) f_{n+2}^2 f_n - 2(x^4 + x^2 - 3) f_{n+2} f_n^2 \\
&\qquad + x^2 f_{n+2} f_n f_{n-2} - 2(x^2 + 1) f_n^3 \\
&\qquad + (x^4 + x^2) f_n^2 f_{n-2} - x^2 f_n f_{n-2}^2 . \qquad (21)
\end{aligned}$$

This implies

$$\begin{aligned}
L_{3n+2} &= 3F_{n+2}^3 - 4F_{n+2}^2 F_n + 2F_{n+2} F_n^2 + F_{n+2} F_n F_{n-2} - 4F_n^3 \\
&\qquad + 2F_n^2 F_{n-2} - F_n F_{n-2}^2 . \qquad (22)
\end{aligned}$$

With these tools, we now express $\Delta^2 x^2 f_{3n}$, $\Delta^2 x^3 f_{3n-1}$, $\Delta^2 x^3 f_{3n+1}$, and $\Delta^2 x^2 f_{3n+2}$ as gibonacci sums with the desired properties.

2.10 A Gibonacci Sum for $\Delta_2 x^2 f_{3n}$: Since $l_{n+1} + l_{n-1} = \Delta^2 f_n$, it follows from identities (17) and (19) that

$$\begin{aligned}
\Delta^2 x^2 f_{3n} &= x^2 l_{3n+1} + x^2 l_{3n-1} \\
&= [f_{n+2}^3 + 3 f_{n+2}^2 f_n - (2x^2 + 7) f_{n+2} f_n^2 + (x^2 + 1) f_n^3 + (x^2 + 2) f_n^2 f_{n-2}] \\
&= [-f_{n+2}^3 + (x^2 + 9) f_{n+2}^2 f_n - (7x^2 + 13) f_{n+2} f_n^2 - x^2 f_{n+2} f_n f_{n-2} \\
&\qquad + (x^2 + 1)(x^2 + 3) f_n^3 + 2(x^2 + 1) f_n^2 f_{n-2} + x^2 f_n f_{n-2}^2]
\end{aligned}$$

$$= (x^2+12)f_{n+2}^2 f_n - (9x^2+20)f_{n+2}f_n^2 - x^2 f_{n+2}f_n f_{n-2}$$
$$+(x^2+1)(x^2+4)f_n^3 + (3x^2+4)f_n^2 f_{n-2} + x^2 f_n f_{n-2}^2. \tag{23}$$

Consequently,

$$5F_{3n} = 13F_{n+2}^2 F_n - 29F_{n+2}F_n^2 - F_{n+2}F_n F_{n-2} + 10F_n^3 + 7F_n^2 F_{n-2} + F_n F_{n-2}^2. \tag{24}$$

2.11 A Gibonacci Sum for $\Delta^2 x^3 f_{3n+1}$: Using the identities $l_{n+2} + l_n = \Delta^2 f_{n+1}$, and (1) and (21), we have

$$\Delta^2 x^3 f_{3n+1} = x^3 l_{3n+2} + x^3 l_{3n}$$
$$= [(x^2+2)f_{n+2}^3 + 2(x^2-3)f_{n+2}^2 f_n - 2(x^4+x^2-3)f_n + 2f_n^2$$
$$+ x^2 f_{n+2}f_n f_{n-2} - 2(x^2+1)f_n^3 + (x^4+x^2)f_n^2 f_{n-2} - x^2 f_n f_{n-2}^2]$$
$$+[2f_{n+2}^3 - (x^2+6)f_{n+2}^2 f_n + (5x^2+6)f_{n+2}f_n^2 + x^2 f_{n+2}f_n f_{n-2}$$
$$-(x^2+1)(x^2+2)f_n^3 - x^2 f_n^2 f_{n-2} - x^2 f_n f_{n-2}^2]$$
$$= (x^2+4)f_{n+2}^3 + (x^2-12)f_{n+2}^2 f_n - (2x^4-3x^2-12)f_{n+2}f_n^2$$
$$+2x^2 f_{n+2}f_n f_{n-2} - (x^2+1)(x^2+4)f_n^3$$
$$+x^4 f_n^2 f_{n-2} - 2x^2 f_n f_{n-2}^2. \tag{25}$$

It then follows that

$$5F_{3n+1} = 5F_{n+2}^3 - 11F_{n+2}^2 F_n + 13F_{n+2}F_n^2 + 2F_{n+2}F_n F_{n-2} - 10F_n^3$$
$$+F_n^2 F_{n-2} - 2F_n F_{n-2}^2. \tag{26}$$

2.12 A Gibonacci Sum for $\Delta^2 x^3 f_{3n-1}$: It follows by gibonacci recurrence, and identities (23) and (25) that

$$\Delta^2 x^3 f_{3n-1} = \Delta^2 x^3 f_{3n+1} - \Delta^2 x^4 f_n^3$$
$$= [(x^2+4)f_{n+2}^3 + (x^2-12)f_{n+2}^2 f_n - (2x^4-3x^2-12)f_{n+2}f_n^2$$

$$+2x^2 f_{n+2}f_n f_{n-2} - (x^2+1)(x^2+4)f_n^3 + x^4 f_n^2 f_{n-2} - 2x^2 f_n f_{n-2}^2]$$
$$-x^2[(x^2+12)f_{n+2}^2 f_n - (9x^2+20)f_{n+2}f_n^2 - x^2 f_{n+2}f_n f_{n-2}$$
$$+(x^2+1)(x^2+4)f_n^3 + (3x^2+4)f_n^2 f_{n-2} + x^2 f_n f_{n-2}^2]$$
$$= (x^2+4)f_{n+2}^3 - (x^4+11x^2+12)f_{n+2}^2 f_n + (7x^4+23x^2+12)f_{n+2}f_n^2$$
$$+(x^4+2x^2)f_{n+2}f_n f_{n-2} - (x^2+1)^2(x^2+4)f_n^3$$
$$-2(x^4+2x^2)f_n^2 f_{n-2} - (x^4+2x^2)f_n f_{n-2}^2. \tag{27}$$

This yields

$$5F_{3n-1} = 5F_{n+2}^3 - 24F_{n+2}^2 F_n + 42F_{n+2}F_n^2 + 3F_{n+2}F_n F_{n-2}$$
$$-20F_n^3 - 6F_n^2 F_{n-2} - 3F_n F_{n-2}^2. \tag{28}$$

2.13 A Gibonacci Sum for $\Delta^2 x^2 f_{3n+2}$**:** Using identities (23) and (25), we have

$$\Delta^2 x^2 f_{3n+2} = \Delta^2 x^3 f_{3n+1} + \Delta^2 x^2 f_{3n}$$
$$= [(x^2+4)f_{n+2}^3 + (x^2-12)f_{n+2}^2 f_n - (2x^4-3x^2-12)f_{n+2}f_n^2$$
$$+2x^2 f_{n+2}f_n f_{n-2} - (x^2+1)(x^2+4)f_n^3 + x^4 f_n^2 f_{n-2} - 2x^2 f_n f_{n-2}^2]$$
$$+[(x^2+12)f_{n+2}^2 f_n - (9x^2+20)f_{n+2}f_n^2 - x^2 f_{n+2}f_n f_{n-2}$$
$$+(x^2+1)(x^2+4)f_n^3 + (3x^2+4)f_n^2 f_{n-2} + x^2 f_n f_{n-2}^2]$$
$$= (x^2+4)f_{n+2}^3 + 2x^2 f_{n+2}^2 f_{n-2}(x^4+3x^2+4)f_{n+2}f_n^2 + x^2 f_{n+2}f_n f_{n-2}$$
$$+(x^4+3x^2+4)f_n^2 f_{n-2} - x^2 f_n f_{n-2}^2. \tag{29}$$

In particular, we have

$$5F_{3n+2} = 5F_{n+2}^3 + 2F_{n+2}^2 F_n - 16F_{n+2}F_n^2 + F_{n+2}F_n F_{n-2}$$
$$+8F_n^2 F_{n-2} - F_n F_{n-2}^2. \tag{30}$$

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A FAMILY OF SUMS OF GIBONACCI POLYNOMIAL PRODUCTS OF ORDER 3: IMPLICATIONS

Abstract: We explore the Pell, Jacobsthal, Vieta, and Chebyshev implications of the sums of gibonacci polynomial products of order 3, studied in [10].

Keyword: Gibonacci Polynomials, Pell, Lucas, Jacobsthal Polynomials.

Mathematical Subject Classification (2010) No.: Primary 05A19, 11B37, 11B39, 11Cxx.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary complex variable; $a(x), b(x), z_0(x)$, and $z_1(x)$ are arbitrary complex polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 6, 8].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [8].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial* [3, 6]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$ and $j_n(1) = L_n$.

Let $a(x) = x$ and $b(x) = -1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = V_n(x)$, the nth *Vieta polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = v_n(x)$, the nth *Vieta-Lucas polynomial* [4, 6].

Finally, let $a(x) = 2x$ and $b(x) = -1$. When $z_0(x) = 1$ and $z_1(x) = x$, $z_n(x) = T_n(x)$, the nth *Chebyshev polynomial* of the first kind; and when $z_0(x) = 1$ and $z_1(x) = 2x$, $z_n(x) = U_n(x)$, the nth *Chebyshev polynomial of the second kind* [4, 6].

The Jacobsthal, Vieta, and Chebyshev subfamilies are closely linked by the relationships in Table 1, where $i = \sqrt{-1}$ [4, 6].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is no ambiguity; so z_n will mean $z_n(x)$. We also let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $c_n = J_n(x)$ or $j_n(x)$, $d_n = V_n$ or v_n, and $e_n = T_n$ or U_n; and correspondingly, $G_n = F_n$ or L_n, $B_n = P_n$ or Q_n, and $C_n = J_n$ or j_n. We also omit a lot of basic algebra.

A *gibonacci polynomial product of order* m is a product of gibonacci polynomials g_{n+k} of the form $\prod_{k\in Z} g_{n+k}^{sj}$, where $\sum_{sj\geq 1} s_j = m$ [9, 11].

Table 1: Links Among the Subfamilies

$J_n(x) = x^{(n-1)/2} f_n(1\sqrt{x})$	$j_n(x) = x^{n/2} l_n(1/\sqrt{x})$
$V_n(x) = i^{n-1} f_n(-ix)$	$v_n(x) = i^n l_n(-ix)$
$V_n(2x) = U_{n-1}(x)$	$v_n(2x) = 2T_n(x)$.

1.1 Sums of Gibonacci Polynomial Products of Order 3: In [10], we studied the following sums of gibonacci polynomial products of order 3:

$$x^2 f_{3n} = 3f_{n+2}^2 f_n - (2x^2+5)f_{n+2}f_n^2 + (x^2+1)f_n^3 + (x^2+1)f_n^2 f_{n-2}; \tag{1}$$

$$\begin{aligned} x^3 f_{3n-1} = {} & f_{n+2}^3 - (x^2+3)f_{n+2}^2 f_n + 3(x^2+1)f_{n+2}f_n^2 + x^2 f_{n+2}f_n f_{n-2} \\ & -(x^2+1)2f_n^3 - x^2 f_n f_{n-2}^2 . \end{aligned} \tag{2}$$

$$x^3 f_{3n+1} = f_{n+2}^3 - 3f_{n+2}^2 f_n + (2x^2+3)f_{n+2}f_n^2 - (x^2+1)f_n^3 - x^2 f_n^2 f_{n-2}; \tag{3}$$

$$x^2 f_{3n+2} = f_{n+2}^3 - 2f_{n+2}f_n^2 + f_n^2 f_{n-2}; \tag{4}$$

$$\begin{aligned} x^3 l_{3n} = {} & 2f_{n+2}^3 - (x^2+6)f_{n+2}^2 f_n + (5x^2+6)f_{n+2}f_n^2 + x^2 f_{n+2}f_n f_{n-2} \\ & -(x^2+1)(x^2+2)f_n^3 - x^2 f_n^2 f_{n-2} - x^2 f_n f_{n-2}^2; \end{aligned} \tag{5}$$

$$\begin{aligned} x^2 l_{3n-1} = {} & -f_{n+2}^3 + (x^2+9)f_{n+2}^2 f_n - (7x^2+13)f_{n+2}f_n^2 - x^2 f_{n+2}f_n f_{n-2} \\ & +(x^2+1)(x^2+3)f_n^3 + 2(x^2+1)f_n^2 f_{n-2} + x^2 f_n f_{n-2}^2; \end{aligned} \tag{6}$$

$$x^2 l_{3n+1} = f_{n+2}^3 + 3f_{n+2}^2 f_n - (2x^2+7)f_{n+2}f_n^2 + (x^2+1)f_n^3 + (x^2+2)f_n^2 f_{n-2}; \tag{7}$$

$$\begin{aligned} x^3 l_{3n+2} = {} & (x^2+2)f_{n+2}^3 + 2(x^2-3)f_{n+2}^2 f_{n-2}(x^4+x^2-3)f_{n+2}f_n^2 \\ & + x^2 f_{n+2}f_n f_{n-2} - 2(x^2+1)f_n^3 + (x^4+x^2)f_n^2 f_{n-2} - x^2 f_n f_{n-2}^2; \end{aligned} \tag{8}$$

$$\Delta^2 x^2 f_{3n} = (x^2+12)f_{n+2}^2 f_n - (9x^2+20)f_{n+2}f_n^2 - x^2 f_{n+2}f_n f_{n-2}$$
$$+(x^2+1)(x^2+4)f_n^3 + (3x^2+4)f_n^2 f_{n-2} + x^2 f_n f_{n-2}^2; \quad (9)$$

$$\Delta^2 x^3 f_{3n-1} = (x^2+4)f_{n+2}^3 - (x^4+11x^2+12)f_{n+2}^2 f_n + (7x^4+23x^2+12)f_{n+2}f_n^2$$
$$+(x^4+2x^2)f_{n+2}f_n f_{n-2} - (x^2+1)^2(x^2+4)f_n^3$$
$$-2(x^4+2x^2)f_n^2 f_{n-2} - (x^4+2x^2)f_n f_{n-2}^2; \quad (10)$$

$$\Delta^2 x^3 f_{3n+1} = (x^2+4)f_{n+2}^3 + (x^2-12)f_{n+2}^2 f_n - (2x^4-3x^2-12)f_{n+2}f_n^2$$
$$+2x^2 f_{n+2}f_n f_{n-2} - (x^2+1)(x^2+4)f_n^3 + x^4 f_n^2 f_{n-2} - 2x^2 f_n f_{n-2}^2; \quad (11)$$

$$\Delta^2 x^2 f_{3n+2} = (x^2+4)f_{n+2}^3 + 2x^2 f_{n+2}^2 f_n - 2(x^4+3x^2+4)f_{n+2}f_n^2 + x^2 f_{n+2}f_n f_{n-2}$$
$$+(x^4+3x^2+4)f_n^2 f_{n-2} - x^2 f_n f_{n-2}^2, \quad (12)$$

where $g_n = g_n(x)$.

With this background, we now explore the implications of this family of gibonacci sums to the Pell, Jacobsthal, Vieta, and Chebyshev subfamilies.

2. Sums of Pell Polynomial Products of Order 3

Since $b_n(x) = g_n(2x)$, it follows from identities (1) through (12) that

$$4x^2 p_{3n} = 3p_{n+2}^2 p_n - (8x^2+5)p_{n+2}p_n^2 + (4x^2+1)p_n^3 + (4x^2+1)p_n^2 p_{n-2};$$

$$8x^3 p_{3n-1} = p_{n+2}^3 - (4x^2+3)p_{n+2}^2 p_n + 3(4x^2+1)p_{n+2}p_n^2 + 4x^2 p_{n+2}p_n p_{n-2}$$
$$-(4x^2+1)^2 p_n^3 - 4x^2 p_n p_{n-2}^2;$$

$$8x^3 p_{3n+1} = p_{n+2}^3 - 3p_{n+2}^2 p_n + (8x^2+3)p_{n+2}p_n^2 - (4x^2+1)p_n^3 - 4x^2 p_n^2 p_{n-2};$$

$$4x^2 p_{3n+2} = p_{n+2}^3 - 2p_{n+2}p_n^2 + p_n^2 p_{n-2};$$

$$4x^3 q_{3n} = p_{n+2}^3 - (2x^2+3)p_{n+2}^2 p_n + (10x^2+3)p_{n+2}p_{n+2}^2 x^2 p_{n+2}p_n p_{n-2}$$
$$-(2x^2+1)(4x^2+1)p_n^3 - 2x^2 p_n^2 p_{n-2} - 2x^2 p_n p_{n-2}^2;$$

$$
\begin{aligned}
4x^4 q_{3n-1} = -p_{n+2}^3 + (4x^2+9)p_{n+2}^2 p_n - (28x^2+13)p_{n+2}p_n^2 - 4x^2 p_{n+2}p_n p_{n-2} \\
+ (4x^2+1)(4x^2+3)p_n^3 + 2(4x^2+1)p_n^2 p_{n-2} + 4x^2 p_n p_{n-2}^2 ;
\end{aligned}
$$

$$
x^2 q_{3n+1} = p_{n+2}^3 + 3p_{n+2}^2 p_n - (8x^2+7)p_{n+2}p_n^2 + (4x^2+1)p_n^3 + 2(2x^2+1)p_n^2 p_{n-2} ;
$$

$$
\begin{aligned}
8x^3 q_{3n+2} = 2(2x^2+1)p_{n+2}^3 + 2(4x^2-3)p_{n+2}^2 p_n - 2(16x^4+4x^2-3)p_{n+2}p_n^2 \\
+ 4x^2 p_{n+2}p_n p_{n-2} - 2(4x^2+1)p_n^3 \\
+ 4(4x^4+x^2)p_n^2 p_{n-2} - 4x^2 p_n p_{n-2}^2 ;
\end{aligned}
$$

$$
\begin{aligned}
4(x^4+x^2)p_{3n} = (x^2+3)p_{n+2}^2 p_n - (9x^2+5)p_{n+2}p_n^2 - x^2 p_{n+2}p_n p_{n-2} \\
+(x^2+1)(4x^2+1)p_n^3 + (3x^2+1)p_n^2 p_{n-2} + x^2 p_n p_{n-2}^2 ;
\end{aligned}
$$

$$
\begin{aligned}
32(x^5+x^3)p_{3n-1} = 4(x^2+1)p_{n+2}^3 - 4(4x^4+11x^2+3)p_{n+2}^2 p_n \\
+ (144x^4+92x^2+5)p_{n+2}p_n^2 + 8(2x^4+x^2)p_{n+2}p_n p_{n-2} \\
- 4(x^2+1)^2(4x^2+1)p_n^3 - 16(2x^4+x^2)p_n^2 p_{n-2} \\
- 8(2x^4+x^2)p_n p_{n-2}^2 ;
\end{aligned}
$$

$$
\begin{aligned}
16(x^4+x^2)p_{3n+1} = 4(x^2+1)p_{n+2}^3 + 4(x^2-3)p_{n+2}^2 p_n - 4(8x^4-3x^2-3)p_{n+2}p_n^2 \\
+ 8x^2 p_{n+2}p_n p_{n-2} - 4(x^2+1)(4x^2+1)p_n^3 \\
+16x^4 p_n^2 p_{n-2} - 8x^2 p_n p_{n-2}^2 ;
\end{aligned}
$$

$$
\begin{aligned}
16(x^4+x^2)p_{3n+2} = 4(x^2+1)p_{n+2}^3 + 8x^2 p_{n+2}^2 p_n - 8(4x^4+3x^2+1)p_{n+2}p_n^2 \\
+ 4x^2 p_{n+2}p_n p_{n-2} + 4(4x^4+3x^2+1)p_n^2 p_{n-2} - 4x^2 p_n p_{n-2}^2 ,
\end{aligned}
$$

where $b_n = b_n(x)$.

Consequently, we have

$$
4P_{3n} = 3P_{n+2}^2 Pn - 13P_{n+2}P_n^2 + 5P_n^3 + 5P_n^2 P_{n-2} ;
$$

$$
8P_{3n-1} = P_{n+2}^3 - 7P_{n+2}^2 P_n + 15P_{n+2}P_n^2 + 4P_{n+2}P_n P_{n-2} - 25P_n^3 - 4P_n P_{n-2}^2 ;
$$

$$8P_{3n+1} = P_{n+2}^3 - 3P_{n+2}^2 P_n + 11P_{n+2}P_n^2 - 5P_n^3 - 4P_n^2 P_{n-2};$$

$$4P_{3n+2} = P_{n+2}^3 - 2P_{n+2}P_n^2 + P_n^2 P_{n-2};$$

$$8Q_{3n} = P_{n+2}^3 - 5P_{n+2}^2 P_n + 13P_{n+2}P_n^2 + 2P_{n+2}P_n P_{n-2} - 15p_n^3 \\ -2P_n^2 P_{n-2} - 2P_n P_{n-2}^2 \;;$$

$$8Q_{3n-1} = -P_{n+2}^3 + 13P_{n+2}^2 P_n - 41P_{n+2}P_n^2 - 4P_{n+2}P_n P_{n-2} + 35P_n^3 \\ +10P_n^2 P_{n-2} + 4P_n P_{n-2}^2;$$

$$2Q_{3n+1} = P_{n+2}^3 + 3P_{n+2}^2 P_n - 15P_{n+2}P_n^2 + 5P_n^3 + 6P^2 P_{n-2};$$

$$16Q_{3n+2} = 6P_{n+2}^3 + 2P_{n+2}^2 P_n - 34P_{n+2}P_n^2 + 4P_{n+2}P_n P_{n-2} - 10P_n^3 \\ +20P^2 P_{n-2} - 4P_n P_{n-2}^2;$$

$$8P_{3n} = 4P_{n+2}^2 P_n - 14P_{n+2}P_n^2 - P_{n+2}P_n P_{n-2} + 10P_n^3 + 4P_n^2 P_{n-2} + P_n P_{n-2}^2;$$

$$64P_{3n-1} = 8P_{n+2}^3 - 72P_{n+2}^2 P_n + 241P_{n+2}P_n^2 + 24P_{n+2}P_n P_{n-2} - 80P_n^3 \\ -48P_n^2 P_{n-2} - 24P_n P_{n-2}^2;$$

$$32P_{3n+1} = 8P_{n+2}^3 - 8P_{n+2}^2 P_n - 8P_{n+2}P_n^2 + 8P_{n+2}P_n P_{n-2} - 40P_n^3 \\ +16P_n^2 P_{n-2} - 8P_n P_{n-2}^2;$$

$$32P_{3n+2} = 8P_{n+2}^3 + 8P_{n+2}^2 P_n - 64P_{n+2}P_n^2 + 4P_{n+2}P_n P_{n-2} + 32P_n^2 P_{n-2} 4P_n P_{n-2}^2.$$

Next we explore the Jacobsthal implications of the gibonacci sums.

3. Sums of Jacobsthal Polynomial Products of Order 3

The identities (1) through (12), coupled with the gibonacci-Jacobsthal relationships in Table 1, can be established the following results involving Jacobsthal polynomial products of order 3:

$$J_{3n} = 3J_{n+2}^2 J_n - (5x+2)J_{n+2}J_n^2 + (x^2+x)J_n^3 + (x^3+x^2)J_n^2 J_{n-2}; \tag{13}$$

$$\begin{aligned} xJ_{3n-1} &= J_{n+2}^3 - (3x+1)J_{n+2}^2 J_n + 3(x^2+x)J_{n+2}J_n^2 + x^2 J_{n+2}J_n J_{n-2} \\ &\quad - (x^3+2x^2+x)J_n^3 - x^4 J_n J_{n-2}^2 . \end{aligned} \tag{14}$$

$$J_{3n+1} = J_{n+2}^3 - 3xJ_{n+2}^2 J_n + (3x^2+2x)J_{n+2}J_n^2 - (x^3+x^2)J_{n-x}^3 3J_n^2 J_{n-2}; \tag{15}$$

$$J_{3n+2} = J_{n+2}^3 - 2x^2 J_{n+2}J_n^2 + x^4 J_n^2 J_{n-2}; \tag{16}$$

$$\begin{aligned} j_{3n} &= 2J_{n+2}^3 - (6x+1)J_{n+2}^2 J_n + (6x^2+5x)J_{n+2}J_n^2 + x^2 J_{n+2}J_n J_{n-2} \\ &\quad - (2x^3+3x^2+x)J_n^3 - x^3 J_n^2 J_{n-2} - x^4 J_n J_{n-2}^2; \end{aligned} \tag{17}$$

$$\begin{aligned} xj_{3n-1} &= -J_{n+2}^3 + (9x+1)J_{n+2}^2 J_n - (13x^2+7x)J_{n+2}J_n^2 - x^2 J_{n+2}J_n J_{n-2} \\ &\quad + (3x^3+4x^2+x)J_n^3 + 2(x^4+x^3)J_n^2 J_{n-2} + x^4 J_n J_{n-2}^2; \end{aligned} \tag{18}$$

$$\begin{aligned} j_{3n+1} &= J_{n+2}^3 + 3xJ_{n+2}^2 J_n - (7x^2+2x)J_{n+2}J_n^2 + (x^3+x^2)J_n^3 \\ &\quad + (2x^4+x^3)J_n^2 J_{n-2}; \end{aligned} \tag{19}$$

$$\begin{aligned} j_{3n+2} &= (2x+1)J_{n+2}^3 - 2(3x^2-x)J_{n+2}^2 J_{n+2}(3x^3-x^2-x)J_{n+2}J_n^2 \\ &\quad + x^3 J_{n+2}J_n J_{n-2} - 2(x^4+x^3)J_n^3 + (x^4+x^3)J_n^2 J_{n-2} - x^5 J_n J_{n-2}^2; \end{aligned} \tag{20}$$

$$\begin{aligned} D^2 J_{3n} &= (12x+1)J_{n+2}^2 J_n - (20x^2+9x)J_{n+2}J_n^2 - x^2 J_{n+2}J_n J_{n-2} \\ &\quad + (4x^3+5x^2+x)J_n^3 + (4x^4+3x^3)J_n^2 J_{n-2} + x^4 J_n J_{n-2}^2; \end{aligned} \tag{21}$$

$$\begin{aligned} D^2 xJ_{3n-1} &= D^2 J_{n+2}^3 - (12x^2+11x+1)J_{n+2}^2 J_n + (12x^3+23x^2+7x)J_{n+2}J_n^2 \\ &\quad + (2x^3+x^2)J_{n+2}J_n J_{n-2} - (x+1)^2(4x^2+x)J_n^3 - 2(2x^4+x^3)J_n^2 J_{n-2} \\ &\quad - (2x^2+x)J_n J_{n-2}^2; \end{aligned} \tag{22}$$

$$\begin{aligned} D^2 J_{3n+1} &= D^2 J_{n+2}^3 - (12x^2-x)J_{n+2}^2 J_n + (12x^3+3x^2-2x)J_{n+2}J_n^2 \\ &\quad + 2x^3 J_{n+2}J_n J_{n-2} - (4x^4+5x^3+x^2)J_n^3 + x^3 J_n^2 J_{n-2} - 2x^5 J_n J_{n-2}^2; \end{aligned} \tag{23}$$

$$D^2J_{3n+2} = D^2J_{n+2}^3 + 2xJ_{n+2}^2J_n - 2(4x^3+3x^2+x)J_{n+2}J_n^2 + x^3J_{n+2}J_nJ_{n-2} + (4x^5+3x^4+x^3)J_n^2J_{n-2} - x^5J_nJ_{n-2}^2\,, \tag{24}$$

where $c_n = c_n(x)$.

To establish identity (13), for example, replace x with $1\sqrt{x}$ in equation (1) and multiply the resulting equation with $x^{(3n+1)/2}$. We then get

$$x^{3n-1}f_{3n} = 3\left[x^{(n+1)/2}f_{n+2}\right]^2 \left[x^{(n-1)/2}f_n\right] - (5x+2)\left[x^{(n+1)/2}f_{n+2}\right]\left[x^{(n-1)/2}f_n\right]^2 (x^2+x)\left[x^{(n-1)/2}f_n\right]^3 + (x^3+x^2)\left[x^{(n-1)/2}f_n\right]^2\left[x^{(n-3)/2}f_{n-2}\right]$$

$$J_{3n} = 3J_{n+2}^2J_n - (5x+2)J_{n+2}J_n^2 + (x^2+x)J_n^3 + (x^3+x^2)J_n^2J_{n-2}\,,$$

as desired, where $f_n = f_n(1\sqrt{x})$ and $J_n = J_n(x)$.

The other results can be confirmed similarly.

