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GIBONACCI EXTENSIONS OF A CATALAN DELIGHT WITH GRAPH-THEORETIC CONFIRMATIONS

Abstract: We explore the polynomial extensions of the well known Catalan identity $F_{n+k}F_{n-k} - F_n^2 = (-1)^{n+k+1}F_k^2$ [4] and their Jacobsthal counterparts; and then confirm them using graph-theoretic techniques.

Keywords: Gibonacci, Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, and Jacobsthal-Lucas Polynomials; and Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, and Jacobsthal-Lucas Numbers; Fubini's Principle, Weighted Digraph, Weighted Adjacency Matrix, Walk, and Q-matrix.

Mathematical Subject Classification (2010) No.: Primary 05A19, 11B37, 11B39, 11Cxx.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_n(x) = a(x)z_{n-1}(x) + b(x)z_{n-2}(x)$, where x is a complex variable; $a(x)$, $b(x)$, $z_0(x)$ and $z_1(x)$ are arbitrary complex polynomials; and $n \geq 2$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*.

Clearly, $f_n(x) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [5, 8].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. The Pell *numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [5, 8].

Finally, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial* [5, 8]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$.

Fibonacci and Jacobsthal polynomials, and Lucas and Jacobsthal-Lucas polynomials are closely related by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [5, 8].

Fibonacci and Lucas polynomials can also be defined explicitly by the Binet-like formulas

$$f_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$l_n(x) = \alpha^n + \beta^n,$$

where $\alpha = \alpha(x) = \dfrac{x + \Delta}{2}$ and $\beta = \beta(x) = \dfrac{x - \Delta}{2}$ are the solutions of the equation $t^2 - xt - 1 = 0$, and $\Delta = \Delta(x) = \sqrt{x^2 + 4} = \alpha - \beta$ [7].

Jacobsthal and Jacobsthal-Lucas polynomials also can be defined explicitly:

$$J_n(x) = \frac{u^n - v^n}{u - v} \quad \text{and} \quad j_n(x) = u^n + v^n,$$

where $u = u(x) = \dfrac{1+D}{2}$ and $v = v(x) = \dfrac{1-D}{2}$ are the solutions of the equation $t^2 - t - x = 0$, and $D = D(x) = \sqrt{4x+1} = u - v$ [7].

In the interest of brevity, clarity, and convenience, we *omit* the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$; we also let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $c_n = J_n(x)$ or $j_n(x)$; and correspondingly, $G_n = F_n$ or L_n, $B_n = P_n$ or Q_n, and $C_n = J_n$ or j_n.

Using the Binet-like formulas, we can easily establish the following results:

$$\begin{aligned} l_n &= f_{n+1} + f_{n-1} & \qquad j_n(x) &= J_{n+1}(x) + xJ_{n-1}(x) \\ l_{2n} &= \Delta^2 f_n^2 + 2(-1)^n & \qquad j_{2n}(x) &= D^2 J_n^2(x) + 2(-x)^n \\ l_{2n} &= l_n^2 - 2(-1)^n & \qquad j_{2n}(x) &= j_n^2(x) - 2(-x)^n. \end{aligned}$$

2. Gibonacci Extensions of a Catalan Delight

In 1879, the Belgian mathematician, E. C. Catalan established the charming identity

$$F_{n+k}F_{n-k} - F_n^2 = (-1)^{n+k+1} F_k^2,$$

where $n \geq k$; it is a generalization of the Cassini's formula $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ [4]. Its Lucas companion is

$$L_{n+k}L_{n-k} - L_n^2 = 5(-1)^{n+k} F_k^2 .$$

Their gibonacci extensions are given by the identity

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1} f_k^2 & \text{if } g_n = f_n \\ (-1)^{n+k} \Delta^2 f_k^2 & \text{if } g_n = l_n. \end{cases} \tag{1}$$

We can establish this using the Binet-like formulas for f_n and l_n.

3. Graph-theoretic Confirmations

Next we confirm identity (1) using graph-theoretic techniques. First, we present some basic facts.

Consider the *weighted digraph* D_1 with vertices v_1 and v_2 in Figure 1. A *weight* is assigned to each edge.

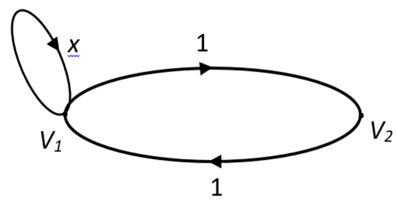

Figure 1: Weighted Digraph D_1

Its *weighted adjacency matrix* is the Q-matrix

$$Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix},$$

where $Q = Q(x)$. It then follows by induction that

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $n \geq 1$ [3].

A *walk* from vertex v_i to vertex v_j is a sequence $v_i - e_i - v_{i+1} - \cdots - v_{j-1} - e_{j-1} - v_j$ of vertices v_k and edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The walk is *closed* if $v_i = v_j$; otherwise, it is *open*. The *length* of a walk is the number of edges in the walk. The *weight* of a walk is the product of the weights of the edges along the walk.

The following theorem provides a powerful tool for computing the weight of a walk of length n from v_i to v_j [2, 3].

Theorem 1: *Let A be the weighted adjacency matrix of a weighted and connected digraph with vertices $v_1, v_2, \ldots, v_k$. Then the ijth entry of the matrix A^n gives the sum of the weights of all walks of length n from v_i to v_j, where $n \geq 1$.* □

The next result follows from this theorem.

Corollary 1: *The ijth entry of Q^n gives the sum of the weights of all walks of length n from v_i to v_j in the weighted digraph D_1, where $1 \leq i, j \leq n$.* □

Consequently, the sum of the weights of all closed walks of length n originating at v_1 is f_{n+1}, and that of walks of length n originating at v_2 is f_{n-1}. So the sum of the weights of all closed walks of length n is $f_{n+1} + f_{n-1} = l_n$. These facts play a pivotal role in our graph-theoretic proofs. So does the "weighted" version of *Fubini's principle*: Two different ways of adding the elements of a finite set yields the same result [1].

3.1: Proof of Identity (1): The proof depends on the next lemma.

Lemma 1: *Let $n \geq k$. Then*

$$g_{n+k} g_{n-k} = g_n^2 - (-1)^{n-k} \gamma f_k^2 ,$$

where

$$\gamma \begin{cases} 1 & \text{if } g_n = f_n \\ -\Delta^2 & \text{otherwise.} \end{cases}$$

Proof: Let $g_n = f_n$. Using the Binet-like formula for f_n and the identity $l_{2n} = \Delta^2 f_n^2 + 2(-1)^n$, we have

$$\begin{aligned} \Delta^2 f_{n+k} f_{n-k} &= (\alpha^{n+k} - \beta^{n+k})(\alpha^{n-k} - \beta^{n-k}) \\ &= l_{2n} - (-1)^n (\alpha^k \beta^{-k} - \alpha^{-k} \beta^k) \end{aligned}$$

$$= l_{2n} - (-1)^{n-k} l_{2k}$$
$$= [\Delta^2 f_n^2 + 2(-1)^n] - (-1)^{n-k}[\Delta^2 f_k^2 + 2(-1)^k]$$
$$= \Delta^2[f_n^2 - (-1)^{n-k} f_k^2]$$
$$f_{n+k}f_{n-k} = f_n^2 - (-1)^{n-k} f_k^2.$$

The case $g_n = l_n$ follows similarly, using the Binet-like formula for l_n, and the identities $l_{2n} = l_n^2 - 2(-1)^n$ and $l_{2n} = \Delta^2 f_n^2 + 2(-1)^n$. □

We are now ready for the graph-theoretic confirmations.

Part I: Suppose $g_n = f_n$. Let A, B, and C be the sets of closed walks of lengths $n+k-1$, $n-k-1$, and $k-1$ originating at v_1, respectively. By Corollary 1, the sums of the weights of the walks in A, B, and C are f_{n+k} , f_{n-k}, and f_k, respectively. Consequently, the sums S_1 and S_2 of the weights of the elements in $A \times B$ and $C \times C$ are $f_{n+k}f_{n-k}$ and f_k^2, respectively.

Let $$S = S_1 + (-1)^{n-k} S_s .$$

Then
$$S = f_{n+k}f_{n-k} + (-1)^{n-k} f_k^2 \quad . \tag{2}$$

We will now compute S in a different way.

Suppose (v,w) be an arbitrary element of $A \times B$. If both v and w begin with a loop, the sum of the weights of such pairs (v,w) is $(xf_{n+k-1})(xf_{n-k-1}) = x^2 f_{n+k-1}f_{n-k-1}$; if v begins with a loop and w does *not*, the corresponding sum is $(xf_{n+k-1})(1 \cdot 1 \cdot f_{n-k-2}) = xf_{n+k-1}f_{n-k-2}$; if v does *not* begin with a loop and w does, the resulting sum is $(1 \cdot 1 \cdot f_{n+k-2})(xf_{n-k-1}) = xf_{n+k-2}f_{n-k-1}$; and if neither begins with a loop, the resulting sum is $(1 \cdot 1 \cdot f_{n+k-2})(1 \cdot 1 \cdot f_{n+k-2}) = f_{n+k-2}f_{n-k-2}$.

Using Lemma 1, the sum S_1 of the weights of all elements in $A \times B$ is then given by

$$\begin{aligned} S_1 &= x^2 f_{n+k-1} f_{n-k-1} + x f_{n+k-1} f_{n-k-2} + x f_{n+k-2} f_{n-k-1} + f_{n+k-2} f_{n-k-2} \\ &= x f_{n+k-1}(x f_{n-k-1} + f_{n-k-2}) + f_{n+k-2}(x f_{n-k-1} + f_{n-k-2}) \\ &= (x f_{n+k-1} + f_{n+k-2}) f_{n-k} \\ &= f_{n+k} f_{n-k} \\ &= f_n^2 - (-1)^{n-k} f_k^2 . \end{aligned}$$

Now let (v, w) be an arbitrary element of $C \times C$. If both v and w begin with a loop, the sum of the weights of such pairs is $(x f_{k-1})(x f_{k-1}) = x^2 f_{k-1}^2$; if v begins with a loop, but w does *not*, the corresponding sum is $(x f_{k-1})(1 \cdot 1 \cdot f_{k-2}) = x f_{k-1} f_{k-2}$; if v does *not* begin with a loop, but w does, the resulting sum is $(1 \cdot 1 \cdot f_{k-2})(x f_{k-1}) = x f_{k-1} f_{k-2}$; and if neither does, the resulting sum is $(1 \cdot 1 \cdot f_{k-2})(1 \cdot 1 \cdot f_{k-2}) = f_{k-2}^2$.

Thus, the sum S_2 of the weights of all elements in $C \times C$ is given by

$$\begin{aligned} S_2 &= x^2 f_{k-1}^2 + 2x f_{k-1} f_{k-2} + f_{k-2}^2 \\ &= (x f_{k-1} + f_{k-2})^2 \\ &= f_k^2 . \end{aligned}$$

Thus,

$$\begin{aligned} S &= S_1 + (-1)^{n-k} S_2 \\ &= [f_n^2 - (-1)^{n-k} f_k^2] + (-1)^{n-k} f_k^2 \\ &= f_n^2 . \end{aligned}$$

The desired result now follows by equating the two values of S.

Part II: Suppose $g_n = l_n$. Let A and B be the sets of closed walks of lengths $n+k$ originating at v_1 and v_2, respectively; and R and S the sets of closed walks of lengths $n-k$ originating at v_1 and v_2, respectively. Let E be the set of closed walks of length $k-1$ originating at v_1. Then the sum S_1 of the weights of all elements in $(A \cup \mathrm{B}) \times (R \cup S)$ is $S_1 = l_{n+k} l_{n-k}$, and the sum S_2 of the weights of all elements in $E \times E$ is $S_2 = f_k^2$. Let $S = S_1 - (-1)^{n-k} \Delta^2 S_2$. Then

$$S = l_{n+k} l_{n-k} - (-1)^{n-k} \Delta^2 f_k^2 \ . \tag{3}$$

We now compute $S = S_1 - (-1)^{n-k} \Delta^2 S_2$ in a different way. To this end, Let (v, w) be an arbitrary element of $(A \cup \mathrm{B}) \times (R \cup S)$.

Case 1: Suppose $v \in A$ and $w \in R$. If both v and w begin with a loop, the sum of the weights of such pairs (v, w) is $(xf_{n+k})(xf_{n-k}) = x^2 f_{n+k} f_{n-k}$; if v begins with a loop and w does *not*, the sum of the weights of the pairs is $(xf_{n+k})(1 \cdot 1 \cdot f_{n-k-1}) = xf_{n+k} f_{n-k-1}$; if v does *not* begin with a loop, but w does, the resulting sum is $(1 \cdot 1 \cdot f_{n+k-1})(xf_{n-k}) = xf_{n+k-1} f_{n-k}$; and if neither does, the resulting sum is $(1 \cdot 1 \cdot f_{n+k-1})(1 \cdot 1 \cdot f_{n-k-1}) = f_{n+k-1} f_{n-k-1}$. Thus, the sum T_1 of the weights of all pairs (v, w) in Case 1 is given by

$$T_1 = x^2 f_{n+k} f_{n-k} + xf_{n+k} f_{n-k-1} + xf_{n+k-1} f_{n-k} + f_{n+k-1} f_{n-k-1}.$$

Case 2: Suppose $v \in A$ and $w \in S$. If v begins with a loop, the sum of the weights of such pairs is $(xf_{n+k}) f_{n-k-1} = xf_{n+k} f_{n-k-1}$; and if v does *not*, the resulting sum is $(1 \cdot 1 \cdot f_{n+k-1})(xf_{n-k-1}) = xf_{n+k-1} f_{n-k-1}$. So the sum T_2 of all pairs in Case 2 is

$$T_2 = xf_{n+k} f_{n-k-1} + f_{n+k-1} f_{n-k-1} \ .$$

Case 3: Suppose $v \in B$ and $w \in R$. If w begins with a loop, the sum of the weights of such pairs is $f_{n+k-1}(xf_{n-k}) = xf_{n+k-1} f_{n-k}$; and if v does *not*, the resulting sum is $f_{n-k-1}(1 \cdot 1 \cdot f_{n-k-1}) = f_{n+k-1}f_{n-k-1}$. So the sum T_2 of all pairs in Case 3 is

$$T_3 = xf_{n+k-1}f_{n-k} + f_{n+k-1}f_{n-k-1}\,. \tag{4}$$

Case 4: Suppose $v \in B$ and $w \in S$. The sum of the weights of such pairs is $T_4 = f_{n+k-1}f_{n-k-1}$.

By combining all four cases and Lemma 1, we have

$$\begin{aligned}
S_1 &= T_1 + T_2 + T_3 + T_4 \\
&= x^2 f_{n+k}f_{n-k} + 2xf_{n+k}f_{n-k-1} + 2xf_{n+k-1}f_{n-k} + 4f_{n+k-1}f_{n-k-1} \\
&= xf_{n+k}(xf_{n-k} + f_{n-k-1}) + f_{n-k-1}(xf_{n+k} + f_{n+k-1}) \\
&\qquad\qquad + 2f_{n+k-1}(xf_{n-k} + f_{n-k-1}) + f_{n+k-1}f_{n-k-1} \\
&= xf_{n+k}f_{n-k-1} + f_{n+k+1}f_{n-k-1} + 2f_{n+k-1}f_{n-k+1} + f_{n+k-1}f_{n-k-1} \\
&= (xf_{n+k} + f_{n+k-1})f_{n-k+1} + (f_{n+k+1} + f_{n+k-1})f_{n-k-1} + f_{n+k-1}f_{n-k+1} \\
&= f_{n+k+1}f_{n-k+1} + l_{n+k}f_{n-k-1} + f_{n+k-1}f_{n-k+1} \\
&= (f_{n+k+1} + f_{n+k-1}f_{n-k+1}) + l_{n+k}f_{n-k-1} \\
&= l_{n+k}f_{n-k+1} + l_{n+k}f_{n-k-1} \\
&= l_{n+k}(f_{n-k+1} + f_{n-k-1}) \\
&= l_{n+k}l_{n-k} \\
&= l_n^2 + (-1)^{n-k}\Delta^2 f_k^2\,.
\end{aligned}$$

Now let (v, w) be an arbitrary element of $E \times E$. The sum S_2 of the weights of the pairs of elements in $E \times E$ is $S_2 = f_k^2$.

Thus,

$$\begin{aligned} S &= S_1 - (-1)^{n-k} \Delta^2 S_2 \\ &= [l_n^2 + (-1)^{n-k} \Delta^2 f_k^2] - (-1)^{n-k} \Delta^2 f_k^2 \\ &= l_n^2. \end{aligned}$$

Equating this with the value of S in equation (3) gives the desired result. □

Next we explore the Jacobsthal companion of the gibonacci extension (1).

4. Jacobsthal Counterparts

Let $g_n = f_n$ in identity (1), and $u = 1/\sqrt{x}$. Multiplying the resulting equation with x^{n-1} yields

$$\left[x^{(n+k-1)/2} f_{n+k}\right] \left[x^{(n-k-1)/2} f_{n-k}\right] - \left[x^{(n-1)/2} f_n^2\right] = x^{n-k} \cdot (-1)^{n+k+1} \left[x^{(k-1)/2} f_k\right]^2$$

$$J_{n+k}(x) J_{n-k}(x) - J_n^2(x) = -(-x)^{n-k} J_k^2(x),$$

where $f_n = f_n(u)$.

On the other hand, let $g_n = l_n$. Multiplying the corresponding equation with x^n, we get

$$\left[x^{(n+k)/2} l_{n+k}\right] \left[x^{(n-k)/2} l_{n-k}\right] - \left[x^{n/2} l_n\right]^2 = (-1) x^{n+k} \left(\frac{D^2}{x}\right) x^{n-k} \left[x^{(k-1)/2} f_k\right]^2$$

$$j_{n+k}(x) j_{n-k}(x) - j_n^2(x) = D^2 (-x)^{n-k} j_k^2(x),$$

where $f_n = f_n(u)$

and $l_n = l_n(u)$.

We thus, have

$$c_{n+k}c_{n-k} - c_n^2 = \begin{cases} -(-x)^{n-k} J_k^2(x) & \text{if } c_n = J_n(x) \\ D^2(-x)^{n-k} J_k^2(x) & \text{if } c_n = j_n(x). \end{cases} \tag{5}$$

Obviously, this can be established independently.

In particular, we have

$$C_{n+k}C_{n-k} - C_n^2 = \begin{cases} -(-2)^{n-k} J_k^2 & \text{if } C_n = J_n \\ 9(-2)^{n-k} J_k^2 & \text{if } C_n = j_n. \end{cases}$$

For example, $j_{17}j_7 - j_{12}^2 = -139{,}392 = (-1)^{19} \cdot 9 \cdot 2^7 \cdot J_5^2$.

Next we confirm the Jacobsthal delight in (5) using graph-theoretic tools.

5. Graph-theoretic Confirmation of Identity (5)

Consider the weighted digraph D_2 in Figure 2 with vertices v_1 and v_2. Its weighted adjacency matrix is given by

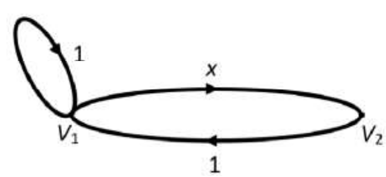

Figure 2: Weighted Digraph D_2

$$M = \begin{bmatrix} 1 & x \\ 1 & 0 \end{bmatrix}.$$

Since,

$$M^n = \begin{bmatrix} J_{n+1}(x) & xJ_n(x) \\ J_{n(x)} & xJ_{n-1(x)} \end{bmatrix},$$

by induction [6], it follows that the sum of the closed walks of length n from v_1 to itself is $J_{n+1}(x)$, and that from v_2 to itself is $xJ_{n-1}(x)$. Consequently, the sum of the weights of all closed walks of length n is $J_{n+1}(x) + xJ_{n-1}(x) = j_n(x)$. As can be expected, these facts play a central role in our graph-theoretic proofs.

In addition, the proofs hinge on equation (5).

We are now ready for the proofs.

Part I: Suppose $c_n = J_n(x)$. Let A, B, and C be the sets of closed walks of length $n+k-1$, $n-k-1$, and $k-1$, all originating at v_1. Then the sum S_1 of the pairs of walks in $A \times B$ is given by $S_1 = J_{n+k}(x)J_{n-k}(x)$; and the sum S_2 of the pairs of walks in $C \times C$ is given by $S_2 = J_k^2(x)$. Then

$$S_1 + (-x)^{n-k} S_2 = J_{n+k}(x)J_{n-k}(x) + (-x)^{n-k} J_k^2(x). \tag{6}$$

Next we compute $S_1 + (-x)^{n+k} S_2$ in a different way. We begin with S_1. Let (v, w) be an arbitrary element of $A \times B$. If both v and w begin with a loop, the sum of the weights of such pairs is $\left[1 \cdot J_{n+k-1}(x)\right]\left[1 \cdot J_{n-k-1}(x)\right] = J_{n+k-1}(x)J_{n-k-1}(x)$; if v begins with a loop and w does *not*, the corresponding sum is $\left[1 \cdot J_{n+k-1}(x)\right]\left[x \cdot 1 \cdot J_{n-k-2}(x)\right] = xJ_{n+k-1}(x)J_{n-k-2}(x)$; if v does *not* begin with a loop and w does, the corresponding sum is $\left[x \cdot 1 \cdot J_{n+k-2}(x)\right]\left[1 \cdot J_{n-k-1}(x)\right] = xJ_{n+k-2}(x)J_{n-k-1}(x)$; and if neither does, the resulting sum is $\left[x \cdot 1 \cdot J_{n+k-2}(x)\right]\left(x \cdot 1 \cdot J_{n-k-2}(x)\right] = x^2 J_{n+k-2}(x)J_{n-k-2}(x)$.

Using equation (5), the sum S_1 of the weights of such pairs is given by

$$\begin{aligned} S_1 &= J_{n+k-1}(x)J_{n-k-1}(x) + xJ_{n+k-1}(x)J_{n-k-2}(x) + xJ_{n+k-2}(x)J_{n-k-1}(x) \\ &\qquad + x^2 J_{n+k-2}(x)J_{n-k-2}(x) \\ &= J_{n+k-1}(x)\left[J_{n-k-1}(x) + xJ_{n-k-2}(x)\right] + xJ_{n+k-2}(x)\left[J_{n-k-1}(x) + xJ_{n-k-2}(x)\right] \end{aligned}$$

$$= \left[J_{n+k-1}(x) + xJ_{n+k-2}(x)\right]J_{n-k}(x)$$

$$= J_{n+k}(x)J_{n-k}(x)$$

$$= J_n^2(x) - (-x)^{n-k}J_k^2(x) \ .$$

To compute S_2, assume (v, w) is an arbitrary element of $C \times C$. If both v and w begin with a loop, the sum of the weights of such pairs is $\left[1 \cdot J_{k-1}(x)\right]\left[1 \cdot J_{k-1}(x)\right] = J_{k-1}^2(x)$; if v begins with a loop and w does *not*, the corresponding sum is $\left[1 \cdot J_{k-1}(x)\right]\left[x \cdot 1 \cdot J_{k-2}(x)\right] = xJ_{k-1}(x)J_{k-2}(x)$; if v does *not* begin with a loop and w does, the corresponding sum is $\left[x \cdot 1 \cdot J_{k-2}(x)\right]\left[1 \cdot J_{k-1}(x)\right] = xJ_{k-1}(x)J_{k-2}(x)$; and if neither does, the resulting sum is $\left[x \cdot 1 \cdot J_{k-2}(x)\right]\left(x \cdot 1 \cdot J_{k-2}(x)\right] = x^2J_{k-2}^2(x)$.