It follows from equations (13) through (24) that

$$J_{3n} = 3J_{n+2}^2J_n - 12J_{n+2}J_n^2 + 6J_n^3 + 12J_n^2J_{n-2}\,; \tag{25}$$

$$2J_{3n-1} = J_{n+2}^3 - 7J_n^2 + 2J_n + 18J_{n+2}J_n^2 + 4J_{n+2}J_nJ_{n-2} - 18J_n^3 - 16J_nJ_{n-2}^2\,.$$

$$J_{3n+1} = J_{n+2}^3 - 6J_{n+2}^2J_n + 16J_{n+2}J_n^2 - 12J_n^3 - 8J_n^2J_{n-2}\,;$$

$$J_{3n+2} = J_{n+2}^3 - 8J_{n+2}J_n^2 + 16J_n^2J_{n-2}\,; \tag{26}$$

$$j_{3n} = 2J_{n+2}^3 - 13J_{n+2}^2J_n + 34J_{n+2}J_n^2 + 4J_{n+2}J_nJ_{n-2} - 30J_n^3 - 8J_n^2J_{n-2} - 16J_nJ_{n-2}^2\,; \tag{27}$$

$$2j_{3n-1} = -J_{n+2}^3 + 19J_{n+2}^2J_n - 66J_{n+2}J_n^2 - 4J_{n+2}J_nJ_{n-2} + 42J_n^3 + 48J_n^2J_{n-2} + 16J_nJ_{n-2}^2\,;$$

$$j_{3n+1} = J_{n+2}^3 + 6J_{n+2}^2 J_n - 32J_{n+2}J_n^2 + 12J_n^3 + 40J_n^2 J_{n-2};$$

$$\begin{aligned} j_{3n+2} &= 5J_{n+2}^3 - 20J_{n+2}^2 J_n + 36J_n + 2J_n^2 + 8J_{n+2}J_n J_{n-2} - 48J_n^3 \\ &\quad + 24J_n^2 J_{n-2} - 32J_n J_{n-2}^2 ; \end{aligned} \tag{28}$$

$$9J_{3n} = 25J_{n+2}^2 J_n - 98J_{n+2}J_n^2 - 4J_{n+2}J_n J_{n-2} + 54J_n^3 + 88J_n^2 J_{n-2} + 16J_n J^2;$$

$$\begin{aligned} 18J_{3n-1} &= 9J_{n+2}^3 - 71J_{n+2}^2 J_n + 202J_{n+2}J_n^2 + 20J_{n+2}J_n J_{n-2} - 162J_n^3 \\ &\quad - 80J_n^2 J_{n-2} - 80J_n J_{n-2}^2; \end{aligned}$$

$$\begin{aligned} 9J_{3n+1} &= 9J_{n+2}^3 - 46J_{n+2}^2 J_n + 104J_{n+2}J_n^2 + 16J_{n+2}J_n J_{n-2} - 108J_n^3 \\ &\quad + 8J_n^2 J_{n-2} - 64J_n J_{n-2}^2; \end{aligned}$$

$$\begin{aligned} 9J_{n+2}^3 &= 9J_{n+2}^3 + 4J_{n+2}^2 J_n - 92J_{n+2}J_n^2 + 8J_{n+2}J_n J_{n-2} \\ &\quad + 184J_n^2 J_{n-2} - 32J_n J_{n-2}^2, \end{aligned}$$

It follows from identities (25), (26), and (28) that $J_{3n} \equiv 3J_{n+2}^2 J_n$ (mod 6), $J_{n+2}^3 \equiv J_{n+2}^3$ (mod 8), and $j_{n+2}^3 \equiv J_{n+2}^3$ (mod 8), respectively.

Next we pursue the Vieta and Chebyshev consequences.

4. Vieta and Chebyshev Consequences

4.1 Vieta Implications: Using the gibonacci-Vieta relationships in Table 1, equations (1) through (12) yield the following results. In the interest of brevity, we omit their proofs.

$$x^2 V_{3n} = 3V_{n+2}^2 V_n - (2x^2 - 5)V_{n+2}V_n^2 - (x^2 - 1)V_n^3 + (x^2 - 1)V_n^2 V_{n-2};$$

$$\begin{aligned} x^3 V_{3n-1} &= -V_{n+2}^3 + (x^2 - 3)V_{n+2}^2 V_n + 3(x^2 - 1)V_{n+2}V_n^2 - x^2 V_{n+2}V_n V_{n-2} \\ &\quad - (x^2 - 1)^2 V_n^3 + x^2 V_n V_{n-2}^2. \end{aligned}$$

$$x^3 V_{3n+1} = V_{n+2}^3 + 3V_{n+2}^2 V_n - (2x^2 - 3)V_{n+2}V_n^2 - (x^2 - 1)V_n^3 + x^2 V_n^2 V_{n-2};$$

$$x^2V_{3n+2} = V_{n+2}^3 - 2V_{n+2}V_n^2 + V_n^2V_{n-2};$$

$$\begin{aligned}x^3v_{3n} = 2V_{n+2}^3 &- (x^2-6)V_{n+2}^2V_n - (5x^2-6)V_{n+2}V_n^2 + x^2V_{n+2}V_nV_{n-2} \\ &+ (x^2-1)(x^2-2)V_n^3 + x^2V_n^2V_{n-2} - x^2V_nV_{n-2}^2;\end{aligned}$$

$$\begin{aligned}x^2v_{3n-1} = -V_{n+2}^3 &+ (x^2-9)V_{n+2}^2V_n + (7x^2-13)V_{n+2}V_n^2 - x^2V_{n+2}V_nV_{n-2} \\ &- (x^2-1)(x^2-3)V_{n-2}^3(x^2-1)V_n^2V_{n-2} + x^2V_nV_{n-2}^2;\end{aligned}$$

$$x^2v_{3n+1} = V_{n+2}^3 - 3V_{n+2}^2V_n + (2x^2-7)V_{n+2}V_n^2 + (x^2-1)V_n^3 - (x^2-2)V_n^2V_{n-2};$$

$$\begin{aligned}x^3v_{3n+2} = (x^2-2)V_{n+2}^3 &- 2(x^2+3)V_{n+2}^2V_n + 2(x^4-x^2-3)V_{n+2}V_n^2 \\ &- x^2V_{n+2}V_nV_{n-2} + 2(x^2-1)V_n^3 - (x^4-x^2)V_n^2V_{n-2} + x^2V_nV_{n-2}^2;\end{aligned}$$

$$\begin{aligned}(x^4-4x^2)V_{3n} = (x^2-12)V_{n+2}^2V_n &+ (9x^2-20)V_{n+2}V_n^2 - x^2V_{n+2}V_nV_{n-2} \\ &-(x^2-1)(x^2-4)V_n^3 - (3x^2-4)V_n^2V_{n-2} + x^2V_nV_{n-2}^2;\end{aligned}$$

$$\begin{aligned}(x^5-4x^3)V_{3n-1} = &-(x^2-4)V_{n+2}^3 + (x^4-11x^2+12)V_{n+2}^2V_n \\ &+ (7x^4-23x^2+12)V_{n+2}V_n^2 - (x^4-2x^2)V_{n+2}V_nV_{n-2} \\ &- (x^2-1)^2(x^2-4)V_n^3 - 2(x^4-2x^2)V_n^2V_{n-2} \\ &+ (x^4-2x^2)V_nV_{n-2}^2;\end{aligned}$$

$$\begin{aligned}(x^7-4x^3)V_{3n+1} = (x^2-4)V_{n+2}^3 &- (x^2+12)V_{n+2}^2V_n \\ &+ (2x^4+3x^2-12)V_{n+2}V_n^2 - 2x^2V_{n+2}V_nV_{n-2} - (x^2-1)(x^2-4)V_n^3 \\ &- x^4V_n^2V_{n-2} + 2x^2V_nV_{n-2}^2;\end{aligned}$$

$$\begin{aligned}(x^4-4x^2)V_{3n-1} = (x^2-4)V_{n+2}^3 &- 2x^2V_{n+2}^2V_n + 2(x^4-3x^2+4)V_{n+2}V_n^2 \\ &- x^2V_{n+2}V_nV_{n-2} - (x^4-3x^2+4)V_n^2V_{n-2} + x^2V_nV_{n-2}^2,\end{aligned}$$

where $d_n = d_n(x)$.

4.2 Chebyshev Implications: Using the Vieta-Chebyshev relationships, these Vieta properties yield the following Chebyshev factoids; again, in the interest of brevity, we omit their justifications.

$$4x^2U_{3n} = 3U_{n+2}^2U_n - (8x^2-5)U_{n+2}U_n^2 - (4x^2-1)U_n^3 + (4x^2-1)U_n^2U_{n-2}\,;$$

$$\begin{aligned}8x^3U_{3n-1} &= -U_{n+2}^3 + (4x^2-3)U_{n+2}^2U_n + 3(4x^2-1)U_{n+2}U_n^2\\ &\quad - 4x^2U_{n+2}U_nU_{n-2} - (4x^2-1)^2U_n^3 + 4x^2U_nU_{n-2}^2\,.\end{aligned}$$

$$8x^3U_{3n+1} = U_{n+2}^3 + 3U_{n+2}^2U_n - (8x^3-3)U_{n+2}U_n^2 - (4x^2-1)U_n^3 + 4x^2U_n^2U_{n-2}\,;$$

$$4x^2U_{3n+2} = U_{n+2}^3 - 2U_{n+2}U_n^2 + U_n^2U_{n-2}\,;$$

$$\begin{aligned}8x^3T_{3n} &= U_{n+1}^3 - (2x^2-3)U_n^2 + 1U_{n-1} + (10x^2-3)U_{n+1}U_{n-1}^2\\ &\quad +2x^2U_{n+1}U_{n-1}U_{n-3} + (2x^2-1)(4x^2-1)U_{n-1}^3\\ &\quad +2x^2U_{n-1}^2U_{n-3} - 2x^2U_{n-1}U_{n-3}^2\,;\end{aligned}$$

$$\begin{aligned}8x^2T_{3n-1} &= -U_{n+1}^3 + (4x^2-9)U_{n+1}^2U_{n-1} + (28x^2-13)U_{n+1}U_{n-1}^2\\ &\quad - 4x^2U_{n+1}U_{n-1}U_{n-3} - (4x^2-1)(4x^2-3)U_{n-1}^3\\ &\quad -2(4x^2-1)U_{n-1}^2U_{n-3} + 4x^2U_{n-1}U_{n-3}^2\,;\end{aligned}$$

$$\begin{aligned}8x^2T_{3n+1} &= U_{n+1}^3 - 3U_{n+1}^2U_{n-1} + (8x^2-7)U_{n+1}U_{n-1}^2 + (4x^2-1)U_{n-1}^3\\ &\quad + 2(2x^2-1)U_{n-1}^2U_{n-3}\,;\end{aligned}$$

$$\begin{aligned}8x^3T_{3n+2} &= (2x^2-1)U_{n+1}^3 - (4x^2+3)U_{n+1}^2U_{n-1} + (16x^4-4x^2-3)U_{n+1}U_{n-1}^2\\ &\quad - 2x^2U_{n+1}U_{n-1}U_{n-3} + (4x^2-1)U_{n-1}^3 - 2(4x^4-x^2)U_{n-1}^2U_{n-3}\\ &\quad + 2x^2U_{n-1}U_{n-3}^2\,;\end{aligned}$$

$$\begin{aligned}4(x^4-x^2)U_{3n} &= (x^2-3)U_{n+2}^2U_n + (9x^2-5)U_{n+2}U_n^2 - x^2U_{n+2}U_nU_{n-2}\\ &\quad -(x^2-1)(4x^2-1)U_n^3 - (3x^2-1)U_n^2U_{n-2} + x^2U_nU_{n-2}^2\,;\end{aligned}$$

$$
\begin{aligned}
8(x^5 - x^3)U_{3n-1} &= -(x^2-1)U_{n+2}^3 + (4x^4 - 11x^2 + 3)U_{n+2}^2 U_n \\
&\quad + (28x^4 - 23x^2 + 3)U_{n+2}U_n^2 - 2(2x^4 - x^2)U_{n+2}U_nU_{n-2} \\
&\quad - (x^2-1)(4x^2-1)^2 U3n - 4(2x^4 - x^2)U_n^2U_{n-2} + 2(2x^4 - x^2)U_nU_{n-2}^2\,;
\end{aligned}
$$

$$
\begin{aligned}
8(4x^7 - x^3)U_{3n+1} &= (x^2-4)U_{n+2}^3 - (x^2+3)U_{n+2}^2U_n + (8x^4 + 3x^2 - 3)U_{n+2}U_n^2 \\
&\quad - 2x^2U_{n+2}U_nU_{n-2} - (x^2-1)(4x^2-1)U_n^3 \\
&\quad - 4x^4U_n^2U_{n-2} + 2x^2U_nU_{n-2}^2\,;
\end{aligned}
$$

$$
\begin{aligned}
4(x^4 - x^2)U_{3n+2} &= (x^2-1)U_{n+2}^3 - 2x^2U_{n+2}^2U_n + 2(4x^4 - 3x^2 + 1)U_{n+2}U_n^2 \\
&\quad - x^2U_{n+2}U_nU_{n-2} - (4x^4 - 3x^2 + 1)U_n^2U_{n-2} + x^2U_nU_{n-2}^2\,,
\end{aligned}
$$

where $e_n = e_n(x)$.

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ON SUFFICIENT CONDITIONS FOR STRONGLY STARLIKENESS OF ORDER α AND TYPE β

Abstract: By making use of differential subordination technique, we derive certain conditionsfor p-valent strongly starlike functions of order α and type β.The results presented here are sharp.

Keyword: Starlike Function, Analytic Fuction.

Mathematical Subject Classification (2010) No.: 30C45, 30C50.

1. Introduction and Definitions

Let A_p denote the class of functions $f(z)$ normalized by

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \tag{1.1}$$

which are analytic in the open unit disc U = {z □ C; |z| < 1}.

For two functions f and g analytic in U, we say that the function f is subordinate to$g$ in U (denoted by$f$ g), if there exists a function ω(z) analytic in U with ω(0) = 0 and |ω(z)| < 1 (z ∈ U), such that f(z) = f (ω(z)) in U. Also, if f and g analytic in U with f(0) = g(0) and g(z) is univalent in U, we say then f(z) g(z) in U provided that f(U) ⊂ g(U).

A function f(z)∈A_p is called starlike in U if it satisfies $R\left(\frac{zf'(z)}{f(z)}\right) > 0 \ \ (z \in U)$

A function f(z)∈A_p is called starlike function of order α in U if it satisfies

$$R\left(\frac{zf'(z)}{f(z)}\right) > \alpha.$$

A function f(z)∈A_p is called strongly starlike of order α (0< α ≤ 1) if it satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi\alpha}{2}$$

A function f(z)∈A_p is called strongly starlike of order α and type β if it satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)} \quad \beta\right)\right| < \frac{\pi\alpha}{2} \tag{1.2}$$

In this present paper we shall derive certain sufficient condition for p-valent strongly starlike functions. In order to prove main results, we need the following Lemma.

Lemma 1.1: Let the function $g(z)$ be analytic and univalent in U and let the function $\theta(\omega)$ and $\varphi(\omega)$ be analytic in domain U containing $g(U)$ with $\varphi(\omega) \neq 0, \omega \in g(U)$. Set $Q(z) = zg'(z)\varphi(g(z))$ and $(z) = \theta(g(z)) + q(z)$ and suppose that

i. $Q(z)$ is univalently starlike in U and

ii. $R\left(\frac{zh'(z)}{Q(z)}\right) = R\left(\frac{\theta'(g(z))}{\varphi(g(z))} + \frac{zQ'(z)}{Q(z)}\right) > 0 \quad (z \in U)$

If $q(z)$ is analytic in U with $q(0) = g(0), g(U) \subset D$ and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \quad \theta(g(z)) + zq'(z)\varphi(g(z)) = \quad (z) \quad q(U) \in D \tag{1.3}$$

then $$q(z) \quad g(z), \qquad (z \in U)$$

and $g(z)$ is the best dominant of (1.3).

2. Sufficient Conditions for Strongly Starlike Functions of Order α and Type β

In this section, we assume that $\alpha, \lambda_0, a, b \in R$ and $\mu \in C$

Theorem 2.1: Let

$0 < \alpha \leq 1, \lambda_0 a \geq 0, |b+1| \leq \frac{1}{\alpha}, |a \quad b \quad 1| \leq \frac{1}{\alpha}, if\ f(z) \in A_p$
satisfies

$$f(z).f^{'}(z) \neq 0 \qquad (z \in U\{0\}) \tag{2.1}$$

and

$$\lambda_0 \left(\frac{1}{p(1-\beta)} \left(\frac{zf^{'}(z)}{f(z)} \quad \beta \right) \right)^{\alpha} + \left(\frac{1}{p(1-\beta)} \left(\frac{zf^{'}(z)}{f(z)} \quad \beta \right) \right)^{'} \left(\frac{1}{p(1-\beta)} \left(\frac{zf^{'}(z)}{f(z)} \quad \beta \right) \right)^{b} \quad (z) \tag{2.2}$$

where

$$(z) = \lambda_0 \left(\frac{1+Az}{1 \quad z} \right)^{a\alpha} + \frac{\alpha(1+A)z(1+Az)^{\alpha(1+b)-1}}{(1 \quad z)^{\alpha(1+b)+1}} \tag{2.3}$$

then the function $f(z)$ is p-valent strongly starlike of order α and type β in U. The number α is sharp for the function $f(z)$ defined by

$$\frac{1}{p(1 \quad \beta)} \left(\frac{zf^{'}(z)}{f(z)} \quad \beta \right) = \lambda_0 \left(\frac{1+Az}{1 \quad z} \right)^{\alpha} \tag{2.4}$$

Proof: We choose $q(z) = \frac{1}{p(1-\beta)} \left(\frac{zf^{'}(z)}{f(z)} \quad \beta \right)$

and

$$g(z) = \left(\frac{1+Az}{1-z} \right)^{\alpha}, \theta(\omega) = \lambda_0 \omega^{\alpha}\ and\ \varphi(\omega) = \omega^{b} \text{ in Lemma.}$$

Clearly the function $g(z)$ is analytic and univalently convex in U and

$$|\arg(g(z))| < \frac{\alpha}{2}\pi \leq \frac{\pi}{2}, (z \in U), \qquad (0 < \alpha \leq 1, 0 < \beta \leq 1) \tag{2.5}$$

The function $q(z)$ is analytic in U with $q(0) = g(0) = 1$ and $g(z) \neq 0$, $(z \in U)$ and the function $\theta(\omega)$ and $\varphi(\omega)$ are analytic in a domain D containing $g(U)$ and $q(U)$, with $\varphi(\omega) \neq 0$ when $\omega \in g(U)$. For $\frac{-1}{\alpha} \leq b+1 \leq \frac{1}{\alpha}$.

Then function $Q(z)$ is given by

$$Q(z) = zq^{'}(z)\varphi(q(z)) = \frac{\alpha(1+A)z(1+Az)^{\alpha(1+b)-1}}{(1 \quad z)^{\alpha(1+b)+1}}$$

is univalently starlike in U because

$$R\left(\frac{zf'(z)}{f(z)}\right) = R\left(1 + (\alpha(1+b) \quad 1)(1 + Az) + (\alpha(1+b)+1)\frac{z}{1 \quad z}\right) \tag{2.6}$$

$$> \frac{(1+3|A|)(1 \quad \alpha|(1+b)|)}{2(1+|A|)}$$

Now $R\left(\frac{zQ'(z)}{Q(z)}\right) > 0$, provided that $\frac{(1+3|A|)(1-\alpha|(1+b)|)}{2(1+|A|)} > 0$

or $|b+1| \leq \frac{1}{\alpha}$

Further, we have

$$\theta(g(z)) + Q(z) = \lambda_0\left(\frac{1+Az}{1 \quad z}\right)^{a\alpha} + \frac{\alpha(1+A)z(1+Az)^{\alpha(1+b)-1}}{(1 \quad z)^{\alpha(1+b)+1}} = \quad (z)$$

where (z) is given by (2.3), and so

$$\frac{z \quad '(z)}{Q(z)} = \frac{\theta'(g(z))}{\varphi(g(z))} + \frac{zQ'(z)}{Q(z)} \tag{2.7}$$

$$= \lambda_0 a(g(z))^{a-b-1} + \frac{zQ'(z)}{Q(z)}$$

Also for $|a \quad b \quad 1| \leq \frac{1}{\alpha}$

$$|\arg(g(z)^{a-b-1})| \leq |a \quad b \quad 1|\frac{\alpha}{2}\pi \leq \frac{\pi}{2} \tag{2.8}$$

Therefore, it follows from (2.1) and (2.5)-(2.8) that

$$R\left(\frac{z \quad '(z)}{Q(z)}\right) > 0 \qquad (z \in U)$$

The other condition of lemma are also satisfied, hence we conclude that

$$q(z) = \frac{1}{p(1 \quad \beta)}\left(\frac{zf'(z)}{f(z)} \quad \beta\right) \quad \left(\frac{1+Az}{1 \quad z}\right)^{\alpha} = g(z)(z \in U)$$

and $g(z)$ is the best dominant of (2.2). by (2.5) we see that the function $f(z)$ is univalent stronglystarlike of order α and type β in U. Furthermore, for the function $f(z)$ defined by (2.4) we have

$$\lambda_0\big(q(z)\big)^{\alpha} + zq'(z)\big(q(z)\big)^{b} = \quad (z)$$

which shows that the number α is sharp, the proof of theorem is now completed.

Theorem 2.2: Let

$$0 < \alpha \leq 1, \lambda_0(b+2) \geq 0, (b+1)R\{\mu\} \geq 0, |b+1| \leq \frac{1}{\alpha} \text{if } f(z) \in A_p$$

satisfies

$$f(z).f'(z) \neq 0 \quad (z \in U\backslash 0)$$

and $$\lambda_0\left(\frac{1}{p(1-\beta)}\left(\frac{zf'(z)}{f(z)} \quad \beta\right)\right)^{(b+2)} + \mu\left(\frac{1}{p(1-\beta)}\left(\frac{zf'(z)}{f(z)} \quad \beta\right)\right)^{b+1}$$

$$+z\left(\frac{1}{p(1 \quad \beta)}\left(\frac{zf'(z)}{f(z)} \quad \beta\right)\right)'\left(\frac{1}{p(1 \quad \beta)}\left(\frac{zf'(z)}{f(z)} \quad \beta\right)\right)^{b} \qquad (z) \qquad (2.10)$$

where

$$(z) = \left(\frac{1+Az}{1 \quad z}\right)^{(b+1)\alpha}\left(\mu + \lambda\left(\frac{1+Az}{1 \quad z}\right)^{\alpha} + \frac{\alpha(1+Az)z}{(1 \quad z)(1+Az)}\right), \qquad (2.11)$$

then the function $f(z)$ is p-valent strongly starlike of order α and type β in U. The number α is sharp for the function $f(z)$ defined by (2.4).

Proof: Let $q(z) = \frac{1}{p(1-\beta)}\left(\frac{zf'(z)}{f(z)} \quad \beta\right)$ and $g(z) = \left(\frac{1+Az}{1-z}\right)^{\alpha}, \theta(\omega) = \lambda_0\omega^{\alpha}$ and $\varphi(\omega) = \omega^b$ in Lemma, clearly the function $g(z), q(z), \theta(\omega), \varphi(\omega)$ and $Q(z) = zg'(z)\varphi(g(z))$ satisfies the condition of Lemma respectively. Further we have:

$$\theta(g(z)) + Q(z) = \left(\frac{1+Az}{1 \quad z}\right)^{(b+1)\alpha}\left(\mu + \lambda\left(\frac{1+Az}{1 \quad z}\right)^{\alpha} + \frac{\alpha(1+Az)z}{(1 \quad z)(1+Az)}\right)$$

where (z) is given by (2.11) and so on

$$\frac{z\ '(z)}{Q(z)} = \frac{\theta(g(z))}{\varphi(g(z))} + \frac{zQ'(z)}{Q(z)} = \lambda(b+2)g(z) + \mu(b+1) + \frac{zQ'(z)}{Q(z)}$$

Now for

$$\lambda(b+2) \geq 0 \text{ and } \mu(b+1)R\{\mu\} \geq 0$$

We have

$$R\left(\frac{zQ'(z)}{Q(z)}\right) > 0 \qquad (z \in U)$$

The other condition of lemma is also satisfied. Hence we obtain the desired result of theorem. Further, for the function $f(z)$ defined by (2.4), we have

$$\lambda\big(q(z)\big)^{(b+2)} + \mu\big(q(z)\big)^{(b+1)} + zq'(z)\big(q(z)\big)^{b} = \quad (z)$$

which shows that the number $\alpha$is sharp. The proof of theorem 2.2 is now completed.

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COMPATIBILITY OF MAPS AND COMMON FIXED POINT THEOREMS IN COMPLEX VALUED METRIC SPACES

Abstract: Some common fixed point theorems in complete complex valued metric spaces satisfying rational inequality are established by using the notion of compatibility and weak compatibility of mappings. Results of this paper generalize the result of Azam [1] and some earlier results.

Keywords: Common Fixed Point, Compatibility, Weak Compatibility, Complex Valued Metric Spaces.

Mathematical Subject Classification No.: 47H10, 54H25.

1. Introduction

Banach contraction principle (BCP) is a benchmark result established by Banach [2] in fixed point theory. According to this principle, a contraction map on a complete metric space always possesses a unique fixed point. After this interesting result and its various applications, number of generalization of this result are available in the literature by using different types of contractive conditions in various abstract spaces.

By generalizing the Banach contraction principle, Jungck [7] set out tradition of common fixed point of mappings for two commuting mappings on complete metric space. After the result of Jungck [7] many authors introduced many concepts namely weak commutativity, compatibility, weak compatibility of maps (Sessa [12], Jungck [6, 8], Jungck and Rhoades [9] etc.) and established results regarding common fixed point. In fact commutativity of maps ⇒ weak commutativity of maps ⇒ compatibility of maps ⇒ weak compatibility of maps, ⇒ but the converse of these implications are not true.

In 2011, Azam *et. al* [1] introduced a generalization of classical metric space which is known as complex valued metric space. He established sufficient condition for the existence of some common fixed point for a pair of maps satisfying rational inequality.

In this article we prove some results regarding common fixed point of maps, by using the notion of compatibility and weak compatibility of maps in complex valued metric space satisfying contractive conditions involving rational expression. We apply our result to find the solution of Uryshon's integral equations as an application.

2. Preliminaries

We recall some basic definition and results which will utilize in our subsequent discussion.

Definition 2.1 [1]: Let be the set of complex numbers and $z_1, z_2 \in$. Define a partial order on as : z_1 z_2 iff $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im\,(z_2)$. It follows that z_1 z_2 if one of the following conditions hold:

(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im\,(z_2)$

(ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im\,(z_2)$

(iii) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im\,(z_2)$

(iv) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im\,(z_2)$

We write z_1 z_2 if $z_1 \neq z_2$ and one of (ii) and (iii) is satisfied and we write z_1 z_2 if only (iv) is satisfied.

Here we note the following holds trivially:

(i) If z_1 z_2 then $|z_1| < |z_2|$;

(ii) If z_1 z_2 and z_2 z_3 then z_1 z_3;

(iii) If $a, b \in$ and $a \leq b$ then az bz for all $z \in$;

(iv) If $a, b \in$ and $0 \leq a \leq b$ and z_1 z_2 implies az_1 bz_2.

Definition 2.2 [1]: Let X be a nonempty set. A function $d: X \times X \rightarrow \mathbb{C}$ is called a complex valued metric on X if for all $z_1, z_2, z_3 \in X$ the following conditions are satisfied.

(CVM 1) $0 \precsim d(z_1, z_2)$ and $d(z_1, z_2) = 0$ if and only if $z_1 = z_2$;

(CVM 2) $d(z_1, z_2) = d(z_2, z_1)$;

(CVM3) $d(z_1, z_2) \precsim d(z_1, z_3) + d(z_3, z_2)$.

Then the pair (X, d) is called a complex valued metric space.

Example 2.3 [5]: Let $X = \mathbb{C}$. Define the mapping $d: X \times X \rightarrow \mathbb{C}$ by

$$d(z_1, z_2) = i|z_1 - z_2|, \forall\, z_1, z_2 \in X.$$

Then (X, d) is a complex valued metric space.

Example 2.4 [11]: Let $X = \mathbb{C}$. Define the mapping $d: X \times X \rightarrow \mathbb{C}$ by

$$d(z_1, z_2) = e^{ik}|z_1 - z_2|, \quad \text{where} \quad k \in \mathbb{R}, \forall\, z_1, z_2 \in X.$$

Then (X, d) is a complex valued metric space.

Example 2.5: Let $X = [0,1]$. Define the mapping $d: X \times X \rightarrow \mathbb{C}$ by

$$d(x, y) = |x - y| + i|x - y|$$

then (X, d) is a complex valued metric space.

Example 2.6 [13]: Let $X = \mathbb{C}$. Define a function $d: X \times X \rightarrow \mathbb{C}$ such that

$$d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|, \forall\, z_1, z_2 \in X,$$

where

$$z_1 = x_1 + iy_1, z_2 = x_2 + iy_2.$$

then (X, d) is a complete complex valued metric space.

Definition 2.7 [10]: Let (X, d) be a complex valued metric space. Consider the following,

(i) A point $x \epsilon X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \quad r \in \quad$ such that $B(x,r) = \{y \in X : d(x,y) \quad r\} \subseteq A$.

(ii) A point $x \in X$ is called a limit point of a set A whenever, for every $0 \quad r \in \quad$, $B(x,r) \cap (A \quad X) \neq \quad$.

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A.

(iv) A subset $A \subseteq X$ is called closed whenever each limit point of A belongs to A.

(v) The family $F = \{B(x,r): x \epsilon X,$ and $0 \quad r\}$ is a sub basis for a topology on X.This topology is denoted by τ_c. Indeed, the topology τ_c is Hausdorff.

Definition 2.8 [4]: Let (X,d) be a complex valued metric space and $\{z_n\}$ a sequence in X and $z \in X$. Consider the following:

(i) If for every $c \in \quad$ with $0 \quad c$ there is $\mathrm{N} \in \quad$ such that, for all $n \geq N, d(z_n, z) \quad c$, then $\{z_n\}$ is said to be convergent, $\{z_n\}$ converges to z and z is the limit point of $\{z_n\}$. We denote this $\lim_{n\to\infty} z_n = z$ or $z_n \to z$ as $n \to \infty$.

(ii) If for every $c \in \quad$ with $0 \quad r$ there is $\mathrm{N} \in \quad$ such that, for all $n > N, d(z_n, z_{n+m}) \quad c$, where $\mathrm{m} \in \quad$, then $\{z_n\}$ is said to be a Cauchy sequence.

(iii) If every Cauchy sequence in (X,d) is convergent, then (X,d) is said to be a complete complex valued metric space.

Lemma 2.9 [10]: Let (X,d) be a complex valued metric space and let $\{z_n\}$ be a sequence in X. Then $\{z_n\}$ converges to z if and only if $|d(z_n, z)| \to 0$, as $n \to \infty$.

Lemma 2.10 [10]: Let (X,d) be a complex valued metric space and let $\{z_n\}$ be a sequence in X. Then $\{z_n\}$ is a Cauchy Sequence if and only if $|d(z_n, z_{n+m})| \to 0$ as $n \to \infty$, where $m \in N$.

Definition 2.11: Two self maps S and T of a complex valued metric space (X,d) are weakly commuting iff $|d(STz, TSz)| \leq |d(Sz, Tz)|, \forall z \in X$.

Definition 2.12: Two self-maps S and T of a complex valued metric space (X, d) are compatible iff $\lim_{n\to\infty}|d(STz_n, TSz_n)| \to 0$ whenever $\{z_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} Tz_n = z \text{ for some } z \in X.$$

Definition 2.13 [3]: Two self maps S and T of a complex valued metric space (X, d) are weakly compatible iff $Sz = Tz$ implies that $TSz = STz$.

Definition 2.14: A function T defined on a complex valued metric space (X, d) is called continuous at a point $z_0 \in X$ if for every $\varepsilon > 0$ there exist $\delta > 0$ such that $|d(Tz, Tz_0)| < \varepsilon$ for all $z \in X$ with $|d(z, z_0)| < \delta$. i.e. $\lim_{z\to z_0}|d(Tz, Tz_0)| = 0$.

Proposition 2.15: Let S and T be two self mappings defined on a complex valued metric space (X, d).Then the commutativity of S and T implies weak commutativity but the converse is not always true.

Proof: If S and T are two self maps on a complex valued metric space (X, d) If S and T are commuting maps then $STx = TSx \quad \forall\, x \in X$, therefore $|d(STx, TSx)| = 0$. Then we have

$$0 = |d(STx, TSx)| \leq |d(Sx, Tx)|$$

is true, i.e. S and T are weakly commuting maps.

For the converse part, we consider the following example:

Let (X, d) be a complex valued metric space, where $X = [0, 1]$ and $d: X \times X \to$ defined by $d(x, y) = |x \quad y| + i|x \quad y|$, $\forall\, x, y \in X$.

Define self maps S and T on X by $Tx = \frac{x}{x+1}$ and $Sx = \frac{x}{x+2}$ $\forall\, x \in X$,

Then we see that $T(Sx) = T\left(\frac{x}{x+2}\right) = \frac{x}{2x+2}$ and $S(Tx) = S\left(\frac{x}{x+1}\right) = \frac{x}{3x+2}$.

Therefore, $ST \neq TS$ i.e. the mappings S and T are not commuting. Now

$$d(STx, TSx) = \left|\frac{x}{3x+2} \quad \frac{x}{2x+2}\right| + i\left|\frac{x}{3x+2} \quad \frac{x}{2x+2}\right| = \left|\frac{x}{3x+2} \quad \frac{x}{2x+2}\right|(1+i).$$

$$\Rightarrow \; |d(STx, TSx)| = \frac{1}{2}\left|\left|\frac{-x^2}{(3x+2)(x+1)}\right|(1+i)\right| .$$

Also, $d(Sx,Tx) = \left|\frac{x}{(x+2)} \quad \frac{x}{(x+1)}\right| + i\left|\frac{x}{(x+2)} \quad \frac{x}{(x+1)}\right|$

$\Rightarrow |d(Sx,Tx)| = \left|\frac{x}{(x+2)} \quad \frac{x}{(x+1)}\right| |1+i| = \left|\left|\frac{-x}{(x+2)(x+1)}\right|(1+i)\right|$

Hence, $|d(STx,TSx)| \le |d(Sx,Tx)| \ \forall\, x \in X$. i.e. S and T are weakly commuting maps. Therefore weakly commutativity does not imply commutativity of maps.

Proposition 2.16: Let S and T be two self mappings defined on a complex valued metric space (X,d).

Proof: If S and T are two self maps of a complex valued metric space (X,d). If S and T are weakly commuting maps then $|d(STx,TSx)| \le |d(Sx,Tx)|, \forall\, x \in X$.

Now we take a sequence $\{x_n\}$ such that $Sx_n, Tx_n \to t$ as $n \to \infty$ for some $t \in X$. Then

$|d(STx_n, TSx_n)| \le |d(Sx_n, Tx_n)| \to 0$ as $n \to \infty$. i.e. S and T are compatible maps.

For the converse part, we consider the following example:

Let (X,d) be a complex valued metric space where $X = [0,1]$ and $d: X \times X \to$ defined by

$$d(x,y) = |x \quad y| + i|x \quad y|, \ \forall\, x, y \in X.$$

Define self maps S an T on X by

$$Tx = x^2 \text{ and } Sx = 2x^2 \ \forall\, x \in X.$$

Then we see that

$T(Sx) = T(2x^2) = 4x^4$ and $S(Tx) = S(x^2) = 2x^4$

$d(STx,TSx) = d(2x^4,4x^4) = |2x^4 \quad 4x^4| + i|2x^4 \quad 4x^4| = | \quad 2x^4|(1+i)$

$|d(STx,TSx)| = 2\sqrt{2}x^4$.

$d(Sx,Tx) = d(x^2,2x^2) = |x^2 \quad 2x^2| + i|x^2 \quad 2x^2| = | \quad x^2|(1+i)$

$|d(Sx,Tx)| = \sqrt{2}x^2$.

Therefore we have $|d(STx, TSx)| \quad |d(Sx, Tx)|$ i.e. S and T are not weakly commuting.

But if we take a sequence defined by $x_n = \frac{2}{n}$ then $x_n \to 0$ as $n \to \infty$.

Now

$$Sx_n = 2(x_n)^2 = 2\left(\frac{2}{n}\right)^2 \to 0 \text{ as } n \to \infty \text{ and } Tx_n = (x_n)^2 = \left(\frac{2}{n}\right)^2 \to 0 \text{ as } n \to \infty$$

$$d(STx_n, TSx_n) = |2{x_n}^4 \quad 4{x_n}^4| + i|2{x_n}^4 \quad 4{x_n}^4| = | \quad 2{x_n}^4|(1 + i)$$

$$|d(STx_n, TSx_n)| = \frac{32\sqrt{2}}{n^4} \to 0 \text{ as } n \to \infty.$$

Hence, S and T are compatible maps.

Proposition 2.17: Let S and T be two self mappings defined on a complex valued metric space (X, d). Then the compatibility of S and T implies weak compatibility but the converse is not always true.

Proof: Let S and T be two self maps defined on a complex valued metric space (X, d).Suppose that S and T are compatible maps and $Sx = Tx$ for some $x \in X$. For every $x \in X$, consider the constant sequence $x_n = x$ for all $n \in$. then $Sx_n = Tx_n \to Sx$ or Tx as $n \to \infty$ and by the compatibility of S and T we have $|d(STx, TSx)| = |d(STx_n, TSx_n)| \to 0$ as $n \to \infty$. Hence $STx = TSx$ i.e. S and T are weakly compatible maps, for the converse part we consider the following example.

Let (X, d) be a complex valued metric space with the mapping where $X = [0, 2]$ and

$d: X \times X \to$ defined by $d(x, y) = |x \quad y| + i|x \quad y|$, $\forall\, x, y \in X$.

Define self maps S and T on X by

$$Sx = \begin{cases} 1, & x = 1 \\ 2, & ot\ \ erwise \end{cases} \quad \text{and} \quad Tx = \begin{cases} 1, & x = 1,2 \\ x, & ot\ \ erwise \end{cases}$$

Then 1 is the only coincidence point of S and T i.e. $S(1) = 1 = T(1)$ and we see that

$ST(1) = S(1) = 1,\ TS(1) = T(1) = 1$

i.e. $ST(1) = TS(1)$ the maps S and T are weakly compatible.

On the other hand, if we take a sequence $\{x_n\}$, defined by $x_n = \left(2 \quad \frac{1}{n}\right) \to 2$ as $n \to \infty$ and

$$Sx_n \to 2\ ,\ Tx_n = x_n \to 2 \text{ as } n \to \infty\ .$$

But,

$$STx_n = S\left(2 \quad \frac{1}{n}\right) = 2 \text{ and } TSx_n = T(2) = 1 \text{ and so}$$

$$|d(STx_n, TSx_n)| = |2 \quad 1| + i|2 \quad 1| = \sqrt{2} \quad\quad 0 \text{ as } n \to \infty.$$

Hence, S and T are not compatible maps.

Also one cannot find a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x \,\epsilon\, X \quad \text{for some } x \,\epsilon\, X,$$

such that

$$|d(STx_n, TSx_n)| \to 0 \text{ as } n \to \infty\ .$$

Hence, S and T are not compatible maps.

Lemma 2.18: Let S and T be compatible mappings from a complex valued metric space (X, d) into itself. Suppose that $\lim_{n\to\infty} Sx_n = x$ for some $x \in X$ and if S is continuous. Then $\lim_{n\to\infty} T\,Sx_n = Sx$.

Proof: If $\lim_{n\to\infty} Tx_n = x$, $\lim_{n\to\infty} S\,Tx_n = Sx$ by continuity of S. But if $\lim_{n\to\infty} Sx_n = x$.

Then since $d(TSx_n, Sx) \quad [d(TSx_n, STx_n) + d(STx_n, Sx)]$ implies that

$$|d(TSx_n, Sx)| \le [|d(TSx_n, STx_n)| + |d(STx_n, Sx)|] \tag{i}$$

Now by the compatibility of S and T we have

$|d(TSx_n, STx_n)| \to 0$ as $n \to \infty$, and $\lim_{n\to\infty} ST\,x_n = Sx$.

Then letting $n \to \infty$ in (i) ,we have $|d(TSx_n, Sx)| \to 0$ yields that

$$\lim_{n\to\infty} T\,Sx_n = Sx.$$

Azam *et al*. [1] proved the following result.

Theorem 2.19 [1]: Let (X,d) be a complete complex valued metric space and $S,T\colon X \to X$ be mapping satisfying

$$d(Sz\,,Tw) \quad \lambda d(z,w) + \mu \frac{d(z,Sz)\; d(w,Tw)}{1+d(z,w)}$$

for all $z,w \in X$,where λ,μ are non negative real numbers with $\lambda+\mu<1$. then S,T have a unique common fixed point in X.

3. Main Results

Here by using the notion of compatibility and weak compatibility of maps, we generalize the above results by taking four maps as opposed to two maps.

Theorem 3.1: Let (X,d) be a complete complex valued metric space and mappings A,B,S and T satisfying.

(3.1.1) $S(X) \subseteq B(X)\,, T(X) \subseteq A(X)$

(3.1.2) $$d(Sz\,,Tw) \quad \alpha d(Az,Bw) + \beta \frac{d(Az,Sz)d(Bw,Tw)}{1+d(Az,Bw)}$$
$$+\gamma \frac{d(Az,Sz)d(Az,Tw)+d(Bw,Sz)d(Bw,Tw)}{d(Az,Tw)+d(Bw,Sz)}$$

where, $d(Az,Tw)+d(Bw,Sz) \neq 0$

$$d(Sz\,,Tw) = 0 \text{ if } d(Az,Tw)+d(Bw,Sz) = 0.$$

for all z,w in X where α,β,γ are non negative reals with $\alpha+\beta+\gamma<1$.

(3.1.3) suppose that A is continuous, pair (S,A) is compatible and (T,B) is weakly compatible. OR

(3.1.4) T is continuous, pair (S,A) is weakly compatible and (T,B) is compatible.

Then A,B,S and T have unique common fixed point in X.

Proof: Suppose z_0 be an arbitrary point in X we define a sequence $\{w_{2n}\}$ in X such that

$$w_{2n} = Sz_{2n} = Bz_{2n+1};\; w_{2n+1} = Tz_{2n+1} = Az_{2n+2},\; \text{for } n = 0,1,2, \qquad (3.1)$$

Now from (3.1.2), we have

$$d(w_{2n,}w_{2n+1}) = d(Sz_{2n}, Tz_{2n+1})$$

$$\alpha d(Az_{2n}, Bz_{2n+1}) + \beta \frac{d(Az_{2n}, Sz_{2n})d(Bz_{2n+1}, Tz_{2n+1})}{1 + d(Az_{2n}, Bz_{2n+1})}$$

$$+\gamma \frac{d(Az_{2n}, Sz_{2n})d(Az_{2n}, Tz_{2n+1}) + d(Bz_{2n+1}, Sz_{2n})d(Bz_{2n+1}, Tz_{2n+1})}{d(Az_{2n}, Tz_{2n+1}) + d(Bz_{2n+1}, Sz_{2n})}$$

Since

$$d(Az_{2n}, Tz_{2n+1}) + d(Bz_{2n+1}, Sz_{2n}) = d(w_{2n-1}, w_{2n+1}) + d(w_{2n}, w_{2n}) \neq 0.$$

which implies that

$$|d(w_{2n}, w_{2n+1})| \leq \alpha|d(w_{2n-1}, w_{2n})| + \beta \frac{|d(w_{2n-1}, w_{2n})||d(w_{2n}, w_{2n+1})|}{|1 + d(w_{2n-1}, w_{2n})|}$$

$$+\gamma \frac{|d(w_{2n-1}, w_{2n})||d(w_{2n-1}, w_{2n+1})| + |d(w_{2n}, w_{2n})||d(w_{2n}, w_{2n+1})|}{|d(w_{2n-1}, w_{2n+1}) + d(w_{2n}, w_{2n})|}$$

$$|d(w_{2n}, w_{2n+1})| \leq \alpha|d(w_{2n-1}, w_{2n})| + \beta|d(w_{2n}, w_{2n+1})|$$

$$+\gamma|d(w_{2n-1}, w_{2n})|$$

$$|d(w_{2n}, w_{2n+1})| \leq \left(\frac{\alpha+\gamma}{1-\beta}\right)|d(w_{2n-1}, w_{2n})|.$$

Similarly

$$d(w_{2n+1,}w_{2n+2}) \quad \alpha d(Az_{2n+1}, Bz_{2n+2}) + \beta \frac{d(Az_{2n+1}, Sz_{2n+1})d(Bz_{2n+2}, Tz_{2n+2})}{1 + d(Az_{2n+1}, Bz_{2n+2})}$$

$$+\gamma \frac{d(Az_{2n+1}, Sz_{2n+1})d(Az_{2n+1}, Tz_{2n+2}) + d(Bz_{2n+2}, Sz_{2n+1})d(Bz_{2n+2}, Tz_{2n+2})}{d(Az_{2n+1}, Tz_{2n+2}) + d(Bz_{2n+2}, Sz_{2n+1})}$$

Since

$$d(Az_{2n+1}, Tz_{2n+2}) + d(Bz_{2n+2}, Sz_{2n+1})$$

$$= d(w_{2n}, w_{2n+2}) + d(w_{2n+1}, w_{2n+1}) \neq 0.$$

$$\left|d\left(w_{2n+1,}w_{2n+2}\right)\right| \leq \alpha|d(w_{2n},w_{2n+1})| + \beta\frac{|d(w_{2n},w_{2n+1})||d(w_{2n+1},w_{2n+2})|}{|1+d(w_{2n},w_{2n+1})|}$$

$$+\gamma\frac{|d(w_{2n},w_{2n+1})||d(w_{2n},w_{2n+2})| + |d(w_{2n+1},w_{2n+1})||d(w_{2n+1},w_{2n+2})|}{|d(w_{2n},w_{2n+2}) + d(w_{2n+1},w_{2n+1})|}$$

$$\left|d\left(w_{2n+1,}w_{2n+2}\right)\right| \leq \alpha|d(w_{2n},w_{2n+1})| + \beta|d(w_{2n+1},w_{2n+2})|$$

$$+\gamma|d(w_{2n},w_{2n+1})|$$

$$\left|d\left(w_{2n+1,}w_{2n+2}\right)\right| \leq \left(\frac{\alpha+\gamma}{1-\beta}\right)|d(w_{2n},w_{2n+1})|.$$

Since if $\alpha+\beta+\gamma<1$, then $\frac{(\alpha+\gamma)}{1-\beta}<1$ or $\delta<1$ $\left(\delta=\frac{\alpha+\gamma}{1-\beta}\right)$,

Therefore for all $n \geq 0$, we have

$$\left|d\left(w_{2n+1,}w_{2n+2}\right)\right| \leq \delta|d(w_{2n},w_{2n+1})| \leq \delta^2|d(w_{2n-1},w_{2n})|$$

$$\leq \ldots \leq \delta^{2n+1}|d(w_0,w_1)| \tag{3.2}$$

By using (3.2) for all $n, m\ \epsilon$ and $m > n$ we have

$$\left|d\left(w_{2n,}w_{2m}\right)\right| \leq \delta^{2n}|d(w_0,w_1)| + \delta^{2n+1}|d(w_0,w_1)| + \delta^{2n+2}|d(w_0,w_1)| +$$

$$\ldots + \delta^{2m-1}|d(w_0,w_1)|$$

$$\leq \sum_{i=1}^{2m-2n}\delta^{i+2n-1}|d(w_0,w_1)| = \sum_{t=2n}^{2m-1}\delta^t|d(w_0,w_1)|$$

$$\leq \sum_{t=2n}^{\infty}(\delta)^t|d(w_0,w_1)| \leq \frac{(\delta)^{2n}}{(1-\delta)}|d(w_0,w_1)| \to 0 \quad \text{as } m,n\to\infty$$

(since $\delta < 1$)

Hence, $\{w_{2n}\}$ is a Cauchy sequence in X. Since X is complete, therefore $\{w_{2n}\}$ converges to point t in X and its subsequences $\{Sz_{2n}\}$, $\{Tz_{2n+1}\}$, $\{Az_{2n+2}\}$, $\{Bz_{2n+1}\}$ are also converge to t.

Case I: Suppose that A is continuous. Then $A^2z_{2n} = AAz_{2n} \to At$.

Also by the compatibility of S and A, from Lemma (2.18) $SAz_{2n} \to At$.

Using (3.1.2), we have

$$d(SAz_{2n}, Tz_{2n+1}) \quad \alpha d(A^2 z_{2n}, Bz_{2n+1}) + \beta \frac{d(A^2 z_{2n}, SAz_{2n}) d(Bz_{2n+1}, Tz_{2n+1})}{1 + d(A^2 z_{2n}, Bz_{2n+1})}$$

$$+ \gamma \frac{d(A^2 z_{2n}, SAz_{2n}) d(A^2 z_{2n}, Tz_{2n+1}) + d(Bz_{2n+1}, SAz_{2n}) d(Bz_{2n+1}, Tz_{2n+1})}{d(A^2 z_{2n}, Tz_{2n+1}) + d(Bz_{2n+1}, SAz_{2n})}$$

Since, $d(A^2 z_{2n}, Tz_{2n+1}) + d(Bz_{2n+1}, SAz_{2n}) = d(At, t) + d(At, t) \neq 0$.

$$|d(SAz_{2n}, Tz_{2n+1})| \leq \alpha |d(A^2 z_{2n}, Bz_{2n+1})| + \beta \frac{|d(A^2 z_{2n}, SAz_{2n})||d(Bz_{2n+1}, Tz_{2n+1})|}{|1 + d(A^2 z_{2n}, Bz_{2n+1})|}$$

$$+ \gamma \frac{|d(A^2 z_{2n}, SAz_{2n})||d(A^2 z_{2n}, Tz_{2n+1})| + |d(Bz_{2n+1}, SAz_{2n})||d(Bz_{2n+1}, Tz_{2n+1})|}{|d(A^2 z_{2n}, Tz_{2n+1}) + d(Bz_{2n+1}, SAz_{2n})|}$$

Letting $n \to \infty$, we get

$$|d(At, t)| \leq \alpha |d(At, t)| + \beta \frac{|d(At, At)||d(t, t)|}{|1 + d(At, t)|} + \gamma \frac{|d(At, At)||d(At, t)| + |d(t, At)||d(t, t)|}{|d(At, t) + d(t, At)|}$$

$\Rightarrow (1 \quad \alpha)|d(At, t)| \leq 0$ yields $At = t$.

Again using (3.1.2), we get

$$d(St, Tz_{2n+1}) \quad \alpha d(At, Bz_{2n+1}) + \beta \frac{d(At, St) d(Bz_{2n+1}, Tz_{2n+1})}{1 + d(At, Bz_{2n+1})}$$

$$+ \gamma \frac{d(At, St) d(At, Tz_{2n+1}) + d(Bz_{2n+1}, St) d(Bz_{2n+1}, Tz_{2n+1})}{d(At, Tz_{2n+1}) + d(Bz_{2n+1}, St)}$$

Since, $d(At, Tz_{2n+1}) + d(Bz_{2n+1}, St) = d(t, t) + d(t, St) \neq 0$.

Letting $n \to \infty$, we have

$$|d(St, t)| \leq \alpha |d(t, t)| + \beta \frac{|d(t, St)||d(t, t)|}{|1 + d(t, t)|} + \gamma \frac{|d(t, St)||d(t, t)| + |d(t, St)||d(t, t)|}{|d(t, t) + d(t, St)|}$$

$\Rightarrow |d(St, t)| \leq 0$ yields $St = t = At$.

Now from (3.1.1) since $S(X) \subseteq B(X)$, there exists a point u in X such that $t = St = Bu = At$.

Then from (3.1.2), we have

$$|d(t,Tu)| = |d(St,Tu)| \leq \alpha|d(At,Bu)| + \beta\frac{|d(At,St)||d(Bu,Tu)|}{|1+d(At,Bu)|}$$

$$+\gamma\frac{|d(At,St)||d(At,Tu)| + |d(Bu,St)||d(Bu,Tu)|}{|d(At,Tu)+d(Bu,St)|}$$

$\Rightarrow |d(t,Tu)| \leq 0$ yields $t = Tu = Bu = St = At$.

Now by weak compatibility of T and B, we have $TBu = BTu \Rightarrow Tt = Bt$.

Again from (3.1.2), we get

$$d(St\,,Tt) \quad \alpha d(At,Bt) + \beta\frac{d(At,St)d(Bt,Tt)}{1+d(At,Bt)} + \gamma\frac{d(At,St)d(At,Tt)+d(Bt,St)d(Bt,Tt)}{d(At,Tt)+d(Bt,St)}$$

$$|d(t,Bt)| = |d(St,Tt)| \leq \alpha|d(At,Bt)| + \beta\frac{|d(At,St)||d(Bt,Tt)|}{|1+d(At,Bt)|}$$

$$+\gamma\frac{|d(At,St)||d(At,Tt)| + |d(Bt,St)||d(Bt,Tt)|}{|d(At,Tt)+d(Bt,St)|}$$

$\Rightarrow |d(t,Bt)| \leq \alpha|d(t,Bt)|$ or $(1 \quad \alpha)|d(t,Bt)| \leq 0$ yields $Bt = t$.

Hence, $At = Bt = St = Tt = t$, i.e. t is the common fixed point of A, B, S and T.

Case II: For the 'or' part, let T is continuous. Then $T^2 z_{2n} = TTz_{2n} \to Tt$.

Also by the compatiblility of T and B, from Lemma (2.18) $BTz_{2n} \to Tt$.

Using (3.1.2), we have

$$d(Sz_{2n}, T^2 z_{2n}) \quad \alpha d(Az_{2n}, BTz_{2n}) + \beta\frac{d(Az_{2n}, Sz_{2n})d(BTz_{2n}, T^2 z_{2n})}{1 + d(Az_{2n}, BTz_{2n})}$$

$$+\gamma\frac{d(Az_{2n}, Sz_{2n})d(Az_{2n}, T^2 z_{2n}) + d(BTz_{2n}, Sz_{2n})d(BTz_{2n}, T^2 z_{2n})}{d(Az_{2n}, T^2 z_{2n}) + d(BTz_{2n}, Sz_{2n})}$$

Since, $d(Az_{2n}, T^2 z_{2n}) + d(BTz_{2n}, Sz_{2n}) = d(t,Tt) + d(t,Tt) \neq 0$.

$$\Rightarrow |d(Sz_{2n}, T^2 z_{2n})| \leq \alpha |d(Az_{2n}, BTz_{2n})| + \beta \frac{|d(Az_{2n}, Sz_{2n})||d(BTz_{2n}, T^2 z_{2n})|}{|1 + d(Az_{2n}, BTz_{2n})|}$$

$$+\gamma \frac{|d(Az_{2n}, Sz_{2n})||d(Az_{2n}, T^2 z_{2n})| + |d(BTz_{2n}, Sz_{2n})||d(BTz_{2n}, T^2 z_{2n})|}{|d(Az_{2n}, T^2 z_{2n}) + d(BTz_{2n}, Sz_{2n})|}$$

Letting $n \to \infty$, we have

$$|d(t, Tt)| \leq \alpha |d(t, Tt)| + \beta \frac{|d(t,t)||d(Tt,Tt)|}{|1 + d(t,Tt)|} + \gamma \frac{|d(t,t)||d(t,Tt)| + |d(Tt,t)||d(Tt,Tt)|}{|d(t,Tt) + d(Tt,t)|}$$

$\Rightarrow (1 \quad \alpha)|\, d(t, Tt)| \leq 0$ yields $Tt = t$.

Now from (3.1.1) since $T(X) \subseteq A(X)$, there exist a point v in X such that $t = Tt = Av$.

Then from (3.1.2), we have

$$d(Sv, T^2 z_{2n}) \quad \alpha d(Av, BTz_{2n}) + \beta \frac{d(Av, Sv) d(BTz_{2n}, T^2 z_{2n})}{1 + d(Av, BTz_{2n})}$$

$$+\gamma \frac{d(Av, Sv) d(Av, T^2 z_{2n}) + d(BTz_{2n}, Sv) d(BTz_{2n}, T^2 z_{2n})}{d(Av, T^2 z_{2n}) + d(BTz_{2n}, Sv)}$$

Since, $d(Av, T^2 z_{2n}) + d(BTz_{2n}, Sv) = d(t, Tt) + d(Tt, Sv) \neq 0$.

$$|d(Sv, T^2 z_{2n})| \leq \alpha |d(Az_{2n}, BTz_{2n})| + \beta \frac{|d(Av, Sv)||d(BTz_{2n}, T^2 z_{2n})|}{|1 + d(Av, BTz_{2n})|}$$

$$+\gamma \frac{|d(Av, Sv)||d(Av, T^2 z_{2n})| + |d(BTz_{2n}, Sv)||d(BTz_{2n}, T^2 z_{2n})|}{|d(Av, T^2 z_{2n}) + d(BTz_{2n}, Sv)|}$$

Letting $n \to \infty$, we have

$$|d(Sv, Tt)| \leq \alpha |d(t, Tt)| + \beta \frac{|d(t, Sv)||d(Tt, Tt)|}{|1 + d(t, Tt)|}$$

$$+\gamma \frac{|d(t, Sv)||d(t, Tt)| + |d(Tt, Sv)||d(Tt, Tt)|}{|d(t, Tt)| + |d(Tt, Sv)|}$$

$$|d(Sv,t)| \leq \alpha|d(t,t)| + \beta\frac{|d(t,Sv)||d(t,t)|}{|1+d(t,t)|}$$

$$+\gamma\frac{|d(t,Sv)||d(t,t)| + |d(t,Sv)||d(t,t)|}{|d(t,t)| + |d(t,Sv)|}$$

$\Rightarrow |d(Sv,t)| \leq 0$, yields $Sv = t = Tt$.