So

$$\begin{aligned} S_2 &= J_{k-1}^2(x) + 2xJ_{k-1}(x)J_{k-2}(x) + x^2J_{k-2}^2(x) \\ &= \left[J_{k-1}(x) + xJ_{k-2}(x)\right]^2 \\ &= J_k^2(x) \, . \end{aligned}$$

Thus,

$$\begin{aligned} S_1 + (-x)^{n-k}S_2 &= \left[J_n^2(x) - (-x)^{n-k}J_k^2(x)\right] + (-x)^{n-k}J_k^2(x) \\ &= J_n^2(x) \, . \end{aligned}$$

This, together with the sum in equation (6), confirms the identity.

Part 2: Suppose $c_n = j_n(x)$. Let A and B be the sets of closed walks of length $n+k$ originating at v_1 and v_2, respectively; let R and S be the sets of closed walks of length $n-k$ originating at v_1 and v_2, respectively; and let E be the set of closed walks of length $k-1$ originating at v_1. Then the sum S_1 of the weights of pairs in $(A \cup B) \ \times (R \cup S)$ is $S_1 = j_{n+k}(x)j_{n-k}(x)$; and the sum S_2 of the weights of pairs in $E \times E$ is $S_2 = J_k^2$.

Then

$$S_1 - D^2(-x)^{n-k} S_2 = j_{n+k}(x) j_{n-k}(x) - D^2(-x)^{n-k} J_k^2(x) . \qquad (7)$$

We now compute the sum $S_1 - D_2(-x)^{n-k} S_2$ in a different way.

First, let (v, w) be any element of $(A \cup B) \times (R \cup S)$.

Case 1: Suppose $v \in A$ and $w \in R$. If both v and w begin with a loop, the sum of the weights of such pairs is $\left[1 \cdot J_{n+k}(x)\right]\left[1 \cdot J_{n-k}(x)\right] = J_{n+k}(x) J_{n-k}(x)$; if v begins with a loop and w does *not*, the corresponding sum is $\left[1 \cdot J_{n+k}(x)\right]\left[x \cdot 1 \cdot J_{n-k-1}(x)\right] = x J_{n+k}(x) J_{n-k-1}(x)$; if v does *not* begin with a loop and w does, the corresponding sum is $\left[x \cdot 1 \cdot J_{n+k-1}(x)\right]\left[1 \cdot J_{n-k}(x)\right] = x J_{n+k-1}(x) J_{n-k}(x)$; and if neither does, the resulting sum is $\left[x \cdot 1 \cdot J_{n+k-1}(x)\right]\left(x \cdot 1 \cdot J_{n-k-1}(x)\right] = x^2 J_{n+k-1}(x) J_{n-k-1}$.

So the sum of the weights of all elements in Case 1 is $J_{n+k}(x) J_{n-k}(x) + x J_n$ $+\ k(x) J_{n-k-1}(x) + x J_{n+k-1}(x) J_{n-k}(x) + x^2 J_{n+k-1}(x) J_{n-k-1}$.

Case 2: Suppose $v \in A$ and $w \in S$. If v begins with a loop, the sum of the weights of such pairs is $\left[1 \cdot J_{n+k}(x)\right]\left[x \cdot J_{n-k-1}(x)\right] = x J_{n+k}(x) J_{n-k-1}(x)$; and if v does *not*, the resulting sum is $\left[x \cdot 1 \cdot J_{n+k-1}(x)\right]\left[x \cdot J_{n-k-1}(x)\right] = x^2 J_{n+k-1}(x) J_{n-k-1}(x)$. Thus, the sum of the weights of all elements in Case 2 is $J_{n+k}(x) J_{n-k-1}(x) + x^2 J_{n+k-1}(x) J_{n-k-1}(x)$.

Case 3: Suppose $v \in B$ and $w \in R$. If w begins with a loop, the sum of the weights of such pairs is $\left[x J_{n+k-1}(x)\right]\left[1 \cdot J_{n-k}(x)\right] = x J_{n+k-1}(x) J_{n-k}(x)$; and if w does *not*, the resulting sum is $\left[x \cdot J_{n+k-1}(x)\right]\left[x \cdot 1 \cdot J_{n-k-1}(x)\right] = x^2 J_{n+k-1}(x) J_{n-k-1}(x)$. So the sum of the weights of all elements in Case 3 is $x J_{n+k-1}(x) J_{n-k}(x) + x^2 J_{n+k-1}(x) J_{n-k-1}(x)$.

Case 4: Suppose $v \in B$ and $w \in S$. The sum of the weights of such pairs is $\left[x \cdot J_{n+k-1}(x)\right]\left[x \cdot J_{n-k-1}(x)\right] = x^2 J_{n+k-1}(x) J_{n-k-1}(x)$.

Combining these cases and using equation (5), the sum S_1 is given by

$$\begin{aligned}
S_1 &= J_{n+k}(x)J_{n-k}(x) + 2xJ_{n+k}(x)J_{n-k-1}(x) + 2xJ_{n+k-1}(x)J_{n-k}(x) \\
&\qquad + 4x^2 J_{n+k-1}(x)J_{n-k-1}(x) \\
&= J_{n+k}(x)\left[J_{n-k}(x) + xJ_{n-k-1}(x)\right] + xJ_{n-k-1}(x)\left[J_{n+k}(x) + xJ_{n+k-1}(x)\right] \\
&\qquad + 2xJ_{n+k-1}(x)\left[J_{n-k}(x) + xJ_{n-k-1}(x)\right] + x^2 J_{n+k-1}(x)J_{n-k-1}(x) \\
&= J_{n+k}(x)J_{n-k+1}(x) + xJ_{n+k-1}(x)J_{n-k-1}(x) + 2xJ_{n+k-1}(x)J_{n-k+1}(x) \\
&\qquad + x^2 J_{n+k-1}(x)J_{n-k-1}(x) \\
&= \left[J_{n+k}(x) + xJ_{n+k-1}(x)\right]J_{n-k+1}(x) + xJ_{n-k-1}(x)\left[J_{n+k+1}(x) + xJ_{n+k-1}(x)\right] \\
&\qquad + xJ_{n+k-1}(x)J_{n-k+1}(x) \\
&= J_{n+k+1}(x)J_{n-k+1}(x) + xJ_{n-k-1}(x)j_{n+k}(x) + xJ_{n+k-1}(x)J_{n-k+1}(x) \\
&= \left[J_{n+k+1}(x) + xJ_{n+k-1}(x)\right]J_{n-k+1}(x) + xJ_{n-k-1}(x)j_{n+k}(x) \\
&= j_{n+k}(x)J_{n-k+1}(x) + xJ_{n-k-1}(x)j_{n+k}(x) \\
&= j_{n+k}(x)\left[J_{n-k+1}(x) + xJ_{n-k-1}(x)\right] \\
&= j_{n+k}(x)j_{n-k}(x) \\
&= j_n^2(x) + D^2(-x)^{n-k}J_k^2(x).
\end{aligned}$$

Now let (v, w) be any element of $E \times E$. If both v and w begin with a loop, the sum of the weights of such pairs is $\left[1 \cdot J_{k-1}(x)\right]\left[1 \cdot Jk-1(x)\right] = J_{k-1}^2(x)$;

if v begins with a loop and w does *not*, the corresponding sum is $\left[1 \cdot J_{k-1}(x)\right]\left[x \cdot 1 \cdot J_{k-2}(x)\right] = xJ_{k-1}(x)J_{k-2}(x)$; if v does *not* begin with a loop and w does, the corresponding sum is $\left[x \cdot 1 \cdot J_{k-2}(x)\right]\left[1 \cdot J_{k-1}(x)\right] = xJ_{k-1}(x)J_{k-2}(x)$; and if neither does, the resulting sum is $\left[x \cdot 1 \cdot J_{k-2}(x)\right]\left(x \cdot 1 \cdot J_{k-2}(x)\right] = x^2 J_{k-2}^2(x)$.

Consequently, the sum S_2 of the weights of all elements in $E \times E$ is given by

$$\begin{aligned} S_2 &= J_{k-1}^2(x) + 2xJ_{k-1}(x)J_{k-2}(x) + x^2 J_{k-2}^2(x) \\ &= \left[J_{k-1}(x) + xJ_{k-2}(x)\right]^2 \\ &= J_k^2(x). \end{aligned}$$

Thus,

$$\begin{aligned} S_1 - D^2(-x)^{n-k} S_2 &= \left[j_n^2(x) + D^2(-x)^{n-k} J_k^2(x)\right] - D^2(-x)^{n-k} J_k^2(x) \\ &= j_n^2(x). \end{aligned}$$

This, coupled with the sum in equation (7), yields the required result. □

6. Additional Byproducts

Identities (1) and (5) have additional implications. It follows from equation (1) that

$$g_{n+k+1}g_{n-k-1} + g_{n+k}g_{n-k} = \begin{cases} 2g_n^2 + (-1)^{n-k} f_{k+2}f_{k-1} & \text{if } g_n = f_n \\ 2g_n^2 - (-1)^{n-k} \Delta^2 f_{k+2}f_{k-1} & \text{if } g_n = l_n; \end{cases}$$

$$g_{n+k+1}g_{n-k-1} - g_{n+k}g_{n-k} = \begin{cases} (-1)^{n-k} f_{2k+1} & \text{if } g_n = f_n \\ -(-1)^{n-k} \Delta^2 f_{2k+1} & \text{if } g_n = l_n; \end{cases}$$

For example, $L_{19}L_7 + L_{18}L_8 = 542,\ 687 = 2L_{13}^2 - (-1)^{13+5} 5F_7F_4$; and $L_{19}L_7 - L_{18}L_8 = -445 = -(-1)^{13+5} 5F_{11}$.

These results have Jacobsthal companions:

$$c_{n+k+1}c_{n-k-1} + c_{n+k}c_{n-k} = \begin{cases} 2c_n^2 - (-x)^{n-k-1}[J_{k+1}^2(x) - xJ_k^2(x)] & \text{if } c_n = J_n(x) \\ 2c_n^2 + D^2(-x)^{n-k-1}[J_{k+1}^2(x) - xJ_k^2(x)] & \text{if } c_n = j_n(x); \end{cases}$$

$$c_{n+k+1}c_{n-k-1} - c_{n+k}c_{n-k} = \begin{cases} -(-x)^{n-k-1}[J_{k+1}^2(x) + xJ_k^2(x)] & \text{if } c_n = J_n(x) \\ D^2(-x)^{n-k-1}[J_{k+1}^2(x) + xJ_k^2(x)] & \text{if } c_n = j_n(x). \end{cases}$$

For example, $j_{14}j_6 + j_{13}j_7 = 2{,}105{,}282 = 2j_{10}^2 + 9(-2)^6(J_4^2 - 2J_3^2)$ and $j_{19}j_7 - j_{18}j_8 = -786{,}816 = 9(-2)^{13-5-1}(J_6^2 + 2J_5^2)$.

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GIBONACCI IMPLICATIONS OF A DELIGHTFUL CATALAN IDENTITY

Abstract: We employ a gibonacci extension of the Catalan identity $F_{n+k}F_{n-k} - F_n^2 = (-1)^{n+k+1}F_k^2$ to extract its implications to the gibonacci, Pell, Pell-Lucas, Jacobsthal, Vieta, and Chebyshev polynomials; and give graph-theoretic interpretations of the gibonacci and Jacobsthal versions.

Keywords: Gibonacci, Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas Polynomials, Jacobsthal-Lucas Numbers, Catalan Identity.

Mathematical Subject Classification (2010) No.: 05A19, 11B37, 11B39, 11Cxx.

1. Introduction

Extended gibonacci polynomials $g_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is a complex variable; $a(x)$, $b(x)$, $z_0(x)$ and $z_1(x)$ are arbitrary complex polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*.

Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 9, 10].

In particular, *Pell polynomials* $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. The Pell numbers P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [5, 9, 10].

Let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial* [3, 9]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$.

Suppose $a(x) = x$ and $b(x) = -1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = V_n(x)$, the nth *Vieta polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = v_n(x)$, the nth *Vieta-Lucas polynomial* [4, 7].

Let $a(x) = 2x$ and $b(x) = -1$. When $z_0(x) = 1$ and $z_1(x) = x$, $z_n(x) = T_n(x)$, the nth *Chebyshev polynomial* of the first kind; and when $z_0(x) = 1$ and $z_1(x) = 2x$, $z_n(x) = U_n(x)$, the nth *Chebyshev polynomial of the second kind* [5, 9].

1.1 Bridges Among the Subfamilies: Fibonacci, Pell, and Jacobsthal polynomials, and Chebyshev polynomials of the second kind are closely linked; and so are Lucas, Pell-Lucas, and Jacobsthal-Lucas polynomials, and Chebyshev polynomials of the first kind [4, 9, 10]:

$$J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x}) \qquad j_n(x) = x^{n/2} l_n(1/\sqrt{x})$$

$$V_n(x) = i^{n-1} f_n(-ix) \qquad v_n(x) = i^n l_n(-ix)$$

$$V_n(x) = U_{n-1}(x/2) \qquad v_n(x) = 2T_n(x/2),$$

where $i = \sqrt{-1}$.

In the interest of brevity, clarity, and convenience, we *omit* the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. We also let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $c_n = J_n(x)$ or $j_n(x)$, $d_n = V_n$ or v_n, and $e_n = T_n$ or U_n; correspondingly, $G_n = F_n$ or L_n, $B_n = P_n$ or Q_n, and $C_n = J_n$ or j_n.

2. Polynomial Extensions of the Catalan Identity

In 1879, E.C. Catalan established the delightful identity

$$F_{n+k}F_{n-k} - F_n^2 = (-1)^{n+k+1}F_k^2 \text{ [8, 10].}$$

More generally, we have [9, 10]

$$g_{n+k}g_{n-k} - g_n^2 = (-1)^{n+k+1}\mu f_k^2,$$

where

$$\mu = \mu(x) = \begin{cases} 1 & \text{if } g_n = f_n \\ -(x^2+4) & \text{if } g_n = l_n. \end{cases}$$

It then follows that

$$g_{n+k}g_{n-k} + (-1)^{n+k}\mu f_k^2 = g_n^2$$

$$g_{n+k+1}g_{n-k-1} - (-1)^{n+k}\mu f_{k+1}^2 = g_n^2 .$$

Consequently,

$$x^2 g_{n+k}g_{n-k} + g_{n+k+1}g_{n-k-1} - (-1)^{n+k}\mu f_{k+2}f_{k-1} = (x^2+1)g_n^2$$

$$x^2 g_{n+k}g_{n-k} - g_{n+k+1}g_{n-k-1} + (-1)^{n+k}\mu(f_{k+1}^2 + x^2 f_k^2) = (x^2-1)g_n^2 .$$

In particular,

$$x^2 g_{n+1}g_{n-1} + g_{n+2}g_{n-2} = (x^2+1)g_n^2$$

$$x^2 g_{n+1}g_{n-1} - g_{n+2}g_{n-2} = (x^2-1)g_n^2 + 2(-1)^n\mu x^2.$$

These two identities imply that

$$2(g_{n+2}^2 g_{n-2}^2 + g_{n+1}^2 g_{n-1}^2) = (x^2+1)^2 g_n^4 + [(x^2-1)g_n^2 + 2(-1)^n \mu x^2]^2 \qquad (1)$$

$$4x^2 g_{n+2} g_{n+1} g_{n-1} g_{n-2} = (x^2+1)^2 g_n^4 - [(x^2-1) g_n^2 + 2(-1)^n \mu x^2]^2 . \qquad (2)$$

It then follows from that

$$G_{n+2}^2 G_{n-2}^2 + G_{n+1}^2 G_{n-1}^2 = 2[G_n^4 + \mu^2(1)]; \qquad (3)$$

$$G_{n+2} G_{n+1} G_{n-1} G_{n-2} = G_n^4 + \mu^2(1);$$

$$2(b_{n+2}^2 b_{n-2}^2 + 16x^4 b_{n+1}^2 b_{n-1}^2) = (4x^2+1)^2 b_n^4 + [(4x^2-1)b_n^2 + 8(-1)^n \mu(2x)x^2]^2;$$

$$2(B_{n+2}^2 B_{n-2}^2 + 16 B_{n+1}^2 B_{n-1}^2) = 25 B_n^4 + [3B_n^2 + 8(-1)^n \mu(2)]^2;$$

$$16x^2 b_{n+2} b_{n+1} b_{n-1} b_{n-2} = (4x^2+1)^2 b_n^4 - [(4x^2-1)b_n^2 + 8(-1)^n \mu(2x)x^2]^2;$$

$$16 B_{n+2} B_{n+1} B_{n-1} B_{n-2} = 25 B_n^4 - [3B_n^2 + 8(-1)^n \mu(2)]^2 .$$

G. E. Ganis discovered identity (3) with $G_n = F_n$ in 1959 [2, 9].

Next we explore the Jacobsthal implications of identities (1) and (2).

3. Jacobsthal Implications

Replacing x with $1/\sqrt{x}$, equation (1) yields

$$2(x^2 g_{n+2}^2 g_{n-2}^2 + g_{n+1}^2 g_{n-1}^2) = (x+1)^2 g_n^4 + [(1-x)g_n^2 + 2(-1)^n \mu(1/\sqrt{x})]^2 , \qquad (4)$$

where $g_n = g_n(1/\sqrt{x})$.

Suppose $g_n = f_n$. Multiplying both sides with x^{2n-2}, we get

$$2[x^2 J_{n+2}^2(x) J_{n-2}^2(x) + J_{n+1}^2(x) J_{n-1}^2(x)] = (x+1)^2 J_n^4(x) + [(1-x)J_n^2(x) - 2(-x)^{n-1}]^2 .$$

When $g_n = l_n$, multiply both sides of equation (4) with x^{2n}. This yields

$$2[x^2 j_{n+2}^2(x) j_{n-2}^2(x) + j_{n+1}^2(x) j_{n-1}^2(x)] = (x+1)^2 j_n^4(x) + [(1-x) j_n^2(x) + 2(4x+1)(-x)^{n-1}]^2 .$$

Combining the two cases, we get

$$2(x^2 c_{n+2}^2 c_{n-2}^2 + c_{n+1}^2 c_{n-1}^2) = (x+1)^2 c_n^4 + [(x-1)c_n^2 + 2c(-x)^{n-1}]^2, \tag{5}$$

where

$$c = \begin{cases} 1 & \text{if } c_n = J_n(x) \\ -(4x+1) & \text{if } c_n = j_n(x). \end{cases}$$

In particular, we have

$$2(4C_{n+2}^2 C_{n-2}^2 + C_{n+1}^2 C_{n-1}^2) = 9C_n^4 + [C_n^2 - (-2)^n c(2)]^2 .$$

Now consider identity (2). Replacing x with $1/\sqrt{x}$, it yields

$$4x g_{n+2} g_{n+1} g_{n-1} g_{n-2} = (x+1)^2 g_n^4 - [(1-x) g_n^2 + 2(-1)^n \mu(/\sqrt{x})]^2,$$

where $g_n = g_n(1/\sqrt{x})$.

Suppose $g_n = f_n$. Multiplying the resulting equation with x^{2n-2}, we get

$$4x J_{n+2}(x) J_{n+1}(x) J_{n-1}(x) J_{n-2}(x) = (x+1)^2 J_n^4(x) - [(x-1) J_n^2(x) + 2(-x)^{n-1}]^2 .$$

Now let $g_n = l_n$. Multiplying the corresponding equation with x^{2n} yields

$$4x j_{n+2}(x) j_{n+1}(x) j_{n-1}(x) j_{n-2}(x) = (x+1)^2 j_n^4(x) - [(x-1) j_n^2(x) + 2(4x+1)(-x)^{n-1}]^2 .$$

Combining the two cases, we get

$$4x c_{n+2} c_{n+1} c_{n-1} c_{n-2} = (x+1)^2 c_n^4 - [(x-1)c_n^2 + 2(-x)^{n-1} c]^2 . \tag{6}$$

This implies

$$8C_{n+2}C_{n+1}C_{n-1}C_{n-2} = 9C_n^4 - [C_n^2 - (-2)^n c(2)]^2 \ .$$

Next we investigate the consequences of identities (1) and (2) to the Vieta and Chebyshev subfamilies. In the interest of brevity, we omit all details.

4. Vieta and Chebyshev Implications

Using the relationships $V_n(x) = i^{n-1} f_n(-ix)$ and $v_n(x) = i^n l_n(-ix)$, they yield

$$2(d_{n+2}^2 d_{n-2}^2 + x^4 d^{n+1} d_{n-1}^2) = (x^2-1)d_n^4 + [(x^2+1)d_n^2 - 2x^2 d]^2 ;$$

$$4x^2 d_{n+2}d_{n+1}d_{n-1}d_{n-2} = -(x^2-1)^2 d_n^4 + [(x^2+1)d_n^2 - 2x^2 d^*]^2 ;$$

$$2(e_{n+2}^2 e_{n-2}^2 + 16x^4 e_{n+1}^2 e_{n-1}^2) = (4x^2-1)e_n^4 + [(4x^2+1)e_n^2 - 8x^2 e]^2 ;$$

$$16x^2 e_{n+2}e_{n+1}e_{n-1}e_{n-2} = -(4x^2-1)^2 e_n^4 + [(4x^2+1)e_n^2 - 8x^2 e^*]^2 ,$$

where

$$d = \begin{cases} 1 & \text{if } d_n = V_n \\ x^2 - 4 & \text{if } d_n = v_n; \end{cases} \qquad d^* = \begin{cases} 1 & \text{if } d_n = V_n \\ 4 - x^2 & \text{if } d_n = v_n; \end{cases}$$

$$e = \begin{cases} 1 & \text{if } e_n = U_n \\ x^2 - 1 & \text{if } e_n = T_n; \end{cases} \qquad e^* = \begin{cases} 1 & \text{if } e_n = U_n \\ 1 - x^2 & \text{if } e_n = T_n. \end{cases}$$

5. Graph-theoretic Interpretations

Next we interpret identities (1), (2), (5), and (6) using digraphs. To this end, first we lay some basic ground work.

5.1 Fibonacci Digraph: Consider the *digraph* D_1 in Figure 1 with two vertices v_1 and v_2, where a *weight* is assigned to each edge [9, 10]. Its *weighted adjacency matrix* Q is given by

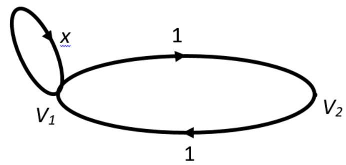

Figure 1: Weighted Digraph D_1

$$Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix},$$

where $Q = Q(x)$. It then follows by induction that

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $n \geq 1$.

A *walk* from vertex v_i to vertex v_j is a sequence $v_i - e_i - v_{i+1} - \cdots - v_{j-1} - e_{j-1} - v_j$ of vertices v_k and edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The walk is *closed* if $v_i = v_j$; otherwise, it is *open*. The *length* of a walk is the number of edges in the walk. The *weight* of a walk is the product of the weights of the edges along the walk.