Since, S and A are weakly compatible on X and $Sv = Av$ and $SAv = ASv \Rightarrow St = At$.

Using (3.1.2), we have

$$d(St,Tz_{2n+1}) \quad \alpha d(At,Bz_{2n+1}) + \beta\frac{d(At,St)d(Bz_{2n+1},Tz_{2n+1})}{1+d(At,Bz_{2n+1})}$$

$$+\gamma\frac{d(At,St)d(At,Tz_{2n+1}) + d(Bz_{2n+1},St)d(Bz_{2n+1},Tz_{2n+1})}{d(At,Tz_{2n+1}) + d(Bz_{2n+1},St)}$$

Since, $d(At,Tz_{2n+1}) + d(Bz_{2n+1},St) = d(At,t) + d(t,St) \neq 0$.

$$|d(St,Tz_{2n+1})| \leq \alpha|d(At,Bz_{2n+1})| + \beta\frac{|d(At,St)||d(Bz_{2n+1},Tz_{2n+1})|}{|1+d(At,Bz_{2n+1})|}$$

$$+\gamma\frac{|d(At,St)||d(At,Tz_{2n+1})| + |d(Bz_{2n+1},St)||d(Bz_{2n+1},Tz_{2n+1})|}{|d(At,Tz_{2n+1}) + d(Bz_{2n+1},St)|}$$

Letting $n \to \infty$, we have

$$|d(St,t)| \leq \alpha|d(St,t)| + \beta\frac{|d(At,At)||d(t,t)|}{|1+d(At,t)|}$$

$$+\gamma\frac{|d(At,At)||d(At,t)| + |d(t,St)||d(t,t)|}{|d(At,t)| + |d(t,St)|}$$

$\Rightarrow (1 \quad \alpha)|d(St,t)| \leq 0$, yields $St = t = At = Tt = Sv$.

Since, $S(X) \subseteq B(X)$, there exists a point w in X such that $t = St = Bw$.

Now, from (3.1.2), we have

$$d(t,Tw) = d(St,Tw)$$

$$\alpha d(At,Bw) + \beta \frac{d(At,St)d(Bw,Tw)}{1 + d(At,Bt\ \ )}$$

$$+\gamma \frac{d(At,St)d(At,Tw) + d(Bw,St)d(Bw,Tr)}{d(At,Tw) + d(Bw,St)}$$

$$|d(t,Tw)| \leq \alpha|d(t,t)| + \beta \frac{|d(At,t)||d(t,Tw)|}{|1 + d(At,t)|}$$

$$+\gamma \frac{|d(At,t)||d(At,Tw)| + |d(t,t)||d(t,Tw)|}{|d(At,Tw)| + |d(t,St)|}$$

$$|d(t,Tw)| \leq \alpha|d(t,t)| + \beta \frac{|d(t,t)||d(t,Tw)|}{|1 + d(t,t)|}$$

$$+\gamma \frac{|d(t,t)||d(t,Tw)| + |d(t,t)||d(t,Tw)|}{|d(t,Tw)| + |d(t,St)|}$$

$\Rightarrow |d(t,Tw)| \leq 0$, yields $t = Tw$. Hence, $t = St = Bw = Tw = Tt = At$.

Since T and B are compatible on X and $Tw = Bw = t$ then by proposition (2.18), $d(BTw,TBw) = 0$.

This implies $Bt = BTw = TBw = Tt$. Hence, $St = Tt = At = Bt = t$.

Therefore, t is a common fixed point of A, B, S and T.

Now for the uniqueness of t, suppose that $t \ \ \neq t$ be another common fixed point of A, B, S and T.

Then, from (3.1.2), we have

$$d(t,t\ \ ) = d(St,Tt\ \ ) \quad \alpha d(At,Bt\ \ ) + \beta \frac{d(At,St)d(Bt\ \ ,Tt\ \ )}{1 + d(At,Bw)}$$

$$+\gamma \frac{d(At,St)d(At,Tt\ \ ) + d(Bt\ \ ,St)d(Bt\ \ ,Tt\ \ )}{d(At,Tt\ \ ) + d(Bt\ \ ,St)}$$

$$|d(t,t\ \ )| \leq \alpha|d(t,t\ \ )| + \beta\frac{|d(t,t)||d(t\ \ ,t\ \ )|}{|1+d(t,t\ \ )|}$$

$$+\gamma\frac{|d(t,t)||d(t,t\ \ )| + |d(t\ \ ,t)||d(t\ \ ,t\ \ )|}{|d(t,t\ \ )| + |d(t\ \ ,t)|}$$

$\Rightarrow (1 \quad \alpha)|d(t,t\ \ )| \leq 0$ which is contradiction. Hence $t = t\ \ $.

i.e. t is the unique common fixed point of A, B, S and T.

On setting $A = B = I$ and $\gamma = 0$ in the inequality (3.1.2), we have the following results (Theorem 4 of [1]) as a corollary

Corollary 3.2: Let (X, d) be a complete complex valued metric space and $S, T: X \to X$ be mapping satisfying

$$d(Sz\ ,Tw) \qquad \alpha d(z,w) + \beta\ \frac{d(z,Sz)\ d(w,Tw)}{1 + d(z,w)}$$

for all $z, w \in X$, where $\alpha\ ,\beta$ are non negative real numbers with $\alpha + \beta < 1$. then S, T have a unique common fixed point in X.

On setting $A = B = I$ and $\alpha = \beta = 0$ in the inequality (3.1.2), we have the following result.

Corollary 3.3: Let (X, d) be a complete complex valued metric space and let $S, T: X \to X$ be mappings satisfying:

$$d(Sz\ ,Tw) \qquad \gamma\ \frac{d(z,Sz)\ d(z,Tw) + d(z,Sw)\ d(w,Tw)}{d(z,Tw) + d(w,Sz)}$$

for all $z, w \in X$, where γ non negative real number with $\gamma \in [0,1)$. Then S, T have a unique common fixed point in X.

Example 3.1: Let $X = [0,1]$ and $d: X \times X \to \qquad$ defined by

$$d(z,w) = \ |z \quad w| + i|z \quad w|\ \ \forall\ z\ ,w \in X.$$

Then (X, d) be a complete complex valued metric space. Now, we define the self mappings $A, B, S, T: X \to X$ by

$$Sz = \frac{z}{2},\ Bz = z \text{ and } Tz = \frac{z}{4},\ Az = \frac{2z}{3} \text{ for all } z, w \in X.$$

$$S(X) = \left[0, \frac{1}{2}\right] \subseteq [0,1] = B(X),\ T(X) = \left[0, \frac{1}{4}\right] \subseteq \left[0, \frac{2}{3}\right] = A(X).$$

By the definition of self mappings we get condition (3.1.1) of the Theorem 3.1.

Now consider

$$d(Sz, Tw) = |Sz \quad Tw|(1+i) = \left|\frac{z}{2} \quad \frac{w}{4}\right|(1+i)$$

$$d(Az, Bw) = |Az \quad Bw|(1+i) = \left|\frac{2z}{3} \quad w\right|(1+i)$$

$$d(Az, Sz) = |Az \quad Sz|(1+i) = \left|\frac{2z}{3} \quad \frac{z}{2}\right|(1+i)$$

$$d(Bw, Tw) = |Bw \quad Tw|(1+i) = \left|w \quad \frac{w}{4}\right|(1+i)$$

$$d(Az, Tw) = |Az \quad Tw|(1+i) = \left|\frac{2z}{5} \quad \frac{w}{4}\right|(1+i)$$

$$d(Bw, Sz) = |Bw \quad Sz|(1+i) = \left|w \quad \frac{z}{2}\right|(1+i)$$

For the verification of inequality (3.1.2), it sufficient to show that $d(Sz, Tw) \quad \alpha d(Az, Bw)$

At $z = 0$ and $w = 0$ the result is obvious.

At $z = 0$ and $w = 1$, $d(Sz, Tw) = \frac{1}{4} = 0.25$ and $d(Az, Bw) = 1$.

At $z = 1$ and $w = 0$, $d(Sz, Tw) = \frac{1}{2} = 0.5$ and $d(Az, Bw) = \frac{2}{3} = 0.66$.

At $z = 1$ and $w = 1$, $d(Sz, Tw) = \frac{1}{4} = 0.25$ and $d(Az, Bw) = \frac{1}{3} = 0.33$.

Hence, the inequality

$$d(Sz, Tw) \quad \alpha d(Az, Bw) + \beta \frac{d(Az, Sz)d(Bw, Tw)}{1 + d(Az, Bw)}$$

$$+\gamma \frac{d(Az, Sz)d(Az, Tw) + d(Bw, Sz)d(Bw, Tw)}{d(Az, Tw) + d(Bw, Sz)}$$

holds good for all z, w in X where $\alpha = \frac{3}{4}, \beta = \frac{9}{100}$ and $\gamma = \frac{6}{100}$. i.e. $\alpha + \beta + \gamma < 1$.

Since the commutativity of pairs (S, A) and (T, B) yields the compatibility of (S, A) and weak compatibility of (T, B).

4. Urysohn Integral Equations

In this section, we applied our main result (Theorem 3.1) to the existence and uniqueness of a common solution of the system of the Urysohn's integral equations.

$$z(t) = \psi_i(t) + \int_a^b K_i(t, s, z(s))ds \tag{4.1}$$

where $i = 1, 2, 3, 4, a, b \in$ with $a \leq b, t \in [a, b],\ z, \psi_i \in C([a, b],\ ^n)$ and $K_i\colon [a, b] \times [a, b] \times\ ^n \to\ ^n$ is a given mapping for each $i = 1, 2, 3, 4$.

Throughout this section, for each $i = 1, 2, 3, 4$ and K_i in equation (4.1) we make use the following symbols

$$\delta_i\big(z(t)\big) = \int_a^b K_i(t, s, z(s))ds$$

Theorem 4.1: Consider the Urysohn's integral equation (4.1). Assume the following conditions hold for each $t \in [a, b]$:

(i) $\alpha + \beta + \gamma < 1$,

(ii) $\delta_1 z(t) + \psi_1(t) + \psi_4(t) \quad \delta_4[\delta_1 z(t) + \psi_1(t) + \psi_4(t)] = 0$ and

$\delta_2 z(t) + \psi_2(t) + \psi_3(t) \quad \delta_3[\delta_2 z(t) + \psi_2(t) + \psi_3(t)] = 0$

(iii) $\delta_1\big(\delta_3 z(t) + \psi_3(t)\big) + \psi_1(t) \quad \big[\delta_3\big(\delta_1 z(t) + \psi_1(t)\big) + \psi_3(t)\big] = 0$ and
$\delta_2\big(\delta_4 z(t) + \psi_4(t)\big) + \psi_2(t) \quad \big[\delta_4\big(\delta_2 z(t) + \psi_2(t)\big) + \psi_4(t)\big] = 0$

(iv) $\psi_1(t) + 3\psi_3(t) + \delta_1\big(2z(t) \quad \psi_1(t)\big) + 2\delta_3 z(t)$
$+\delta_3\big(2z(t) \quad \delta_3 z(t) \quad \psi_3(t)\big) = 4z(t)$ and

$\psi_2(t) + 3\psi_4(t) + \delta_2[\delta_2 z(t) + \psi_2(t)] + 2\delta_4 z(t)$
$+ \delta_4[2z(t) \quad \delta_4 z(t) \quad \psi_4(t)] = 4z(t)$

(v) $2z(t) \quad \delta_3 z(t) \quad \psi_3(t) \quad \delta_2 w(t) \quad \psi_2(t) \neq 0$ and
$2w(t) \quad \delta_4 w(t) \quad \psi_4(t) \quad \delta_1 w(t) \quad \psi_1(t) \neq 0$

where,

$$K_{zw}(t) = \|\delta_1 z(t) + \psi_1(t) \quad \delta_2 w(t) \quad \psi_2(t)\|_\infty \sqrt{1+a^2}\, e^{i\tan^{-1} a}\ ;$$

$$L_{zw}(t) = \|2z(t) \quad \delta_3 z(t) \quad \psi_3(t) \quad 2w(t) + \delta_4 w(t) + \psi_4(t)\|_\infty \sqrt{1+a^2}\, e^{i\tan^{-1} a}$$

$$M_{zw}(t) =$$

$$\frac{[\|2z(t)-\delta_3 z(t)-\psi_3(t)-\delta_1 z(t)-\psi_1(t)\|_\infty \|2w(t)-\delta_4 w(t)-\psi_4(t)-\delta_3 w(t)-\psi_2(t)\|_\infty]\sqrt{1+a^2}\, e^{i\, tan^{-1} a}}{1+\|2z(t)-\delta_3 z(t)-\psi_3(t)-2w(t)+\delta_4 w(t)+\psi_4(t)\|_\infty \sqrt{1+a^2}\, e^{i\, tan^{-1} a}}$$

$$N_{zw}(t) =$$

$$\frac{\begin{bmatrix} \|2z(t)-\delta_3 z(t)-\psi_3(t)-\delta_1 z(t)-\psi_1(t)\|_\infty \|2z(t)-\delta_3 z(t)-\psi_3(t)-\delta_2 w(t)-\psi_2(t)\|_\infty \\ +\|2w(t)-\delta_4 w(t)-\psi_4(t)-\delta_1 w(t)-\psi_1(t)\|_\infty \|2w(t)-\delta_4 w(t)-\psi_4(t)-\delta_2 w(t)-\psi_2(t)\|_\infty \end{bmatrix}\sqrt{1+a^2}\, e^{i\, tan^{-1} a}}{[\|2z(t)-\delta_3 z(t)-\psi_3(t)-\delta_2 w(t)-\psi_2(t)\|_\infty + \|2w(t)-\delta_4 w(t)-\psi_4(t)-\delta_1 w(t)-\psi_1(t)\|_\infty]\sqrt{1+a^2}\, e^{i\, tan^{-1} a}}$$

Then the system of equations (4.1) have a unique common solution.

Proof: Let $X = C([a,b], \quad ^n)$, $a > 0$ and $d: X \times X \to \quad$ be defined by

$$d(z,w) = \max_{t\in[a,b]} \|z(t) \quad w(t)\|_\infty \sqrt{1+a^2}\, e^{i\tan^{-1} a} \tag{4.2}$$

Then (X,d) be a complete complex valued metric space.

Define mappings A, B, S, and $T : X \to X$ by

$$Sz(t) = \delta_1 z(t) + \psi_1(t) = \int_a^b K_1(t,s,z(s))ds + \psi_1(t);$$

$$Tz(t) = \delta_2 z(t) + \psi_2(t) = \int_a^b K_2(t,s,z(s))ds + \psi_2(t)\,;$$

$$Az(t) = 2z(t) \quad \delta_3 z(t) \quad \psi_3(t) = 2z(t) \quad \int_a^b K_3(t,s,z(s))ds + \psi_3(t)\,;$$

$$Bz(t) = 2z(t) \quad \delta_4 z(t) \quad \psi_4(t) = 2z(t) \quad \int_a^b K_4(t,s,z(s))ds + \psi_4(t)\,;$$

Let $z, w \in X$, then we get

$$d(Sz,Tw) = \max_{t\in[a,b]} \|\delta_1 z(t) + \psi_1(t) \quad \delta_2 w(t) \quad \psi_2(t)\|_\infty \sqrt{1+a^2}\, e^{i\tan^{-1} a}$$

$$d(Az,Bw) =$$
$$\max_{t\in[a,b]} \|2z(t) \quad \delta_3 z(t) \quad \psi_3(t) \quad 2w(t) + \delta_4 w(t) + \psi_4(t)\|_\infty \sqrt{1+a^2}\, e^{i\tan^{-1} a}$$

$d(Az, Sz) =$
$\max_{t\in[a,b]} \|2z(t) \quad \delta_3 z(t) \quad \psi_3(t) \quad \delta_1 z(t) \;\; \psi_1(t)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a}$

$d(Az, Tw) =$
$\max_{t\in[a,b]} \|2z(t) \quad \delta_3 z(t) \quad \psi_3(t) \quad \delta_2 w(t) \;\; \psi_2(t)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a}$

$d(Bw, Sz) =$
$\max_{t\in[a,b]} \|2w(t) \quad \delta_4 w(t) \quad \psi_4(t) \quad \delta_1 w(t) \;\; \psi_1(t)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a}$

$d(Bw, Tw) =$
$\max_{t\in[a,b]} \|2w(t) \quad \delta_4 w(t) \quad \psi_4(t) \quad \delta_2 w(t) \;\; \psi_2(t)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a}$ (4.3)

From Theorem (3.1) for $t \in [a, b]$, we have

$$\text{Max}_{t\in[a,b]} K_{zw}(t) \quad \alpha \max_{t\in[a,b]} L_{zw}(t) + \beta \max_{t\in[a,b]} M_{zw}(t) + \gamma \max_{t\in[a,b]} N_{zw}(t)$$

By the above equation we get

$$d(Sz, Tw)\gamma$$
$$\alpha d(Az, Bw) + \beta \frac{d(Az,Sz)d(Bw,Tw)}{1+d(Az,Bw)} + +\gamma \frac{d(Az,Sz)d(Az,Tw)+d(Bw,Sz)d(Bw,Tw)}{d(Az,Tw)+d(Bw,Sz)}$$

Now we shall show that $S(X) \subseteq B(X)$. For this

$$B\big(Sz(t)+\psi_4(t)\big) = 2[Sz(t)+\psi_4(t)] \quad \delta_4[Sz(t)+\psi_4(t)] \quad \psi_4(t)$$
$$= Sz(t) + Sz(t) + \psi_4(t) \quad \delta_4[Sz(t)+\psi_4(t)]$$
$$= Sz(t) + \delta_1 z(t) + \psi_1(t) + \psi_4(t) \quad \delta_4[\delta_1 z(t) + \psi_1(t) + \psi_4(t)]$$

Using the given condition of Theorem (4.1) we get

$$B\big(Sz(t)+\psi_4(t)\big) = Sz(t)$$

which shows that

$$S(X) \subseteq B(X).$$

Similarly we show that $T(X) \subseteq A(X)$.

Now, we shall prove that the pair (S, A) and (T, B) are compatible.

Let $\{z_n\}$ be a sequence such that

$$\lim_{n\to\infty} Sz_n(t) = \lim_{n\to\infty} Az_n(t) = z\,(t) \text{ for some } z(t) \in X,$$

for each $t \in [a, b]$. Then we have,

$$\|SAz_n(t) \quad ASz_n(t)\| = \left\|S\big(2z_n(t) \quad \delta_3 z_n(t) \quad \psi_3(t)\big) \quad A\big(\delta_1 z_n(t) + \psi_1(t)\big)\right\|$$

$$= \big\|\delta_1\big(2z_n(t) \quad \delta_3 z_n(t) \quad \psi_3(t)\big) + \psi_1(t) \quad 2\big(\delta_1 z_n(t) + \psi_1(t)\big) \quad \big(\delta_1 z_n(t) + \psi_1(t)\big) \quad \psi_3(t)\big\|$$

$$= \big\|\delta_1\big(2z(t) \quad \delta_3 z(t) \quad \psi_3(t)\big) + \psi_1(t) \quad 2\big(\delta_1 z(t) + \psi_1(t)\big) \quad \psi_3(t)\big\|$$

$$\|SAz_n(t) \quad ASz_n(t)\| = \big\|\delta_1\big(\delta_3 z(t) + \psi_3(t)\big) + \psi_1(t) \quad \delta_3\big(\delta_1 z(t) + \psi_1(t)\big) \quad \psi_3(t)\big\|\,.$$

From condition (iii) we get $\|SAz_n(t) \quad ASz_n(t)\| = 0$ whenever

$$\lim_{n\to\infty} Sz_n(t) = \lim_{n\to\infty} Az_n(t) = z(t)$$

for some $z(t) \in X$, for each $t \in [a, b]$.

Hence, the pair (S, A) is compatible. Similarly we can show that (T, B) is compatible.

Next we shall prove that the pair (S, A) and (T, B) are weakly compatible.

For each $t \in [a, b]$, we get

$$\|ASz(t) \quad SAz(t)\| = \big\|A\big(\delta_1 z(t) + \psi_1(t)\big) \quad S\big(2z(t) \quad \delta_3 z(t) \quad \psi_3(t)\big)\big\|$$

$$= \big\|2\big(\delta_1 z(t) + \psi_1(t)\big) \quad \delta_3\big(\delta_1 z(t) + \psi_1(t)\big) \quad \psi_3(t) \quad \delta_1\big(2z(t) \quad \delta_3 z(t) \quad \psi_3(t)\big) + \psi_1(t)\big\| \qquad (4.4)$$

If $Sz = Az$ for some $z \in X$, then we have

$$\delta_1 z(t) + \psi_1(t) = 2z(t) \quad \delta_3 z(t) \quad \psi_3(t) \text{ for } t \in [a, b]$$

Therefore, from the (4.4), we get

$$\|ASz(t) \quad SAz(t)\| = \left\|4z(t) \quad 2\delta_3 z(t) \quad 3\psi_3(t) \quad \delta_3\big(2z(t) \quad \delta_3 z(t) \quad \psi_3(t)\big) \quad \delta_1\big(2z(t) \quad \psi_1(t)\big) \quad \psi_1(t)\right\| \text{ for all } t \in [a, b].$$

From condition (iv) $\|ASz(t) \quad SAz(t)\| = 0$, that is $ASz(t) = SAz(t)$ for all $t \in [a, b]$.

Therefore, $ASz = SAz$ whenever $Sz = Az$.

Hence, the pair (S, A) is weakly compatible. Similarly we can show that (T, B) is weakly compatible.

Thus, all the conditions of Theorem (3.1) are satisfied. Therefore there exists a unique common fixed point of A, B, S and T in X and consequently there exist a unique common solution of the system (4.1).

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UPPER LOWER CONTRA β-CONTINUOUS INTUITIONISTIC FUZZY MULTIFUNCTIONS

Abstract: In this paper we introduce the concepts of upper and lower contra α-continuous intuitionistic fuzzy multifunctions from a topological space to an intuitionistic fuzzy topological space and obtain some of its properties and characterization.

Keywords: Intuitionistic Fuzzy Sets, Intiitionistic Fuzzy Topology, Intuitionistic Fuzzy Multifunctions, Lower Contra B-Continuous and Upper Contra β-Continuous Intutionistic Fuzzy Multifunctions.

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1. Introduction

After the introduction of fuzzy sets by Zadeh [65] in 1965 and fuzzy topology by Chang [17], several research studies were conducted on the generalization of the notions of fuzzy sets and fuzzy topology. The concept of intuitionistic fuzzy sets was introduced by Atanassov [2], [3], [4] as a generalization of fuzzy sets. In the last 32 years various concepts of fuzzy mathematics have been extended for intuitionistic fuzzy sets. In 1997 Coker [18] introduced the concept of intuitionistic fuzzy topological spaces as a generalization of fuzzy topological spaces.

In 1999, Ozbakir and Coker [46] introduced the concept intuitionistic fuzzy multifunctions and studied their lower and upper intuitionistic fuzzy semi continuity from a topological space to an intuitionistic fuzzy topological space.

Recently many weak and strong forms of upper and lower semi continuous Intuitionistic fuzzy multifunctions such as Intuitionistic fuzzy lower and upper β-continuous [31] Intuitionistic fuzzy lower and upper quasi continuous [59] Intuitionistic fuzzy lower and upper irrosolute Intuitionistic fuzzy upper and lower β-irresolute [63] have been appeared in the liturecture.

In this present paper we introduce and characterize the concepts of upper and lower contra β-continuous intuitionistic fuzzy multifunctions from a topological space to an intuitionistic fuzzy topological space.

2. Preliminaries

Through out this paper (X, τ) and (Y, Γ) represents a topological space and an intuitionistic fuzzy topological space respectively.

Definition 2.1 [30], [45]: A subset a of a topological space(X, τ) is called:

(a) Semi-open if A ⊂ Cl(Int(A)).

(b) Semi-closed if its complement is semi-open.

(c) β-open if A ⊂ Int(Cl(Int(A))).

(d) β-closed if its complement is α-open.

Remark 2.1 [40]: Every open (resp.closed) set is β-open (resp.β-closed) and every β-open (resp.β-closed) set is semi-open (resp.semi-closed) but not the converse may not be true.

The family of all β-open (resp. semi-open) subsets of a topological space (X, τ) is denoted by βO(X)(resp.SO(X)), similarly for the family of all β-closed (resp.semi-closed,) subsets of topological space (X, τ) is denoted by βC(X)(resp. SC(X)). The intersection of all β-closed (resp. semi-closed) sets of X containing a set A of X is called the β-closure [32](resp. semi-closed) of A. It is denoted by βCl(A) (resp.sCl(A)).The union of all α-open (resp.semi-open) subsets of A of X is called the β-interior [16] (resp. semi-interior) of A. It is denoted by β-Int(A)(resp.sInt(A)). A subset A of X is β-closed(resp.semi-closed) if and only if A ⊃ Cl(Int(Cl(A))), (resp.A ⊃ Int(Cl(A)). A subset N of a topological space (X, τ) is called a β-neighborhood [31] of a point x of X if there exits an β-open set O of X such that $x \in O \subset N$. A is an β-open in X, if and only if it is a β-neighborhood of each of its points. A subset V of X is called an β-neighbourhood of a subset a of X if there exists U ∈ β OX such that $AetU \subset V$. A mapping f from a topological space

(X, τ) to another topological space $(X\ ,\tau\ )$ is said to be β-continuous [31], [32] if the inverse image of every open set of X is β-open in X.

Lemma 2.1 [52]: The following properties hold for a subset A of a topological space(X, τ):

(a) A is β-closed in X ⇔ sInt(Cl(A)) ⊂ A ;

(b) sInt(Cl(A)) = Cl(Int(Cl(A)));

(c) βCl(A) = A ∪Cl(Int(Cl(A))).

Lemma 2.2 [52]: The following properties hold for a subset A of a topological space(X, τ):

(a) A ∈ βO(X),

(b) U ⊂ A ⊂ Int(Cl(U)) for open set U of X.

(c) U ⊂ A ⊂ sCl(U) for some open set U of X.

(d) A ⊂ sCl(Int(A)).

Definition 2.2 [2], [3], [4]: Let Y be a nonempty fixed set. An intuitionistic fuzzy set $\tilde{A}$ in Y is an object having the form $\tilde{A}$ = $\{< x, \mu_{\tilde{A}}(y), \nu_{\tilde{A}}(y) >: y \in Y\}$. where the functions $\mu_{\tilde{A}}(y)$: Y → I and $\nu_{\tilde{A}}(y)$: Y → I denotes the degree of membership (namely $\mu_{\tilde{A}}(y)$) and the degree of non membership (namely $\nu_{\tilde{A}}(y)$) of each elemently $y \in Y$ to the set $\tilde{A}$ respectively, and 0 $\leq \mu_{\tilde{A}}(y) + \nu_{\tilde{A}}(y) \leq$ 1 for each y ∈ Y.

Definition 2.3 [2], [3], [4]: Let Y be a non-empty fixed set. An intuitionistic fuzzy set $\tilde{A}$ in Y is an object having the form $\tilde{A}$ = $\left\{\left(y, \mu_{\tilde{A}}(y), \nu\tilde{A}(y)\right) : y \in Y\right\}$ where the functions $\mu_{\tilde{A}}(y) : Y \rightarrow I$ and $\nu_{\tilde{A}}(y) : Y \rightarrow I$ where I = [0, 1], denotes the degree of membership (namely $\mu_{\tilde{A}}(y)$) and the degree of non membership (namely $\nu_{\tilde{A}}(y)$) of each element $y \in Y$ to the set $\tilde{A}$ respectively, and 0 ≤ $\mu_{\tilde{A}}(y) + \nu_{\tilde{A}}(y) \leq$ 1 for each $y \in$ Y.

Definition 2.4 [2], [3], [4]: Let Y be a non- empty set and the intuitionistic fuzzy sets $\tilde{A}$ and $\tilde{B}$ be in the form $\tilde{A}$ = $\{(y, \mu_{\tilde{A}}, \nu_{\tilde{A}}) : y \in Y\}$, $\tilde{B}$ = $\{(y, \mu_{\tilde{B}}(y)), \nu_{\tilde{B}}(y): y \in Y\}$ and let $\{\tilde{A}_{\beta}: \beta \in\ \}$ be an arbitrary family of intuitionistic fuzzy sets in Y, then :

(a) $\tilde{A} \subseteq \tilde{B}$ if $\forall$ y ∈ Y [$\mu_{\tilde{A}}$(y) ≤ $\mu_{\tilde{B}}$(y) and $\nu_{\tilde{A}}$(y) ≥ $\nu_{\tilde{A}}$(y) ≥ $\nu_{\tilde{B}}$(y)

(b) $\tilde{\boldsymbol{A}} = \tilde{B}$ if $\tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{A}$;

(c) $\tilde{\boldsymbol{A}}^{c}$={ $< x,\ \boldsymbol{\nu_{\tilde{A}}(y), \mu_{\tilde{A}}(y)} > : y \in Y$;

(d) $\tilde{0} = \{< y, 0,1 > : y \in Y\}$ and $\tilde{1} = \{ < y, 1.0 > : y \in Y\}$

(e) $\cap \tilde{A}_{\beta} = \{ < x, \vee\ \mu_{\tilde{A}}(y), \quad \vee\ \nu_{\tilde{A}}(y) > : y \in Y\}$

(f) $\cup \tilde{A}_{\beta} = \{ < x, \vee\ \mu_{\tilde{A}}(y), \quad \wedge\ \nu_{\tilde{A}}(y) > : y \in Y$

Definition: 2.5 [19]: Two intuitionistic fuzzy sets $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{B}}$ of Y are said to be quasi-coincident $\tilde{\boldsymbol{A}}\, \boldsymbol{q} \tilde{\boldsymbol{B}}$ for short) if $\exists\ y \in Y$ such that

$\mu_{\tilde{A}}$ (y) $> \nu_{\tilde{B}}$(y)

$\nu_{\tilde{A}}$(y) $< \mu_{\tilde{B}}$(y).

Lemma 2.3 [19]: For any two intuitionistic fuzzy sets $\tilde{\boldsymbol{A}}$ **and** $\tilde{\boldsymbol{B}}$ of Y,

$$\sim(\tilde{A} q \tilde{B}) \qquad \tilde{A} \subset \tilde{B}^{c}.$$

Definition 2.6 [18]: An intuitionistic fuzzy topology on a non-empty set Y is a family **Γ** of intuitionistic fuzzy sets in Y which satisfy the following axioms:

(a) $\tilde{\boldsymbol{0}}, \tilde{1} \in \Gamma$,

(b) $\tilde{\boldsymbol{A}}_{\boldsymbol{1}} \cap \tilde{\boldsymbol{A}}_{\boldsymbol{2}} \in \boldsymbol{\Gamma}$ for any $\tilde{A}_1, \tilde{A}_2 \in \Gamma$,

(c) $\tilde{A}_{\beta} \in \Gamma$ for arbitrary family $\{\tilde{\boldsymbol{A}}_{\boldsymbol{\beta}} : \boldsymbol{\beta} \in \boldsymbol{\Lambda}\} \in \boldsymbol{\Gamma}$.

In this case the pair **(Y, Γ)** is called an intuitionistic fuzzy topological space and each intuitionistic fuzzy set in Γ, is known as an intuitionistic fuzzy open set in Y.

The complement $\tilde{\boldsymbol{B}}^{\boldsymbol{c}}$ of an intuitionistic fuzzy open set $\tilde{\boldsymbol{B}}$ is called an intuitionistic fuzzy closed set is Y.

Definition 2.7 [19]: Let Y be a non- empty set and c ∈ Y a fixed element in Y, if $\boldsymbol{\alpha} \in (\ \boldsymbol{0}, \boldsymbol{1}]$ and $\boldsymbol{\beta} \in [\boldsymbol{0}, \boldsymbol{1})$ are two real numbers such that $\boldsymbol{\alpha} + \boldsymbol{\beta} < 1$ then,

(a) $\mathbf{c}(\boldsymbol{\alpha},\boldsymbol{\beta}) = (y, c_\alpha, c_{1-\beta})$ is called an intuitionistic fuzzy point (IFP in short) in Y, where α denotes the degree of membership of $\mathbf{c}(\boldsymbol{\alpha},\boldsymbol{\beta})$, and $\boldsymbol{\beta}$ denotes the degree of non membership of **$\mathbf{c}(\boldsymbol{\alpha},\boldsymbol{\beta})$.**

(b) $\mathbf{c}(\boldsymbol{\beta}) = (y, 0,1 \quad c_{1-\beta})$ is called a vanishing intuitionistic fuzzy point (VIFP in short) in Y, where $\boldsymbol{\beta}$ denotes the degree of non membership of **$\mathbf{c}(\boldsymbol{\beta})$.**

Definition 2.8 [18]: Let $(\mathbf{Y},\boldsymbol{\Gamma})$ be an intuitionistic fuzzy topological space and $\widetilde{\boldsymbol{A}}$ be an intuitionistic fuzzy set in Y. Then the interior and closure of $\widetilde{\boldsymbol{A}}$ are defined by:

(a) $\mathbf{cl}(\widetilde{\boldsymbol{A}}) = \quad \{\widetilde{K}: \widetilde{K}$is an intuitionistic fuzzy closed set in Y and $\widetilde{\boldsymbol{A}} \subseteq \widetilde{\boldsymbol{K}}\}$.

(b) $\mathbf{Int}(\widetilde{\boldsymbol{A}}) = \quad \{\widetilde{\boldsymbol{G}}: \widetilde{G}$ is an intuitionistic fuzzy open set in Y and $\widetilde{\boldsymbol{G}} \subseteq \widetilde{\boldsymbol{A}}\}$.

Lemma 2.4 [17]: For any intuitionistic fuzzy set in $\widetilde{\boldsymbol{A}}$**in** $(\mathbf{Y},\boldsymbol{\Gamma})$ we have:

(a) $\tilde{A}$ is an intuitionistic fuzzy closed set in $Y \quad Cl(\tilde{A})=(\tilde{A})$

(b) $\widetilde{\boldsymbol{A}}$ is an intuitionistic fuzzy open set in **Y** $\quad \boldsymbol{Int}(\widetilde{\boldsymbol{A}}) = \tilde{A}$

(c) $\mathbf{Cl}(\widetilde{A}^{c}) = (\boldsymbol{Int}\widetilde{\boldsymbol{A}})^{\wedge}\boldsymbol{c}$

(d) $\mathbf{Int}(\widetilde{A}^{c}) = (\boldsymbol{Cl}\,\widetilde{\boldsymbol{A}})^{\boldsymbol{c}}$

Definition 2.9 [46]: Let X and Y are two non- empty sets. A function $\mathbf{F} : (\mathbf{X},\boldsymbol{\tau}) \to (Y, \Gamma)$ is called intuitionistic fuzzy multifunctions, if $\mathbf{F}(\boldsymbol{x})$ is an intuitionistic fuzzy set in Y, $\forall\ x \in X$.

Definition 2.10 [58]: Let $\mathbf{F} : (\mathbf{X},\boldsymbol{\tau}) \to (\boldsymbol{Y},\boldsymbol{\Gamma})$ is an intuitionistic fuzzy multifunction and A be a subset of X, then $\mathbf{F(A)} = {}_{\boldsymbol{x}\in\boldsymbol{A}}\ \boldsymbol{F}(\boldsymbol{x})$.

Definition 2.11 [58]: Let F: (X, τ) $\to$ (Y, Γ) be an intuitionistic fuzzy multifunction. Then

(a) $\mathbf{A} \subseteq \boldsymbol{B} \Rightarrow \boldsymbol{F}(\boldsymbol{A}) \subseteq \boldsymbol{F}(\boldsymbol{B})$ for any subsets A and B of X.

(b) $\boldsymbol{F}(\boldsymbol{A} \cap \boldsymbol{B}) \subseteq \boldsymbol{F}(\boldsymbol{A}) \cap \boldsymbol{F}(\boldsymbol{B})$ for any subsets A and B of X.

(c) $\mathbf{F}({}_{\boldsymbol{\alpha}\in\boldsymbol{\Lambda}})\boldsymbol{A}_{\boldsymbol{\alpha}} = \quad \{\boldsymbol{F}(\boldsymbol{A}_{\boldsymbol{\alpha}}): \boldsymbol{\alpha}\in\boldsymbol{\Lambda}\}$ for any family of subsets in X. $\{(A_\alpha): \alpha \in \Lambda\}$ in X.

Definition 2.12 [46]: Let $\boldsymbol{F} : (\boldsymbol{X}, \boldsymbol{\tau}) \to (\boldsymbol{Y}, \boldsymbol{\Gamma})$ is an intuitionistic fuzzy multifunction, then the upper inverse $\boldsymbol{F}^{+}(\widetilde{\boldsymbol{A}})$ and lower $\boldsymbol{F}^{-}(\widetilde{\boldsymbol{A}})$ of an intuitionistic fuzzy set $\widetilde{\boldsymbol{A}}$ in Y are defined as follows:

(a) $F^{+}(\widetilde{A}) = \{ x \in X : F(x) \subseteq (\tilde{A}) \}$

(b) $F^{-}(\widetilde{A}) = \{ x \in X : F(x) q \widetilde{A} \}$

Lemma 2.5 [58]: Let $\boldsymbol{F} : (\boldsymbol{X}, \boldsymbol{\tau}) \to (\boldsymbol{Y}, \boldsymbol{\Gamma})$ be an intuitionistic fuzzy multifunction and $\widetilde{\boldsymbol{A}}$, $\widetilde{B}$ be intuitionistic fuzzy sets in Y. Then

(a) $F^{+}(\tilde{1}) = F^{-}(\tilde{1}) = \mathrm{X}$,

(b) $F^{+}(\tilde{A}) \subseteq F^{-}(\tilde{A})$

(c) $[\boldsymbol{F}^{-}(\widetilde{\boldsymbol{A}})]^{\boldsymbol{c}} = [\boldsymbol{F}^{+}(\widetilde{\boldsymbol{A}})^{\boldsymbol{c}}]$

(d) $[\boldsymbol{F}^{+}(\widetilde{\boldsymbol{A}})^{\boldsymbol{c}} = [\boldsymbol{F}^{-}(\widetilde{\boldsymbol{A}})^{\boldsymbol{c}}$

(e) If $\widetilde{\boldsymbol{A}} \subseteq \widetilde{\boldsymbol{B}}$, then $\boldsymbol{F}^{+}(\widetilde{\boldsymbol{A}}) \subseteq \boldsymbol{F}^{-}(\widetilde{\boldsymbol{B}})$

(f) If $\tilde{A} \subseteq \widetilde{B}$, then $F^{-}(\tilde{A}) \subseteq F^{-}(\widetilde{B})$

Definition 2.13 [58]: An Intutionistic fuzzy multifunction $\boldsymbol{F}(\boldsymbol{X}, \boldsymbol{\tau}) \to (\boldsymbol{Y}, \boldsymbol{\Gamma})$ is said to be:

(a) Intuitionistic fuzzy upper semi -continuous at a point $\boldsymbol{x_0} \in \boldsymbol{X}$, if for any intuitionistic fuzzy open set $\widetilde{\boldsymbol{W}} \subset \boldsymbol{Y}$ such that $\boldsymbol{F}(\boldsymbol{x_0}) \subset \widetilde{\boldsymbol{W}}$ there exists an open set $\mathbf{U} \subset \boldsymbol{X}$ containing x_0 such $F(U) \subset \widetilde{W}$.

(b) Intuitionistic fuzzy lower semi continuous at a point $\boldsymbol{x_0} \in \boldsymbol{X}$,if for any intuitionistic fuzzy open set $\widetilde{\boldsymbol{W}} \subset \boldsymbol{Y}$ such that $\boldsymbol{F}(\boldsymbol{x_0}) \boldsymbol{q} \widetilde{\boldsymbol{W}}$ there exists an open set $\boldsymbol{U} \subset \boldsymbol{X}$ containing x_0 such that $\boldsymbol{F}(\boldsymbol{x}) \boldsymbol{q} \widetilde{\boldsymbol{W}}, \ \forall\, \boldsymbol{x} \in \boldsymbol{X}$.

(c) Intuitionistic fuzzy upper semi-continuous (intuitionistic fuzzy lower semi-continuous) if it is intuitionistic fuzzy upper semi-continuous (Intuitionistic fuzzy lower semi-continuous) at each point of X.

Definition 2.14 [12]: An Intuitionisic fuzzy multifunction $\mathbf{F}\ (\boldsymbol{X}, \boldsymbol{\tau}) \to (\boldsymbol{Y}, \boldsymbol{\Gamma})$ is said to be:

(a) Intuitionistic fuzzy upper $\boldsymbol{\beta}$-continuious at a point $\boldsymbol{x_0} \in \boldsymbol{X}$, if for any intuitionistic fuzzy open set $\widetilde{\boldsymbol{W}} \subset \boldsymbol{Y}$ such that $\boldsymbol{F(x_0)} \subset \widetilde{\boldsymbol{W}}$ there exists $\mathbf{U} \in \boldsymbol{\beta}$O(X) containing $\boldsymbol{x_0}$ such that $\mathbf{F(U)} \subseteq \widetilde{\boldsymbol{W}}$.

(b) Intuitionistic fuzzy lower $\boldsymbol{\beta}$-continuous at a point $x_0 \in X$, if for any intuitionistic fuzzy open set $\widetilde{W} \subset Y$ such that $\boldsymbol{F(x_0) q \widetilde{W}}$ there exists $\mathbf{U} \in \boldsymbol{\beta}$O(X) containing $\boldsymbol{x_0}$ such that F$(x) q \widetilde{W}$, $\forall\, x \in X$.

(c) Intuitionistic fuzzy upper $\boldsymbol{\beta}$-continuous (resp. Intuitionistic fuzzy lower β-continuous) if it is intuitionistic fuzzy upper $\boldsymbol{\beta}$-continuous (resp. intuitionistic fuzzy lower β-continuous) at every point of X.

Remark 2.2 [12]: Every intuitionistic fuzzy lower semi-continuous (resp. intuitionistic upper semi continuous) multifunction is intuitionistic fuzzy lower $\boldsymbol{\beta}$-continuous (resp.intuitionistic fuzzy upper β-continuous) but the converse may not be true.

Definition 2.15 [9]: An Intuitionistic fuzzy multifunction F : (X,τ) $\rightarrow (Y, \Gamma)$ is said to be:

(a) Intutionistic fuzzy upper contra continuous at a point $x_0 \in X$, if for any intuitionistic fuzzy closed set $\widetilde{\boldsymbol{W}} \subset Y$ of Y such that F $(x_0) \subset \widetilde{W}$ there exist an open set Uof X containing $\boldsymbol{x_0}$such that F(U)$\subset \widetilde{W}$.

(b) Intuitionistic fuzzy lower contra continuous at a point $\boldsymbol{x_0} \in \boldsymbol{X}$, if for any intuitionistic fuzzy closed set $\widetilde{\boldsymbol{W}} \subset \boldsymbol{Y}$ of Y such that $\boldsymbol{F(x_0) q \widetilde{W}}$ there exist an open set U of X containing $\boldsymbol{x_0}$ such that F$(x) q \widetilde{W}$, $\forall\, x \in U$.

(c) Intuitionistic fuzzy upper contra continuous (intuitionistic fuzzy lower contra continuous) if it is intuitionistic fuzzy upper contra β-continuous (Intuitionistic fuzzy lower contra $\boldsymbol{\beta}$-continuous)at each point of X.

3. Upper(lower) Contra β-continuous Intutionistic Fuzzy Multifunctions

Definition 3.1 [10]: An Intuitionistic fuzzy multifunction F : (X, τ) → (Y, $\boldsymbol{\Gamma}$) is said to be:

(a) Intutionistic fuzzy upper contra α-continuous at a point $\boldsymbol{x_0} \in$ X, if for any intuitionistic fuzzy closed set $\widetilde{\boldsymbol{W}} \subset \boldsymbol{Y}$ of Y such that F$(x_0) \subset \widetilde{W}$ there exist an β-open set U of X containing $\boldsymbol{x_0}$ such that F(U) $\subset \widetilde{W}$.

(b) Intuitionistic fuzzy lower contra $\boldsymbol{\beta}$-continuous at a point $\boldsymbol{x_0} \in \boldsymbol{X}$, if for any intuitionistic fuzzy closed set $\widetilde{\boldsymbol{W}}$ of Y such that $\boldsymbol{F(x_0)q\widetilde{W}}$ there exist an α-open set U of X containing $\boldsymbol{x_0}$ such that $\mathrm{F}(x)q\widetilde{W},\ \forall x \in U$.

(c) Intuitionistic fuzzy upper contra $\boldsymbol{\beta}$-continuous (intuitionistic fuzzy lower contra β-continuous) if it is intuitionistic fuzzy upper contra β-continuous (Intuitionistic fuzzy lower contra β-continuous) at each point of X.

Remark 3.1 [12]: Every intuitionistic fuzzy lower (resp.upper) contra continuous multifunction is intuitionistic fuzzy lower (resp.upper) contra β-continuous. But the converse may be not true.

The concepts of intuitionistic fuzzy lower (resp. upper) β-continuous and intuitionistic fuzzy lower (upper) contra $\boldsymbol{\beta}$-continuous multifunctions are independent.

Theorem 3.1: Let $\boldsymbol{F}: (\boldsymbol{X}, \boldsymbol{\tau}) \to (\boldsymbol{Y}, \boldsymbol{\Gamma})$, be an intuitionistic fuzzy multifunction then following conditions are equivalent:

(a) F is intuitionistic fuzzy upper contra $\boldsymbol{\beta}$-continuous.

(b) For each point $x \in X$ and any intuitionistic fuzzy closed set $\widetilde{\boldsymbol{B}}$ of Y, such that $F(\,x) \subseteq \tilde{B}$, $\exists$ an $\boldsymbol{\beta}$-neighborhood U of x such that $\boldsymbol{U} \subseteq \boldsymbol{F}^{+}(\widetilde{\boldsymbol{B}})$.

(c) $\boldsymbol{F}^{+}(\widetilde{\boldsymbol{B}})$ is an $\boldsymbol{\beta}$-open set in X for every intuitionistic fuzzy closed set $\widetilde{\boldsymbol{B}}$ of Y.

(d) $\boldsymbol{F}^{-}(\widetilde{\boldsymbol{B}})$ is an $\boldsymbol{\beta}$-closed set in X for every intuitionistic fuzzy open set $\widetilde{\boldsymbol{B}}$ in Y.

Proof: (a) (b): Obvious.

(b) (c): Let $\widetilde{\boldsymbol{B}}$ be any intuitionistic fuzzy closed set of Y and let $\boldsymbol{x} \in \boldsymbol{F}^{+}\,(\widetilde{\boldsymbol{B}}.)$ Then $\boldsymbol{F(x)} \subset \widetilde{\boldsymbol{B}}$ and so by (b) an $\boldsymbol{\beta}$-neighboubhood U of $\boldsymbol{x}$ such that $\mathrm{U} \subset F^{+}\,(\,\tilde{B})$. It follows that $F^{+}(\tilde{B})$ is the union of $\boldsymbol{\beta}$ open sets of X is $\boldsymbol{\beta}$-open in X.

(c) (b): Let $\boldsymbol{x} \in$X and $\widetilde{\boldsymbol{B}}$ be an intuitionistic fuzzy closed set of Y such that $F^{+}(\tilde{B})$. Then U $= F^{+}(\tilde{B})$ is an $\boldsymbol{\beta}$-neighborhood of x such that $\boldsymbol{U} \subseteq \boldsymbol{F}^{+}(\widetilde{\boldsymbol{B}})$. Hence, F is intuitionistic fuzzy upper contra β-continuous.

(c) (d): It follows from the fact that $[F^{+}(\tilde{B})]^{c} = [\boldsymbol{F}^{-}(\widetilde{\boldsymbol{B}})]^{\boldsymbol{c}}$.

Definition 3.2: The kernel of an intuitionistic fuzzy set $\widetilde{\boldsymbol{B}}$ in intuitionistic fuzzy topological space $(\boldsymbol{Y}, \boldsymbol{\Gamma})$ given by $\mathrm{Ker}(\tilde{B}) = \cap \{\widetilde{A} : \tilde{A} \in \Gamma \text{ and } \tilde{B} \subseteq \tilde{A}\}$

Lemma 3.1 [11]: For an intuitionistic fuzzy set $\widetilde{\boldsymbol{B}}$ in an intuitionistic fuzzy topological space $(\boldsymbol{Y}, \boldsymbol{\Gamma})$, if $\widetilde{\boldsymbol{B}} \in \boldsymbol{\Gamma}$, then $\widetilde{\boldsymbol{B}} = \mathbf{Ker}(\widetilde{\boldsymbol{B}})$

Theorem 3.2: Let $F:(\boldsymbol{X}, \boldsymbol{\tau}) \to (\boldsymbol{Y}, \boldsymbol{\Gamma})$ be an intuitionistic fuzzy multifunction, if

$$\boldsymbol{\beta Cl}(\boldsymbol{F}^{-}(\widetilde{\boldsymbol{B}}) \subseteq \boldsymbol{F}^{-}(\boldsymbol{Ker}(\widetilde{\boldsymbol{B}}))$$

for any intuitionistic fuzzy set $\tilde{B}$ of Y, then F is intuitionistic fuzzy upper contra $\boldsymbol{\beta}$-continuous multifunction.

Proof: Suppose that $\boldsymbol{\beta Cl}\left(\boldsymbol{F}^{-}(\widetilde{\boldsymbol{B}})\right) \subseteq \boldsymbol{F}^{-}(\boldsymbol{Ker}(\widetilde{\boldsymbol{B}}))$ for any intuitionistic fuzzy set $\widetilde{\boldsymbol{B}}$ of Y. Let $\widetilde{\boldsymbol{A}} \in \boldsymbol{\Gamma}$, then hypothesis and lemma 3.1 $\boldsymbol{\beta Cl}\left(\boldsymbol{F}^{-}(\widetilde{\boldsymbol{A}})\right) \subseteq \boldsymbol{F}^{-}(\boldsymbol{Ker}(\widetilde{\boldsymbol{A}}) = F^{-}(\tilde{A})$. This implies that $\boldsymbol{\beta Cl}\left(\boldsymbol{F}^{-}(\widetilde{\boldsymbol{A}})\right) \subseteq \boldsymbol{F}^{-}(\widetilde{\boldsymbol{A}})$, but we know $\boldsymbol{F}^{-}(\widetilde{\boldsymbol{A}}) \subseteq \boldsymbol{\beta Cl}\,(\boldsymbol{F}^{-}(\widetilde{\boldsymbol{A}}))$, Hence, $\boldsymbol{F}^{-}(\widetilde{\boldsymbol{A}})$ is $\boldsymbol{\beta}$-closed set in X. Thus, by Theorem 3.1, F is intuitionistic fuzzy upper contra $\boldsymbol{\beta}$-continuous.

Theorem 3.3: Let F be an intuitionistic multifunction $\boldsymbol{F}: (\boldsymbol{X}, \boldsymbol{\tau}) \to (\boldsymbol{Y}, \boldsymbol{\Gamma})$, then following conditions are equivalent:

(a) F is intuitionistic fuzzy lower contra $\boldsymbol{\beta}$-continuous.

(b) For any intuitiontistic fuzzy closed set $\widetilde{\boldsymbol{B}}$ of Y such that $\boldsymbol{F}(\boldsymbol{x})\boldsymbol{q}\widetilde{\boldsymbol{B}}$, $\exists$ an $\boldsymbol{\beta}$-neighbourhood U of x such that $U \subseteq F^{-}(\tilde{B})$.

(c) $\boldsymbol{F}^{-}(\widetilde{\boldsymbol{B}})$ is $\boldsymbol{\beta}$-open in X for every intuitiontistic fuzzy closed set $\widetilde{\boldsymbol{B}}$ of Y.

(d) $\boldsymbol{F}^{+}(\widetilde{\boldsymbol{B}})$ is an $\boldsymbol{\beta}$-closed in X for every intuitionistic fuzzy open set $\tilde{B}$ in Y.

Proof: (a) (b): Obvious.

(a) (c): Let $\widetilde{\boldsymbol{B}}$ be any intuitiontistic fuzzy closed set of Y and let $x \in F^{+}(\tilde{B})$. Then $\boldsymbol{F}(\boldsymbol{x}) \subset \widetilde{\boldsymbol{B}}$ and so an β-neighborhood U of $\boldsymbol{x}$ such that $U \subset F^{-}(\tilde{B})$. It follows that $\boldsymbol{F}^{-}(\widetilde{\boldsymbol{B}})$ is the union of β-open sets of X is $\boldsymbol{\beta}$-open in X.

(c) (a): Let $x \in X$ and $\widetilde{\boldsymbol{B}}$ be an intuitionistic fuzzy closed set of Y such that $\boldsymbol{F}^{-}(\widetilde{\boldsymbol{B}})$ is $\boldsymbol{\beta}$-open in X. Then $U = F^{-}(\tilde{B})$is a$\boldsymbol{\beta}$-neighbourhood of x such that $\mathbf{U} \subset \boldsymbol{F}^{-}(\widetilde{\boldsymbol{B}})$.

Hence, F is intuitionistic fuzzy lower contra $\boldsymbol{\beta}$-continuous.

$(\boldsymbol{c})$ **(d):** It follows from the fact that $\left[\boldsymbol{F}^{+}(\widetilde{\boldsymbol{B}})\right]^{\boldsymbol{c}} = [\boldsymbol{F}^{-}(\widetilde{\boldsymbol{B}})^{\boldsymbol{c}}$.

Theorem 3.4: Let $\boldsymbol{F} : (\boldsymbol{X}, \boldsymbol{\tau}) \to (\boldsymbol{Y}, \boldsymbol{\Gamma})$ be an intuitiontistic fuzzy multifunction. If $\boldsymbol{\beta Cl}\left(\boldsymbol{F}^{+}(\widetilde{\boldsymbol{B}})\right) \subseteq \boldsymbol{F}^{+}\left(\boldsymbol{Ker}(\widetilde{\boldsymbol{B}})\right)$ for any intuitiontistic fuzzy set $\tilde{B}$ of Y, then F is intuitiontistic fuzzy lower contra $\boldsymbol{\beta}$-continuous multifunction.

Proof: Suppose that $\boldsymbol{\beta Cl}\left(\boldsymbol{F}^{+}(\widetilde{\boldsymbol{B}}) \subseteq \boldsymbol{F}^{+}\boldsymbol{Ker}(\widetilde{\boldsymbol{B}})\right)$ for any intuitionistic fuzzy set $\widetilde{\boldsymbol{B}}$ of Y. Let $\widetilde{\boldsymbol{A}} \in \boldsymbol{\Gamma}$, by lemma 3.1 $\boldsymbol{\beta Cl}\left(\boldsymbol{F}^{+}(\widetilde{\boldsymbol{A}})\right) \subseteq \boldsymbol{F}^{+}(\boldsymbol{Ker}(\widetilde{\boldsymbol{A}}) = \boldsymbol{F}^{+}(\widetilde{\boldsymbol{A}})$.

This implies that $\boldsymbol{\beta Cl}\left(\boldsymbol{F}^{+}(\widetilde{\boldsymbol{A}})\right) \subseteq \boldsymbol{F}^{+}(\widetilde{\boldsymbol{A}})$.

But we know $\boldsymbol{F}^{+}(\widetilde{\boldsymbol{A}}) \subseteq \boldsymbol{\beta Cl}(\boldsymbol{F}^{-}(\widetilde{\boldsymbol{A}}))$, Hence, $\boldsymbol{F}^{+}(\widetilde{\boldsymbol{A}})$ is$\boldsymbol{\beta}$-closed set in X. Thus, by Theorem 3.2, F is an intuitionistic fuzzy lower contra $\boldsymbol{\beta}$-continuous.