The following theorem gives a mechanism for computing the weight of a walk of length n from any vertex v_i to any vertex v_j [5, 10].

Theorem 1: *Let A be the weighted adjacency matrix of a weighted and connected digraph with vertices $v_1, v_2, \ldots, v_k$. Then the ijth entry of the matrix A^n gives the sum of the weights of all walks of length n from v_i to v_j, where $n \geq 1$.* □

The next result follows from this theorem.

Corollary 1: *The ijth entry of* Q^n *gives the sum of the weights of all walks of length* n *from* v_i *to* v_j *in the weighted digraph* D_1, *where* $1 \le i,\ j \le n$. □

It then follows that the sum of the weights of all closed walks of length n originating at v_1 is f_{n+1}, and that of walks of length n originating at v_2 is f_{n-1}. So the sum of the weights of all closed walks of length n is $f_{n+1} + f_{n-1} = l_n$.

We are now ready for the interpretations.

5.2 Identities (1) and (2): Case 1: Let $g_n = f_n$. By Corollary 1, the sum of weights of the closed walks x of length n from v_1 to v_1 is f_{n+1}, and that of the walks y of length $n-1$ from v_1 to v_2 is f_{n-1}. So the sum of the weights of pairs (x, y) of such walks is $S_1 = f_{n+1}f_{n-1}$.

Similarly, let v be an arbitrary closed walk of length $n+1$ originating at v_1, and w an arbitrary walk of length $n-2$ from v_1 to v_2. Then $S_2 = f_{n+2}f_{n-2}$ gives the sum of the weights of pairs (v, w) of such walks.

Thus, by the Catalan identity, we have

$$\begin{aligned}
x^4S_1^2 + S_2^2 &= x^4(f_{n+1}f_{n-1})^2 + (f_{n+2}f_{n-2})2 \\
&= x^4[f_n^2 + (-1)^n]^2 + [f_n^2 - (-1)^n x^2]^2 \\
&= (x^4+1)f_n^4 + 2(-1)^n x^2(x^2-1)f_n^2 + 2x^4 \\
2(x^4S_1^2 + S_2^2) &= 2(x^4+1)f_n^4 + 4(-1)^n x^2(x^2-1)f_n^2 + 4x^4 \\
&= (x^2+1)f_n^4 + [(x^2-1)f_n^2 + 2(-1)^n x^2]^2; \\
S_1S_2 &= (f_{n+1}f_{n-1})(f_{n+2}f_{n-2}) \\
&= [f_n^2 + (-1)^n][f_n^2 - (-1)^n x^2]
\end{aligned}$$

$$\begin{aligned}
&= f_n^4 - (-1)^n (x^2-1) f_n^2 - x^2 \\
4x^2 S_1 S_2 &= 4x^2 f_n^4 - 4(-1)^n x^2 (x^2-1) f_n^2 - 4x^4 \\
&= (x^2+1)^2 f_n^4 - [(x^2-1) f_n^2 + 2(-1)^n x^2]^2 \ ,
\end{aligned}$$

as expected.

Case 2: Let $g_n = l_n$. Again, by Corollary 1, the sum of the weights of the closed walks x of length $n+1$ in the digraph is l_{n+1}, and that of those y of length $n-1$ is l_{n-1}. So the sum of the weights of the pairs (x, y) of such walks is $S_1 = l_{n+1} l_{n-1}$. Likewise, $S_2 = l_{n+2} l_{n-2}$ gives the sum of the weights of pairs (v, w) of closed walks v of length $n+2$ at v_1 and closed walks w of length $n-2$ at v_2.

Then

$$\begin{aligned}
x^4 S_1^2 + S_2^2 &= x^4 (l_{n+1} l_{n-1})^2 + (l_{n+2} l_{n-2})^2 \\
&= x^4 [l_n^2 + (-1)^n \mu]^2 + [l_n^2 - (-1)^n \mu x^2]^2 \\
&= (x^4+1) l_n^4 + 2(-1)^n \mu x^2 (x^2-1) l_n^2 + 2\mu^2 x^4 \\
2(x^4 S_1^2 + S_2^2) &= 2(x^4+1) l_n^4 + 4(-1)^n \mu x^2 (x^2-1) l_n^2 + 4\mu^2 x^4 \\
&= (x^2+1)^2 l_n^4 + [(x^2-1) l_n^2 + 2(-1)\mu x^2]^2; \\
S_1 S_2 &= (l_{n+1} l_{n-1})(l_{n+2} l_{n-2}) \\
&= [l_n^2 + (-1)^n \mu][l_n^2 - (-1)^n \mu x^2] \\
&= l_n^4 - (-1)^n \mu (x^2-1) l_n^2 - \mu^2 x^2 \\
4x^2 S_1 S_2 &= 4x^2 l_n^4 - 4(-1)^n \mu x^2 (x^2-1) l_n^2 - 4\mu^2 x^4 \\
&= (x^2+1)^2 l_n^4 - [(x^2-1) l_n^2 + 2(-1)\mu x^2]^2 ,
\end{aligned}$$

as desired.

Next we give graph-theoretic interpretations of identities (5) and (6). To interpret them, we employ an appropriate digraph.

5.3 Jacobsthal Digraph: Consider the weighted digraph D_2 in Figure 2 [10]. Its weighted adjacency matrix is

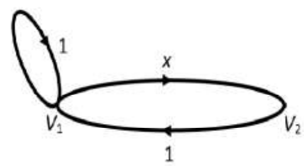

Figure 2: Jacobsthal Digraph D_2

$$M = \begin{bmatrix} 1 & x \\ 1 & 0 \end{bmatrix},$$

Then

$$M^n = \begin{bmatrix} J_{n+1}(x) & xJ_n(x) \\ J_n(x) & xJ_{n-1}(x) \end{bmatrix},$$

where $n \geq 1$.

The sum of the weights of the closed walks of length n originating at v_1 in D_2 is $J_{n+1}(x)$, and that of those originating at v_2 is xJ_{n-1}. Consequently, the sum of all closed walks of length n in the digraph is $J_{n+1}(x) + xJ_{n-1}(x) = j_n(x)$. As before, these facts play a pivotal role in the graph-theoretic pursuits.

5.4 Identities (5) and (6): Case 1: Suppose $c_n = J_n(x)$. The sum of the weights of the walks α of length $n-1$ from v_2 to v_1 is $J_{n-1}(x)$, and that of closed walks β of length n originating at v_1 is $J_{n+1}(x)$. So the sum of the weights of pairs (α, β) of such walks is $S_1 = J_{n+1}(x)J_{n-1}(x)$.

The sum of the weights of the walks v of length $n-2$ from v_2 to v_1 is $J_{n-2}(x)$, and that of walks w of length $n+1$ from v_1 to itself is $J_{n+2}(x)$. Consequently, the sum of the weights of pairs (v, w) of such walks is $S_2 = J_{n+2}(x)J_{n-2}(x)$.

By the Catalan-like identity $J_{n+k}(x)J_{n-k}(x) - J_n^2 = -(-x)^{n-k} J_k^2$, we then have

$$
\begin{aligned}
S_1^2 + x^2 S_2^2 &= [J_{n+1}(x)J_{n-1}(x)]^2 + x^2[J_{n+2}(x)J_{n-2}(x)]^2 \\
&= [J_n^2(x) - (-x)^{n-1}]^2 + x^2[J_n^2(x) - (-x)^{n-2}]^2 \\
&= (x^2+1)J_n^4(x) + 2(x-1)(-x)^{n-1}J_n^2(x) + 2x^{2n-2} \\
2(S_1^2 + x^2 S_2^2) &= 2(x^2+1)J_n^4(x) + 4(x-1)(-x)^{n-1}J_n^2(x) + 4x^{2n-2} \\
&= (x+1)^2 J_n^4(x) + [(x-1)J_n^2(x) + 2(-x)^{n-1}]^2; \\
S_1 S_2 &= [J_{n+1}(x)J_{n-1}(x)][J_{n+2}(x)J_{n-2}(x)] \\
&= [J_n^2(x) - (-x)^{n-1}][J_n^2(x) - (-x)^{n-2}] \\
&= J_n^4(x) + (x-1)(-x)^{n-2}J_n^2(x) + (-x)^{2n-3} \\
4xS_1S_2 &= 4xJ_n^4(x) - 4(x-1)(-x)^{n-1}J_n^2(x) - 4(-x)^{2n-2} \\
&= (x+1)^2 J_n^4(x) - [(x-1)J_n^2(x) + 2(-x)^{n-1}]^2,
\end{aligned}
$$

as desired.

Case 2: Let $c_n = j_n(x)$. The sum of the weights of closed walks α of length $n+1$ in D_2 is $j_{n+1}(x)$, and that of those β of length $n-1$ is $j_{n-1}(x)$. So the sum of the weights of pairs (α, β) of such walks is given by $S_1 = j_{n+1}(x)j_{n-1}(x)$. Likewise, the sum of the pairs (v, w) of closed walks is $S_2 = j_{n+2}(x)j_{n-2}(x)$, where v is an arbitrary closed walk of length $n+2$ and w an arbitrary closed walk of length n 2.

By the Catalan-like identity $j_{n+k}(x)j_{n-k}(x) - j_n^2 = -c(-x)^{n-k} J_k^2$, we then have

$$
\begin{aligned}
S_1^2 + x^2 S_2^2 &= [j_{n+1}(x)j_{n-1}(x)]^2 + x^2[j_{n+2}(x)j_{n-2}(x)]^2 \\
&= [j_n^2(x) - c(-x)^{n-1}]^2 + x^2[j_n^2(x) - c(-x)^{n-2}]^2
\end{aligned}
$$

$$
\begin{aligned}
&= (x^2+1)j_n^4(x) + 2c(x-1)(-x)^{n-1}\,j_n^2(x) + 2c^2x^{2n-2}\\
2(S_1^2 + x^2S_2^2) &= 2(x^2+1)j_n^4(x) + 4c(x-1)(-x)^{n-1}\,j_n^2(x) + 4c^2x^{2n-2}\\
&= (x+1)^2\,j_n^4(x) + [(x-1)j_n^2(x) + 2c(-x)^{n-1}]^2;\\
S_1S_2 &= [j_{n+1}(x)j_{n-1}(x)][j_{n+2}(x)j_{n-2}(x)]\\
&= [j_n^2(x) - c(-x)^{n-1}][j_n^2(x) - c(-x)^{n-2}]\\
&= j_n^4(x) + c(x-1)(-x)^{n-2}\,j_n^2(x) + c^2(-x)^{2n-3}\\
4xS_1S_2 &= 4xj_n^4(x) - 4c(x-1)(-x)^{n-1}\,j_n^2(x) - 4c^2(-x)^{2n-2}\\
&= (x+1)^2\,j_n^4(x) - [(x-1)J_n^2(x) + 2c(-x)^{n-1}]^2,
\end{aligned}
$$

again as expected, where $c = -(4x+1)$.

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GENERALIZATION OF MULTIPLICATIVE TRIPLE FIBONACCI SEQUENCES

Abstract: Sequence has been most interesting and fascinating topic for Mathematician and research scholars for centuries. The Fibonacci sequence is the source of many acceptable interesting identities. In this paper, Coupled Fibonacci Sequences involve two sequences of integers in which the elements of one sequence are part of the generalization of the other and vice-versa .We present some results of multiplicative triple Fibonacci Sequences of second order under one specific schemes.

Keywords: Fibonacci sequence, Multiplicative Triple Fibonacci Sequence.

Mathematical Subject Classification: 11B39, 11B37.

1. Introduction

Triple Fibonacci Sequence gave a new path in generalization of coupled Fibonacci Sequences. It is most interesting and fascinating topic and much work has been done on Fibonacci sequence. Fibonacci sequence first introduced by J. Z. Lee and J. S. Lee [1]. He described new ideas for Fibonacci-Triple Sequence involves three sequences or 3-F Sequences.

The Fibonacci–Triple Sequence involves three sequences of integers in which the elements of one sequence are part of the generalization of the other and vice-versa. J. Z. Lee and J. S. Lee first introduced additive triple Fibonacci sequence. He described specific scheme and derived recurrent formula. In [2] and [3], B. Singh and O. Sikhwal have studied multiplicative coupled Fibonacci Sequences and additive Fibonacci triple sequences with some fundamental properties.

2. Fibonacci-triple Sequence of Second Order

Fibonacci Triple sequence is the explosive development in the region of Fibonacci sequence. Let $\{P_i\}_{i=0}^{\infty}$, $\{Q_i\}_{i=0}^{\infty}$ and $\{R_i\}_{i=0}^{\infty}$, be three infinite sequences and six arbitrary real numbers a, b, c, d, e and f be given. Then J. Z. Lee and J. S. Lee [2] defined following nine deferent schemes of multiplicative triple Fibonacci sequences are as follows:

First scheme

$$P_{n+2} = Q_{n+1}R_n$$
$$Q_{n+2} = R_{n+1}P_n$$
$$R_{n+2} = P_{n+1}Q_n$$

Second Scheme

$$P_{n+2} = R_{n+1}Q_n$$
$$Q_{n+2} = P_{n+1}R_n$$
$$R_{n+2} = Q_{n+1}P_n$$

Third Scheme

$$P_{n+2} = P_{n+1}Q_n$$
$$Q_{n+2} = Q_{n+1}R_n$$
$$R_{n+2} = R_{n+1}P_n$$

Fourth Scheme

$$P_{n+2} = Q_{n+1}P_n$$
$$Q_{n+2} = R_{n+1}Q_n$$
$$R_{n+2} = P_{n+1}R_n$$

Fifth Scheme

$$P_{n+2} = P_{n+1}R_n$$
$$Q_{n+2} = Q_{n+1}P_n$$
$$R_{n+2} = R_{n+1}Q_n$$

Sixth Scheme

$$P_{n+2} = R_{n+1}P_n$$
$$Q_{n+2} = P_{n+1}Q_n$$
$$R_{n+2} = Q_{n+1}R_n$$

Seventh Scheme

$$P_{n+2} = P_{n+1}P_n$$
$$Q_{n+2} = Q_{n+1}Q_n$$
$$R_{n+2} = R_{n+1}R_n$$

Eighth Scheme

$$P_{n+2} = Q_{n+1}Q_n$$
$$Q_{n+2} = R_{n+1}R_n$$
$$R_{n+2} = P_{n+1}P_n$$

Ninth Scheme

$$P_{n+2} = R_{n+1}R_n$$
$$Q_{n+2} = P_{n+1}P_n$$
$$R_{n+2} = Q_{n+1}Q_n$$

Now we obtain some results of multiplicative triple Fibonacci sequence of second order for Eighth scheme.

3. Main Results

First few terms of eighth schemes are as follows:

n	P_n	Q_n	R_n
0	a	b	c
1	d	e	f
2	be	cf	Ad
3	cef	adf	Bde
4	$acdf^2$	abd^2e	bce^2f
5	a^2bd^2e	b^2cde^3f	ac^2df^3e

If we set a = b = c and d = e = f then the following result is true.

Theorem (3.1): For every integer $n \geq 0$

$$\text{(a) } P_{n+6} = P_{n+2}Q_{n+2}R_{n+3}^3,$$
$$\text{(b) } Q_{n+6} = Q_{n+2}R_{n+2}P_{n+3}^3$$
$$\text{(c) } R_{n+6} = P_{n+2}R_{n+2}Q_{n+3}^3$$

Proof: We prove the above results by induction method.

(a) If n = 0 then

$$P_6 = Q_5.Q_4 = (R_4R_3).(R_3R_2) = R_4.(R_3^2).R_2 \quad \text{(By Eighth Scheme)}$$

$$=(P_3.P_2)(P_2.P_1)^2.(P_1.P_0) \quad \text{(By Eighth Scheme)}$$

$$=P_3.(P_2P_1).(P_2P_1)^2.P_0 \quad \text{(By Eighth Scheme)}$$

$$=(Q_2.Q_1).R_3^3.P_0 \quad \text{(By Eighth Scheme)}$$

$$= (Q_2.Q_1).R_3^3.Q_0 \quad \text{(By Induction hypo.)}$$

$$=Q_2.(Q_1Q_0).R_3^3. \quad \text{(By Eighth Scheme)}$$

$$=Q_2.P_2.R_3^3$$

$$P_6 = P_2.Q_2.R_3^3$$

Thus, the assume that the result is true for some integer $n = 0$.

Let us assume that the result is true for some integer $n \geq 1$.

Then

$$P_{n+7} = Q_{n+6}.Q_{n+5} = (R_{n+5}R_{n+4}).(R_{n+4}R_{n+3}) = R_{n+5}.(R_{n+4}^2).R_{n+3}$$
(By Eighth Scheme)

$$=(P_{n+4}.P_{n+3})(P_{n+3}.P_{n+2})^2.(P_{n+2}.P_{n+1}) \quad \text{(By Eighth Scheme)}$$

$$= P_{n+4}.(P_{n+3}P_{n+2}).(P_{n+3}P_{n+2})^2.P_{n+1} \quad \text{(By Eighth Scheme)}$$

$$=(Q_{n+3}.Q_{n+2}).R_{n+4}^3.P_{n+1} \quad \text{(By Eighth Scheme)}$$

$$=Q_{n+3}.(Q_{n+2}P_{n+1}).R_{n+4}^3. \quad \text{(By Eighth Scheme)}$$

$$=Q_{n+3}.(Q_{n+2}Q_{n+1}).R_{n+4}^3. \quad \text{(By Induction hypo.)}$$

$$= Q_{n+3}.P_{n+3}.R_{n+4}^3. \quad \text{(By Eighth Scheme)}$$

$$P_{n+7} = P_{n+3}Q_{n+3}R_{n+4}^3$$

Hence result is true for all integers $n \geq 0$ similar proofs can be given for remaining parts (b) and (c).

Theorem (3.2): For every integer $n \geq 0$

$$P_nQ_nR_n = (P_0Q_0R_0)^{F_{n-1}}.(P_1Q_1R_1)^{F_n}.$$

Proof: To prove this, we shall use induction method.

If $n = 1$ the result is obviously true, since

$$P_1Q_1R_1 = (P_0Q_0R_0)^{F_1-1}.\left(P_1Q_1R_1\right)^{F_1} = P_1Q_1R_1.$$

Now suppose that the result is true for some integer $n \geq 0$. Then

$$P_{n+1}Q_{n+1}R_{n+1} = (Q_nQ_{n-1})(R_nR_{n-1})(P_nP_{n-1}) \quad \text{(By Eighth Scheme)}$$

$$=(P_{n-1}Q_{n-1}R_{n-1}).(P_nQ_nR_n)$$

$$=(P_0Q_0R_0)^{F_{n-2}}.(P_1Q_1R_1)^{F_{n-1}}.(P_0Q_0R_0)^{F_{n-1}}(P_1Q_1R_1)^{F_n}$$

(By Induction hypo.)

$$= \left(P_0 Q_0 R_0\right)^{F_{n-2}+F_{n-1}} . \left(P_1 Q_1 R_1\right)^{F_n+F_{n-1}}$$

$$= \left(P_0 Q_0 R_0\right)^{F_n} . \left(P_1 Q_1 R_1\right)^{F_{n+1}}$$

Hence, by induction method result is true for all $n \geq 1$.

Theorem (3.3): For every integer $n \geq 0$

$$P_{n+2} Q_{n+2} R_{n+2} = F_{n+1}\left(P_0 Q_0 R_0\right).F_{n+2}\left(P_1 Q_1 R_1\right)$$

Proof: The statement is obviously true if $n = 0$ and $n = 1$. let us assume that the statement is true for all integers less than or equal to some integer $n \geq 0$. Then by (ninth scheme):

$$P_{n+3} Q_{n+3} R_{n+3} = \left(Q_{n+2} Q_{n+1}\right)\left(R_{n+2} R_{n+1}\right)\left(P_{n+2} P_{n+1}\right) \qquad \text{(By Eighth scheme)}$$

$$= \left(P_{n+2} Q_{n+2} R_{n+2}\right)\left(P_{n+1} Q_{n+1} R_{n+1}\right)$$

$$= [F_{n+1}\left(P_0 . Q_0 . R_0\right).F_{n+2}\left(P_1 . Q_1 . R_1\right)]..\left[F_n\left(P_0 . Q_0 . R_0\right)..F_{n+1}\left(P_1 . Q_1 . R_1\right)\right]$$

(By ind. hypo.)

$$P_{n+3} Q_{n+3} R_{n+3} = F_{n+3}\left(P_1 . Q_1 . R_1\right).F_{n+2}\left(P_0 . Q_0 . R_0\right)$$

Hence, the statement is true for all integers $n \geq 0$.

4. Conclusion

Much work has been done on multiplicative coupled and triple Fibonacci sequences. In this paper, we have described some results of multiplicative triple Fibonacci sequence of second order under one specific scheme.

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A GENERAL COMPARATIVE STUDY OF SOME ASPECTS OF GRAPH ISOMORPHISM

Abstract: Mathematical modeling and simulation for the graph isomorphism problems is to apply whether two graphs are isomorphic or not isomorphic. Finding a simple and efficient criterion for detection of isomorphism is still actively pursued and is an important unsolved problem in graph theory. In this paper, we will present a review in the form of a general comparative study of some aspects of graph isomorphism [1, 3, and 5]. We will also discuss graph isomorphism problem with impetus of Markovian property [1, 4] using MATLAB.

Keywords: Graph Isomorphism, Mathematical Modeling and Simulation, Markovian Property.

Mathematical Subject Classifications (2010) No.: 60A05, 60C05, 60E05, 68R10, 05C10, 97K30.

1. Introduction

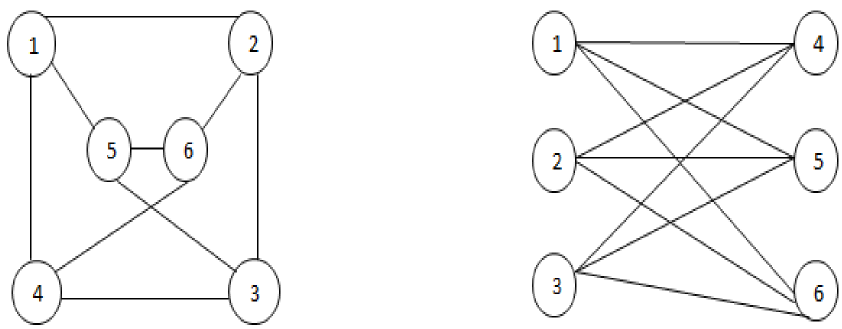
Two graphs H_1 and H_2 are isomorphism or not can be determined by the many methods. There is some necessary and sufficient condition for graph isomorphism. Necessary condition for two isomorphism- (1) V (H_1) = V (H_2), E (H_1) = E (H_2), Degree sequence of H_1 and H_2 are same (The degree sequence of a graph is the sequence of the degrees of the vertices in ascending order). Sufficient condition is [3, 5]

(1) Adjacency matrix of H_1 and H_2 are same.

(2) All column vector of P_U^I[x] are different in Probability Distribution Matrix.

2. Illustrative Example

Example 1: Show the graphs H_1 and H_2 are isomorphism or not.