Definition 3.2: Given a family $\{\boldsymbol{F}_{\boldsymbol{\beta}}: (\boldsymbol{X}, \boldsymbol{\tau}) \to (\boldsymbol{Y}, \boldsymbol{\Gamma}): \boldsymbol{\beta} \in \boldsymbol{\Lambda}\}$, of intuitionistic fuzzy multifunction, we define the union $_{\alpha\epsilon\Lambda} F_{\alpha}$ and intersection $_{\boldsymbol{\alpha\epsilon\Lambda}} \boldsymbol{F}_{\boldsymbol{\alpha}}$ as,

(a) $_{\beta\epsilon\Lambda} F_{\beta}: (X, \tau) \to (Y, \Gamma), (\ _{\beta\epsilon\Lambda} F_{\beta})(x) = \ _{\beta\epsilon\Lambda} F_{\beta}(x)$.

(b) $_{\beta\epsilon\Lambda} F_{\beta}: (X, \tau) \to (Y, \Gamma), (\ _{\beta\epsilon\Lambda} F_{\beta})(x) = \ _{\beta\epsilon\Lambda} F_{\beta})(x)$.

Theorem 3.5: If $\boldsymbol{F}_{\boldsymbol{\beta}} : (\boldsymbol{X}, \boldsymbol{\tau}) \to (\boldsymbol{Y}, \boldsymbol{\Gamma})$, for $\beta \in \lambda$ $\boldsymbol{\beta} =$ 1,2,3,...n is an intuitionistic fuzzy upper contra $\boldsymbol{\beta}$-continuous then $_{\boldsymbol{\beta\epsilon\Lambda}} \boldsymbol{F}_{\boldsymbol{\beta}}$ is intuitionistic fuzzy upper $\boldsymbol{\beta}$-continuous.

Proof: Let $\tilde{B}$ be an intuitionistic fuzzy closed set in Y. To show that $\left(\cup_{\boldsymbol{\beta=1}}^{\boldsymbol{n}} \boldsymbol{F}_{\boldsymbol{\beta}}\right)^{+}(\widetilde{\boldsymbol{B}}) = \{\boldsymbol{x} \in \boldsymbol{X} : \cup_{\boldsymbol{\beta=1}}^{\boldsymbol{n}} \boldsymbol{F}_{\boldsymbol{\beta}}(\boldsymbol{x}) \subseteq \widetilde{\boldsymbol{B}}\ \}$ ***is*** $\boldsymbol{\beta}$-open in X. Let $x \in \left(\cup_{\beta=1}^{n} F_{\beta}\right)^{+}(\tilde{B})$ then $\boldsymbol{F}_{\boldsymbol{\beta}}(\boldsymbol{x}) \subseteq (\widetilde{\boldsymbol{B}})$ for $\boldsymbol{\beta}$ = 1,2 ,3...n , Since $\boldsymbol{F}_{\boldsymbol{\beta}}: (\boldsymbol{X}, \boldsymbol{\tau}) \to (\boldsymbol{Y}, \boldsymbol{\Gamma})$ is an intuitionistic fuzzy upper contra $\boldsymbol{\beta}$-continuous multifunction, for $\boldsymbol{\beta}$ = 1,2,3,..., then ∃ β-open set U containing x such that $\forall \boldsymbol{y} \in \boldsymbol{U}_{\boldsymbol{x}}, \boldsymbol{F}_{\boldsymbol{\beta}}(\boldsymbol{y}) \subseteq \widetilde{\boldsymbol{B}}$. let U$=\cup_{\boldsymbol{\beta=1}}^{\boldsymbol{n}} \boldsymbol{U}$, then $\boldsymbol{U} \subset \left(\cup_{\boldsymbol{\beta=1}}^{\boldsymbol{n}} \boldsymbol{F}_{\boldsymbol{\beta}}\right)^{+}(\widetilde{\boldsymbol{B}})$ **.** Therefore $\left(\cup_{\boldsymbol{\beta=1}}^{\boldsymbol{n}} \boldsymbol{F}_{\boldsymbol{\beta}}\right)^{+}(\widetilde{\boldsymbol{B}})$ is β-open.

Hence, $_{\boldsymbol{\beta\epsilon\Lambda}} \boldsymbol{F}_{\boldsymbol{\beta}}$is an intuitionistic fuzzy upper contra $\boldsymbol{\beta}$-continuous.

Theorem 3.6: If $\boldsymbol{F}_{\boldsymbol{\beta}} : (\boldsymbol{X}, \boldsymbol{\tau}) \to (\boldsymbol{Y}, \boldsymbol{\Gamma})$, for $\boldsymbol{\beta}$ = 1, 2, 3,...n is an intuitionistic fuzzy upper contra $\boldsymbol{\beta}$-continuous then $_{\boldsymbol{\beta\epsilon\Lambda}} \boldsymbol{F}_{\boldsymbol{\beta}}$ is an intuitionistic fuzzy lower $\boldsymbol{\beta}$-continuous.

Proof: Let $\widetilde{\boldsymbol{B}}$ be an intuitionistic fuzzy closed set in Y. To show that $\left(\cup_{\beta=1}^{n} F_{\beta}\right)(\widetilde{B})^{-} = \{x \in X : \cup_{\beta=1}^{n} F_{\beta}(x) q \widetilde{B}\}$ is $\boldsymbol{\alpha}$-open in X. Let $x \in \left(\cup_{\beta=1}^{n} F_{\beta}\right)^{-}(\widetilde{B})$ then $F_{\beta}(x) q(\widetilde{B})$ for $\beta = 1,2,3...n$. Since $F_{\beta}: (X, \tau) \rightarrow (Y, \Gamma)$ is an intuitionistic fuzzy lower contra β-continuous multifunction, for $\beta = 1,2,3,...$, then $\exists \boldsymbol{\beta}$-open set U containing x such that $\forall y \in U_x, F_{\beta}(y) q \widetilde{B}$. let $\mathbf{U} = \cup_{\beta=1}^{n} F_{\beta}\mathbf{U}$, then $\mathrm{U} \subset \left(\cup_{\beta=1}^{n} F_{\beta}\right)^{-}(\tilde{B})$. Therefore $\left(\cup_{\beta=1}^{n} F_{\beta}\right)^{-}(\widetilde{B})$is β-open. Hence, $_{\beta \in \lambda} F_{\beta}$ is an intuitionistic fuzzy lower contra $\boldsymbol{\beta}$-continuous.

Theorem 3.7: Let $\{U_{\beta} : \beta \epsilon \Lambda\}$ be an β-open cover of a topological space (X, τ). An intuitionistic fuzzy mulifuntion $F: (X, \tau) \rightarrow (Y, \Gamma)$ is an intuitionistic fuzzy upper contra $\boldsymbol{\beta}$-continuous if and only if restriction $F \backslash U_{\beta}: U_{\beta} \rightarrow Y$ is an intuitionistic fuzzy upper contra $\boldsymbol{\beta}$-continuous for each $\beta \in \Lambda$.

Proof: Necessity: Suppose that F is an intuitionistic fuzzy upper $\boldsymbol{\beta}$-continuous. Let $\beta \epsilon \Lambda$, $x \in U_{\alpha}$ and $\tilde{V}$ be any intuitionistic fuzzy closed set in Y such that $(F \backslash U_{\beta}(x) \subseteq \widetilde{V}$. Since F is intuitionistic fuzzy upper contra $\boldsymbol{\beta}$-continuous and $F(x) = (\mathrm{F} \backslash U_{\beta})(x)$, there exists $\boldsymbol{\alpha}$-open set G of X containing x such that $F(G) \subseteq \widetilde{V}$. Let $U = G \quad U_{\beta}$, then $x \in U$ is $\boldsymbol{\beta}$-open set in X and $(F \backslash U_{\beta})(\mathrm{U}) = \mathrm{F}(\mathrm{U}) \subseteq \widetilde{V}$. Therefore, it follows that $\left(F \backslash U_{\beta}\right)$ is an intuitionistic fuzzy upper contra $\boldsymbol{\beta}$-continuous.

Sufficiency: Let $x \in X$ and $\widetilde{V}$ be any intuitionistic fuzzy closed set in Y such that $F(x) \subseteq \widetilde{V}$ there exists $\beta \in \lambda$ and $x \in U_{\beta}$. Since $F \backslash U_{\beta}: U_{\beta} \rightarrow Y$is anintuituionistic fuzzy upper contra $\boldsymbol{\beta}$-continuous. and $F(x) = (F \backslash U_{\beta})(x)$, there exists $\boldsymbol{\beta}$-open set U$\in U_{\beta}$ containing x such that $\left(F \backslash U_{\beta}\right)(U) \subseteq \widetilde{V}$. We have $\boldsymbol{\beta}$-open set $U \in U_{\beta}$ containing x and $F(U) \subseteq \widetilde{V}$. Therefore F is an intuitionistic fuzzy upper contra β-continuous.

Theorem 3.8: Let $\{U_{\beta} : \beta \in \Lambda\}$ be an α -open cover of a topological space $(\mathbf{X}, \tau)$. An Intuitionistic fuzzy multifunction $F: (X, \tau) \rightarrow (Y, \Gamma)$ is an intuitionistic fuzzy lower contra $\boldsymbol{\alpha}$-continuous if and only if the restriction $F \backslash U_{\beta}: U_{\beta} \rightarrow Y$ is intuitionistic fuzzy lower contra $\boldsymbol{\alpha}$-continuous for each $\beta \in \lambda$.

Proof: Necessity: Suppose that F is intuitionistic fuzzy lower contra $\boldsymbol{\beta}$-continuous. Let $\beta \in \Lambda$ and $x \in U_{\beta}$, let $\widetilde{V}$ be any intuitionistic fuzzy closed set in Y such that $\left(F \backslash U_{\beta}\right)(x) q \widetilde{V}$. Since F is intuitionistic fuzzy lower contra β-continuous and $F(x)(F \backslash U_{\beta})(x)$, there exists $\boldsymbol{\beta}$-open set U_0 of X containing x such that $\mathrm{F}(U_0) q \widetilde{V}$. Let $U = U_0 \cap U_{\beta}$, then $x \in U$ is $\boldsymbol{\beta}$-open in X and $(\mathrm{F} \backslash U_{\beta})(\mathrm{U}) = \mathrm{F}(\mathrm{U}) \mathrm{q} \widetilde{V}$.

Therefore, it follows that $(F\backslash U_\beta)$ is an intutionistic fuzzy lower contra $\boldsymbol{\beta}$-continuous.

Sufficiency: Let $\boldsymbol{x} \in \boldsymbol{X}$ and $\widetilde{\boldsymbol{V}}$ be any intuitionistic fuzzy closed set in Y such that $\boldsymbol{F}(\boldsymbol{x})\boldsymbol{q}\widetilde{\boldsymbol{V}}$ there exists $\boldsymbol{\beta} \in \boldsymbol{\Lambda}$ and $\in U_\beta$. Since $\boldsymbol{F}\backslash\boldsymbol{U}_{\boldsymbol{\beta}}: \boldsymbol{U}_{\boldsymbol{\beta}} \to \boldsymbol{Y}$ is an intutionistic fuzzy lower contra $\boldsymbol{\beta}$-continuous and $\boldsymbol{F}(\boldsymbol{x}) = (\boldsymbol{F}\backslash\boldsymbol{U}_{\boldsymbol{\beta}})(\boldsymbol{x})$,there exists open set $\boldsymbol{U}_{\boldsymbol{0}} \in \boldsymbol{U}_{\boldsymbol{\beta}}$ containing $\boldsymbol{x}$ such that $(\boldsymbol{F}\backslash\boldsymbol{U}_{\boldsymbol{\beta}})(\boldsymbol{U}_{\boldsymbol{0}})\boldsymbol{q}\widetilde{\boldsymbol{V}}$, we have open set $\boldsymbol{U}_{\boldsymbol{0}} \in \boldsymbol{U}$ containing $\boldsymbol{x}$ and $\boldsymbol{F}(\boldsymbol{U}_{\boldsymbol{0}})\boldsymbol{q}\widetilde{\boldsymbol{V}}$. Therefore F is an intuitionistic fuzzy lower contra $\boldsymbol{\beta}$-continuous.

Definition 3.3: An intuitionistic fuzzy multifunction $\boldsymbol{F}: (\boldsymbol{X}, \boldsymbol{\tau}) \to (\boldsymbol{Y}, \boldsymbol{\Gamma})$, then the intuitionistic fuzzy multifunction $\boldsymbol{\beta Cl F}: (\boldsymbol{X}, \boldsymbol{\tau}) \to (\boldsymbol{Y}, \boldsymbol{\Gamma})$ is defined by $(\boldsymbol{\beta Cl F})(\boldsymbol{x}) = \boldsymbol{\beta Cl}(\boldsymbol{F}(\boldsymbol{x}))$ for every $x \in Y$.

Lemma 3.2 [31]: For an intuitionistic fuzzy multifunction $\boldsymbol{F}: (\boldsymbol{X},\boldsymbol{\tau}) \to (\boldsymbol{Y}, \boldsymbol{\Gamma})$ it follows that $(\beta Cl(F))^{-}(\tilde{V}) = F^{-}(\tilde{V})$, for each intuitionistic fuzzy open set $\tilde{V}$ of Y.

Theorem 3.9: An intuitionistic fuzzy multifunction $\boldsymbol{F}: (\boldsymbol{X}, \boldsymbol{\tau}) \to (\boldsymbol{Y}, \boldsymbol{\Gamma})$ is intuitionistic fuzzy lower contra $\boldsymbol{\beta}$-continuous if and only if $\boldsymbol{\beta Cl}(\boldsymbol{F}): (\boldsymbol{X}, \boldsymbol{\tau}) \to (\boldsymbol{Y}, \boldsymbol{\Gamma})$ is intuitionistic fuzzy lower contra β-continuous.

Proof: Necessity: Suppose that F is an intuitionistic fuzzy lower $\boldsymbol{\beta}$-continuous. Let $x \in X$ and let $\tilde{V}$ be any intuitionistic fuzzy open set of Y such that $\beta Cl(F(x))q\tilde{V}$. By lemma 3.2 we have $\boldsymbol{x} \in (\boldsymbol{Cl}(\boldsymbol{F}))^{-}(\widetilde{\boldsymbol{V}}) = \boldsymbol{F}^{-}(\widetilde{\boldsymbol{V}})$ and hence, $F(x)q\tilde{V}$. Since F is intuitionistic fuzzy lower contra β-continuous, there exists a $\boldsymbol{\beta}$-closed set U of X containing $\boldsymbol{x}$ such that $\boldsymbol{F}(\boldsymbol{u})\boldsymbol{q}\widetilde{\boldsymbol{V}}, \forall \boldsymbol{u} \in \boldsymbol{U}$.

Hence, **Cl(F)(u)q**$\widetilde{\boldsymbol{V}}$ for each $\boldsymbol{u} \in \boldsymbol{U}$.

This show that **Cl(F)** is an intuitionistic fuzzy lower contra $\boldsymbol{\beta}$-continuous.

Sufficiency: Suppose $\boldsymbol{\beta}$Cl(F) is an intuitionistic fuzzy lower contra β-continuous. Let $x \in X$ and let $\tilde{V}$ be any intuitionistic fuzzy open set of Y such that $\boldsymbol{F}(\boldsymbol{x})\boldsymbol{q}\widetilde{\boldsymbol{V}}$, by lemma 3.2, we have $\boldsymbol{x} \in \boldsymbol{F}^{-}(\widetilde{\boldsymbol{V}}) = (\boldsymbol{\beta Cl}(\boldsymbol{F}))^{-}(\widetilde{\boldsymbol{V}})$ and Hence, $\beta Cl(F)(x)q\tilde{V}$. Since $\boldsymbol{\beta}$Cl(F) is an intuitionistic fuzzy lower contra $\boldsymbol{\beta}$-continuous, there exists a $\boldsymbol{\beta}$-closed set U of X containing $\boldsymbol{x}$ such that $\boldsymbol{\alpha Cl}(\boldsymbol{F}(\boldsymbol{u}))\boldsymbol{q}\widetilde{\boldsymbol{V}}$ for each $u \in U$. Since $\widetilde{\boldsymbol{V}}$ be an intuitionistic fuzzy open set of Y, hence $\boldsymbol{F}(\boldsymbol{u})\boldsymbol{q}\widetilde{\boldsymbol{V}}$ for each $\boldsymbol{u} \in \boldsymbol{U}.$

This shows that F is intuitionistic fuzzy lower contra $\boldsymbol{\beta}$-continuous.

Definition 3.4: An intuitionistic fuzzy set $\widetilde{\boldsymbol{A}}$ in intuitionistic fuzzy topological space (Y, Γ)i s called cl-neighbourhood of an intuitionistic fuzzy set $\widetilde{\boldsymbol{V}}$ in Y, if there exists an intuitionistic fuzzy closed set $\widetilde{U}$ in Y such that $\widetilde{V} \subseteq \widetilde{U} \subseteq \widetilde{A}$.

Theorem 3.10: If $\boldsymbol{F}: (\boldsymbol{X}, \boldsymbol{\tau}) \to (\boldsymbol{Y}, \boldsymbol{\Gamma})$ is an intuitionistic fuzzy upper contra $\boldsymbol{\beta}$-continuous multifunction then for each point $\boldsymbol{x} \in \boldsymbol{X}$ and each intuitionistic fuzzy cl-neighbourhood $\widetilde{\boldsymbol{V}}$ of $F(x), \boldsymbol{F}^{+}(\widetilde{\boldsymbol{V}})$ is an $\boldsymbol{\beta}$-neighbourhood of $\boldsymbol{x}$.

Proof: Let $\boldsymbol{x} \in \boldsymbol{X}$ and $\widetilde{\boldsymbol{V}}$ be an intuitionistic fuzzy cl-neighbourhood of $\boldsymbol{F}(\boldsymbol{x})$, then $\exists$ an intuitionistic fuzzy closed set $\widetilde{\boldsymbol{A}}$ in Y such that $\boldsymbol{F}(\boldsymbol{x}) \subseteq \widetilde{\boldsymbol{A}} \subseteq \widetilde{\boldsymbol{V}}$. We have $\boldsymbol{x} \in \boldsymbol{F}^{+}(\widetilde{\boldsymbol{A}}) \subseteq \boldsymbol{F}^{+}(\widetilde{\boldsymbol{V}})$ and Since $\boldsymbol{F}^{+}(\widetilde{\boldsymbol{A}})$ is $\boldsymbol{\beta}$-open set, $\boldsymbol{F}^{+}(\widetilde{\boldsymbol{V}})$ is a $\boldsymbol{\beta}$-neighbourhood of x.

Theorem 3.11: For an intuitionistic fuzzy multifunction F : (X,τ) $\to (Y, \Gamma)$ the following are equivalent:

(a) F is intuitionistic fuzzy lower contra $\boldsymbol{\beta}$-continuous.

(b) For any and any net $\boldsymbol{x} \in \boldsymbol{X}$ and any net $(x_i)_{i \in I}$is **β**-eventually in $\boldsymbol{F}^{-}(\widetilde{\boldsymbol{B}})$.

Proof: (a) ⇒ (b): Let $(\boldsymbol{x_i})_{\boldsymbol{i \in I}}$ be net $(\boldsymbol{x_i})_{\boldsymbol{i \in I}}$ $\boldsymbol{\beta}$-coverging to $\boldsymbol{x}$ in X and $\widetilde{\boldsymbol{B}}$ be any intuitionistic fuzzy closed set Y with $\boldsymbol{x} \in \boldsymbol{F}^{-}(\widetilde{\boldsymbol{B}})$. Since F is intuitionistic fuzzy lower contra **β**-continuous ∃an **β**-open set $\boldsymbol{A} \subset \boldsymbol{X}$ containing $\boldsymbol{x}$ such that $\boldsymbol{A} \subset \boldsymbol{F}^{-}(\widetilde{\boldsymbol{B}})$. Since $x_i \to x$, ∃ an index $\boldsymbol{i_0} \in \boldsymbol{I}$ such that $\boldsymbol{x_i} \in \boldsymbol{A}$ for every $\boldsymbol{i} \geq \boldsymbol{i_0}$ we have $\boldsymbol{x_i} \in \boldsymbol{A} \subset \boldsymbol{F}^{-}(\widetilde{\boldsymbol{B}})$, $\forall\ i \geq i_0$. Hence, $(\boldsymbol{x_i})_{\boldsymbol{i \in I}}$ is **β**-eventually in $F^{-}(\widetilde{B})$.

(b) ⇒ (a): Suppose that F is not intuitionistic fuzzy lower contra **β**-continuous ∃ a point $x \in X$ and an intuitionistic fuzzy closed set $\widetilde{\boldsymbol{B}}$ with $\boldsymbol{x} \in \boldsymbol{F}^{-}(\widetilde{\boldsymbol{B}}$)such that B $F^{-}(\widetilde{B})$for any **β**-open set B $\subset X$ containing x. Let $(x_i) \in B$ and $(\boldsymbol{x_i})$ $F^{-}(\widetilde{B})$ for each **β**-open set B ⊂ X containing x. Then the **β**-neighbourhood net (x_i) **β** -converges to $\boldsymbol{x}$ but $(\boldsymbol{x_i})_{\boldsymbol{i_{\in I}}}$ is not β-eventually in $F^{-}(\widetilde{B})$. Thus, is a contradiction.

Theorem 3.12: For an intuitionistic fuzzy multifunction **F : ((X, $\boldsymbol{\tau}$) → (Y, Γ)** the following are equivalent:

(a) F is intuitionistic fuzzy upper contra **β**-continuous.

(b) For any $\boldsymbol{x} \in \boldsymbol{X}$ and any net $(\boldsymbol{x_i})_{\boldsymbol{i \in I}}$ $\boldsymbol{\beta}$-converging to x in X and each intuitionistic fuzzy closed set $\widetilde{B}$ of Y with $\boldsymbol{x} \in \boldsymbol{F}^{+}(\widetilde{\boldsymbol{B}})$, the net$(\boldsymbol{x_i})_{\boldsymbol{i \in I}}$ is **β**-eventually in $\boldsymbol{F}^{+}(\widetilde{\boldsymbol{B}})$.

Proof: The proof of this theorem is similar to that of Theorem 3.3.

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ON $\boldsymbol{\beta}$ CONTINUOUS INTUITIONISTIC FUZZY MULTIFUNCTIONS

Abstract: In this paper we introduce and characterize the concepts of β-continuous intuitionistic fuzzy multifunctions from a topological space to an intuitionistic fuzzy topological space.

Keywords: Intuitionistic Fuzzy Sets, Intuitionistic Fuzzy Topology, Intuitionistic Fuzzy Multifunctions, Upper β-Continuous and Lower β-Continuous Intuitionistic Fuzzy Multifunctions.

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1. Introduction and Preliminaries

After the introduction of fuzzy sets by Zadeh [29] in 1965 and fuzzy topology by Chang [6] in 1967, several researches were conducted on the generalizations of the notions of fuzzy sets and fuzzy topology. The concept of intuitionistic fuzzy sets was introduced by Atanassov [2, 3, 4] as a generalization of fuzzy sets. In the last 27 years various concepts of fuzzy mathematics have been extended for intuitionistic fuzzy sets. In 1997 Coker [7] introduced the concept of intuitionistic fuzzy topological spaces as a generalization of fuzzy topological spaces. In 1999, Ozbakir and Coker [23] introduced the concept intuitionistic fuzzy multifunctions and studied their lower and upper intuitionistic fuzzy semi continuity from a topological space to an intuitionistic fuzzy topological space. In the present paper we introduce the concepts of intuitionistic fuzzy β-continuous multifunctions and obtain some of their characterizations and properties.

Throughout this paper $(X, \mathcal{T})$ and (Y, Γ) represents a topological space and an intuitionistic fuzzy topological space respectively. A subset A of a topological space $(X,\mathcal{T})$ is called Semi open [11] (resβ-open [19]) if A$\subset Cl(Int(A))$ (resp.A$\subset Int\left(Cl(Int(A))\right)$. The complement of a semi open (resp. β-open) set is called semi closed (resp. β-closed). Every open (resp. closed) set is β-open (resp. β-closed) and every β-open (resp. β-closed) set is semi open (resp. semi closed) ,but the converses may not be true. The family of all β-open (resp.β-closed) subsets of topological space$(X,\mathcal{T})$ is denoted by βO(X) (resp. βC(X)). The intersection of all β-closed (resp. semi closed) sets of X containing a set A of X is called the β-closure [14] (resp. semi closure) of A. It is denoted by βCl(A) (resp. sCl(A)). The union of all β-open (resp. semi open) sub sets of A of X is called the α-interior [14] (resp. semi interior) of A. It is denoted by βInt(A) (resp. sInt(A)). A subset A of X is β-closed (resp. semi closed) if and only if A $\supset Cl(Int(Cl(A)))$ $(resp. A \supset Int\left(Cl(A)\right)$. A subset N of a topological space $(X,\mathcal{T})$ is called a β-neighborhood [14] of a point x of X if there exists a β-open set O of X such that x $\in O \subset$ N. A is a α-open in X if and only if it is a α-neighborhood of each of its points. A subset V of X is called a β-neighborhood of a subset A of X if there exists $U \in \beta$O(X) such that $A \subset U \subset V$. A mapping f from a topological space $(X, \mathcal{T})$ to another topological space (X*, $\mathcal{T}$) is said to be β-continuous [15, 16] if the inverse image of every open set of X* is β-open in X. Every continuous mapping isβ-continuous but the converse may not be true [15]. A multifunction F from a topological space $(X, \mathcal{T})$ to another topological space (X*, $\mathcal{T}$) is said to be lower β-continuous [18] (resp.upper β-continuous [18]) at a point $x_0 \in X$ if for every α-neighborhood U of x_0 and for any open set W of X* such that F$(x_0) \cap W \neq$ (resp. F$(x_0) \subset W$) there is a β-neighborhood U of x_0 such that F$(x) \cap W \neq$ (resp. F$(x) \subset W$) for every x $\in U$.

Lemma 2.1 [25]: Let A be a subset of a topological space $(X, \mathcal{T})$. Then:

(a) A is β-closed in X sInt(Cl(A) $\subset$ A;

(b) sInt(Cl(A)) = Cl(Int(Cl(A)));

(c) βCl(A) = A Cl(Int(Cl(A))).

Lemma 2.2 [25]: Let A be a subset of a topological space $(X, \mathcal{T})$. Then the following conditions are equivalent:

(a) A $\in \beta O$(X)

(b) $U \subset A \subset Int(Cl(U))$ for some open set U.

(c) $U \subset A \subset sCl(U)$ for some open set U.

(d) $A \subset sCl(Int(A))$.

Definition 2.1 [2, 3, 4]: Let Y be a nonempty fixed set. An intuitionistic fuzzy set $\widetilde{A}$ in Y is an object having the form

$$\widetilde{A} = \{< x, \mu_{\widetilde{A}}(y),\ \upsilon_{\widetilde{A}}(y) > y \in Y\ \}$$

where the functions $\mu_{\widetilde{A}} : Y \rightarrow I$ and $\upsilon_{\widetilde{A}} : Y \rightarrow I$ denotes the degree of membership (namely $\mu_{\widetilde{A}}(y)$) and the degree of non membership (namely $\upsilon_{\widetilde{A}}(y)$) of each element $y \in Y$ to the set $\widetilde{A}$ respectively, and $0 \leq \mu_{\widetilde{A}}(x) + \upsilon_{\widetilde{A}}(x) \leq 1$ for each $y \in Y$.

Definition 2.2 [2, 3, 4]: Let Y be a nonempty set and the intuitionistic fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ be in the form $\widetilde{A} = \{< y, \mu_{\widetilde{A}}(y), \upsilon_{\widetilde{A}}(y) > : y \in Y\}$, $\widetilde{B} = \{< y, \mu_{\widetilde{B}}(y), \upsilon_{\widetilde{B}}(y) > : y \in Y\}$ and let

$\{\widetilde{A}_{\alpha} : \beta \in \ \}$ be an arbitrary family of intuitionistic fuzzy sets in Y. Then:

(a) $\widetilde{A} \subseteq \widetilde{B}$ if $\forall y \in Y\,[\ \mu_{\widetilde{A}}(y) \leq \mu_{\widetilde{B}}(y)$ and $\upsilon_{\widetilde{A}}(y) \geq \upsilon_{\widetilde{B}}(y)]$;

(b) $\widetilde{A} = \widetilde{B}$ if $\widetilde{A} \subseteq \widetilde{B}$ and $\widetilde{B} \subseteq \widetilde{A}$;

(c) $\tilde{A}^{c} = \{ < y, \upsilon_{\widetilde{A}}(y), \mu_{\widetilde{A}}(y) > : y \in Y\}$;

(d) $\widetilde{0} = \{< y, 0,1 > : y \in Y\}$ and $\widetilde{1} = \{< y, 1,0 > : y \in Y\}$

(e) $\cap\ \widetilde{A}_{\alpha} = \{< y,\ \wedge\ \mu_{\widetilde{A}}(y),\ \vee \upsilon_{\widetilde{A}}(y) > : y \in Y\}$;

(f) $\cup\ \widetilde{A}_{\alpha} = \{< y,\ \vee\ \mu_{\widetilde{A}}(y),\ \wedge \upsilon_{\widetilde{A}}(y) > : y \in Y\}$;

Definition 2.3 [8]: Two Intuitionistic Fuzzy Sets $\widetilde{A}$ and $\widetilde{B}$ of Y are said to be quasi coincident ($\widetilde{A}\, q\, \widetilde{B}$ for short) if $\exists\ y \in Y$ such that

$$\mu_{\widetilde{A}}(y) > \upsilon_{\widetilde{B}}(y) \quad \text{or} \quad \upsilon_{\widetilde{A}}(y) < \mu_{\widetilde{B}}(y).$$

Lemma 2.3 [8]: For any two intuitionistic fuzzy sets $\tilde{A}$ and $\tilde{B}$ of Y, $\rceil(\tilde{A}q\tilde{B}) \Leftrightarrow \tilde{A} \subset \tilde{B}^c$.

Definition 2.4 [7]: An intuitionistic fuzzy topology on a non empty set Y is a family Γ of intuitionistic fuzzy sets in Y which satisfy the following axioms:

(O_1). $\tilde{0}, \tilde{1} \in \Gamma$,

(O_2). $\tilde{A}_1 \cap \tilde{A}_2 \in \Gamma$, for any $\tilde{A}_1, \tilde{A}_2 \in \Gamma$,

(O_3). $\cup \tilde{A}_\alpha$ for any arbitrary family $\{\tilde{A}_\alpha : \alpha \in \Lambda\} \in \Gamma$.

In this case the pair (Y, Γ) is called an intuitionistic fuzzy topological space and each intuitionistic fuzzy set in Γ, is known as an intuitionistic fuzzy open set in *Y*. The complement $\tilde{B}^c$ of an intuitionistic fuzzy open set $\tilde{B}$ is called an intuitionistic fuzzy closed set in *Y*.

Definition 2.5 [7]: Let (Y, Γ) be an intuitionistic fuzzy topological space and $\tilde{A}$ be an intuitionistic fuzzy set in Y. Then the interior and closure of $\tilde{A}$ are defined by:

cl($\tilde{A}$) = $\cap\{\tilde{K} : \tilde{K}$ is an intuitionistic fuzzy closed set in *Y* and $\tilde{A} \subseteq \tilde{K}\}$,

int($\tilde{A}$) = $\cup\{\tilde{G} : \tilde{G}$ is an intuitionistic fuzzy open set in *Y* and $\tilde{G} \subseteq \tilde{G}\}$.

Definition 2.6 [23]: Let X and Y are two non empty sets. A function F: X → Y is called intuitionistic fuzzy multifunction if F(x) is an intuitionistic fuzzy set in Y, $\forall$ $x \in X$.

Definition 2.7 [27]: Let F : X → Y is an intuitionistic fuzzy multifunction and A be a subset of X. Then F(A) = ${}_{x\in A} F(x)$.

Definition 2.8 [23]: Let F : X → Y be an intuitionistic fuzzy multifunction. Then the upper inverse $F^+(\tilde{A})$ and lower inverse $F^-(\tilde{A})$ of an intuitionistic fuzzy set $\tilde{A}$ in Y are defined as follows:

$$F^+(\tilde{A}) = \{x \in X : F(x) \subseteq \tilde{A}\}$$

$$F^-(\tilde{A}) = \{x \in X : F(x)q\tilde{A}\}.$$

Definition 2.9 [23]: An Intuitionistic fuzzy multifunction F: $(X,\mathcal{T}) \to (Y,\Gamma)$ is said to be:

(a) Intuitionistic fuzzy upper β-continuous [28] (Intuitionistic fuzzy upper semi continuous [23]) at a point $x_0 \in X$ if for any intuitionistic fuzzy open set $\widetilde{W} \subset Y$ such that $F(x_0) \subset \widetilde{W}$ there exists an $U \in \beta O(X)$ (resp. open set $U \subset X$) containing x_0 such that $F(U) \subset \widetilde{W}$.

(b) Intuitionistic fuzzy lower β-continuous (resp. Intuitionistic fuzzy lower semi continuous) at a point $x_0 \in X$ if for any intuitionistic fuzzy open set $\widetilde{W} \subset Y$ such that $F(x_0)q\widetilde{W}$ there exists an $U \in \beta O(X)$ (resp. open set $U \subset X$) containing x_0 such that $F(x)q\widetilde{W}$, $\forall\, x \in \widetilde{W}$.

(c) Intuitionistic fuzzy upper β-continuous (resp. intuitionistic fuzzy lower β-continuous Intuitionistic fuzzy upper semi-continuous, intuitionistic fuzzy lower semi-continuous) if it is intuitionistic fuzzy upper β-continuous (resp. intuitionistic fuzzy lower β-continuous intuitionistic fuzzy upper semi-continuous, intuitionistic fuzzy lower semi-continuous) at each point of X.

β-Continuous Intuitionistic Fuzzy Multifunctions

Definition 3.1: An Intuitionistic fuzzy multifunction F: $(X,\mathcal{T}) \to (Y,\Gamma)$ is said to be:

(a) Intuitionistic fuzzy β-continuous at a point $x_0 \in X$ if for any $\tilde{G}_1, \tilde{G}_2 \in IFO(Y)$ such that $F(x_0) \subset \tilde{G}_1$ and $F(x_0)q\tilde{G}_2$ there exists $U \in \beta O(X)$ containing x_0 such that $F(u) \subset \tilde{G}_1$ and $F(u)q\tilde{G}_2, \forall\, u \in U$.

(b) Intuitionistic fuzzy β-continuous if it has this property at each point of X.

Theorem 3.1: If F : $(X,\mathcal{T}) \to (Y,\Gamma)$ is intuitionistic fuzzy β-continuous then F is intuitionistic fuzzy upper β-continuous and intuitionistic fuzzy lower β-continuous .

Proof: Obvious.

Theorem 3.2: Let F: $(X,\mathcal{T}) \to (Y,\Gamma)$ be an intuitionistic fuzzy multifunction, Then the following statements are equivalent:

(a) F is intuitionistic fuzzy β-continuous at a point $x \in X$;

(b) for any $\tilde{G}_1, \tilde{G}_2 \in IFO(Y)$ such that $F(x) \subset \tilde{G}_1$ and $F(x)q\tilde{G}_2$, there result the relation $x \in sCl(\, Int(F^+(\tilde{G}_1) \cap F^-(\tilde{G}_2)))$.

(c) for every $\tilde{G}_1, \tilde{G}_2 \in IFO(Y)$ such that $F(x) \subset \tilde{G}_1$ and $F(x)q\tilde{G}_2$, and for any semi-open set U of X containing x, there exists a non-empty open set $G_U \subset U$, such that $F(G_U) \subset \tilde{G}_1$ and $F(u)q\tilde{G}_2, \forall\, u \in G_U$.

Proof: (a) (b): Let $\tilde{G}_1, \tilde{G}_2 \in IFO(Y)$ with $F(x) \subset \tilde{G}_1$ and $F(x)q\tilde{G}_2$, $\exists\, U \in \beta O(X)$ containing x such that $F(U) \subset \tilde{G}_1$ and $F(u)q\tilde{G}_2, \forall\, u \in U$. Thus, $x \in U \subset F^+(\tilde{G}_1)$ and $x \in U \subset F^-(\tilde{G}_2)$. Therefore $x \in U \subset F^+(\tilde{G}_1) \cap F^-(\tilde{G}_2)$. Since $U \in \alpha O(X)$. By Lemma 2.2 we have

$$x \in U \subset sCl(IntU) \subset sCl(Int\left(F^+(\tilde{G}_1) \cap F^-(\tilde{G}_2)\right)).$$

(b) (c): Let $\tilde{G}_1, \tilde{G}_2 \in IFO(Y)$ with $F(x) \subset \tilde{G}_1$ and $F(x)q\tilde{G}_2$. Then $x \in sCl\left(Int\left(F^+(\tilde{G}_1) \cap F^-(\tilde{G}_2)\right)\right)$. Let U be any semi-open subset of X containing x. Then $U \cap Int\left(F^+(\tilde{G}_1) \cap F^-(\tilde{G}_2)\right) \neq$. Put $G_U = Int(Int(F^+(\tilde{G}_1) \cap F^-(\tilde{G}_2)) \cap U)$, then $G_U \neq \phi$, $G_U \subset U$, $G \subset Int(F^+(\tilde{G}_1)) \subset F^+(\tilde{G}_1)$ and $G_U \subset Int(F^-(\tilde{G}_2)) \subset F^-(\tilde{G}_2)$. And thus, $F(G_U) \subset \tilde{G}_1$ and $F(u)q\tilde{G}_2, \forall u \in G_U$.

(c) (a): Let $\{U_x\}$ be the family of semi-open sets of X containing x. For any semi-open set U of X containing x and for every $\tilde{G}_1, \tilde{G}_2 \in IFO(Y)$ with $F(x) \subset \tilde{G}_1$ and $F(x)q\tilde{G}_2$, there exists a non-empty open set $G_U \subset U$ such that $F(G_U) \subset \tilde{G}_1$ and $F(u)q\tilde{G}_2, \forall u \in G_U$. Let $W = \cup\{G_U : U \in U_x\}$. Then W is open in X, $x \in sCl(W)$, $F(w) \subset \tilde{G}_1$ and $F(w)q\tilde{G}_2$, for every $w \in W$. Put $S = W \cup \{x\}$, then $W \subset S \subset sCl(W)$ thus, $W \in \alpha O(X)$, $x \in S, F(S) \subset \tilde{G}_1$ and $F(t)q\tilde{G}_2, \forall\, t \in S$. Hence, F is intuitionistic fuzzy β-continuous at x.

Definition 3.2: Let $\tilde{A}$ be an intuitionistic fuzzy set of an intuitionistic fuzzy topological space (Y, Γ). Then $\tilde{V}$ is said to be a neighbourhood of $\tilde{A}$ in Y if there exists an intuitionistic fuzzy open set $U\ of\ Y$ such that $\tilde{A} \subset \tilde{U} \subset \tilde{V}$.

Definition 3.3: Let (Y, Γ) be an intuitionistic fuzzy topological space, an intuitionistic fuzzy set $\tilde{V}$ is called a semi q-neighbourhood of an intuitionistic fuzzy set $\tilde{A}$ of Y if $\exists\, \beta\, \tilde{U} \in IFSO(Y)$ such that $\tilde{A}q\tilde{U} \subset \tilde{V}$.

Theorem 3.3: Let F: $(X, \mathcal{T}) \to (Y, \Gamma)$ be an intuitionistic fuzzy multifunction, Then the following statements are equivalent:

(a) F is intuitionistic fuzzy β-continuous.

(b) $F^{+}(\tilde{G}_1) \cap F^{-}(\tilde{G}_2) \in \beta O(X)$, for every $\tilde{G}_1, \tilde{G}_2 \in IFO(Y)$

(c) $F^{+}(\tilde{V}_2) \cup F^{-}(\tilde{V}_1) \in \beta C(X)$, for any $\tilde{V}_1, \tilde{V}_2 \in IFC(Y)$.

(d) $sInt(Cl(F^{-}(\tilde{B}_1) \cup F^{+}(\tilde{B}_2))) \subset F^{-}(Cl\tilde{B}_1) \cup F^{+}(Cl\tilde{B}_2)$, for any pair of intuitionistic fuzzy sets $\tilde{B}_1, \tilde{B}_2$ of Y.

(e) $\beta Cl(F^{-}(\tilde{B}_1) \cup F^{+}(\tilde{B}_2)) \subset F^{-}(Cl\tilde{B}_1) \cup F^{+}(Cl\tilde{B}_2)$, for any pair of intuitionistic fuzzy sets $\tilde{B}_1, \tilde{B}_2$ of Y.

(f) $\beta Int(F^{-}(\tilde{B}_1) \cap F^{+}(\tilde{B}_2))) \supset F^{-}\left(Int(\tilde{B}_1)\right) \cap F^{+}(Int(\tilde{B}_2))$, for any pair of intuitionistic fuzzy sets $\tilde{B}_1, \tilde{B}_2$ of Y.

(g) For each point x of X for each neighbourhood $\tilde{V}_1$ of F(x) and for each q-neighbourhood $\tilde{V}_2$ of F(x) , $F^{+}(\tilde{V}_1) \cap F^{-}(\tilde{V}_2)$ is a α-neighbourhood of x.

Proof: (a) **(b):** Let any $\tilde{G}_1, \tilde{G}_2 \in IFO(Y)$ and $x \in F^{+}(\tilde{G}_1) \cap F^{-}(\tilde{G}_2)$, thus, $F(x) \subset \tilde{G}_1$ and $F(x) q \tilde{G}_2$, Since F being intuitionistic fuzzy β-continuous according to the theorem 3.2 (b). There follows that $x \in sCl(Int(F^{+}(\tilde{G}_1) \cap F^{-}(\tilde{G}_2)))$. And as x is chosen arbitrarily in $F^{+}(\tilde{G}_1) \cap F^{-}(\tilde{G}_2)$, we have $F^{+}(\tilde{G}_1) \cap F^{-}(\tilde{G}_2) \subset sCl(Int\left(F^{+}(\tilde{G}_1) \cap F^{-}(\tilde{G}_2)\right))$ and thus, $F^{+}(\tilde{G}_1) \cap F^{-}(\tilde{G}_2) \in \beta O(X)$ by Lemma 2.2.

(b) **(c):** It follows from Theorem 3.2 [27] (c) and (d).

(c) **(d):** Suppose that (c) holds and let $\tilde{B}_1, \tilde{B}_2$ be two intuitionistic fuzzy sets of Y. Then $Cl(\tilde{B}_1) \in IFC(Y)$, $Cl(\tilde{B}_2) \in IFC(Y)$ and thus, by (c) $F^{-}(Cl(\tilde{B}_1)) \cup F^{+}(Cl(\tilde{B}_2)) \in \alpha C(X)$. Hence, by Lemma 2.1(a), $sInt[\, Cl(F^{-}(Cl(\tilde{B}_1)) \cup F^{+}\left(Cl(\tilde{B}_2)\right)] \subset F^{-}\left(Cl(\tilde{B}_1)\right) \cup F^{+}(Cl(\tilde{B}_2)$. Now $\tilde{B}_1 \subset Cl(\tilde{B}_1)$ and $\tilde{B}_2 \subset Cl(\tilde{B}_2)$. By Theorem 3.2 [27] (e) and (f)$F^{+}(\tilde{B}_2) \subset F^{+}(Cl(\tilde{B}_2))$ and $F^{-}(\tilde{B}_1) \subset F^{-}(Cl(\tilde{B}_1))$.

Consequently, $sInt(\, Cl(F^{-}(\tilde{B}_1)) \cup F^{+}(\tilde{B}_2)) \subset F^{-}\left(Cl(\tilde{B}_1)\right) \cup F^{+}(Cl(\tilde{B}_2)$.

(d) **(e):** Suppose (d) hold. Since $\beta Cl(A) = A \cup sInt(Cl(A))$ for each subset A of X, it follows that, $\alpha Cl(F^{-}(\tilde{B}_1) \cup F^{+}(\tilde{B}_2)) = (F^{-}(\tilde{B}_1) \cup F^{+}(\tilde{B}_2)) \cup sInt(\, Cl(F^{-}(\tilde{B}_1) \cup F^{+}(\tilde{B}_2)))$ $\subset (F^{-}(\tilde{B}_1) \cup F^{+}(\tilde{B}_2)) \cup \ (F^{-}(Cl\tilde{B}_1)) \cup F^{+}(Cl\tilde{B}_2)) \subset F^{-}\left(Cl(\tilde{B}_1)\right) \cup F^{+}(Cl(\tilde{B}_2)$.

(e) **(f):** $(\beta Int\left(F^{-}(\tilde{B}_1) \cap F^{+}(\tilde{B}_2)\right))^c = \beta Cl((F^{-}(\tilde{B}_1) \cap F^{+}(\tilde{B}_2))^c)$

$$= \beta Cl((F^{-}(\tilde{B}_1)^c) \cup (F^{+}(\tilde{B}_2)^c) = \beta Cl\,(F^{+}(\tilde{B}_1)^c \cup F^{-}(\tilde{B}_2)^c)$$

$$\subset \; F^{+}\left(Cl(\tilde{B}_1)^c\right) \cup F^{-}\left(Cl(\tilde{B}_2)^c\right) = F^{+}\left((Int(\tilde{B}_1))^c\right) \cup F^{-}\left((Int(\tilde{B}_2))^c\right)$$

$$= (F^{-}(Int\tilde{B}_1))^c \cup (F^{+}(Int\tilde{B}_2))^c = (F^{-}(Int\tilde{B}_1)) \cap F^{+}(Int\tilde{B}_2))^c .$$

and thus , $\beta Int(F^{-}(\tilde{B}_1) \cap F^{+}(\tilde{B}_2)) \supset F^{-}(Int\tilde{B}_1) \cap F^{+}(Int\tilde{B}_2)$.

$(\boldsymbol{a})$ **(g):** Let $x \in X$, $\tilde{V}_1$ is a neighbourhood of F(x) and $\tilde{V}_2$ is a q-neighbourhood of F(x). Then $\exists\, \tilde{U}_1, \tilde{U}_2 \in IFO(Y)$ such that $F(x) \subset \tilde{U}_1 \subset \tilde{V}_1$ and F(x) q$\tilde{U}_2 \subset \tilde{V}_2$. Therefore, $x \in F^{+}(\tilde{U}_1) \cap F^{-}(\tilde{U}_2)$. Therefore, by hypothesis

$$x \in F^{+}(\tilde{U}_1) \cap F^{-}(\tilde{U}_2) = F^{+}(Int(\tilde{U}_1)) \cap F^{-}\left(Int\,(\tilde{U}_2)\right)$$
$$\subset \beta Int\,(F^{+}(\tilde{U}_1) \cap F^{-}(\tilde{U}_2)) \subset \beta Int(F^{+}(\tilde{V}_1) \cap F^{-}(\tilde{V}_2))$$

$\subset (F^{+}(\tilde{V}_1) \cap F^{-}(\tilde{V}_2))$. It follows that $F^{+}(\tilde{V}_1) \cap F^{-}(\tilde{V}_2)$ is β-neighbourhood of x.

(g) **(a):** Obvious**.**

Definition 3.4: An intuitionistic fuzzy multifunction F: $(X, \mathcal{T}) \rightarrow (Y, \Gamma)$ is called :

(a) intuitionistic fuzzy strongly lower semi- continuous $F^{-}(\tilde{B})$ is a open set in X if for each intuitionistic fuzzy set $\tilde{B}\ of\ Y$.

(b) intuitionistic fuzzy strongly upper semi-continuous if $F^{+}(\tilde{B})$ is a open set in X if for each intuitionistic fuzzy set $\tilde{B}\ of\ Y$.

Theorem 3.4: Let F: $(X, \mathcal{T}) \rightarrow (Y, \Gamma)$ be an intuitionistic fuzzy upper β-continuous and intuitionistic fuzzy strongly lower semi-continuous intuitionistic fuzzy multifunction then F is intuitionistic fuzzy β-continuous.

Proof: Let $\tilde{G}_1, \tilde{G}_2 \in IFO(Y)$ Now F being intuitionistic fuzzy upper β continuous, and $\tilde{G}_1 \in IFO(Y)$, $F^{+}(\tilde{G}_1) \in \beta O(X)$ by Theorem 4.1 [28]. Again F being intuitionistic fuzzy strongly lower semi-continuous,$F^{-}(\tilde{G}_2)$ is an open set in X. Hence, $F^{+}(\tilde{G}_1) \cap F^{-}(\tilde{G}_2) \in \beta O(X)$ and by Theorem 3.3, F is intuitionistic fuzzy β continuous.

Theorem 3.5: Let $F:(X,\mathcal{T}) \rightarrow (Y,\Gamma)$ be an intuitionistic fuzzy lower β-continuous and intuitionistic fuzzy strongly upper semi-continuous intuitionistic fuzzy multifunction then F is intuitionistic fuzzy β-continuous.

Proof: Obvious.

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UPPER (LOWER) $\boldsymbol{\beta}$ CONTINUOUS INTUITIONISTIC FUZZY MULTIFUNCTIONS

Abstract: In this paper we introduce and characterize the concepts of upper(lower) β-continuous intuitionistic fuzzy multifunctions from a topological space to an intuitionistic fuzzy topological space.

Keywords: Intuitionistic Fuzzy Sets, Intuitionistic Fuzzy Topology, Intuitionistic Fuzzy Multifunctions, Upper β-Continuous and Lower β-Continuous Intuitionistic Fuzzy Multifunctions.

Mathematical Subject Classification (2010) No.: 54A99, 03E99.

1. Introduction

After the introduction of fuzzy sets by Zadeh [30] in 1965 and fuzzy topology by Chang [7] in 1967, several researches were conducted on the generalizations of the notions of fuzzy sets and fuzzy topology. The concept of intuitionistic fuzzy sets was introduced by Atanassov [3, 4, 5] as a generalization of fuzzy sets. In the last 27 years various concepts of fuzzy mathematics have been extended for intuitionistic fuzzy sets. In 1997 Coker [8] introduced the concept of intuitionistic fuzzy topological spaces as a generalization of fuzzy topological spaces. In 1999, Ozbakir and Coker [24] introduced the concept intuitionistic fuzzy multifunctions and studied their lower and upper intuitionistic fuzzy semi continuity from a topological space to an intuitionistic fuzzy topological space. Abd El-Monsef *et al*. [1] defined β-continuous functions as a generalization of semi-continuity [16] and precontinuity [18]. In the present paper we introduce the concepts of intuitionistic fuzzy upper(lower) β-continuous multifunctions and obtain some of their characterizations and properties.

2. Preliminaries

Throughout this paper $(X, \mathcal{T})$ and (Y, Γ) represents a topological space and an intuitionistic fuzzy topological space respectively. A subset A of a topological space $(X, \mathcal{T})$ is called Semi open [12] (resβ-open[20]) if A$\subset Cl(Int(A))$(resp.A$\subset Int\left(Cl(Int(A))\right)$. The complement of a semi open (resp. β-open) set is called semi closed (resp. β-closed). Every open (resp. closed) set is β-open (resp. β-closed) and every β-open (resp. β-closed) set is semi open (resp. semi closed) ,but the converses may not be true..The family of all β-open (resp. β-closed) subsets of topological space$(X, \mathcal{T})$ is denoted by βO(X) (resp. βC(X)). The intersection of all β-closed (resp. semi closed) sets of X containing a set A of X is called the β-closure [15] (resp. semi closure) of A. It is denoted by βCl(A) (resp. sCl(A)). The union of all β-open (resp. semi open) sub sets of A of X is called the β-interior [15] (resp. semi interior) of A. It is denoted by βInt(A) (resp. sInt(A)) . A subset A of X is β-closed (resp. semi closed) if and only if A $\supset Cl(Int(Cl(A)))$ $(resp. A \supset Int(Cl(A))$. A subset N of a topological space $(X, \mathcal{T})$ is called a β-neighborhood [15] of a point x of X if there exists a β-open set O of X such that x $\in O \subset$ N. A is a β -open in X if and only if it is a β-neighborhood of each of its points. A subset V of X is called a β-neighborhood of a subset A of X if there exists $U \in \beta$O(X) such that $A \subset U \subset V$. A mapping f from a topological space (X, $\mathcal{T}$) to another topological space (X*, $\mathcal{T}$) is said to be β-continuous [16, 17] if the inverse image of every open set of X* is β-open in X. Every continuous mapping is β-continuous but the converse may not be true [16]. A multifunction F from a topological space (X, $\mathcal{T}$) to another topological space (X*, $\mathcal{T}$) is said to be lower β-continuous [19] (resp. upper β-continuous[18]) at a point $x_0 \in X$ if for every α-neighborhood U of x_0 and for any open set W of X* such that F$(x_0) \cap W \neq$ (resp. F$(x_0) \subset W$) there is a β-neighborhood U of x_0 such that F$(x) \cap W \neq$ (resp. F$(x) \subset W$) for every x $\in U$.

Lemma 2.1 [26]: Let A be a subset of a topological space (X, $\mathcal{T}$). Then:

(a) A is β-closed in X sInt(Cl(A) $\subset$ A;

(b) sInt(Cl(A)) = Cl(Int(Cl(A)));

(c) βCl(A) = A Cl(Int(Cl(A))).

Lemma 2.2 [26]: Let A be a subset of a topological space (X, $\mathcal{T}$). Then the following conditions are equivalent :

(a) $A \in \beta O(X)$

(b) $U \subset A \subset Int(Cl(U))$ for some open set U.

(c) $U \subset A \subset sCl(U)$ for some open set U.

(d) $A \subset sCl(Int(A))$.

Definition 2.1 [3, 4, 5]: Let Y be a nonempty fixed set. An intuitionistic fuzzy set $\widetilde{A}$ in Y is an object having the form

$$\widetilde{A} = \{< x, \mu_{\widetilde{A}}(y), \upsilon_{\widetilde{A}}(y) > y \in Y\}$$

where the functions $\mu_{\widetilde{A}}: Y \to I$ and $\upsilon_{\widetilde{A}}: Y \to I$ denotes the degree of membership (namely $\mu_{\widetilde{A}}(y)$) and the degree of non membership (namely $\upsilon_{\widetilde{A}}(y)$) of each element $y \in Y$ to the set $\widetilde{A}$ respectively, and $0 \le \mu_{\widetilde{A}}(x) + \upsilon_{\widetilde{A}}(x) \le 1$ for each $y \in Y$.

Definition 2.2 [3, 4, 5]: Let Y be a nonempty set and the intuitionistic fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ be in the form $\widetilde{A} = \{< y, \mu_{\widetilde{A}}(y), \upsilon_{\widetilde{A}}(y) > : y \in Y\}$, $\widetilde{B} = \{< y, \mu_{\widetilde{B}}(y), \upsilon_{\widetilde{B}}(y) > : y \in Y\}$ and let $\{\widetilde{A}_\alpha : \beta \in \ \}$ be an arbitrary family of intuitionistic fuzzy sets in Y. Then:

(a) $\widetilde{A} \subseteq \widetilde{B}$ if $\forall y \in Y\, [\mu_{\widetilde{A}}(y) \le \mu_{\widetilde{B}}(y)$ and $\upsilon_{\widetilde{A}}(y) \ge \upsilon_{\widetilde{B}}(y)]$;

(b) $\widetilde{A} = \widetilde{B}$ if $\widetilde{A} \subseteq \widetilde{B}$ and $\widetilde{B} \subseteq \widetilde{A}$;

(c) $\tilde{A}^c = \{< y, \upsilon_{\widetilde{A}}(y), \mu_{\widetilde{A}}(y) > : y \in Y\}$;

(d) $\widetilde{0} = \{< y, 0, 1 > : y \in Y\}$ and $\widetilde{1} = \{< y, 1, 0 > : y \in Y\}$

(e) $\cap \widetilde{A}_\alpha = \{< y, \wedge \mu_{\widetilde{A}}(y), \vee \upsilon_{\widetilde{A}}(y) > : y \in Y\}$;

(f) $\cup \widetilde{A}_\alpha = \{< y, \vee \mu_{\widetilde{A}}(y), \wedge \upsilon_{\widetilde{A}}(y) > : y \in Y\}$;

Definition 2.3 [9]: Two Intuitionistic Fuzzy Sets $\widetilde{A}$ and $\widetilde{B}$ of Y are said to be quasi coincident ($\widetilde{A}\, q\, \widetilde{B}$ for short) if $\exists\, y \in Y$ such that

$$\mu_{\widetilde{A}}(y) > \upsilon_{\widetilde{B}}(y) \quad \text{or} \quad \upsilon_{\widetilde{A}}(y) < \mu_{\widetilde{B}}(y).$$

Lemma 2.3 [9]: For any two intuitionistic fuzzy sets $\tilde{A}$ and $\tilde{B}$ of Y, $\rceil(\tilde{A}q\tilde{B}) \Leftrightarrow \tilde{A} \subset \tilde{B}^c$.

Definition 2.4 [8]: An intuitionistic fuzzy topology on a non empty set Y is a family Γ of intuitionistic fuzzy sets in Y which satisfy the following axioms:

(O$_1$). $\tilde{0}, \tilde{1} \in \Gamma$,

(O$_2$). $\tilde{A}_1 \cap \tilde{A}_2 \in \Gamma$, for any $\tilde{A}_1, \tilde{A}_2 \in \Gamma$,

(O$_3$). $\cup \tilde{A}_\alpha$ for any arbitrary family $\{\tilde{A}_\alpha : \alpha \in \Lambda\} \in \Gamma$.