$H_1 H_2$

Now check above given all condition [2, 5]

(i) $V(H_1) = (1, 2, 3, 4, 5, 6)$, $E(H_1) = (9)$

(ii) $V(H_2) = (1, 2, 3, 4, 5, 6)$, $E(H_2) = (9)$

(ii) Degree sequence $H_1 = (3, 3, 3, 3, 3, 3)$
$H_2 = (3, 3, 3, 3, 3, 3)$

(iii) Adjacent matrix of both graphs

(iv) $A_1 = PA_2P^{-1}$, where P is the permutation matrix.

$$A_1 = \begin{matrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{matrix} \quad \text{and } A_2 = \begin{matrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{matrix}$$

Now we find permutation matrix, cycle of the given graphs are:

$\alpha = (1\ 2\ 3\ 4\ 6\ 5), (1\ 4\ 3\ 6\ 2\ 5)$

$$P = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{matrix} \quad \text{and} \quad P^{-1} = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{matrix}$$

We get the result by given formula:

$$A_1 = PA_2P^{-1}$$

By use the MATLAB,

Now check above given all condition [2, 5]

(i) $V(H_1) = (1, 2, 3, 4, 5, 6)$, $E(H_1) = (9)$

$V(H_2) = (1, 2, 3, 4, 5, 6)$, $E(H_2) = (9)$

(ii) Degree sequence– $H_1 = (3, 3, 3, 3, 3, 3)$

$H_2 = (3, 3, 3, 3, 3, 3)$

(iii) Adjacent matrix of both graphs-

(iv) $A_1 = PA_2P^{-1}$, where P is the permutation matrix.

$$A_1 = \begin{matrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{matrix} \quad \text{and} \quad A_2 = \begin{matrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{matrix}$$

Now we find permutation matrix, cycle of the given graphs are:

$\alpha = (1\ 2\ 3\ 4\ 6\ 5), (1\ 4\ 3\ 6\ 2\ 5)$

$$P = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{matrix} \quad \text{and} \quad P^{-1} = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{matrix}$$

We get the result by given formula:

$$A_1 = PA_2P^{-1}$$

By use the MATLAB,

$$PA_2P^{-1} = \begin{matrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{matrix}$$

We get that the given graphs H_1 isomorphic to H_2.

(v) Now by $B_{ij} = {}^{A_{ij}}\!/_{D_i}$

When using the degree invariance, the multisets of $dgr(H_1) = \{\{3, 3, 3, 3, 3, 3\}\}$ respectively $dgr(H_2) = \{\{3, 3, 3, 3, 3, 3\}\}$, we get $dgr(H_1) = dgr(H_2)$. According to the degree invariance the graphs are isomorphic.

$PAU_1 = (Q_1, \sum, M_1, \Gamma_1)$ by graph H_1 and the probability distribution matrix D_1 is shown below:

$$D_1 = \begin{matrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{matrix}$$

Let state 2 of U_1 is selected as the initial state.

Then the initial state distribution vector Γ_1 is,

$$\Gamma_1 = \begin{bmatrix} \frac{0}{1} & \frac{1}{1} & \frac{0}{1} & \frac{0}{1} & \frac{0}{1} & \frac{0}{1} \end{bmatrix}$$

The probability distribution vector P_1^2(x) for some string x are computed as follows:

$$P_1^2(\varepsilon) = \Gamma_1^2.M_1(\varepsilon)$$

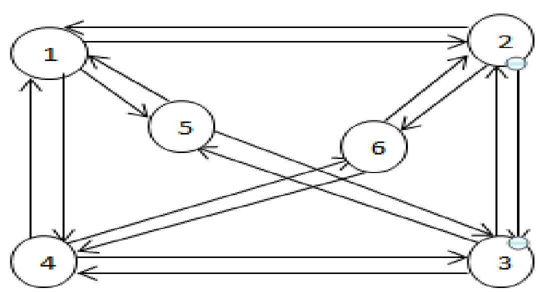
$$P_1^2(\alpha) = \Gamma_1^2.M_1(\alpha)$$

$$P_1^2(\alpha^2) = \Gamma_1^2.M_1(\alpha^2)$$

.
.
.

If $P_U^{i,j}[x] \neq P_U^{i,k}[x]$ for a fixed i, where $1 \le i, k \le n$ and $j \neq k$. The entire column vectors of $P_U^i[x]$ are different. Since all the states in U_1 are "distinguishable" after PAU_1 reach string a^3, we only to compute $P_1^2[a^3]$

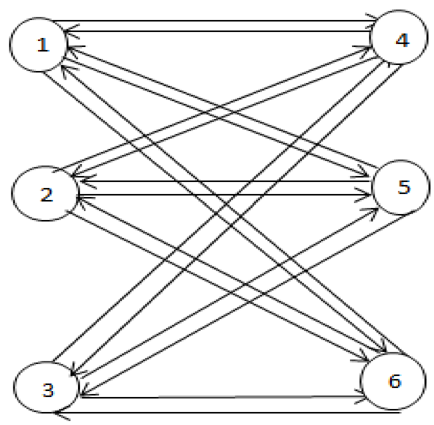
$$P_1^2[a^3] = \begin{matrix} \frac{0}{1} & \frac{1}{1} & \frac{0}{1} & \frac{0}{1} & \frac{0}{1} & \frac{0}{1} \\ \frac{1}{3} & \frac{0}{3} & \frac{1}{3} & \frac{0}{3} & \frac{0}{3} & \frac{1}{3} \\ \frac{0}{3} & \frac{1}{3} & \frac{0}{3} & \frac{1}{3} & \frac{1}{3} & \frac{0}{3} \\ \frac{1}{3} & \frac{0}{3} & \frac{1}{3} & \frac{0}{3} & \frac{0}{3} & \frac{1}{3} \end{matrix}$$



U_1

Similarly to the approach taken as above, the corresponding $PAU_2 = (Q_2, \sum, M_2, \Gamma_2)$ and its probability matrix is computed. Suppose that state 2 in U_2 is selected as the initial state [1, 4].

$$P_2^2[a^3] = \begin{matrix} \frac{0}{1} & \frac{1}{1} & \frac{0}{1} & \frac{0}{1} & \frac{0}{1} & \frac{0}{1} \\ \frac{0}{3} & \frac{0}{3} & \frac{0}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{0}{3} & \frac{0}{3} & \frac{0}{3} \\ \frac{0}{3} & \frac{0}{3} & \frac{0}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{matrix}$$



U_2

We can find that the $P_1^2[a^3]$ is isomorphic to $P_2^2[a^3]$.

3. Conclusion

In this paper we have presented an invariant for graphs. This invariantconsists simultaneously of information about the adjacency condition and degrees of vertices using this invariant. In this direction, the generalized heuristic program by using probability propagation matrix invariance criterion will decrease the existence of graphs for which the invariances will not work [1, 4].

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AN EQUIVALENT CONDITION FOR TRANSITIVE AUTOMORPHIM GROUP OF A FINITELY CONNECTED DOMAIN IN

Abstract: In this note we give an equivalent condition for Transitive automorphism group of a finitely connected smooth domain in . In fact we deduce it by using important results in classical complex function theory.

Keywords: Isotropy Group, Transitive Automorphism Group.

Mathematical Subject Classification (2010) No.: 30C20, 30C35.

1. Introduction

Let Ω be a plane domain of finite connectivity. Further we assume that Ω is also a smooth and bounded open set in $\mathbb{C}$. Therefore, by our assumption the complement of Ω in $\mathbb{C}_\infty$ contains n-connected components and the boundary of $\Omega(=\partial\Omega)$ is the union of n-smooth C^1 Jordan curves.

A conformal automorphism on Ω is a one-to-one and on-to holomorphic map $\phi : \Omega \to \Omega$ with ϕ^{-1}. Under the composition of mappings, the automorphisms of Ω form a group. We denote it by $Aut(\Omega)$. This is a topological group with the topology of uniform convergence on compact sets.

Let $z_0 \in \Omega$. The isotropy group I_{z0} of z_0 in $Aut(\Omega)$, is defined by

$$I_{z0} = \{\phi \in Aut(\Omega) : \phi(z_0) = z_0\}$$

Here z_0 is called a fixed point of the automorphism ϕ.

The group of automorphisms of Ω is said to be transitive if $p, q \in \Omega$, there is an automorphism ϕ such that $\phi(p) = q$.

There are several interesting results in classical complex analysis, the conditions under which a given domain Ω is conformally equivalent to the unit disk.

In this note we relate these results to give an equivalent condition for transitive automorphism group of Ω.

We prove the following:

Theorem 1. *Let Ω be a smooth and bounded domain of finite connectivity. Let $z \in \Omega$. Then I_z is infinite if and only if $Aut(\Omega)$ is transitive group.*

2. Preliminaries

It is known that $Aut(\Omega)$ for a finitely connected domain with connectivity > 2 is finite. Therefore, for such domains neither I_z is infinite for $z \in \Omega$ nor $Aut(\Omega)$ is transitive. In fact, for such domains the number of proper holomorphic maps is also finite. For more details see [1], [2], [3]. So, It is enough to show it for the domains of connectivity ≤ 2. Since Ω is bounded and simply connected domain in $\mathbb{C}$, by Riemann mapping theorem it is conformally equivalent to the unit disc. If it is doubly connected then it is conformally equivalent an annulus $A(r, R)$. Two annuli $A(r_1, R_1)$ and $A(r_2, R_2)$ are conformally equivalent if $R_1/r_1 = R_2/r_2$.

The following result is proved by Aumann and Carathedory [5].

Theorem 2: *Let Ω be a bounded planar domain. Let $z_0 \in \Omega$. If I_{z0} is infinite then Ω is conformally equivalent to the unit disk $\mathbb{D}$.*

The following is another interesting result stated with an elementary proof in [4].

Theorem 3: *Let Ω be a smooth and bounded domain in $\mathbb{C}$ with finite connectivity. If Ω has transitive automorphism group then Ω is conformally equivalent to the unit disk $\mathbb{D}$.*

The extension of the Theorem 3 to higher dimensions known as Bun Wong-Rosay Theorem.

First we see the automorphisms of $\mathbb{D}$ which have 0 as fixed point. This gives the description of the Isotropy group I_0 of $Aut(\mathbb{D})$.

Theorem 4: *Suppose ψ is an automorphism on $\mathbb{D}$ and $\psi(0) = 0$. Then*

$$\psi(z) = \alpha z$$

for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$.

Proof: By Schwarz`s lemma $|\psi'(0)| \leq 1$. Since $\psi \in Aut(\mathbb{D}), \psi^{-1}$ also belongs to $Aut(\mathbb{D})$ and $\psi^{-1}(0) = 0$. Once again by Schwarz`s lemma

$$|(\psi^{-1)'}(0)| \leq 1$$

Since

$$|\psi(z)| \leq |z| \text{ and } |z| = \psi^{-1}(\psi(z)) \leq |\psi(z)|$$

It follows that

$$\frac{|\psi(z)|}{|z|} = 1 \quad \text{for all} \quad z \in \mathbb{D}^*$$

Then $\psi(z) = \alpha z$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ which is a rotation.

This can also be arrived by using chain rule on

$$\psi^{-1}(\psi(z)) = z$$

to obtain

$$(\psi^{-1})'(0) = \frac{1}{\psi'(0)}$$

So, we must have

$$|\psi'(0)| = 1$$

Then $\psi(z) = \alpha z$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ which is a rotation. □

It follows that I_0 is infinite in $Aut(\mathbb{D})$.

3. Proof of Theorem 1

Proof: Let Ω be a smooth and bounded plane domain with finite connectivity. Let $z \in \Omega$ and the Isotropy group I_z, of z is infinite. Then by Theorem 2, Ω is conformally equivalent to the unit disk. Let $f : \Omega \to \mathbb{D}$ be a biholomorphic map.

We know that the $Aut(\mathbb{D})$ consists of the rotations

$$\psi_\theta(z) = e^{i\theta} z, \qquad 0 \le \theta < 2\pi$$

and the Moebious transformations of the form

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}, \qquad 0 \le |a| < 1$$

and also the composition of the above mappings.

Let $p, q \in \Omega$. Suppose $f(p) = a$ and $f(q) = b$ with $a, b \in \mathbb{D}$.

Then there is an automorphism (for example) $g = \varphi_b^{-1} \circ \varphi_a$ maps a to b.

Then $f^{-1} \circ g \circ f$ is an automorphism of Ω which maps p to q. Hence, it proves one part of the result.

Conversely, assume that $Aut(\Omega)$ is transitive. This implies by Theorem 3, Ω is conformally equivalent to the unit disk. Let $f : \Omega \to \mathbb{D}$ be a biholomorphic map. Once again it is enough to prove that the Isotropy group I_a in $Aut(\mathbb{D})$ is infinite. Let $z_0 \in \Omega$ and $f(z_0) = a$. We show that I_a in $Aut(\mathbb{D})$ is infinite.

Since the automorphism group is transitive there are automorphisms φ, ϕ which map a to 0 and 0 to a respectively. It is easy now to verify that there are infinitely many automorphisms of the form $\varphi \circ \psi \circ \phi$ where ψ is a rotation. All these automorphisms have a has a fixed point. This completes the proof of the theorem. □

The proof motivates to state separately the following point.

Let Ω be a bounded domain. Then $Aut(\Omega)$ is transitive if and only if there is a point $p \in \Omega$ such that for every $q \in \Omega$ there is an automorphism ϕ of Ω, with $\phi(q) = p$.

4. Concluding Remarks

These algebraic properties play a considerable role in studying some of the topological and geometric properties of automorphism groups. Here is an instance the transitive property of $Aut(\Omega)$ can be applied to construct an example.

The following result was proved by H. Cartan, and quite useful to characterize a non-compact automorphism group of a bounded domain. The topology is understood to be the open compact topology or often analysts say it topology of uniformly convergence on compact sets.

Theorem 5: *Let Ω be a bounded domain in $\mathbb{C}$. The $Aut(\Omega)$ is noncompact if and only if there exists a point $p \in \Omega$ and q on the boundary of Ω such that*

$$\lim_{k \to \infty} \phi_k(p) = q$$

where ϕ_k is a sequence of automorphisms on Ω.

The automorphism group of the unit disk is non-compact. We can apply Theorem 5 and use the transitive property of the $Aut(\mathbb{D})$ to construct a sequence (ϕ_k) as follows:

$$\lim_{k \to \infty} \phi_k(0) = 1$$

where

$$\phi_k(0) = 1 - \frac{1}{k}$$

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SOFT ALMOST I-REGULAR SPACES

Abstract. In this paper the concept of soft almost I-regular spaces in soft ideal topological spaces have been introduced and studied.

Keywords: Soft set, Soft Ideal, Soft I-regular space, Soft almost I- regular space.

Mathematical Subject Classification No.: Primary 54A40, 54D10, Secondary 06D72.

1. Introduction

The concept of soft set theory was introduced by Molodtsov [10] as a general mathematical tool for dealing with problems that contains uncertainty. In 2011, Shabir and Naz [15] initiated the study of soft topological spaces and derived their basic properties. Singal and Arya [16] have introduced the concept of almost regular spaces and obtained several properties. Recently, Guler and Kale [3] introduced the notion of soft I-regular spaces. The main aim of this paper is to introduce a new soft separation axiom called soft almost I-regularity which is a weak form of soft I-regularity and investigate some of their properties and characterizations.

2. Preliminaries

Throughout this paper X denotes a nonempty set, E denotes the set of parameters and S(X, E) denotes the family of soft sets over X. For definition and basic properties of soft sets, reader should refer [1, 6, 8, 10, 12, 15, 17].

Definition 2.1: [15] A subfamily τ of S (X, E) is called soft topology on X if:

(a) $\tilde{\phi}, \tilde{X}$ belongs to τ.

(b) The union of any number of soft sets in τ belongs to τ.

(c) The intersection of any two of soft sets in τ belongs to τ.

The triplet (X, τ, E) is called a soft topological space over X. The members of τ are called soft open sets in X and their complements called soft closed sets in X.

Lemma 2.1: [15] Let(X, τ, E) be a soft topological space. Then the collection $\tau_\alpha = \{F(\alpha) : (F, E) \ (\in \tau)\}$ for each $\alpha \in E$, defines a topology on X.

Definition 2.2: [15] In a soft topological space$(X, \tau, E)$the intersection of all soft closed super sets of (F, E) is called the soft closure of (F, E). It is denoted by $Cl(F, E)$.

Definition 2.3: [17] In a soft topological space$(X, \tau, E)$the union of all soft open subsets of (F, E) is called soft interior of (F, E). It is denoted by $Int\,(F, E)$.

Lemma 2.2: [15, 17] Let (X, τ, E) be a soft topological space and let $(F, E), (G, E) \in S\,(X, E)$. Then:

(a) (F, E) is soft closed if and only if $(F, E) = Cl(F, E)$

(b) If $(F, E) \subseteq (G, E)$, then $Cl\,(F, E) \subseteq Cl(G, E)$.

(c) (F, E) is soft open if and only if $(F, E) = Int\,(F, E)$.

(d) If $(F, E) \subseteq (G, E)$, then $Int\,(F, E) \subseteq Int\,(G, E)$.

(e) $(Cl\,(F, E))^C = Int\,((F, E)^C)$.

(f) $\left(Int\,(F, E)\right)^C = Cl\,((F, E)^C)$.

Definition 2.4: [4] Let (X, τ, E) be a soft topological space over X and Y be a nonempty subset of X. Then $\tau_Y = \{(F_Y, E) : (F, E) \in \tau\}$ is said to be the soft relative topology on Y and (Y, τ_Y, E) is called a soft subspace of (X, τ, E).

Lemma 2.3: [4] Let (Y, τ_Y, E) be a soft subspace of a soft topological space (X, τ, E) and (F, E) be a soft open set in Y. If $\tilde{Y} \in \tau$ then $(F, E) \in \tau$.

Lemma 2.4: [4] Let (Y, τ_Y, E) be a soft topological subspace of a soft topological space (X, τ, E) and (F, E) be a soft set over X, then:

(a) (F, E) is soft open in Y if and only if $(F, E) = \tilde{Y} \cap (G, E)$ for some soft open set (G, E) in X.

(b) (F, E) is soft closed in Y if and only if $(F, E) = \tilde{Y} \cap (G, E)$ for some soft closed set (G, E) in X.

Lemma 2.5: [14] Let (X, τ, E) be a soft topological space and (Y, τ_Y, E) be a soft subspace of (X, τ, E), then a soft closed set (F_Y, E) of Y is soft closed in X if and only if $\tilde{Y}$ is soft closed in X.

Definition 2.5: [2, 7] The soft set $(F, E) \in S(X, E)$. is called a soft point, if there exists $x \in X$ and $e \in E$ such that $F(e) = \{x\}$ and $F(e^C) = \phi$ for each $e^C \in E \quad \{e\}$, and the soft point (F, E) is denoted by x_e. We denote the family of all soft points over X by $SP(X, E)$.

Definition 2.6: [17] **T**he soft point x_e is said to be in the soft set (G, E) denoted by $x_e \in (G, E)$ if $x_e \subseteq (G, E)$.

Lemma 2.6: [2, 11] Let $(F, E), (G, E) \in S(X, E)$ and $x_e \in SP(X, E)$. Then we have:

(a) $x_e \in (F, E)$ if and only if $x_e \quad (F, E)^c$.

(b) $x_e \in (F, E) \cup (G, E)$ if and only if $x_e \in (F, E)$ or $x_e \in (G, E)$.

(c) $x_e \in (F, E) \cap (G, E)$ if and only if $x_e \in (F, E)$ and $x_e \in (G, E)$.

(d) $(F, E) \subseteq (G, E)$ if and only if $x_e \in (F, E)$ implies $x_e \in (G, E)$.

Definition 2.7: [9] A soft set (F, E) in a soft topological space (X, τ, E) is said to be soft dense in X, if $Cl(F, E) = \tilde{X}$.

Definition 2.8: [13] Let I be a non-empty collection of soft sets over X with the same set of parameters E. Then I ∈ S(X,E) is said to be a soft ideal on X if,

(a) $(F, E) \in I$ and $(G, E) \in I$ implies $(F, E) \cup (G, E) \in I$.

(b) $(F, E) \in I$ and $(G, E) \subseteq (F, E)$ implies $(G, E) \in I$.

A soft topological space (X, τ, E) with a soft ideal I is called soft ideal topological space and is denoted by (X, τ, E, I).

Definition 2.9: [5] Let (X, τ, E, I) be a soft ideal topological space over Xwith the same set of parameters E. Then,

$$(F, E) \ \ (I, \tau) = \tilde{\cup}\{x_e \in \tilde{X} : (U, E) \cap (F, E) \quad I, \forall\, (U, E) \in \tau \text{ contaning } x_e\}$$

is called the soft local function of (F, E) with respect to I and τ.

Definition 2.10: [5] Let (X,τ,E, I) be a soft ideal topological space with the same set of parameters E and $Cl \ : S(X, E) \rightarrow S(X, E)$ be the soft closure operator such that $Cl \ (F, E) = (F, E) \cup (F, E)$. Then, there exists a unique soft topology over X with the same set of parameters E, finer than τ, called the -soft topology, denoted by τ .

Lemma 2.7: [6] If I be a soft ideal on X and Y is a subset of X, then $I_Y = \{\tilde{Y} \cap (I, E) : (I, E) \in I\}$ is a soft ideal on Y.

Definition 2.11: [5] Let (X, τ, E, I) be a soft ideal topological space over X with the same set of parameters E. Then,

$\beta(I, \tau) = \{(F, E)\text{-}(G, E) : (F, E) \in \tau, (G, E) \in I\}$ is a soft basis for the soft topology τ

Definition 2.12: [3] A soft ideal topological space (X, τ, E, I) is said to be a soft I-regular, if for every soft closed set (G, E) of X such that for each soft point x_e (G, E) there exists disjoint soft open sets (U, E) and (V, E) and such that $x_e \in (U, E)$, (G, E) $(V, E) \in I$.

3. Soft Almost I–Regular Spaces

Definition 3.1: A soft ideal topological space (X, τ, E, I) is said to be soft almost I-regular, if for every soft regular closed set (G, E) of X such that for each soft point x_e (G, E) there exists disjoint soft open sets (U, E) and (V, E) such that $x_e \in (U, E)$, (G, E) $(V, E) \in I$.

Remark 3.1: Every soft I-regular space is soft almost I-regular but the converse may not be true. For,

Example 3.1: Let (X, τ, E, I) be a softideal topological space where

$$X = \{a, b, c\}, E = \{e_1, e_2\} \ \tau = \{\phi, X, \{(e_1, \{a\}), (e_2, \{b\})\}, \{(e_1, \{a, c\}), (e_2, \{b, c\})\}, \{(e_1, \{a, b\}), (e_2, \{a, b\})\}\} \text{ and}$$

$$I = \{\phi, \{(e_1, \{b\}), (e_2, \phi)\}, \{(e_1, \phi), (e_2, \{a\})\}, \{(e_1, \{b\}), (e_2, \{a\})\}\}.$$

Then (X, τ, E, I) is soft almost I-regular but not soft I-regular.

Theorem 3.1: A soft ideal topological space (X, τ, E, I) is soft almost I-regular if and only if $(X, \tau\ , E, I)$ is a soft almost I-regular.