In this case the pair (Y, Γ) is called an intuitionistic fuzzy topological space and each intuitionistic fuzzy set in Γ, is known as an intuitionistic fuzzy open set in *Y*. The complement $\tilde{B}^c$ of an intuitionistic fuzzy open set $\tilde{B}$ is called an intuitionistic fuzzy closed set in *Y*.

Definition 2.5 [8]: Let (Y, Γ) be an intuitionistic fuzzy topological space and $\tilde{A}$ be an intuitionistic fuzzy set in Y. Then the interior and closure of $\tilde{A}$ are defined by:

$$\text{cl}(\tilde{A}) = \cap\{\tilde{K} : \tilde{K} \text{ is an intuitionistic fuzzy closed set in } Y \text{ and } \tilde{A} \subseteq \tilde{K}\},$$

$$\text{int}(\tilde{A}) = \cup\{\tilde{G} : \tilde{G} \text{ is an intuitionistic fuzzy open set in } Y \text{ and } \tilde{G} \subseteq \tilde{G}\}.$$

Definition 2.6 [24]: Let X and Y are two non empty sets. A function F: X → Y is called intuitionistic fuzzy multifunction if F(x) is an intuitionistic fuzzy set in Y, $\forall x \in X$.

Definition 2.7 [28]: Let F : X→Y is an intuitionistic fuzzy multifunction and A be a subset of X. Then F(A) = $_{x\in A} F(x)$.

Definition 2.8 [24]: Let F : X → Y be an intuitionistic fuzzy multifunction. Then the upper inverse $F^+(\tilde{A})$ and lower inverse $F^-(\tilde{A})$ of an intuitionistic fuzzy set $\tilde{A}$ in Y are defined as follows:

$$F^+(\tilde{A}) = \{x \in X : F(x) \subseteq \tilde{A}\}$$

$$F^-(\tilde{A}) = \{x \in X : F(x)q\tilde{A}\}.$$

Lemma 2.9 [28]: Let $\boldsymbol{F}: (\boldsymbol{X}, \boldsymbol{\tau}) \to (\boldsymbol{Y}, \boldsymbol{\Gamma})$ be an intuitionistic fuzzy multifunction and $\widetilde{\boldsymbol{A}}$, $\tilde{B}$ be intuitionistic fuzzy sets in Y. Then:

(a) $F^{+}(\tilde{1}) = F^{-}(\tilde{1})$ =X,

(b) $F^{+}(\tilde{A}) \subseteq F^{-}(\tilde{A})$

(c) $[\boldsymbol{F}^{-}(\widetilde{\boldsymbol{A}})]^{\boldsymbol{c}} = [\boldsymbol{F}^{+}(\widetilde{\boldsymbol{A}})^{\boldsymbol{c}}]$

(d) $[\boldsymbol{F}^{+}(\widetilde{\boldsymbol{A}})^{\boldsymbol{c}} = [\boldsymbol{F}^{-}(\widetilde{\boldsymbol{A}})^{\boldsymbol{c}}$

(e) If $\widetilde{\boldsymbol{A}} \subseteq \widetilde{\boldsymbol{B}}$, then $\boldsymbol{F}^{+}(\widetilde{\boldsymbol{A}}) \subseteq \boldsymbol{F}^{-}(\widetilde{\boldsymbol{B}})$

(f) If $\tilde{A} \subseteq \widetilde{B}$, then $F^{-}(\tilde{A}) \subseteq F^{-}(\widetilde{B})$

Definition 2.10 [24]: An Intuitionistic fuzzy multifunction F : $(X, \mathcal{T}) \to Y, \Gamma)$ (is said to be:

(a) Intuitionistic fuzzy upper β-continuous [28] (Intuitionistic fuzzy upper semi continuous [23]) at a point $x_0 \in X$ if for any intuitionistic fuzzy open set $\widetilde{W} \subset Y$ such that $F(x_0) \subset \widetilde{W}$ there exists an $U \in \beta O(X)$ (resp. open set $U \subset X$) containing x_0 such that $F(U) \subset \widetilde{W}$.

(b) Intuitionistic fuzzy lower β-continuous (resp. Intuitionistic fuzzy lower semi continuous) at a point $x_0 \in X$ if for any intuitionistic fuzzy open set $\widetilde{W} \subset Y$ such that $F(x_0) q \widetilde{W}$ there exists an $U \in \beta O(X)$ (resp. open set $U \subset X$) containing x_0 such that $F(x) q \widetilde{W}$, $\forall\, x \in \widetilde{W}$.

(c) Intuitionistic fuzzy upper β-continuous (resp. intuitionistic fuzzy lower β-continuous Intuitionistic fuzzy upper semi-continuous, intuitionistic fuzzy lower semi-continuous) if it is intuitionistic fuzzy upper β-continuous (resp. intuitionistic fuzzy lower β-continuous intuitionistic fuzzy upper semi-continuous, intuitionistic fuzzy lower semi-continuous) at each point of X.

3. Upper $\boldsymbol{\beta}$ Continuous Intuitionistic Fuzzy Multifunctions

Definition 3.1: An Intuitionistic fuzzy multifunction F: $(X, \mathcal{T}) \to (Y, \Gamma)$ is said to be:

(a) intuitionistic fuzzy Upper β-continuous at a point $x_0 \in X$ if for any intuitionistic fuzzy open set $\widetilde{W}$ of Y such that $F(x_0) \subset \widetilde{W}$ there exists $U \in \beta O(X)$ containing x_0 such that $F(U) \subset \widetilde{W}$

(b) intuitionistic fuzzy upperβ-continuous if it has this property at each point of X.

Theorem 3.2: For an intuitionistic fuzzy multifunction F : $(X, \mathcal{T}) \rightarrow (Y, \Gamma)$ and a point $x \in X$ the following conditions are equivalent:

(a) F is intuitionistic fuzzy upper β-continuous at x.

(b) For each intuitionistic fuzzy open set $\tilde{G}$ of Y, $F(x) \subset \tilde{G}$, there results the relation $x \in sCl(Int\left(F^{+}(\tilde{G})\right))$.

(c) For any semi-open set $U \subset X$ containing x and for any intuitionistic fuzzy open set $\tilde{G}$ of Y, $F(x) \subset \tilde{G}$, there exists a non-empty open set $V \subset U$ such that $F(V) \subset \tilde{G}$.

Proof: (a) (b): Let $x \in X$ and $\tilde{B}$ be any intuitionistic fuzzy open set of Y such that $F(x) \subset \tilde{B}$, there is a $U \in \beta O(X)$ such that $x \in U$ and $F(v) \subset \tilde{B}$, $\forall v \in U$. Thus, $x \in U \subset F^{+}(\tilde{B})$. Since, $U \in \beta O(X)$, $U \subset sCl(Int(U)) \subset sCl(Int\ (F^{+}(\tilde{B})))$. Hence, $x \in sCl(Int\ (F^{+}(\tilde{B})))$.

(b) (c): Let $\tilde{B}$ be any intuitionistic fuzzy open set of Y such that $F(x) \subset \tilde{B}$, then $x \in sCl(Int\left(F^{+}(\tilde{B})\right))$. Let $U \subset X$ be any semi-open set such that $x \in U$, then $U \cap Int(F^{+}(\tilde{B})) \neq \phi$. Put $V = U \cap Int\ (F^{+}(\tilde{B}))$, then V is an semi-open set in X, $V \subset U$, $V \neq \phi$ and $F(V) \subset \tilde{B}$.

(c) (a): Let $\{U_x\}$ be the system of the semi-open sets in X containing x. For any semi-open set $U \subset X$ such that $x \in U$ and $\tilde{B}$ be any intuitionistic fuzzy open set of Y such that $F(x) \subset \tilde{B}$, there exists a non empty open set $B_U \subset U$ such that $F(B_U) \subset \tilde{B}$. Let W = $\bigcup_{U \in U_x} B_U$, then W is open, $x \in sCl(W)$ and $F(w) \subset \tilde{B}$, $\forall w \in W$. Put S = W $\cup \{x\}$, then $W \subset S \subset sCl\ (W)$. Thus, S$\in \beta(X)$, $x \in S$ and $F(w) \subset \tilde{B}$, $\forall w \in S$. Hence, F is intuitionistic fuzzy upper β continuous at x.

Theorem 3.3: For an intuitionistic fuzzy multifunction F: $(X, \mathcal{T}) \rightarrow (Y, \Gamma)$ the following conditions are equivalent:

(a) F is intuitionistic fuzzy upper β continuous.

(b) $F^{+}(\tilde{G}) \in \beta O(X)$, for every intuitionistic fuzzy open set$\tilde{G}$ of Y.

(c) $F^{-}(\tilde{B}) \in \beta C(X)$ for each intuitionistic fuzzy closed set $\tilde{B}$ of Y.

(d) For each point $x \in X$ and for each neighborhood $\tilde{V}$ of $F(x)$ in Y , $F^{+}(\tilde{V})$ is a β-neighborhood of x.

(e) For each point $x \in X$ and for each neighborhood $\tilde{V}$ of $F(x)$ in Y, there is an β-neighborhood U of x such that $F(U) \subset \tilde{V}$.

(f) $\beta Cl(F^{-}(\tilde{B})) \subset F^{-}\left(Cl(\tilde{B})\right)$ for each intuitionistic fuzzy set $\tilde{B}$ of Y.

(g) $sInt(Cl\left(F^{-}(\tilde{B})\right)) \subset F^{-}(Cl(\tilde{B}))$ for any intuitionistic fuzzy set $\tilde{B}$ of Y.

Proof: (a) (b): Let $\tilde{V}$ be any intuitionistic fuzzy open set of Y and $x \in F^{+}(\tilde{V})$. By Theorem 3.2, $x \in sCl(Int\, F^{+}(\tilde{B}\,))$. Therefore, we obtain $F^{+}(\tilde{V}) \subset sCl(Int\, F^{+}(\tilde{B}\,))$. Hence by Lemma 2.2, $F^{+}(\tilde{V}) \in \beta O(X)$**.**

(b) (a): Let x be arbitrarily chosen in X and $\tilde{G}$ be any intuitionistic fuzzy open set of Y such that $F(x) \subset \tilde{G}$, so $x \in F^{+}(\tilde{G})$. By hypothesis $F^{+}(\tilde{G}) \in \beta O$(X), we have $x \in F^{+}(\tilde{G}) \subset sCl(Int\,(F^{+}(\tilde{G})))$ and thus, F is intuitionistic fuzzy upper β-continuous at x according to Theorem 3.2 As x was arbitrarily chosen, F is intuitionistic fuzzy upper β-continuous.

(b) (c): This follows from Lemma 2.6 that $[F^{-}(\tilde{A})]^{c} = F^{+}(\tilde{A}^{c})$.

(c) (f): Let $\tilde{B}$ be any intuitionistic fuzzy open set of Y. Then by (c), $F^{-}(Cl(\tilde{B}))$ is an β closed set in X. Thus by Lemma 2.1 we have $F^{-}(Cl(\tilde{B})) \supset sInt\,(Cl\left(F^{-}\left(Cl(\tilde{B})\right)\right)) \supset sInt(Cl(F^{-}(\tilde{B}))) \supset F^{-}(\tilde{B}) \cup$ $sInt(Cl\left(F^{-}(\tilde{B})\right)) \supset \beta Cl\left(F^{-}(\tilde{B})\right)$.

(f) (g): Let $\tilde{B}$ be any intuitionistic fuzzy open set of Y. By Lemma 2.1, we have $\beta Cl\left(F^{-}(\tilde{B})\right) = F^{-}(\tilde{B}) \cup sInt(Cl\left(F^{-}(\tilde{B})\right)) \subset F^{-}(Cl(\tilde{B}))$.

(g) (c): Let $\tilde{B}$ be any intuitionistic fuzzy closed set of Y. Then by (g) we have, $sInt\left(Cl\left(F^{-}(\tilde{B})\right)\right) \subset F^{-}(\tilde{B}) \cup sInt\left(Cl\left(F^{-}(\tilde{B})\right)\right) \subset F^{-}(Cl(\tilde{B})) = F^{-}(\tilde{B})$.

Hence, By Lemma 2.1, $F^{-}(\tilde{B}) \in \beta C(X)$.

(b) $\Rightarrow$ **(d):** Let $x \in X$ and $\tilde{V}$ be a neighborhood of $F(x)$ in Y. Then there is an intuitionistic fuzzy open set $\tilde{G}$ of Y such that $F(x) \subset \tilde{G} \subset \tilde{V}$. Hence, $x \in F^+(\tilde{G}) \subset F^+(\tilde{V})$. Now by hypothesis $F^+(\tilde{G}) \in \beta O(X)$, and thus $F^+(\tilde{V})$ is an β-neighborhood of x.

(d) $\Rightarrow$ **(e):** Let $x \in X$ and $\tilde{V}$ be a neighborhood of $F(x)$ in Y. Put $U = F^+(\tilde{V})$. Then U is an β -neighborhood of x and $F(U) \subset \tilde{V}$.

(e) $\Rightarrow$ $(\boldsymbol{a})$**:** Let $x \in X$ and $\tilde{V}$ be an intuitionistic fuzzy set in Y such that $F(x) \subset \tilde{V}$. $\tilde{V}$, being an intuitionistic fuzzy open set in Y , is a neighborhood of F(x) and according to the hypothesis there is a β-neighborhood U of x such that $F(U) \subset \tilde{V}$. Therefore, there is $A \in \beta O(X)$ such that $x \in A \subset U$ and hence, $F(A) \subset F(U) \subset \tilde{V}$.

Corollary 3.4 [27]: For a fuzzy multifunction F: $(X, \mathcal{T}) \rightarrow (Y, \sigma)$ the following conditions are equivalent:

(a) F is fuzzy upper β continuous.

(b) $F^+(G) \in \beta O(X)$, for every fuzzy open set G of Y.

(c) $F^-(B) \in \beta C(X)$ for each fuzzy closed set B of Y.

(d) For $\forall\, x \in X$ and for each neighborhood V of $F(x)$ in Y , $F^+(V)$ is a β-neighborhood of x.

(e) For $\forall\, x \in X$ and for each neighborhood V of $F(x)$ in Y , there is a β-neighborhood U of x such that $F(U) \subset V$.

(f) $\beta Cl(F^-(B)) \subset F^-\big(Cl(B)\big)$ for each fuzzy set B of Y.

(g) $sInt(Cl\big(F^-(B)\big)) \subset F^-(Cl(B))$ for any fuzzy set B of Y.

Corollary 3.5 [26]: For a multifunction F from a topological space $(X, \mathcal{T})$ to another topological space $(Y, \mathfrak{J})$ the following conditions are equivalent:

(a) F is upper β continuous.

(b) $F^+(G) \in \beta O(X)$, for every open set G of Y.

(c) $F^{-}(B) \in \beta C(X)$ for each closed set B of Y.

(d) For $\forall\, x \in X$ and for each neighborhood V of $F(x)\ in\ Y$, $F^{+}(V)$ is a β-neighborhood of x.

(e) For $\forall\, x \in X$ and for each neighborhood V of $F(x)\ in\ Y$, there is a β-neighborhood U of x such that $F(U) \subset V$.

(f) $\beta Cl(F^{-}(B)) \subset F^{-}\big(Cl(B)\big)$ for each set B of Y.

(g) $sInt(Cl\big(F^{-}(B)\big)) \subset F^{-}(Cl(B))$ for any set B of Y.

3. Lower β-Continuous Intuitionistic Fuzzy Multifunctions

Definition 4.1: An Intuitionistic fuzzy multifunction F : $(X, \mathcal{T}) \to (Y, \Gamma)$ is said to be:

(a) Intuitionistic fuzzy lower β-continuous at a point $x_0 \in X$, if for any Intuitionistic fuzzy open set $\widetilde{W} \subset Y$ such that $F(x_0)q\widetilde{W}$ there exists $U \in \beta O(X)$ containing x_0 such that $F(x)q\widetilde{W}\ \forall x \in U$.

(b) Intuitionistic fuzzy lower β-continuous if it is intuitionistic fuzzy lower β-continuous at every point of X.

Definition 4.2: Let $\widetilde{A}$ be an intuitionistic fuzzy set of an intuitionistic fuzzy topological space (Y, Γ). Then $\tilde{V}$ is said to be a neighbourhood of $\widetilde{A}$ in Y if there exists an intuitionistic fuzzy open set $U\ of\ Y$ such that $\widetilde{A} \subset \tilde{U} \subset \tilde{V}$.

Theorem 4.3: Let F: $(X, \mathcal{T}) \to (Y, \Gamma)$ be an intuitionistic fuzzy multifunction and let $x \in X$. Then the following statements are equivalent:

(a) F is intuitionistic fuzzy lower β-continuous at x.

(b) For each intuitionistic fuzzy open set $\tilde{B}$ of Y with $F(x)q\tilde{B}$, implies $x \in sCl(Int(\, F^{-}(\tilde{B}))$

(c) For any semi-open set $U\ of\ X$ containing x and for any intuitionistic fuzzy open set $\tilde{B}$ of Y with $F(x)q\tilde{B}$, there exists a non empty open set $V \subset U$ such that $F(v)q\tilde{B},\ \forall\, v \in V$.

Proof: (a) **(b):** Let $x \in X$ and $\tilde{B}$ be any intuitionistic fuzzy open set of Y such that $F(x)q\tilde{B}$. Then by (a) $\exists\, U \in \beta O(X)$ such that $x \in U$ and $F(v)q\tilde{B}$,$\forall\, v \in U$. Thus, $x \in U \subset F^{-}(\tilde{B})$. Now $U \in \beta O(X)$ implies $U \subset sCl(Int(U))$.
Hence, $x \in sCl(Int\ F^{-}(\tilde{B}))$.

(b) **(c):** Let $\tilde{B}$ be any intuitionistic fuzzy open set of Y such that $F(x)q\tilde{B}$, then $x \in sCl(intF^{-}(\tilde{B}))$. Let U be any semi-open set of X containing x. Then $U \cap Int(F^{-}(\tilde{B})) \neq \phi$. Put $V = U \cap Int\ (F^{-}(\tilde{B}))$, then V is an semi-open set of X, $V \subset U,\ V \neq \phi$ and $F(v)q\tilde{B},\ \forall\, v \in V$.

(c) $(\boldsymbol{a})$: Let $\{U_x\}$ be the system of the semi-open sets in X containing x. For any semi-open set $U \subset X$ such that $x \in U$ and any intuitionistic fuzzy open set $\tilde{B}$ of Y such that $F(x)q\tilde{B}$, there exists a non empty open set $B_U \subset U$ such that $F(v)q\tilde{B}, \forall v \in B_U$. Let W = $\bigcup_{U \in U_x} B_U$, then W is open in X, $x \in sCl(W)$ and $F(v)q\tilde{B}$, $\forall\, v \in W$. Put S = W $\cup \{x\}$, then $W \subset S \subset sCl\ (W)$. Thus, $S \in \beta O(X)$, $x \in S$ and $F(v)q\tilde{B}$, $\forall\, v \in S$. Hence, F is intuitionistic fuzzy lower β-continuous at x.

Definition 4.4 [25]: Let X and Y are two non empty sets. A function F : X → Y is called fuzzy multifunction if F(x) is a fuzzy set in Y, $\forall$ x $\in$ X.

Theorem 4.5: Let F: $(X,\mathcal{T})$→(Y,Γ) be an intuitionistic fuzzy multifunction, Then the following statements are equivalent:

(a) F is intuitionistic fuzzy lower β-continuous.

(b) $F^{-}(\tilde{G}) \in \beta O(X)$, for every intuitionistic fuzzy open set $\tilde{G}$ of Y.

(c) $F^{+}(\tilde{V}) \in \beta C(X)$ for every intuitionistic fuzzy closed set $\tilde{V}$ of Y.

(d) $sInt(Cl(\ F^{+}(\tilde{B}))) \subset F^{+}(Cl(\widetilde{B}))$, for each intuitionistic fuzzy set $\tilde{B}$ of Y.

(e) $F(sInt\ (Cl(A))) \subset Cl(F(A))$, for each subset A of X.

(f) $F(\beta Cl(A)) \subset Cl(F(A))$, for each subset A of X,

(g) $\beta Cl(F^{+}(\tilde{B})) \subset F^{+}(Cl(\tilde{B}))$, for each Intuitionistic fuzzy set $\tilde{B}$ of Y .

(h) $F(Cl\left(Int(Cl(A))\right)) \subset Cl(F(A))$ for any subset A of X.

Proof: (a) (b): Let $\widetilde{G}$ be any intuitionistic fuzzy open set of Y and $x \in F^{-}(\tilde{G})$, so $F(x)q\tilde{G}$, since F is Intuitionistic Fuzzy lower β-continuous, by Theorem 4.3 it follows that $x \in sCl(Int\, F^{-}(\tilde{B}\,))$. As x is chosen arbitrarily in $F^{-}(\tilde{G})$, we have $F^{-}(\tilde{G}) \subset sCl(Int\, F^{-}(\tilde{G}))$ and thus, $F^{-}(\tilde{G}) \in \beta O(X)$.

(b) (a): Let x be arbitrarily chosen in X and $\widetilde{G}$ be any intuitionistic fuzzy open set of Y such that $F(x)q\tilde{G}$, so $x \in F^{-}(\tilde{G})$. By hypothesis $F^{-}(\tilde{G}) \in \beta O$(X), we have $x \in F^{-}(\tilde{G}) \subset sCl(Int\,(F^{-}(\tilde{G})))$ and thus, F is intuitionistic fuzzy lower β-continuous at x according to Theorem 4.3 As x was arbitrarily chosen, F is intuitionistic fuzzy lower β-continuous.

(b) (c): Obvious.

(c) (d): Let $\tilde{B}$ be any arbitrary intuitionistic fuzzy set of Y. Since $Cl(\tilde{B})$ is intuitionistic fuzzy closed set in Y by hypothesis, $F^{+}(Cl\,(\tilde{B})) \in \beta C(X)$. Hence, by Lemma 2.1, we obtain

$$F^{+}(Cl(\tilde{B})) \supset sInt\,(Cl\left(F^{+}\left(Cl(\tilde{B})\right)\right)) \supset sInt(Cl(F^{+}(\tilde{B}))).$$

(d) (e): Suppose that (d) holds, and let A be an arbitrary subset of X. Let us put $\tilde{B} = F(A)$, then $A \subset F^{+}(\tilde{B})$. Therefore, by hypothesis, we have $sInt\,(Cl(A)) \subset sInt(Cl(F^{+}(\tilde{B}))) \subset F^{+}(Cl(\tilde{B}))$. Therefore,
$F(sInt(Cl(A))) \subset F(F^{+}\left(Cl(\tilde{B})\right)) \subset Cl(\tilde{B}) = Cl(F(A))$.

(e) (c): Suppose that (e) holds, and let $\tilde{B}$ be any intuitionistic fuzzy closed set of Y. Put $A = F^{+}(\tilde{B})$, then $F(A) \subset \tilde{B}$. Therefore, by hypothesis, we have $F(sInt(Cl(A))) \subset Cl(F(A)) \subset Cl(\tilde{B}) = \tilde{B}$ and $F^{+}(F(sInt\,(Cl(A)))) \subset F^{+}(\tilde{B})$. Since we always have $F^{+}\left(F(sInt(Cl(A)))\right) \supset sInt(Cl(A))$, we obtain $F^{+}(\tilde{B}) \supset sInt(Cl(F^{+}(\tilde{B})))$. Hence, by Lemma, 2.1, $F^{+}(\tilde{B}) \in \beta C(X)$.

(c) (f): Since $A \subset F^{+}(F(A))$, we have $A \subset F^{+}(Cl(F(A)))$. Now *Cl(F(A))* is an intuitionistic fuzzy closed set in *Y* and so by hypothesis $F^{+}(Cl(F(A))) \in \beta C(X)$. Thus, $\beta Cl(A) \subset F^{+}(Cl(F(A)))$. Consequently, $F(\beta Cl(A)) \subset F(F^{+}(Cl(F(A)))) \subset Cl(F(A))$.

(f) (c): Let $\tilde{B}$ be any intuitionistic fuzzy closed set of Y. Replacing A by $F^{+}(\tilde{B})$ we get by (f), $F(\beta Cl(F^{+}(\tilde{B}))) \subset Cl(F\left(F^{+}(\tilde{B})\right)) \subset Cl(\tilde{B}) = \tilde{B}$.

Consequently, $\beta Cl(F^+(\tilde{B})) \subset F^+(\tilde{B})$. But $F^+(\tilde{B}) \subset \beta Cl(F^+(\tilde{B}))$. And so, $\beta Cl(F^+(\tilde{B})) = F^+(\tilde{B})$. Thus, $F^+(\tilde{B}) \in \beta C(X)$.

(f) (g): Let $\tilde{B}$ be any intuitionistic fuzzy set of Y. Replacing A by $F^+(\tilde{B})$ we get by (f), $F(\beta Cl(F^+(\tilde{B}))) \subset Cl(F\left(F^+(\tilde{B})\right)) \subset Cl(\tilde{B})$. Thus, $\beta Cl(F^+(\tilde{B})) \subset F^+(Cl(\tilde{B}))$.

(g) (f): Replacing $\tilde{B}$ by F(A), where A is a subset of X, we get by (g), $\beta Cl(A) \subset \beta Cl\left(F^+(F(A))\right) = \beta Cl\left(F^+(\tilde{B})\right) = F^+\left(Cl(\tilde{B})\right) = F^+(Cl(F(A)))$. Thus, $F(\beta Cl(A)) \subset F\left(F^+(Cl(F(A))\right) \subset Cl(F(A))$.

(e) (h): Follows from by Lemma 2.9.

(h) (a): Let $x \in X$ and $\tilde{V}$ be any intuitionistic fuzzy open set in Y such that $F(x)q\tilde{V}$. Then $x \in F^-(\tilde{V})$. We shall show that $F^-(\tilde{V}) \in \beta(X)$. By the hypothesis, we have $F(Cl(Int(Cl(F^+(\tilde{V}^c))))) \subset Cl\left(F\left(F^+(\tilde{V}^c)\right)\right) \subset (\tilde{V}^c)$, which implies that $Cl(Int(Cl(F^+(\tilde{V}^c)))) \subset F^+(\tilde{V}^c) \subset (F^-(\tilde{V}))^c$. Therefore, we obtain $F^-(\tilde{V}) \subset Int(Cl\left(Int\left(F^-(\tilde{V})\right)\right))$. Hence, $F^-(\tilde{V}) \in \beta(X)$. Put U $= F^-(\tilde{V})$. Then $x \in U \in \beta O(X)$ and $F(u)q\tilde{V}$ for every $u \in U$ thus, F is intuitionistic fuzzy lower β continuous.

Corollary 4.6[27]: For a fuzzy multifunction $F: (X, \mathcal{T}) \rightarrow (Y, \sigma)$ the following statements are equivalent:

(a) F is fuzzy lower α-continuous.

(b) $F^-(G) \in \beta O(X)$, for every fuzzy open set G of Y.

(c) $F^+(V) \in \beta C(X)$ for every fuzzy closed set $\tilde{V}$ of Y.

(d) $sInt(Cl(F^+(B))) \subset F^+(Cl(B))$, for each fuzzy set B of Y.

(e) $F(sInt\left(Cl(A)\right)) \subset Cl(F(A))$, for each subset A of X.

(f) $F(\beta Cl(A)) \subset Cl(F(A))$, for each subset A of X,

(g) $\beta Cl(F^+(B)) \subset F^+(Cl(B$ for each fuzzy set $\tilde{B}$ of Y .

(h) $F(Cl\left(Int(Cl(A))\right)) \subset Cl(F(A))$ for any subset A of X.

Corollary 4.7 [26]: For a multifunction F: $(X,\mathcal{T})$ → (Y,σ) the following statements are equivalent:

(a) F is lower β-continuous.

(b) $F^{-}(G) \in \beta O(X),$ for every open set G of Y.

(c) $F^{+}(V) \in \beta C(X)$ for every closed set $\tilde{V}$ of Y.

(d) $sInt(Cl(F^{+}(B))) \subset F^{+}(Cl(B))$, for each set B of Y.

(e) $F(sInt\,(Cl(A))) \subset Cl(F(A))$, for each subset A of X.

(f) $F(\beta Cl(A)) \subset Cl(F(A))$, for each subset A of $X,$

(g) $\beta Cl(F^{+}(B)) \subset F^{+}(Cl(B$ for each set $\tilde{B}$ of Y .

(h) $F(Cl\left(Int(Cl(A))\right)) \subset Cl(F(A))$ for any subset A of X.

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