Proof: Let (X, τ, E, I) be a soft almost I-regular spacesand (F, E) be a soft -regular closed set over X such that x_e (F, E). Since $(F, E)^C$ is a soft -regular open set, $(F, E)^C = (H, E)$ (I, E) where (H, E) is soft regular open and $(I, E) \in I$. Then $(H, E)^C$ is a soft regular closed set such that x_e $(H, E)^C$. Since (X, τ, E, I) is a soft almost I-regular, there exists disjoint soft open sets (U, E) and (V, E) such that $x_e \in (U, E)$,$(H, E)^C$ $(V, E) \in I$.Then $((F, E)$ $(I, E))$-$(V, E) \in I$. By definition of soft ideal, (F, E) $(V, E) \in I$.

Conversely let $(X, \tau\ , E, I)$ be a soft almost I-regular spaces and (F, E) be a soft regular closed set over X such that x_e (F, E). Since, $\tau \subseteq \tau$, (F, E) is a soft -regular closed set over X. By hypothesis, there exists soft -regular open sets, (U, E) and (V, E) such that $x_e \in (U, E)$ and (F, E) $(V, E) \in I$. Since (U, E) is a soft -regular open set, $(U, E) = (H_1, E)$ (I_1, E) where $(H_1, E) \in \tau$ and $(I_1, E) \in I$. Then $x_e \in (H_1, E)$. Similarly, $(V, E) = (H_2, E)$ (I_2, E) where $(H_2, E) \in \tau$ and $(I_2, E) \in I$. By hereditary of I,(F, E) $(H_2, E) \in I$. So, (X, τ, E, I) is soft almost I-regular.

Theorem 3.2: Let (X, τ, E, I) be a soft ideal topological space. Then the following conditions are equivalent:

(i) $(X, \tau, E, I)$is soft almost I-regular.

(ii) For each $y \in Y$ and soft regular open set (U, E)containing y_e, there is a soft open set (V, E) containing y_e such that $Cl(V, E)$ $(U, E) \in I$.

(iii) For each $x \in X$ and soft regular closed set (F, E) not containing x_e , there is a soft open set (V, E) containing x_e such that $Cl(V, E) \tilde{\cap} (F, E) \in I$.

Proof: (i) $\Rightarrow$ (ii) Let $y \in Y$, and (U, E) be a soft regular open set containing y_e. Then there exists disjoint soft open sets (V, E) and (W, E) such that $y_e \in (V, E)$ and $(U, E)^C$ $(W, E) \in I$. Then $(U, E)^C \subseteq (W, E) \cup (I, E)$. Now $(V, E) \tilde{\cap} (W, E) =$ implies that $(V, E) \subseteq (W, E)^C$ and so $Cl(V, E) \subseteq (W, E)^C$. Hence, $Cl(V, E)$ $(U, E) \subseteq (W, E)^C \tilde{\cap} ((W, E) \cup (I, E)) = (W, E)^C \in I$.

(ii) $\Rightarrow$ (iii)Let (F, E) be a soft regular closed set on X such that x_e (F, E) . Then, there exists a soft open set (V, E) containing x_e such that $Cl(V, E)$ $(F, E)^C \in I$, which implies that $Cl(V, E) \tilde{\cap} (F, E) \in I$.

(iii) $\Rightarrow$ (i) Let (F, E) be a soft regular closed set on X such that x_e (F, E) .

Then there exists a soft open set (V, E) containing x_e such that

$$\mathrm{Cl}(\mathrm{V}, \mathrm{E}) \widetilde{\cap} (\mathrm{F}, \mathrm{E}) \in \mathrm{I}. \text{ If} \mathrm{Cl}(\mathrm{V}, \mathrm{E}) \widetilde{\cap} (\mathrm{F}, \mathrm{E}) = (\mathrm{I}, \mathrm{E}) \in \mathrm{I}, \text{ then}$$

(F, E) $(\mathrm{Cl}(\mathrm{V}, \mathrm{E}))^{\mathrm{C}} = (\mathrm{I}, \mathrm{E}) \in \mathrm{I}$. (V, E) and $(\mathrm{Cl}(\mathrm{V}, \mathrm{E}))^{\mathrm{C}}$ are the required disjoint soft open sets such that $x_e \in (\mathrm{V}, \mathrm{E})$ and (F, E) $(\mathrm{Cl}(\mathrm{V}, \mathrm{E}))^{\mathrm{C}} \in \mathrm{I}$.

Hence, $(\mathrm{X}, \tau, \mathrm{E}, \mathrm{I})$ is soft almost I-regular.

Lemma 3.1: If (Y, E) is a soft dense subspace of a soft ideal topological space $(\mathrm{X}, \tau, \mathrm{E}, \mathrm{I})$ then, $\mathrm{Int}_{\mathrm{Y}}\mathrm{Cl}_{\mathrm{Y}}(\mathrm{A}) = \mathrm{Int}(\mathrm{Cl}(\mathrm{A})) \cap \mathrm{Y}$.

Proof: Obvious.

Theorem 3.3: Let $(\mathrm{X}, \tau, \mathrm{E}, \mathrm{I})$ is a soft almost I-regular spaces and $(\mathrm{Y}, \tau_{\mathrm{Y}}, \mathrm{E}, \mathrm{I})$ be a dense subspace ofX then$(\mathrm{Y}, \tau_{\mathrm{Y}}, \mathrm{E}, \mathrm{I})$is a soft almost I_{Y}-regular space.

Proof: Let $y \in \mathrm{Y}$ and (U, E) be a soft regular open set of Y containing y_e. Then by "Lemma 3.1", $(\mathrm{U}, \mathrm{E}) = \mathrm{Int}_{\mathrm{Y}}\mathrm{Cl}_{\mathrm{Y}}(\mathrm{U}, \mathrm{E}) = \mathrm{Int}(\mathrm{Cl}(\mathrm{U}, \mathrm{E})) \cap \mathrm{Y}$.

Thus, $\mathrm{Int}(\mathrm{Cl}(\mathrm{U}, \mathrm{E}))$ is a soft regular open set of X containing y_e.

Since $(\mathrm{X}, \tau, \mathrm{E}, \mathrm{I})$ is soft almost I-regular there exists a soft open set (V, E) containing y_e such that $\mathrm{Cl}(\mathrm{V}, \mathrm{E})$ $(\mathrm{U}, \mathrm{E}) \in \mathrm{I}$.

Consequently, $(\mathrm{Cl}(\mathrm{V}, \mathrm{E}) \cap \mathrm{Y})$ $(\mathrm{U}, \mathrm{E}) \in \mathrm{I}$. Hence, $(\mathrm{Y}, \tau_{\mathrm{Y}}, \mathrm{E}, \mathrm{I})$ is soft almost I_{Y}-regular.

Lemma 3.2: If Y is a soft regular open subspace of X then every soft regular open subset of Y is soft regular open in X.

Theorem 3.4: Let $(\mathrm{X}, \tau, \mathrm{E}, \mathrm{I})$ is a soft almost I-regular spaces and Y is a soft regular open subspace of $(\mathrm{X}, \tau, \mathrm{E}, \mathrm{I})$ then $(\mathrm{Y}, \tau_{\mathrm{Y}}, \mathrm{E}, \mathrm{I})$is soft almost I_{Y}-regular.

Proof: Let $(\mathrm{X}, \tau, \mathrm{E}, \mathrm{I})$is a soft almost I-regular spaces and let $(\mathrm{Y}, \tau_{\mathrm{Y}}, \mathrm{E}, \mathrm{I})$be a soft regular open subspace of $(\mathrm{X}, \tau, \mathrm{E}, \mathrm{I})$. Let (U, E) be a soft regular openset of Ycontaining y_e. Then by "Lemma 3.2", (U, E) be a soft regular open set of Xcontaining y_e. Since $(\mathrm{X}, \tau, \mathrm{E}, \mathrm{I})$ is soft almost I-regular by, "Theorem 3.2", there exists a soft open set (V, E) containing y_e such that $\mathrm{Cl}(\mathrm{V}, \mathrm{E})$ $(\mathrm{U}, \mathrm{E}) \in \mathrm{I}$.

Consequently, $\mathrm{Cl}_{\mathrm{Y}}(\mathrm{V}, \mathrm{E})$ $(\mathrm{U}, \mathrm{E}) \in \mathrm{I}$. Hence, $(\mathrm{Y}, \tau_{\mathrm{Y}}, \mathrm{E}, \mathrm{I})$ is soft almost I_{Y}-regular.

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UPPER(LOWER) δ-PRECONTINUOUS INTUITIONISTIC FUZZY MULTIFUNCTIONS

Abstract: In this paper we introduce the concepts of upper and lower δ-precontinuous intuitionistic fuzzy multifunctions from a topological space to an intuitionistic fuzzy topological space and obtain some of their properties and characterization.

Keywords and phrases: Intuitionistic fuzzy sets, Intiitionistic fuzzy topology, Intuitionistic fuzzy multifunctions, lower δ-precontinuous and upper δ-precontinuous Intutionistic fuzzy multifunctions δ-prenbd of a point, δ-preopen set, Intuitionistic fuzzy upper(lower)-nbd of a intuitionistic fuzzy set.

Mathematical Subject Classification No.: 54A99, 03E99.

1. Introduction

After the introduction of fuzzy sets by zadeh [30] in 1965 and fuzzy topology by Chang [7], several research studies were conducted on the generalization of the notions of fuzzy sets and fuzzy topology. The concept of intuitionistic fuzzy sets was introduced by Atanassov [1, 2, 3] as a generalization of fuzzy sets. In the last 32 years various concepts of fuzzy mathematics have been extended for intuitionistic fuzzy sets. In 1997 Coker [8] introduced the concept of intuitionistic fuzzy topological spaces as a generalization of fuzzy topological spaces. In 1999, Ozbakir and Coker [21] introduced the concept intuitionistic fuzzy multifunctions and studied their lower and upper intuitionistic fuzzy semi continuity from a topological space to an intuitionistic fuzzy topological space.

Recently many weak and strong forms of upper and lower semi continuous Intuitionistic fuzzy multifunctions such as Intuitionistic fuzzy lower and upper α-continuous [16], Intuitionistic fuzzy lower and upper quasi continuous [29], Intuitionistic fuzzy lower and upper α-irresolute multifunction [23], Intuitionistic fuzzy upper and lower α-irresolute fuzzy multifunction [23], have been appeared in the liturecture.

In this paper we introduce and characterize the concepts of upper and lower -precontinuous intuitionistic fuzzy multifunctions from a topological space to an intuitionistic fuzzy topological space.

2. Preliminaries

Throughtout this paper (X, τ) and (Y, Γ) represents a topological space and an intuitionistic fuzzy topological space respectively. The δ-interior [20] of a subset A of X is the union of all regular open sets of x contained in A and is denoted by δ-intA. A subset A of X is called δ-open [20] if A = δ-intA , i.e., a set is δ-open if it is the union of regular open sets. The complement of a δ-open set is called δ-closed. A set A of (X, τ) is δ-closed [20], iff A = δ clA, where

$$\delta\text{-clA}=\{x \in X : A \bigcap (\text{intcl} \bigcup) = \varphi, U \in \tau, x \in U\}.$$

A subse A of a topological space (X, τ) is said to be δ-preopen [22] in X if A$\subseteq (int\delta \quad ClA)$. The family of all δ-preopen sets in X is denoted by δ-PO(X)[22]). The δ-preinterior of a subset A [22] of X is defined to be the union of all δ-preopen sets contained in A and is denoted by δ-IntA.

The complement of a δ-preopen set is called δ-preclosed [22]. The intersection of all δ-preclosed sets containing A in X is called δ-preclosure of A [22] and is denoted by δ-pclA. A set A is δ-preopen(δ-preclosed) iff A = δ-pint A(resp. A = δ-pclA) [22]. A subset U of X is called a δ-preneighbourhood [22] of a point $x \in X$, if there exists a δ-preopen set V in X such that $x \in V \subseteq U$.

Lemma 2.1: [26] Let A be a subset of a space (X, τ) . Then A$\in \delta$-PO(X) iff A∩U$\in \delta$-PO(X) for each regular open δ-open set U of X.

Lemma 2.2: [26] Let A and X_0 be subset's of a space (X,τ). If A $\in \delta$-PO(X) and X_0 is δ-open in (X,τ), then A $\cap X_0 \in \delta\text{-}PO(X_0)$.

Lemma 2.3: [26] Let A $\subset X_0 \subset$ X, if X_0 is δ-open in (X,τ) and A $\in \delta\text{-}PO(X_0)$, then A $\in \delta$-PO(X).

Definition 2.4: [1], [2], [3] Let Y be a nonempty fixed set. An intuitionistic fuzzy set $\tilde{A}$ in Y is an object having the form

$$\tilde{A} = \{< x, \mu_{\tilde{A}}(y), \nu_{\tilde{A}}(y) > : y \in Y\}.$$

where the functions $\mu_{\tilde{A}}(y)$: Y → I and $\nu_{\tilde{A}}(y)$: Y → I denotes the degree of membership (namely $\mu_{\tilde{A}}(y)$) and the degree of non membership (namely $\nu_{\tilde{A}}(y)$) of each elemently $y \in Y$ to the set $\tilde{A}$ respectively, and

$$0 \leq \mu_{\tilde{A}}(y) + \nu_{\tilde{A}}(y) \leq 1 \text{ for each } y \in Y.$$

Lemma 2.5: [9] For any two intuitionistic fuzzy sets $\tilde{\boldsymbol{A}}$ **and** $\tilde{\boldsymbol{B}}$ of Y,

$$\sim(\tilde{A}q\tilde{B}) \quad \tilde{A} \subset \tilde{B}^c.$$

Definition 2.6: [8] An intuitionistic fuzzy topology on a non-empty set Y is a family $\boldsymbol{\Gamma}$ of intuitionistic fuzzy sets in Y which satisfy the following axioms:

(a) $\tilde{\mathbf{0}}, \tilde{1} \in \Gamma$,

(b) $\tilde{\boldsymbol{A}}_{\mathbf{1}} \cap \tilde{\boldsymbol{A}}_{\mathbf{2}} \in \boldsymbol{\Gamma}$ for any $\tilde{A}_1, \tilde{A}_2 \in \Gamma$,

(c) $\tilde{A}_\beta \in \Gamma$ for arbitrary family $\{\tilde{\boldsymbol{A}}_{\boldsymbol{\beta}}: \boldsymbol{\beta} \in \boldsymbol{\Lambda}\} \in \boldsymbol{\Gamma}$.

In this case the pair $(\mathbf{Y}, \boldsymbol{\Gamma})$ is called an intuitionistic fuzzy topological space and each intuitionistic fuzzy set in Γ, is known as an intuitionistic fuzzy open set in Y.

The complement $\tilde{\boldsymbol{B}}^{\boldsymbol{c}}$ of an intuitionistic fuzzy open set $\tilde{\boldsymbol{B}}$ is called an intuitionistic fuzzy closed set is Y.

Definition 2.7: [8] Let $(\mathbf{Y}, \boldsymbol{\Gamma})$ be an intuitionistic fuzzy topological space and $\tilde{\boldsymbol{A}}$ be an intuitionistic fuzzy set in Y. Then the interior and closure of $\tilde{\boldsymbol{A}}$ are defined by:

(a) **cl**$(\widetilde{\boldsymbol{A}})$ = $\{\widetilde{K}: \widetilde{K}$ is an intuitionistic fuzzy closed set in Y and $\widetilde{\boldsymbol{A}} \subseteq \widetilde{\boldsymbol{K}}\}$**.**

(b) **Int**$(\widetilde{A})$ = $\{\widetilde{\boldsymbol{G}}: \widetilde{G}$ is an intuitionistic fuzzy open set in Y and $\widetilde{\boldsymbol{G}} \subseteq \widetilde{\boldsymbol{A}}\}$**.**

Definition 2.8: [23] Let X and Y are two non-empty sets. A function $\mathbf{F} : (\mathbf{X}, \boldsymbol{\tau}) \rightarrow (Y, \Gamma)$ is called intuitionistic fuzzy multifunctions, if $\mathbf{F}(\boldsymbol{x})$ is an intuitionistic fuzzy set in Y, $\forall\ x \in X$.

Definition 2.9: [34] Let $\mathbf{F}:(\mathbf{X}, \boldsymbol{\tau}) \rightarrow (\boldsymbol{Y}, \boldsymbol{\Gamma})$ is an intuitionistic fuzzy multifunction and A be a subset of X, then $\mathbf{F(A)} = {}_{\boldsymbol{x} \in \boldsymbol{A}}\ \boldsymbol{F}(\boldsymbol{x})$.

Definition 2.10: [34] Let F : (X, τ) → (Y, Γ) be an intuitionistic fuzzy multifunction, then

(a) $\mathbf{A} \subseteq \boldsymbol{B} \Rightarrow \boldsymbol{F}(\boldsymbol{A}) \subseteq \boldsymbol{F}(\boldsymbol{B})$ for any subsets A and B of X.

(b) $\boldsymbol{F}(\boldsymbol{A} \cap \boldsymbol{B}) \subseteq \boldsymbol{F}(\boldsymbol{A}) \cap \boldsymbol{F}(\boldsymbol{B})$for any subsets A and B of X.

(c) $\mathbf{F}(\ {}_{\boldsymbol{\alpha \epsilon \Lambda}} \boldsymbol{A}_{\boldsymbol{\alpha}} = \ \{\boldsymbol{F}(\boldsymbol{A}_{\boldsymbol{\alpha}}) : \boldsymbol{\alpha} \boldsymbol{\epsilon} \boldsymbol{\Lambda}\}$ for any family of subsets in X.
$\{(A_{\alpha}) : \alpha \epsilon \Lambda\}$ in X.

Definition 2.11: [23] Let $\boldsymbol{F}: (\boldsymbol{X}, \boldsymbol{\tau}) \rightarrow (\boldsymbol{Y}, \boldsymbol{\Gamma})$ is an intuitionistic fuzzy multifunction, then the upper inverse $\boldsymbol{F}^{+}(\widetilde{\boldsymbol{A}})$ and lower $\boldsymbol{F}^{-}(\widetilde{\boldsymbol{A}})$ of an intuitionistic fuzzy set $\widetilde{\boldsymbol{A}}$ in Y are defined as follows:

(a) $F^{+}(\widetilde{A}) = \{\ x \in X : F(x) \subseteq (\tilde{A}\ )\}$

(b) $F^{-}(\tilde{A}) = \{\ x \in X : F(x) q\ \widetilde{A}\}$

Lemma 2.12: [34] Let $\boldsymbol{F}: (\boldsymbol{X}, \boldsymbol{\tau}) \rightarrow (\boldsymbol{Y}, \boldsymbol{\Gamma})$ be an intuitionistic fuzzy multifunction and $\widetilde{\boldsymbol{A}}$, $\widetilde{B}$ bean intuitionistic fuzzy sets in Y.

Then

(a) $F^{+}(\tilde{1}) = F^{-}(\tilde{1}) = \mathrm{X}$,

(b) $F^{+}(\tilde{A}) \subseteq F^{-}(\tilde{A})$

(c) $[\boldsymbol{F}^{-}(\widetilde{\boldsymbol{A}})]^{\boldsymbol{c}} = [\boldsymbol{F}^{+}(\widetilde{\boldsymbol{A}})^{\boldsymbol{c}}]$

(d) $[\boldsymbol{F}^{+}(\widetilde{\boldsymbol{A}})^{\boldsymbol{c}} = [\boldsymbol{F}^{-}(\widetilde{\boldsymbol{A}})^{\boldsymbol{c}}$

(e) If $\widetilde{\boldsymbol{A}} \subseteq \widetilde{\boldsymbol{B}}$, then $\boldsymbol{F}^{+}(\widetilde{\boldsymbol{A}}) \subseteq \boldsymbol{F}^{-}(\widetilde{\boldsymbol{B}})$

(f) If $\tilde{A} \subseteq \widetilde{B}$, then $F^{-}(\tilde{A}) \subseteq F^{-}(\widetilde{B})$

Definition 2.13: [23] An Intutionistic fuzzy multifunction $\boldsymbol{F}(\boldsymbol{X}, \boldsymbol{\tau}) \to (\boldsymbol{Y}, \boldsymbol{\Gamma})$ is said to be:

(a) Intuitionistic fuzzy upper semi -continuous at a point $\boldsymbol{x_0} \in \boldsymbol{X}$, if for any intuitionistic fuzzy open set $\widetilde{\boldsymbol{W}} \subset \boldsymbol{Y}$ such that $\boldsymbol{F}(\boldsymbol{x_0}) \subset \widetilde{\boldsymbol{W}}$ there exists an open set $\mathbf{U} \subset \boldsymbol{X}$ containing x_0 such $F(U) \subset \widetilde{W}$.

(b) Intuitionistic fuzzy lower semi continuous at a point $\boldsymbol{x_0} \in \boldsymbol{X}$,if for any intuitionistic fuzzy open set $\widetilde{\boldsymbol{W}} \subset \boldsymbol{Y}$such that $\boldsymbol{F}(\boldsymbol{x_0})\boldsymbol{q}\widetilde{\boldsymbol{W}}$ there exists an open set $\boldsymbol{U} \subset \boldsymbol{X}$ containing x_0 such that $\boldsymbol{F}(\boldsymbol{x})\boldsymbol{q}\widetilde{\boldsymbol{W}},\ \forall\, \boldsymbol{x} \in \boldsymbol{X}$.

(c) Intuitionistic fuzzy upper semi-continuous (intuitionistic fuzzy lower semi-continuous) if it is intuitionistic fuzzy upper semi-continuous (Intuitionistic fuzzy lower semi-continuous) at each point of X.

3. Intuitionistic Fuzzy Upper(Lower) $\boldsymbol{\delta}$-Precontinuous Multifunctions

Definitions 3.1: An Intuitionistic fuzzy multifunction F : X→Y is said to be:

(a) An Intuitionistic fuzzy upper δ-precontinuous at a point $x \in$X if for each intuitionistic fuzzy open set V of Y with $F(x) \subseteq$ V, there exists U$\in \delta$-PO(X) such that $x \in$U, F(U) ⊆ V,

(b) An Intuitionistic fuzzy upper δ-precontinuous at a point $x \in$X if for each intuitionistic fuzzy open set V of Y with $F(x)$ qV, there exists U$\in \delta$-PO(X) such that $x \in$U, F(u)q V for all u ∈ U.

(c) Intuitionistic fuzzy upper δ-precontinuous multifunctions if F has this property at each point $x \in$X.

Theorem 3.2: For an intuitionistic fuzzy multifunction F : X→Y, the following statements are equivalent:

(a) F is an intuitionistic fuzzy upper δ-precontinuous.

(b) $F^{+}(V) \in \delta$-PO(X) for any intuitionistic open set V of Y.

(c) F^{-}(V) is δ-Preclosed in X for any intuitionistic closed set V of Y.

(d) δ-pcl$(F^{-}(B)) \subseteq F^{-}$(clB) for any intuitionistic fuzzy open set B of Y.

(e) For each point $x \in$ X and each intuitionistic fuzzy neighbourhood V of $(x), F^{+}(V)$ is a δ-preneighbourhood of x.

(f) For each pint $x \in$X and each intuitionistic fuzzy neighbourhood V of F(x), there exists a δ-preneighbourhood U of x such that F(U) ⊆ V .

(g) F^{+}(int B)$\subseteq \delta$-pint$(F^{+}$(B)),for any intuitionistic fuzzy set B of Y.

(h) F^{+}(B)$\subseteq$ int(δ cl$(F^{+}$(B))),for any intuitionistic fuzzy open set B of Y.

(i) For each point $x \in$X and each intuitionistic fuzzy neighbourhood V of F(x), δ cl$(F^{+}$(V)) is a nbd of x.

Proof: (a)$\Rightarrow$(b): Let V be an intuitionistic fuzzy open set of Y and $x \in F^{+}$(V). Then F$(x)\subseteq$V by (a), there exists U$\in \delta$-PO(X) such that $x \in$U and F(U) ⊆ V. Then U$\subseteq F^{+}$(V). Again U$\in \delta$-PO(X) $\Rightarrow$U $\subseteq$ int(δ clU)$\subseteq$int(δ cl $(F^{+}$(V)))$\Rightarrow x \in$int(δ cl$(F^{(}V)))$, we have F^{+}(V)$\subseteq$int(δ cl$(F^{+}$(V))) and therefore, F^{+}(V)$\in \delta$-PO(X).

(b)$\Rightarrow$(c): Follows from the fact that , for any intuitionistic fuzzy set V of Y.

(c)$\Rightarrow$(d): For any intuitionistic fuzzy set B of Y, clB is an intuitionistic fuzzy closed in Y. By (c), F^{-}(clB) is δ-preclosed in X. Hence, δ-pcl$(F^{-}$(V)) $\subseteq F^{-}$(clV) $\subseteq F^{-}$(clV).

(d)$\Rightarrow$(c): Let V be any closed in, then V = clV. So by (d), δ-pcl$(F^{-}$(V)$\subseteq F^{-}$(V) and hence $F^{-}(V)$ is δ-preclosed in X.

(b)$\Rightarrow$(e): Let $x \in$X and V be an intuitionistic fuzzy nbd of F(x). Then there exists a fuzzy open set G in Y such that F$(x)\subseteq$G$\subseteq$V. Then $x \in F^{+}$(G)$\subseteq F^{+}$(V) and since F^{+}(G)$\in \delta$-PO(X) by (b), F^{+}(V) is a δ-precontinuous neighbourhood of x.

(e) $\Rightarrow$(f): Let $x \in$X and V be any intuitionistics fuzzy nbd of F(x), then by (e), F^+(V) is a δ-pre-nbd of x, put U$=F^+$(V). Then F(U) ⊆ V.

(f) $\Rightarrow$(a): Let $x \in$X and be any intuitionistic fuzy open set in Y such that F(x)⊆V. then V is an intuitionistic fuzzy nbd of F(x). By (f), there exists a δ-prenbd U of x such that F(U) ⊆ V. Therefore, there exists W$\in \delta$-PO(X) such that $x \in$W ⊆ U and hence, F(W) ⊆ F (U) ⊆ V.

(b)$\Rightarrow$(g): Let B be any intuitionistic fuzzy set in Y. Then intB, if intuitionistic fuzzy open set in Y. By (b), F^+(int B)$\in \delta$-PO(X). Hence, F^+(intB)$\subseteq \delta$-Pint$(F^+$(B)).

(g)$\Rightarrow$(b): Let V be any intuitionistic fuzzy open set in Y. Then by (g), F^+(V)$=F^+$(intV) $\subseteq \delta$-pint$(F^+$(V))and hence, F^+(V)$\in \delta$-PO(X).

(b)$\Rightarrow$(h): It follow from the definition of δ-preopen set in X.

(h)$\Rightarrow$(i): Let $x \in$X and V be an intuitionistic fuzzy neighbourhood of F(x). Then there exists an intuitionistic fuzzy open set U in Y such that F(x)qV ⊆ U. Then $x \in F^+$(V) $\subseteq$ int(δ-cl(F^+(V))) $\subseteq \delta$-cl$(F^+$(V)) and hence, δ-cl$(F^+$(V)) is a nbd of x.

(i)$\Rightarrow$(h): Let V be any intuitionistic fuzzy open set in Y and $x \in F^+$(V). By (i),δ-cl$(F^+$(V))is a nbd of x and thus, $x \in$int(δ-cl(F(V))). Hence, $F^+(V) \subseteq$ int(δ-cl$(F^+$(V))).

Theorem 3.3: For an intuitionistic fuzzy multifunction F : X $\rightarrow$ Y, the following statements are equivalent:

(a) F is an intuitionistic fuzzy lower δ-precontinuous.

(b) F^-(V)$\in \delta$-PO(X) for any intuitionistic open set V of Y.

(c) F^+(V) is δ-Preclosed in X for any intuitionistic closed set V of Y.

(d) δ-pcl$(F^+$(B)) $\subseteq F^+$(clB) for any intuitionistic fuzzy open set B of Y.

(e) For each point $x \in$ X and each intuitionistic fuzzy neighbourhood V of F(x), F^-(V) is a δ-preneighbourhood of x.

(f) For each point $x \in$ X and each intuitionistic fuzzy neighbourhood V of F(x), there exists a δ-preneighbourhood U of x such that F(u)qV.

(g) F^-(intB) $\subseteq \delta$-pint(F^-(B)),for any intuitionistic fuzzy set B of Y.

(h) F^-(B) $\subseteq$int(δ cl(F^-(B))), for any intuitionistic fuzzy open set B of Y.

(i) For each point $x \in$ X and each intuitionistic fuzzy neighbourhood q-nbd V of F(x), δ-cl(F^-(V)) is a neighbourhood of x.

Proof: (a)$\Rightarrow$(b): Let V be an intuitionistic fuzzy open set of Y and $x \in F^-$(V). Then F(x)qV by (a), there exists U$\in \delta$-PO(X), such that $x \in$U and F(U)qV, for each u $\in$U. Then U$\subseteq F^-$(V).

Since, U$\in \delta$-PO(X), $x \in$U $\subseteq$ int(δ clU)$\subseteq$int(δ cl(F^-(V))) and so $x \in$ int(δ cl(F^-(V)))$\Rightarrow F^-$(V)$\subseteq$int(δ cl(F^-(V))). Hence, F^-(V)$\in \delta$-PO(X).

(b)$\Rightarrow$(c): Follows from the fact that , for any intuitionistic fuzzy set V of Y.

(c)$\Rightarrow$(d): For any intuitionistic fuzzy set B of Y, clB is an intuitionistic fuzzy closed in Y. By (c), F^+(clB)is δ-preclosed in X. Hence, δ-pcl($F^+(V)$) $\subseteq \delta$-pcl(F^+(clB))$\subseteq F^+$(clB).

(d)$\Rightarrow$(c): Let V be any closed of Y. Then δ-pcl(F^+(V)$\subseteq F^+$(clV) $= F^+$(V) is δ-preclosed in X.

(b)$\Rightarrow$(e): Let $x \in$X and U be a intuitionistic fuzzy q-nbd of F(x). Then there exists a fuzzy open set V in Y such that F(x)qV$\subseteq$U. Then $x \in F^-$(U)$\in \delta$-PO(X)(by(b)) and so $x \in F^-$(V)$\subseteq F^+$(U) and F^-(U) is a δ-prenbd of x.

(e)$\Rightarrow$(f): Let $x \in$X and V be any intuitionistics fuzzy q-nbd of F(x). Then by (e), F^-(V) is a δ-prenbd of x, put U = F^-(V). Then F(u)qV, for each u$\in$U.

(f)$\Rightarrow$(a): Let $x \in$X and V be any intuitionistic fuzy open set in Y such that F(x)⊆V. Then V is an intuitionistic fuzzy neighbourhood of F(x). By (f), there exists a δ-prenbd U of x such that F(U) ⊆ V. Therefore, there exists W$\in \delta$-PO(X) such that $x \in$W ⊆ U and hence F(W) ⊆ F(U) ⊆ V.

(b)$\Rightarrow$(g): Let B be any intuitionistic fuzzy set in Y. Then intB is intuitionistic fuzzy open set in Y. By (b), F^{-}(intB)$\in \delta$-PO(X). Hence, F^{-}(intB)$\subseteq \delta$-Pint$(F^{-}$(intB))$\subseteq \delta$- int$(F^{-}$(B)).

(g)$\Rightarrow$(b): Let V be any intuitionistic fuzzy open set in Y. Then by(g), F^{-}(V) $=F^{-}$(intV)$\subseteq \delta$-pint$(F^{-}$(V)) and hence, F^{-}(V)$\in \delta$-PO(X).

(b)$\Rightarrow$(h): It follow from the definition of δ-preopen set in X.

(h)$\Rightarrow$(i): Let $x \in$X and V be an intuitionistic fuzzy q-nbd of F(x). Then there exists an intuitionistic fuzzy open set U in Y such that F(x)qU ⊆ V. Then $x \in F^{-}$(U)$\subseteq$ int$(\delta$-cl$(F^{-}$(U))) (by(h))$\subseteq \delta$- cl$(F^{-}$(U)) and hence, δ-cl$(F^{-}$(U)) is a neighbourhood of x.

(i)$\Rightarrow$(h): Let V be any intuitionistic fuzzy open set in Y and $x \in F^{-}$(V). Then F(x)qV and so V is an intuitionisic fuzzy open q-nbd of F(x). So by (i), δ-cl$(F^{-}$(V)) is a nbd of x and so $x \in$int $(\delta$-cl$(F^{-}$(V)))S. Thus, F^{-}(V)$\subseteq$int$(\delta$-cl$(F^{-}$(V))).

Theorem 3.4: Let $\{U_\beta : \beta \in \lambda\}$ be a δ-open cover of X. A an intuitionistic fuzzy multifunction F:X$\to$Y is an intuitionistic fuzzy upper δ-precontinuous if and only if the restriction F/U_β: $U_\beta \to$Y is an intuitionistic fuzzy upper δ-precontinuous for each $\beta \in \Lambda$.

Proof: Let $\beta \in \lambda$ and $x \in U_\beta$. Let V be any open set in Y containing $(F/U_\beta)(x)$. Since (F/U_β) (x) = F(x) and F is an intuitionistic fuzzy upper δ-precontinuous, there exists $U_0 \in \delta$-PO(X) containing x such that $F(U_0) \subseteq$ V. Set U $=U_0 \bigcap U_\beta$. By lemma 2.2 U$\in \delta$-PO(U_0) with $x \in$U and $(F/U_\beta)(U)$ = F(U)⊆V. Therefore, (F/U_β) is an intuitionistic fuzzy upper δ-precontinuous.

Conversely: Let $x \in$ X and V be an intuitionistic fuzzy open set in Y such that F(x) $\subseteq$V. Then there exists $\beta \in \lambda$ such that $x \in U_\beta$ and (F/U_β) (x)= F$(x)$$\subseteq$V. Since (F/U_β) : $U_\beta \to$ Y is an intuitionistic fuzzy upper δ-precontinuous, there exists U$\in$ δ-PO(U_β) with $x \in$U such that $(F/U_\beta)(U)$$\subseteq$V. By lemma 2.3, U$\in$ $\delta_P O(X)$ and F(U) =(F/U_β)(U)$\subseteq$ V and Hence, F is an intuitionistic fuzzy upper δ-precontinuous.

Theorem 3.5: Let $\{U_\beta : \beta \in \lambda\}$ be a δ-open cover of X. An intuitionistic fuzzy multifunction F : X $\to$ Y is intuitionistic fuzzy lowerδ-precontinuous iff the restriction F/U_β : $U_\beta \to$ Y is intuitionistic fuzzy lower δ-precontinuous for each $\beta \in \lambda$.

Proof: Let $\beta \in \lambda$ and $x \in U_\beta$. Let V be any intuitionistic fuzzy open set in Y such that $(F/U_\beta)(x)qV$. Since $(F/U_\beta)(x)$ = F(x) and F is an intuitionistic fuzzy lower δ-precontinuous, there exists $U_0 \in \delta$-PO(X) containing x such that F(u)qV for all u $\in U_0$. Set U = $U_0 \bigcap U_\beta$. By lemma 2.2 U$\in$ δ-PO(U_0) with $x \in$U and $(F/U_\beta)(u)qV$ for all u$\in$U. Therefore(F/U_β) is anintuitionistic fuzzy lower δ-precontinuous.

Conversely: Let $x \in$ X and V be an intuitionistic fuzzy open set in Y with F(x)qV. As $\{U_\beta : \beta \in \lambda\}$ is a cover of X, there exists $\beta \in \lambda$ such that $x \in U_\beta$. Then $(F/U_\beta)(x)$ $=F(x)qV$. Since (F/U_β): $U_\beta \to$Y is an intuitionistic fuzzy lower δ-precontinuous, there exists U$\in$ δ-PO(U_β) with $x \in$U, such that $(F/U_\beta)(u)q \subseteq$V for all u$\in$U. Since $U_\beta' s$ are δ-open, By lemma 2.3, U$\in$ $\delta_P O(X)$ with $x \in$U. Moreover, we have F(u)qV, for all u$\in$U and hence, F is an intuitionistic fuzzy lower δ-precontinuous.

Definition 3.6: For an intuitionistic fuzzy multifunction F : X $\to$ Y, intuitionistic multifunction δ-pcl F : X $\to$ Y is given by (δ-pclF)(x)= δ-pcl$F(x)$, for each $x \in$ X.

Lemma 3.7: Let F : X $\to$ Y be an intuitionistic multifunction. Then we have $(\delta\text{-pcl}\,F)^+$ (G) =$F^+(G)$, for each G $\in$ δ-PO(Y).

Proof: Suppose that G is a δ -preopen in Y. Let $x \in (\delta\text{-pcl}\, F)^{+}(G)$.

Then (δ -pclF)(x) ⊆ G ⇒ F(x) ⊆ G ⇒ $x \in F^{+}(G)$. Converse is obvious.

Theorem 3.8: An intuitionistic fuzzy multifunction F:X→Y is intuitionistic upper δ -precontinuous iff δ -pcl F : X→Y is so.

Proof: Suppose that F is an intuitionistic fuzzy upper δ -precontinuous. Let $x \in$ X and G be any intuitionistic fuzzy open set in Y such that (δ -pclF)(x) ⊆(G). Then F(x) ⊆ G. Since F is an intuitionistic fuzzy upper δ -precontinuous, there exists U∈ δ -PO(X) there exists U∈ δ -PO(X) with $x \in$ U such that F(u)qG, for all u∈U. Since, G is fuzzy δ -preopen, by lemma 2.4, u $\in F^{-}(G) = (\delta\text{-pcl}\, F)^{-}(G)$, for all u∈U. Thus, (($\delta$ -pclF)(u)q)G, for all u ∈U⇒ δ -pcl F is an intuitionistic fuzzy lower δ -precontinuous.

Conversely: Suppose that δ -pclF is an intuitionistic fuzzy upper δ -precontinuous Let $x \in$ X and G be any intuitionistic fuzzy open set in Y such that F(x) ⊆ G⇒ $x \in F^{+}(G)$. (as an intuitionistic fuzzy open sets are intuitionistic fuzzy δ -preopen) we have by lemma 3.9 $x \in (\delta\text{-pcl}\, F)^{+}(G)$ i.e.(δ -pclF)(x) ⊆ G, as δ -pclF is an intuitionistic fuzzy lower δ -precontinuous, there exists U∈ δ PO(X) with $x \in$ U such that (δ -pclF)(U) ⊆ G, for all u∈U. Then F(U) ⊆ G. Hence, F is an intuitionistic fuzzy upper δ -precontinuous.

Lemma 3.9: Let F:X→Y be an intuitionistic fuzzy multifunction. Then we have $(\delta\text{-pcl}\, F)^{-}(G) = F^{-}(G)$, for each G∈ δ -PO(Y).

Proof: Obvious.

Theorem 3.10: An intuitionistic fuzzy multifunction F : X → Y is an intuitionitic fuzzy lower δ -precontinuous if and only if δ -pcl F : X → Y is an intuitionistic lower δ -precontinuous.

Proof: Suppose that F : X → Y is an intuitionitic fuzzy lower δ -precontinuous. Let $x \in$ X and G be any intuitionistic fuzzy open set in Y such that (δ -pclF)(x) qG. By Lemma 2.4, $x \in (\delta \;\; pclF)^{-}(G) = F^{-}(G)$ as every an intuitionistic fuzzy open set G of Y is an intuitionistic fuzzy δ -preopen in Y and hence F(x) qG. By an intuitionistic fuzzy lower δ-precontinuouity of F, there exists U$\in \delta$-PO(X) with $x \in$U such that F(u)qG, for all u$\in$U. Since G is fuzzy δ-preopen, by lemma 2.4, u$\in F^{-}(G)$ = (δ-pcl $F)^{-}$(G), for all u$\in$U. Thus, ((δ-pclF)(u)q)G, for all u$\in$U$\Rightarrow \delta$-pclF is anintuitionistic fuzzy lower δ-precontinuous.

Conversely: Suppose that δ-pclF is an intuitionistic fuzzy lower δ-precontinuous. Let $x \in$X and G be any intuitionistic fuzzy open set in Y such that F(x) qG. by lemma 2.4 we have $x \in F^{-}$(G) = $(\delta - pclF)^{-}$(G), (as intuitionistic fuzzy open sets are intuitionistic fuzzy δ-preopen) and hence (δ-pclF)(x) qG. Since (δ-pclF) is an intuitionistic fuzzy lower δ-precontinuous, there exists U$\in \delta$ PO(X) with $x \in$U such that (δ-pclF)(u)q)G, for all u$\in$U. Since G is an intuitionistic fuzzy δ-preopen in Y, By lemma 2.4, u$\in$(δ-pcl $F)^{-}(G)$ = $F^{-}(G)$, for all u$\in$U. Therefore, F(u)qG, for all u$\in$U and Hence, F is an intuitionistic fuzzy lower δ-precontinuous.

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UPPER (LOWER) CONTRA IRRESOLUTE INTUITIONISTIC FUZZY MULTIFUNCTIONS

Abstract: In this paper we introduce the concepts of upper and lower contra -irresolute intuitionistic fuzzy multifunctions from a topological space to an intuitionistic fuzzy topological space and obtain some of their properties and characterization.

Keywords and phrases: Intuitionistic Fuzzy Sets, Intiitionistic Fuzzy Topology, Intuitionistic Fuzzy Multifunctions, Lower Contra-Irresolute and Upper Contra-Irresolute Intutionistic Fuzzy Multifunctions.

Mathematical Subject Classification No.: 54A99, 03E99.

1. Preliminaries

Throughtout this paper (X, τ) and (Y, Γ) represents a topological space and an intuitionistic fuzzy topological space respectively.

Defintion 1.1: [13] A subset A of a topological space (X, τ) is called:

(a) Semi-open if $A \subset Cl(Int(A))$.

(b) Semi-closed if its complement is semi-open.

Remark 1.1: [16] Every open (resp. closed) set is semi-open (resp. semi-closed) but the converse may not be true.

The family of all (semi-open) subsets of a topological space (X, τ) is denoted by SO(X)), similarly for the family of all semi-closed) subsets of topological space (X, τ) is denoted by SC(X)). The intersection of all (semi-closed) sets of X containing a set A of X is called the (semi-closure) of A. It is denoted by α Cl(A) (resp.sCl(A)). The union of all α -open (semi-open) subsets of A of X is called the (semi-interior) of A. It is denoted by sInt(A)). A subset A of X is (semi-closed) if and only if A$\supset$ Int(Cl(A)).

Definition 1.2: [2, 3, 4] Let Y be a non-empty fixed set. An intuitionistic fuzzy set $\widetilde{A}$ in Y is an object having the form

$$\widetilde{A} = \{< x, \mu_{\widetilde{A}}(y), \nu_{\widetilde{A}}(y) >: y \in Y\}$$

where the functions $\mu_{\widetilde{A}}(y)$:Y$\rightarrow$I and $\nu_{\widetilde{A}}(y)$:Y$\rightarrow$I denotes the degree of membership (namely $\mu_{\widetilde{A}}(y)$) and the degree of non membership (namely $\nu_{\widetilde{A}}(y)$) of each elemently $y \in$Y to the set $\widetilde{A}$ respectively, and 0 $\leq \mu_{\widetilde{A}}(y) + \nu_{\widetilde{A}}(y) \leq 1$ for each $y \in$ Y.

Definition 1.3: [2, 3, 4] Let Y be a non-empty set and the intuitionistic fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ be in the form $\widetilde{A}$ = {$(y, \mu_{\widetilde{A}}$, $\nu_{\widetilde{A}})$: $y \in$Y}, $\widetilde{B}$={$(y, \mu_{\widetilde{B}}(y))$, $\nu_{\widetilde{B}}(y)$: $y \in$Y} and let { $\widetilde{A}_{\beta}$: $\beta \in \Lambda$} be an arbitrary family of intuitionistic fuzzy sets in Y.

Then

(a) $\widetilde{A} \subseteq \widetilde{B}$ if $\forall$ $y \in$Y [$\mu_{\widetilde{A}(y)} \leq \mu_{\widetilde{B}}(y)$ and $\nu_{\widetilde{A}}(y) \geq \nu_{\widetilde{B}}(y)$]

(b) $\widetilde{A} = \widetilde{B}$ if $\widetilde{A} \subseteq \widetilde{B}$ and $\widetilde{B} \subseteq \widetilde{A}$;

(c) $\widetilde{A}^c = \{y, \nu_{\widetilde{A}}(y), \mu_{\widetilde{A}}(y) : y \in Y\}$;

(d) $\tilde{0} = \{(y,\ 0,1) : y \in Y\}$ and $\{(y,1,0) : y \in Y\}$

(e) $\bigcap \widetilde{A}_{\beta}$ = $\{(y, \wedge\, \mu_{\widetilde{A}}(y), \vee\, \nu_{\widetilde{A}}(y)) : y \in Y\}$

(f) $\bigcup \widetilde{A}_{\beta}$ = $\{(y, \vee\, \mu_{\widetilde{A}}(y), \wedge\, \nu_{\widetilde{A}}(y) : y \in Y\}$

Definition 1.4: [9] Two an intuitionistic fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ of Y are said to be quasi-coincident ($\widetilde{A}q\widetilde{B}$ for short) if $\exists\ y \in \mathrm{Y}$ such that $\mu_{\widetilde{A}} > \nu_{\widetilde{B}}(y)$ or $\nu_{\widetilde{A}}(y) < \mu_{\widetilde{B}}(y)$

Lemma 1.5: [9] For any two an intuitionistic fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ of Y, $(\widetilde{A}q\widetilde{B}) \Leftrightarrow \widetilde{A} \subset \widetilde{B}^c$.

Definition 1.6: [7] An intuitionistic fuzzy topology on a non-empty set Y is a family Γ of an intuitionistic fuzzy sets in Y which satisfy the following axioms:

(a) $\widetilde{0}, \widetilde{1} \in \Gamma$,

(b) $\widetilde{A}_1 \bigcap \widetilde{A}_2 \in \Gamma$ for any $\widetilde{A}_1, \widetilde{A}_2 \in \Gamma$,

(c) $\bigcup \widetilde{A}_\beta \in \Gamma$ for any arbitrary family $\{\widetilde{A}_\beta : \beta \in \wedge\} \in \Gamma$.

In this case the pair (Y, Γ) is called an intuitionistic fuzzy topological space and each intuitionistic fuzzy set in Γ, is known as an intuitionistic fuzzy open set in Y.

The complement $\widetilde{B}^c$ of an intuitionistic fuzzy open set $\widetilde{B}$ is called an intuitionistic fuzzy closed set.

Definition 1.7: [7] Let (Y, Γ) be an intuitionistic fuzzy topological space and $\widetilde{A}$ be an intuitionistic fuzzy set in Y. Then the interior and closure of $\widetilde{A}$ are defined by:

(a) $\mathrm{cl}(\widetilde{A}) = \bigcap \{\widetilde{K} : \widetilde{K}$ is an intuitionistic fuzzy closed set in Y and $\widetilde{A} \subseteq \widetilde{K}\}$.

(b) $\mathrm{Int}(\widetilde{A}) = \bigcap \{\widetilde{G} : \widetilde{G}$ is an intuitionistic fuzzy open set in Y and $\widetilde{G} \subseteq \widetilde{A}\}$.

Lemma 2.8: [6] For any intuitionistic fuzzy set $\widetilde{A}$ in (Y, Γ) we have:

(a) $\widetilde{A}$ is an intuitionistic fuzzy closed set in $\mathrm{Y} \Leftrightarrow \mathrm{Cl}(\widetilde{A}) = \widetilde{A}$

(b) $\widetilde{A}$ is an intuitionistic fuzzy open set in Y $\Leftrightarrow \mathrm{Int}(\widetilde{A}) = \widetilde{A}$

(c) $\mathrm{Cl}(\widetilde{A}^c) = (\mathrm{Int}\,\widetilde{A})^c$

(d) $\mathrm{Int}(\widetilde{A}^c) = (\mathrm{Cl}\,\widetilde{A})^c$

Definition 1.8: [11] A subset $\widetilde{A}$ of an intuitionistic fuzzy topological space (Y, Γ) is called:

(a) intuitionistic fuzzy semi-open if $\widetilde{A} \subset \mathrm{Cl}(\mathrm{Int}(\widetilde{A}))$.

(b) intuitionistic fuzzy semi-closed if its complements is semi-open.

Remark 1.2: [16] Every IF-open (resp. IF-closed) set is IFsemi-open (resp. IFsemi-closed).

The family of all intuitionistic fuzzy intuitionistic fuzzy semi-open sets of an intuitionistic fuzzy topological space $(\mathrm{X}, \Gamma)$is denoted by SO(X), Similarly the family of all intuitionistic fuzzy semiclosed) sets of intuitionistic fuzzy topological space (X, Γ) is denoted by IFSC(X).

Definition 1.9: [6] Let (Y, Γ) be an intuitionistic fuzzy topological space and $\widetilde{A}$ be an intuitionistic fuzzy set in Y. Then the semi-interior and semi-closure of $\widetilde{A}$ are defined by:

(a) $\mathrm{SCl}(\widetilde{A}) = \bigcap\{\widetilde{K} : \widetilde{K}$ is an intuitionistic fuzzy closed set in Y and $\widetilde{A} \subseteq \widetilde{K}\}$.

(b) $_{\mathrm{S}}\mathrm{Int}(\widetilde{A}) = \bigcup\{\widetilde{G} : \widetilde{G}$ is an intuitionistic fuzzy open set in Y and $\widetilde{G} \subseteq \widetilde{A}$.

Definition 1.10: [17] Let X and Y are two non empty sets. A function F : X $\rightarrow$ Y is called intuitionistic fuzzy multifunctions, if $\mathrm{F}(x)$ is an intuitionistic fuzzy set in Y, $\forall\ x \in \mathrm{X}$.

Definition 1.11: [22] Let F : $(\mathrm{X}, \tau) \rightarrow (\mathrm{Y}, \Gamma)$ is an intuitionistic fuzzy multifunction and A be a subset of X. Then $\mathrm{F(A)} = \bigcup_{x \in A} \mathrm{F}(x)$.

Definition 1.12: [22] Let F : $(\mathrm{X}, \tau) \rightarrow (\mathrm{Y}, \Gamma)$ be an intuitionistic fuzzy multifunction.

Then

(a) A $\subseteq$ B $\Rightarrow$ F(A) $\subseteq$ F(B) for any subsets A and B of X.

(b) F(A $\bigcap$ B) $\subseteq$ F(A) $\bigcap$ F(B) for any subsets A and B of X.

(c) F($\bigcup_{\tilde{\alpha}\in\Lambda} A_\alpha$) = $\bigcup$ {F$(A_\alpha):\alpha\in\Lambda$} for any family of subsets $\{(A_\alpha):\alpha\in\Lambda\}$ in X.

Definition 1.13: [17] Let F : (X, τ) $\rightarrow$ (Y, Γ) is an intuitionistic fuzzy multifunction, Then the upper inverse $F^{+}(\widetilde{A})$ and lower $F^{-}(\widetilde{A})$ of an intuitionistic fuzzy set $\widetilde{A}$ in Y are defined as follows:

(a) $F^{+}(\widetilde{A}) = \{x \in \mathrm{X} : \mathrm{F}(x) \subseteq \widetilde{A}\}$.

(b) $F^{-}(\widetilde{A}) = \{x \in \mathrm{X} : \mathrm{F}(x) \mathrm{q} \widetilde{A}\}$.

Definition 1.14: [17, 12] An Intuitionistic fuzzy multifunction F : (X, τ) $\rightarrow$ (Y, Γ) is said to be:

(a) Intutionistic fuzzy upper contra continuous at a point $x_0 \in$ X, if for any intuitionistic fuzzy closed set $\widetilde{W}$ of Y such that F $(x_0) \subset \widetilde{W}$ there exist an open set U of X containing x_0 such that F(U) $\subset \widetilde{W}$.

(b) Intuitionistic fuzzy lower contra continuous at a point $x_0 \in X$, if for any intuitionistic fuzzy closed set $\widetilde{W}$ of Y such that $F(x_0) q \widetilde{W}$ there exist an open set U of X containing x_0 such that F$(x) q$ $\widetilde{W}$, $\forall$ $x \in$ U.

2. Upper(Lower) Contra Irresolute Intutionistic Fuzzy Multifunctions

Definition 2.1: An intuitionistic fuzzy multifunction F : (X, τ) $\rightarrow$ (Y, Γ) is said to be:

(a) Intuitionistic fuzzy upper contra-irresolute at a point $x_0 \in$ X, if for any intuitionistic fuzzy semi-closed set $\widetilde{W}$ of Y, such that F(x_0) $\subset \widetilde{W}$ there exists U $\in$ SO(X) containing x_o such that F(U) $\subset \widetilde{W}$.

(b) Intuitionistic fuzzy lower irresolute at a point $x_0 \in$ X, if for any intuitionistic fuzzy semi-closed set $\widetilde{W}$ of Y, such that $F(x_0)\, q\, \widetilde{W}$ there exists U$\in SO(X)$, containing x_o such that $F(x)\, q\, \widetilde{W}$ $\forall$ $x \in$ U.

(c) Intuitionistic fuzzy upper contra irresolute (resp.Intuitionistic fuzzy lower contra irresolute) if it is has this property at each point of X.

Theorem 2.2: Let F : (X, τ) $\rightarrow$ (Y, Γ) be an intuitionistic fuzzy multifunction then following conditions are equivalent:

(a) F is an intuitionistic fuzzy upper contra irresolute.

(b) For each point $x \in$ X and any intuitionistic fuzzy semi-closed set $\widetilde{B}$ of Y such that $x \in F^+(\widetilde{B})$, $\exists$ an semi-neighbourhood U of x such that U$\subseteq F^+(\tilde{B})$.

(c) $F^+(\widetilde{B})$ is semi-open set in X for every intuitionistic fuzzy semi-closed set $\widetilde{B}$ of Y.

(d) $F^-(\widetilde{B})$ is an semi-closed set in X for every intuitionistic fuzzy semi-open set $\widetilde{B}$ in Y.

Proof: (a)$\Leftrightarrow$(b). Obvious.

(b) $\Rightarrow$ (c): Let $\widetilde{B}$ be any an intuitionistic fuzzy semi-closed set of Y and let $x \in F^+(\widetilde{B})$. Then $F(x) \subset \widetilde{B}$. And so by (b) $\exists$ is an semi-neighbourhood U of x such that U $\subset F^+(\widetilde{B})$. It follows that $F^+(\widetilde{B})$ is the union of semi-open sets of X is semi-open in X.

(c) $\Rightarrow$ (b) Let $x \in$ X and $\widetilde{B}$ be an intuitionistic fuzzy semi-closed set of Y such that $x \in F(\overline{B})$. Then U = $F^+(\widetilde{B})$ is an semi-neighbourhood of x such that U$\subseteq F^+(\widetilde{B})$.

(c) $\Leftrightarrow$ (d) It follows from the fact that $[F^+(\widetilde{B})]^c = [F^-(\widetilde{B})]^c$.

Definition 2.3: The semi-kernel of an intuitionistic fuzzy set $\widetilde{B}$ in an intuitionistic fuzzy topological space (Y, Γ) given by

$$\text{SKer}(\widetilde{B}) = \bigcap \{\widetilde{A} : \widetilde{A} \in \text{IFSO(X) and } \widetilde{B} \subseteq \widetilde{A}\}.$$

Lemma 2.4: For an intuitionistic fuzzy set $\widetilde{B}$ in an intuitionistic fuzzy topological space (Y, Γ), if $\widetilde{B} \in$ IFSO(X), then $\widetilde{B} = {}_{S}\text{Ker}(\widetilde{B})$.

Proof: Obvious.

Theorem 2.5: Let F : (X, τ) $\rightarrow$ (Y, Γ) be an intuitionistic fuzzy multifunction. If ${}_{S}\text{Cl}(F^{-}(\widetilde{B})) \subseteq F^{-}({}_{S}\text{Ker}(\widetilde{B}))$ for any intuitionistic fuzzy set $\widetilde{B}$ of Y, then F is an intuitionistic fuzzy upper contra irresolute multifunction.

Proof: Suppose that ${}_{S}\text{Cl}(F^{-}(\widetilde{B})) \subseteq F^{-}{}_{S}\text{Ker}(\widetilde{B}))$ for any intuitionistic fuzzy set $\widetilde{B}$ of Y. Let $\widetilde{A} \in \text{IF}_{S}\text{O(X)}$, then by hypothesis and lemma 3.4 ${}_{S}\text{Cl}(F^{-}(\widetilde{A})) \subseteq F^{-}{}_{S}\text{Ker}(\widetilde{A})) = F^{-}(\widetilde{A})$. This implies that $\text{SCl}(F^{-}(\widetilde{A})) \subseteq F^{-}(\widetilde{A})$ But we know $F^{-}(\widetilde{A}) \subseteq {}_{S}\text{Cl}(F^{-}(\tilde{A}))$. Hence, $F^{-}(\widetilde{A})$ is semi-closed set in X. Thus, by theorem 2.2, F is an intuitionistic fuzzy upper contra-irresolute multifunction.

Theorem 2.6: Let F : (X, τ) $\rightarrow$ (Y, Γ), be an intuitionistic fuzzy multifunction then following conditions are equivalent:

(a) F is an intuitionistic fuzzy lower contra-irresolute.

(b) For any intuitiontistic fuzzy semi-closed set $\widetilde{B}$ of Y such that $x \in F^{-}(\widetilde{B})$, $\exists$ an semi-neighbourhood U of x such that U$\subseteq F^{-}(\widetilde{B})$.

(c) $F^{-}(\widetilde{B})$ is semi-open in X for every an intuitiontistic fuzzy semi-closed set $\widetilde{B}$ of Y.

(d) $F^{+}(\widetilde{B})$ is an semi-closed in X for every an intuitionistic fuzzy semi-open set ($\widetilde{B}$) in Y.

Proof: (a)$\Leftrightarrow$(b) Obvious.

(b) $\Rightarrow$ (c) Let $\widetilde{B}$ be any intuitiontistic fuzzy semi-closed set of Y and $x \in F^{-}(\widetilde{B})$. Then by (b) and there exists an semi-neighborhood U of x such that U$\subset F^{-}(\widetilde{B})$. It follows that $F^{-}(\widetilde{B})$ is the union of semi-open sets of X is semi-open in X.

(c) $\Rightarrow$ (a) Let $x_0 \in$X and $\widetilde{B}$ be an intuitionistic fuzzy semi-closed set of Y such that $F(x_0)\,\mathrm{q}\,\widetilde{B}$. Then $x_0 \in F^{-}(\widetilde{B})$ and $F^{-}(\widetilde{B})$ is semi-open in X. And so, U$=F^{-}(\widetilde{B})$ is a semi-neighbourhood of x_0 such that $\mathrm{F}(x)\,\mathrm{q}\,(\widetilde{B})\ \forall\ x \in$U. Hence, F is an intuitionistic fuzzy lower contra-irresolute.

(c) $\Leftrightarrow$(d) It follows from the fact that $[F^{+}(\widetilde{B})]^{c} = [F^{-}(\widetilde{B})^{c}]$.

Theorem 2.7: Let F : (X,τ) $\rightarrow$ (Y,Γ) be an intuitiontistic fuzzy multifunction. If $_{\mathrm{S}}\mathrm{Cl}(F^{+}(\widetilde{B})) \subseteq F^{+}(\mathrm{Ker}(\widetilde{B}))$ for any intuitiontistic fuzzy set $\widetilde{B}$ of Y. Then F is an intuitiontistic fuzzy lower contra-irresolute multifunction.

Proof: Suppose that $_{\mathrm{S}}\mathrm{Cl}(F^{+}(\widetilde{B})) \subseteq F^{+}{}_{(\mathrm{S}}\mathrm{Ker}(\widetilde{B}))$ for any intuitionistic fuzzy set $\widetilde{B}$ of Y.

Let $\widetilde{A} \in$IF$_{\mathrm{S}}$O(X), by lemma 2.4 $_{\mathrm{S}}\mathrm{Cl}(F^{+}(\widetilde{A})) \subseteq F^{+}{}_{(\mathrm{S}}\mathrm{Ker}(\widetilde{A}))=F^{+}(\widetilde{A})$ this implies that $\mathrm{Cl}(F^{+}(\widetilde{A})) \subseteq F^{+}(\widetilde{A})$. But we know $F^{+}(\widetilde{A}) \subseteq \mathrm{Cl}(F^{+}(\widetilde{A}))$, Hence, $F^{+}(\widetilde{A})$ is semi-closed set in X. Thus, by theorem 2.6, F is an intuitionistic fuzzy lower contra -irresolute multifunction.

Definition 2.8: Given a family $\{F_{\beta}$: (X,τ) $\rightarrow$ (Y, Γ) : $\beta \in \wedge\}$, of an intuitionistic fuzzy multifunction, we define the union $\bigcup_{\beta\in\wedge} F_{\beta}$ and intersection $\bigcap_{\beta\in\wedge}$ as,

(a) $\bigcup_{\beta\in\wedge} F_{\beta}$: (X,τ)$\rightarrow$(Y,Γ), $(\bigcup_{\beta\in\wedge} F_{\beta})\,(x) = \bigcup_{\beta\in\wedge} F_{\beta}(x)$

(b) $\bigcap_{\beta\in\wedge} F_{\beta}$: (X,τ)$\rightarrow$(Y,Γ), $(\bigcap_{\beta\in\wedge} F_{\beta})\,(x) = \bigcap_{\beta\in\wedge} F_{\beta}(x)$

Theorem 2.9: If $F_\beta : (X, \tau) \to (Y, \Gamma)$, for each $\beta \in \wedge$ is an intuitionistic fuzzy upper contra irresolute then $\bigcup_{\beta \in \wedge} F_\beta$ is an intuitionistic fuzzy upper contra-irresolute.

Proof: Let $\widetilde{B}$ be an intuitionistic fuzzy semi-closed set in Y. To show that $(\bigcup_{\beta=1}^{n} F_\beta)^+ (\widetilde{B})$ $=\{x \in X : \bigcup_{\beta=1}^{n} F_\beta(x) \subseteq \widetilde{B}\}$ is semi-open in X.

Let $x \in (\bigcup_{\beta=1}^{n} F_\beta)^+ (\widetilde{B})$ then $F_\beta(x) \subseteq (\widetilde{B})$ for β = 1, 2, 3...n,. Since F_β : (X, τ) $\to$ (Y, Γ) is an intuitionistic fuzzy upper contra-irresolute multifunction , for β = 1, 2, 3,..., then $\exists$ semi-open set U containing x such that $\forall\ y \in U_x, F_\beta(y) \subseteq \widetilde{B}$. let U = $\bigcup_{\beta=1}^{n} U$, then U$\subset (\bigcup_{\beta=1}^{n} F_\beta)^+ (\widetilde{B})$. Therefore $(\bigcup_{\beta=1} F_\beta)^+ (\widetilde{B})$ is semi-open. Hence, $\bigcup_{\beta \in \wedge} F_\beta$ is intuitionistic fuzzy upper contra-irresolute.

Theorem 2.10: If F_β: (X, τ) $\to$ (Y, Γ), for β = 1, 2, 3,...n is an intuitionistic fuzzy lower contra-irresolute then $\bigcup_{\beta \in \wedge} F_\beta$ is an intuitionistic fuzzy lower contra-irresolute.

Proof: Let $\widetilde{B}$ be an intuitionistic fuzzy semi-closed set in Y. To show that $(\bigcup_{\beta=1}^{n} F_\beta)^- (\widetilde{B})$ = $\{x \in X : \bigcup_{\beta=1}^{n} F_\beta(x)\, q\, \widetilde{B}\}$ is semi-open in X.

Let $x \in (\bigcup_{\beta=1}^{n} F_\beta)^- (\widetilde{B})$ then $F_\beta(x)\, q\, (\widetilde{B})$ for β = 1, 2, 3...n , Since, F_β : (X, τ) $\to$ (Y, Γ) is intuitionistic fuzzy lower contra -irresolute multifunction , for β = 1, 2, 3,..., then $\exists$ semi-open set U containing x such that $\forall\ y \in U_x$, $F_\beta(y) q \widetilde{B}$. let $U = \bigcup_{\beta=1}^{n} U$, then $U \subset (\bigcup_{\beta=1}^{n} F_\beta)^- (\widetilde{B})$.

Therefore $(\bigcup_{\beta=1} F_\beta)^- (\widetilde{B})$ is open. Hence, $\bigcup_{\beta \in \wedge} F_\beta$ is an intuitionistic fuzzy lower contra-irresolute.

Theorem 2.11: Let $\{\bigcup_{\beta} : \beta \in \wedge\}$, be an α-open cover of a topological space (X, τ). An intuitionistic fuzzy mulifuntion F : (X, τ) $\rightarrow$ (Y, Γ) is an intuitionistic fuzzy upper contra -irresolute if and only if restriction $F \mid \bigcup_{\beta} : \bigcup_{\beta} \rightarrow$ Y is an intuitionistic fuzzy upper contra -irresolute for each $\beta \in \wedge$.

Proof: Necessity: Suppose that F is intuitionistic fuzzy upper contra-irresolute. Let $\beta \in \wedge$, $x \in \bigcup_{\beta}$ and $\widetilde{V}$ be any intuitionistic fuzzy semi-closed set in Y such that $(F \mid \bigcup_{\beta})\,(x) \subseteq \widetilde{V}$. Since F is an intuitionistic fuzzy upper contra-irresolute and $F(x) = (F \mid \bigcup_{\beta})(x)$, there exists semi-open set G of X containing x such that F(G)$\subseteq \widetilde{V}$. Let U = G$\bigcap U_{\beta}$, then $x \in$U$\in SO(X)$ in X and $(F \mid U_{\beta})$(U) = F(U)$\subseteq \widetilde{V}$. Therefore, it follows that $(F \mid U_{\beta})$ is intuitionistic fuzzy upper contra -irresolute.

Sufficiency: Let $x \in$X and $\widetilde{V}$ be any intuitionistic fuzzy semi-closed set in Y such that $F(x) \subseteq \widetilde{V}$ there exists $\beta \in \wedge$ and $x \in U_{\beta}$. Since $F \mid U_{\beta} : U_{\beta} \rightarrow$Y is an intuituionistic fuzzy upper contra -irresolute. and $F(x) = (F \mid U_{\beta})\,(x)$, there exists semi-open set U$\in U_{\beta}$ containing x such that $(F \mid U_{\beta})$(U)$\subseteq \widetilde{V}$. We have semi-open set U$\in U_{\beta}$ containing x and F(U) $\subseteq \widetilde{V}$. Therefore F is intuitionistic fuzzy upper contra-irresolute.

Theorem 2.12: Let $\{U_{\beta} : \beta \in \wedge\}$ be an semi-open cover of a topological space (X, τ). An Intuitionistic fuzzy multifunction F : (X, τ) $\rightarrow$ (Y, Γ) is intuitionistic fuzzy lower contra-irresolute if and only if the restriction $F \mid U_{\beta} : U_{\beta} \rightarrow$ Y is an intuitionistic fuzzy lower contra-irresolute for each $\beta \in \wedge$.

Proof: Necessity: Suppose that F is intuitionistic fuzzy lower contra-irresolute. Let $\beta \in \wedge$ and $x \in U_{\beta}$. Let $\widetilde{V}$ be any intuitionistic fuzzy semi-closed set in Y such that $(F \mid U_{\beta})(x) q \widetilde{V}$. Since F is an intuitionistic fuzzy lower contra-irresolute and $F(x) = (F \mid U_\beta)(x)$, there exists semi-open set U_o of X containing x such that $F(U_o)\, q \widetilde{V}$. Let U $= U_o \bigcap U_\beta$, then $x \in$ U is semi-open in X and $(F \mid U_\beta)(U)$ = F(U) and F(U) $q\widetilde{V}$ for all $x \in U$. Therefore it follows that $(F \mid U_\beta)$ is an intutionistic fuzzy lower contra-irresolute.

Sufficiency: Let $x \in$ X and $\widetilde{V}$ be any semi-closed set in Y such that $F(x)\, q \widetilde{V}$ there exists $\beta \in \wedge$ and $x \in U_\beta$. Since $F \mid U_\beta : U_\beta \rightarrow$ Y is an intutionistic fuzzy lower contra -irresolute and $F(x) = (F \mid U_\beta)(x)$, there exists open set $U_o \in U_\beta$ containing x such that $(F \mid U_\beta)(U_o) qV$. we have semi-open set $U_o \in U_\beta$ containing x and $F(U_o) q\widetilde{V}$. Therefore F is an intuitionistic fuzzy lower contra-irresolute.

Definition 2.13: An intuitionistic fuzzy multifunction F : $(X, \tau) \rightarrow (Y, \Gamma)$, then the intuitionistic fuzzy multifunction $_S$ClF : $(X, \tau) \rightarrow (Y, \Gamma)$ is defined by $_S$Cl(F)(x) = $_S$ Cl(F(x)) for every $x \in$ Y.

Lemma 2.14: For an intuitionistic fuzzy multifunction F : $(X, \tau) \rightarrow (Y, \Gamma)$ it follows that $_S\text{Cl}(F)^{-}(\widetilde{V}) = F^{-}((\widetilde{V}))$, for each intuitionistic fuzzy $_S$-open set $\widetilde{V}$ of Y.

Proof: Obvious.

Theorem 2.15: An intuitionistic fuzzy multifunction F : $(X, \tau) \rightarrow (Y, \Gamma)$ is intuitionistic fuzzy lower contra-irresolute if and only if $_S$Cl(F) : $(X, \tau) \rightarrow (Y, \Gamma)$ is an intuitionistic fuzzy lower contra irresolute.

Proof: Necessity: Suppose that F is an intuitionistic fuzzy lower -irresolute. Let $x \in$ X and $\widetilde{V}$ be any intuitionistic fuzzy semi-open set of such that α Cl(F)$(x)\, q \widetilde{V}$ .Since F is an intuitionistic fuzzy lower contra-irresolute, there exists semi-open set U of X containing x such that F(u)q$\widetilde{V}$, $\forall$ u$\in$U.

Hence, $_S$Cl(F)(u)q$\widetilde{V}$ for each u$\in$U. This show that Cl(F) is an intuitionistic fuzzy contra-irresolute.

Sufficiency: Suppose SCl(F) is intuitionistic fuzzy lower contra-irresolute. Let $x \in$ X and $\widetilde{V}$ be any intuitionistic fuzzy semi-open set of Y such that $\mathrm{F}(x)\,\mathrm{q}\,\widetilde{V}$. by lemma 3.14, we have $x \in F^{-}(\widetilde{V}) = (SCl(F)^{-}(\widetilde{V})$ and hence SCl(F)(x)q$\widetilde{V}$. Since SCl(F) is an intuitionistic fuzzy lower contra-irresolute, there exists a semi-closed set U of X containing x such that SCl(F(u))q$\widetilde{V}$ for each u$\in$U. Since $\widetilde{V}$ be intuitionistic fuzzy semi-open set of Y, hence F(u)q$\widetilde{V}$ for each u$\in$U. This shows that F is an intuitionistic fuzzy lower contra -irresolute.

Definition 2.16: An intuitionistic fuzzy set $\widetilde{A}$ in intuitionistic fuzzy topological space (Y,Γ) is called cl-neighbourhood of an intuitionistic fuzzy set $\widetilde{V}$ in Y, if there exists an intuitionistic fuzzy semi-closed set $\widetilde{U}$ in Y such that $\widetilde{V} \subseteq \widetilde{U} \subseteq \widetilde{A}$.

Theorem 2.17: If F : $(X,\tau) \to (Y,\Gamma)$ is an intuitionistic fuzzy upper contra -irresolute multifunction, then for each point $x \in$ X and each cl-neighbourhood of $\widetilde{V}$ of $\mathrm{F}(x)$, $F^{+}(\widetilde{V})$ is an semi-neighbourhood of x.

Proof: Let $x \in$ X and $\widetilde{V}$ be an cl-neighbourhood of $\mathrm{F}(x)$, then $\exists$ an intuitionistic fuzzy semi-closed set $\widetilde{A}$ in Y such that $\mathrm{F}(x) \subseteq \widetilde{A} \subseteq \widetilde{V}$. We have $x \in F^{+}(\widetilde{A}) \subseteq F^{+}(\widetilde{V})$ and since $F^{+}(\widetilde{A})$ is an semi-open, $F^{+}(\widetilde{V})$ is an semi-neighbourhood of x.

Theorem: 2.18 For an intuitionistic fuzzy multifunction F : $(X,\tau)\to(Y,\Gamma)$ the following are equivalent:

(a) F is an intuitionistic fuzzy lower contra-irresolute.

(b) For any $x \in$ X and any net $(x_i)_{i\in I}$ α-converging to x in X and each an intuitionistic fuzzy semi-closed set $\widetilde{B}$ of Y with $x \in F^{+}(\widetilde{B})$, then the net $(x_i)_{i\in I}$ is eventually in $F^{-}(\widetilde{B})$.

Proof: (a) $\Rightarrow (b)$. let $(x_i)_{i\in I}$ be net α-coverging to x in X and $\widetilde{B}$ be any intuitionistic fuzzy semi-closed set Y with $x \in F^{-}(\widetilde{B})$. Since F is an intuitionistic fuzzy lower contra-irresolute $\exists$ an α-open set A of X containing x such that

A$\subset F^{-}(\widetilde{B})$. Since x_i is converge to x $\exists$ an index $i_{0\in I}$ such that $x_i \in$A for every $i \geq i_0$ we have $x_i \in$A$\subset F^{-}(\widetilde{B})$ $\forall$ $i \geq i_0$. Hence $(x_i)_{i\in I}$ is eventually in $F^{-}(\widetilde{B})$.

(b) $\Rightarrow$ (a) Suppose that F is not intuitionistic fuzzy lower contra-irresolute $\exists$ a point $x \in$X and an intuitionistic fuzzy semi-closed set $\widetilde{B}$ with $x \in F^{-}(\widetilde{B})$ such that B does not subset $F^{-}(\widetilde{B})$ for any semi-open set B $\subset$ X containing x. Let $(x_i) \in$B and (x_i) does not belong to $F^{-}(\widetilde{B})$ for each semi-open set B $\subset$ X containing x. Then the semi-neighbourhood net (x_i) converges to x but $(x_i)_{i\in I}$ is not semi-eventually in $F^{-}(\widetilde{B})$. This is a contradiction.

Theorem 2.19: For an intuitionistic fuzzy multifunction F : $(X, \tau) \to (Y, \Gamma)$ the following are equivalent:

(a) F is an intuitionistic fuzzy upper contra -irresolute.

(b) For any $x \in$X and any net $(x_i)_{i\in I}$ α-converging to x in X and each intuitionistic fuzzy semi-closed set $\widetilde{B}$ of Y with $x \in F^{+}(\widetilde{B})$, then the net $(x_i)_{i\in I}$ is eventually in $F^{+}(\widetilde{B})$.

Proof: The proof of this theorem is similar to that of theorem 2.18.

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FUZZY STRONGLY QUASI CONTINUITY IN FUZZY BITOPOLOGICAL SPACE

Abstract: In the present paper, the concept of Fuzzy strongly quasi continuity has been introduced which is a generalization of Fuzzy quasi continuity for Fuzzy Bitopological spaces and some of their basic properties are studied. Fuzzy strongly quasi continuity is stronger than Fuzzy quasi continuity in Fuzzy Bitopological spaces.

Keywords: Fuzzy topology, Fuzzy Bitopology, Fuzzy Quasi Open and Closed, Fuzzy Quasi and Strongly Quasi Continuous.

Mathematical Subject Classification (2000) No.: Primary 54A40, 54E55.

1. Introduction

Zadeh [7] introduced the concept of Fuzzy set which formed the back bone of Fuzzy mathematics. In 1968, Chang [2] introduced the concept of Fuzzy Topological space as a generalization of Topological space. Later Kandil [3] introduced the concept of Fuzzy Bitopological space as a generalization of Fuzzy topological space. U. D. Tapi, S. S. Thakur and G. P. S. Rathore [5] and S. N. Maheshwari, G. I. Chai and P. C. Jain [4] introduced the concept of Fuzzy quasi open sets in Fuzzy Topological spaces. Recently U. D. Tapi, S. S. Thakur and G. P. S. Rathore [6] introduced the concept of Fuzzy quasi continuity in Fuzzy Bitopological space.

A Fuzzy set is a pair (X, μ), where X is a set and $\mu : X \to [0, 1]$ is a membership function. For each $x \in X$, $\mu(x)$ is called the degree of membership of x in (X, μ), $\mu(A)$ is the membership function of the Fuzzy set $A = (X, \mu(A))$.

A family τ of Fuzzy sets of X is called a Fuzzy topology on X if 0 null Fuzzy set and 1 the whole Fuzzy set belong to τ and τ is closed with respect to any supremum and finite infimum. The members of τ are called τ–Fuzzy open sets of X and their complements are τ–Fuzzy closed sets. For a Fuzzy set λ of X the closure of λ denoted by cl(λ) is the intersection of all Fuzzy closed super sets of λ and interior of λ denoted by int(λ) is the union of all Fuzzy open subsets of λ. A mapping $f: (X, \tau) \to (X\ , \sigma)$ is Fuzzy continuous (respectively Fuzzy open) if the inverse image (resp. image) of every Fuzzy open set of X (resp. X) is Fuzzy open set of X (resp. X). A system (X, τ, σ) consists of a set X with two Fuzzy topologies τ and σ on X is called a Fuzzy Bitopological space.

In the present paper, we studied and modified the result of U. D. Tapi, S. S. Thakur and G. P. S. Rathore [6] and introduced the concept of Fuzzy strongly quasi continuity in Fuzzy Bitopological space, which is stronger than the concept of Fuzzy quasi continuity in Fuzzy Bitopological space. In the section 2, we present some basic definitions in Fuzzy Bitopological spaces, section 3 consists of our main results and section 4 comprises the discussion and conclusion.

2. Preliminaries

In this section, we present some definitions in Fuzzy Bitopological spaces, which will be required in our main results. The definitions are as follows:

Definition 2.1: A mapping $f: (X, \tau, \sigma) \to (X\ , \tau\ , \sigma\ )$ is said to be pairwise Fuzzy continuous if $f: (X, \tau) \to (X\ , \tau\ )$ and $f: (X, \sigma) \to (X\ , \sigma\ )$ are Fuzzy continuous. A mapping $f: (X, \tau, \sigma) \to (X\ , \tau\ , \sigma\ )$ is said to be pair wise Fuzzy open if $f: (X, \tau) \to (X\ , \tau\ )$ and $f: (X, \sigma) \to (X\ , \sigma\ )$ are Fuzzy open.

Definition 2.2 [1]: Let (X, τ) be a Fuzzy topological space and λ be a Fuzzy set of (X, τ), then λ is called

(i) A Fuzzy semi open set of X if there exists a $v \epsilon \tau$ such that $v \leq \lambda \leq cl(v)$.

(ii) A Fuzzy semi closed set of X if there exists a $v' \epsilon \tau$ such that $int(v') \leq \lambda \leq v'$.

Definition 2.3 [1]: Let $f: (X, \tau) \to (X\ , \sigma)$ be a mapping from a Fuzzy topological space X to Fuzzy topological space X , then the mapping f is called

(i) A Fuzzy semi continuous mapping, if $f^{-1}(\lambda)$ is a Fuzzy semi open set of X for each λϵσ.

(ii) A Fuzzy semi open mapping if $f(\lambda)$ is a Fuzzy semi open set for each λϵτ.

(iii) A Fuzzy semi closed mapping if $f(\lambda)$ is a Fuzzy semi closed set for each Fuzzy closed set λ of X.

Definition 2.4 [5]: A Fuzzy set λ in a Fuzzy Bitopological space (X, τ, σ) is said to be Fuzzy quasi open if it is the union of a τ–Fuzzy open set and a σ Fuzzy open set.

Definition 2.5 [5]: A Fuzzy set λ in a Fuzzy Bitopological space (X, τ, σ) is Fuzzy quasi closed if its complement is Fuzzy quasi open.

Definition 2.6 [6]: Let (X, τ, σ) and $(X\ , \tau\ , \sigma\ )$ be Fuzzy Bitopological space and $f: X \rightarrow X$ then f is Fuzzy quasi continuous if for every Fuzzy quasi set B in X , $f^{-1}(B)$ is Fuzzy quasi closed in X.

Definition 2.7 [6]: A mapping f from a Fuzzy Bitopological space (X, τ, σ) to a Fuzzy Bitopological space $(X\ , \tau\ , \sigma\ )$ is pairwise Fuzzy semi continuous if the inverse image of every Fuzzy quasi open set in X is Fuzzy quasi semi open in X.

Proposition 2.1: Every pair wise Fuzzy continuous mapping is Fuzzy quasi continuous. But the converse may not be true.

Example 2.1: Let X = {x, y}, X = {a, b} and U, V, U′, V′ be the Fuzzy sets defined as follows:

U(x) = 0.3, U(y) = 0.6

V(x) = 0.7, V(y) = 0.3

U′ (a) = 0.7, U′ (b) = 0.6

V′ (a) = 0.7, V′ (b) = 0.3

Let τ = {0, U, 1}, σ = {0, V, 1} and τ = {0, U′, 1}, σ = {0, V′, 1}. Then the mapping $f: (X, \tau, \sigma) \rightarrow (X\ , \tau\ , \sigma\ )$ defined by $f(x) = a, f(y) = b$ is Fuzzy quasi continuous, but not pairwise Fuzzy continuous.

3. Main Results

In the following, we define Fuzzy strongly quasi continuity and prove some theorems in Fuzzy Bitopological Space by studying Fuzzy quasi continuity in Fuzzy Bitopological space, which was introduced by U. D. Tapi, S. S. Thakur and G. P. S. Rathore [6].

Definition 3.1: Let (X,τ,σ) and $(X\ ,\tau\ ,\sigma\ )$ be Fuzzy Bitopological space and $f: X \rightarrow X$ then f is Fuzzy strongly quasi continuous if for every Fuzzy quasi open set B in X , $f^{-1}(B)$ is Fuzzy quasi open in X.

Theorem 3.1: Let (X,τ,σ) and $(X\ ,\tau\ ,\sigma\ )$ be Fuzzy Bitopological space and $f: X \rightarrow X$ then the following conditions are equivalent:

a) f is Fuzzy strongly quasi continuous.

b) For every Fuzzy quasi closed set B in X , $f^{-1}(B)$ is Fuzzy quasi closed in X.

c) For every Fuzzy point x_β of X and every Fuzzy quasi open set M such that $f(x_\beta) \in M$, there is a Fuzzy quasi open set A in X such that $x_\beta \in A$ and $f(A) \leq M$.

d) For every Fuzzy point x_β of X and every quasi neighbourhood M of $f(x_\beta)$, $f^{-1}(M)$ is a quasi neighbourhood of x_β.

e) For every Fuzzy point x_β of X and every quasi neighbourhood M of $f(x_\beta)$ there is a quasi neighbourhood N of x_β in X such that $f(N) \leq M$.

f) For every Fuzzy point x_β of X and every Fuzzy quasi open set M of X such that $f(x_\beta)qM$ there is a Fuzzy quasi open set A in X such that $x_\beta qM$ and $f(A) \leq M$.

g) For every Fuzzy point x_β of X and every quasi Q-neighbourhood M of $f(x_\beta)$, $f^{-1}(M)$ is a quasi Q-neighbourhood N of x_β.

h) For every Fuzzy point x_β of X and every quasi Q-neighbourhood M of $f(x_\beta)$, there is a quasi Q-neighbourhood N of x_β such that $f(N) \leq M$.

i) $f(qcl(A)) \leq qcl(f(A))$ for every Fuzzy set A of X.

j) $qcl(f^{-1}(B)) \leq f^{-1}(qcl(B))$ for every Fuzzy set B of X .

k) $f^{-1}(qint(B)) \leq qint(f^{-1}(B))$ for every Fuzzy set B of X .

Proof:

$\underline{\underline{a \Leftrightarrow b}}$ Obvious

$\underline{\underline{a \Rightarrow c}}$ Let x_β be a Fuzzy point of X and M be a Fuzzy quasi open set in X such that $f(x_\beta) \in M$. Put $A = f^{-1}(M)$.

Then by (a), A is a Fuzzy quasi open set in X such that x_β and f(A) ≤ M.

$\underline{\underline{c \Rightarrow a}}$ Let M be a Fuzzy quasi open set in X and x_β then $f(x_\beta)$ ϵM.

Now by (c) there is a Fuzzy quasi open set A in X such that x_β and f(A) ≤ M.

Then x_β ϵ A ≤ $f^{-1}(M)$. Hence by [4], $f^{-1}(M)$ is Fuzzy quasi open in X.

$\underline{\underline{a \Rightarrow d}}$ Let x_β be a Fuzzy point of X and M be a quasi Q-neighbourhood of $f(x_\beta)$ then there is a Fuzzy quasi open set N such that $f(x_\beta)$ ϵN ≤ M.

Now $f^{-1}(N)$ is Fuzzy quasi open in X and x_β ϵ $f^{-1}(N) \leq f^{-1}(M)$.

Thus, $f^{-1}(M)$is a quasi neighbourhood of x_β in X.

$\underline{\underline{d \Rightarrow e}}$ Let x_β be a Fuzzy point of X and M be a quasi neighbourhood of $f(x_\beta)$.

Then N= $f^{-1}(M)$ is a quasi neighbourhood of x_β and $f(N) = f(f^{-1}(M)) \leq$ M.

$\underline{\underline{e \Rightarrow c}}$ Let x_β be a Fuzzy point of X and M be a Fuzzy quasi open set such that $f(x_\beta)$ ϵM then M is a quasi neighbourhood of $f(x_\beta)$ so there is a quasi neighbourhood N of x_β in X such that x_β ϵN and f(N) ≤ M.

Hence, there exists a Fuzzy quasi open set A in X such that $f(x_\beta)$ ϵA ≤ N and so f(A) ≤ f(N) ≤ M.

$\underline{\underline{a \Rightarrow f}}$ Let x_β be a Fuzzy point of X and M be a Fuzzy quasi open set in X such that $f(x_\beta)$ qM

Let A= $f^{-1}(M)$. Then A is Fuzzy quasi open x_β qA, and $f(A) = f(f^{-1}(M)) \leq$ M.

$\underline{\underline{f \Rightarrow a}}$ Let M be a Fuzzy quasi open set in X and x_β ϵ $f^{-1}(M)$

Clearly $f(x_\beta)$ ϵM.

Choose the Fuzzy point x_β as $x_\beta^c(x) = 1 \quad x_\beta(x)$ then$f(x_\beta^c)$ qM and so by (f), there exists a Fuzzy quasi open set A such that $x_\beta^c qA$ and f(A) ≤ M.

Now $x_\beta^c qA \rightarrow x_\beta^c(x) + A(x) = 1 \quad \beta + A(x) > 1$

$$\rightarrow A(x) > \beta \rightarrow x_\beta \epsilon A \leq f^{-1}(M).$$

Hence, by [5], $f^{-1}(M)$ is Fuzzy quasi open.

$\underline{\underline{f \Rightarrow g}}$ Let x_β be a Fuzzy point of X and M be a quasi Q-neighbourhood of $f(x_\beta)$. Then there is a Fuzzy quasi open set U in X such that $f(x_\beta)qU \leq M$.

By hypothesis there is a Fuzzy quasi open set A in X such that $x_\beta qA$ and $f(A) \leq U$. Thus $x_\beta qA \leq f^{-1}(U) \leq f^{-1}(M)$.

Hence, $f^{-1}(M)$ is a quasi Q-neighbourhood of x_β.

$\underline{\underline{g \Rightarrow h}}$ Let x_β be a Fuzzy point of X and M be a quasi neighbourhood of $f(x_\beta)$.

Then $N = f^{-1}(M)$ is quasi Q-neighbourhood of x_β and $f(N) = f\left(f^{-1}(M)\right) \leq M$.

$\underline{\underline{h \Rightarrow f}}$ Let x_β be a Fuzzy point of X and M be a Fuzzy quasi open set such that $f(x_\beta)$ qM. Then M is a quasi Q-neighbourhood of $f(x_\beta)$. So there is a quasi Q-neighbourhood N of x_β such that $f(N) \leq M$. Now N being quasi Q-neighbourhood of x_β there exists a Fuzzy quasi open set A in X such that $x_\beta qA \leq N$.

Hence, $x_\beta qA$ and $f(A) \leq f(N) \leq M$.

$\underline{\underline{b \Leftrightarrow i}}$ Obvious.

$\underline{\underline{i \Leftrightarrow j}}$ Obvious.

$\underline{\underline{a \Leftrightarrow k}}$ Obvious.

Theorem 3.2: Let $(X, \tau, \sigma), (X\ , \tau\ , \sigma\ )$ and $(X\ \ , \tau\ \ , \sigma\ \ )$ be Fuzzy Bitopological spaces. If $f: X \rightarrow X$ and $g: X\ \rightarrow X$ are Fuzzy strongly quasi continuous mappings.

Then $gof: X \rightarrow X$ is Fuzzy strongly quasi continuous.

Proof: Obvious.

Theorem 3.3: Let (X, τ, σ) and $(X\ , \tau\ , \sigma\ )$ be two Fuzzy Bitopological spaces and $f: (X, \tau, \sigma) \rightarrow (X\ , \tau\ , \sigma\ )$ be a mapping if the graph mapping $g: (X, \tau, \sigma) \rightarrow (X \times X\ ,\ \tau\ \ ,\ \sigma\ \ )$ of f is Fuzzy strongly quasi continuous then f is Fuzzy strongly quasi continuous, where $\tau\ \ ,\ \sigma\ \ $ are the product topologies generated by τ and $\tau\ $ (resp. σ and $\sigma\ $).

Proof: Let λ be a Fuzzy quasi open set in $X\ $. Then $1 \times \lambda$ is a Fuzzy quasi open set in $X \times X\ $. Since g is fuzzy strongly quasi continuous $g^{-1}(1 \times \lambda)$ is fuzzy quasi open in X.

Now $f^{-1}(\lambda) = 1\ \ \ f^{-1}(\lambda) = g^{-1}(1 \times \lambda)$.

Hence, $f^{-1}(\lambda)$ is Fuzzy quasi open in X and f is Fuzzy strongly quasi continuous.

Theorem 3.4: Let $f: (X, \tau, \sigma) \rightarrow (X\ , \tau\ , \sigma\ )$ be a Fuzzy strongly quasi continuous mapping and A is non empty subset of X. If χ_A is Fuzzy bi-open in X, then $\frac{f}{A}: (A, \tau_A, \sigma_A) \rightarrow (X\ , \tau\ , \sigma\ )$ is Fuzzy strongly quasi continuous.

Proof: Let λ be a Fuzzy quasi open set in $X\ $. Then $f^{-1}(\lambda)$ is Fuzzy quasi open in X, because f is Fuzzy strongly quasi continuous. Therefore by [5], $(\frac{f}{A})^{-1}(\lambda) = \chi_A\ \ f^{-1}(\lambda)$ is Fuzzy quasi open in (A, τ_A, σ_A). Hence, $\frac{f}{A}$ is Fuzzy strongly quasi continuous.

4. Discussion and Conclusion

In this paper, we introduced the concept of Fuzzy strongly quasi continuity which is a generalization of Fuzzy quasi continuity. Fuzzy strongly quasi continuity is stronger than Fuzzy quasi continuity in Fuzzy Bitopological spaces. We also proved some theorems for Fuzzy strongly quasi continuous mappings in Fuzzy Bitopological spaces.

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FUZZY CODON COMPLEMENTS

Abstract: A codon with different nucleotide bases can be thought as a set of three nucleotide bases and complement for this type of codon is a singleton set. This singleton set can be treated as a fuzzy point when studied under certain situations with suitable restriction. Keeping this point in view, a particular type of fuzzy sets on the set of nucleotide bases are studied and its various properties are explored.

Keywords: Fuzzy Metric Space, Fuzzy Group, Fuzzy Ring, Fuzzy Topological Group, Fuzzy Topological Ring, Fuzzy Codon Complement.

Mathematical Subject Classification No.: 94D05, 20N25, 06D72.

1. Introduction

A polymer is a large macromolecule consisting of many identical or similar building blocks, called its monomers, that are linked by bonds to form a chain. DNA and RNA are linear polymers and their monomeric units are called nucleotides. Each nucleotide consist of three subunits: a phosphate group and a sugar (ribose in case of RNA and deoxyribose in DNA) and nucleotide bases. There are four nucleotide bases of DNA strand-Adenine (A), Cytosine (C), Guanine (G) and Thymine (T) which are sometimes termed as DNA alphabets or Letters. In practice the nucleotides are represented by their bases. A sequence of three nucleotides forms a unit called codon. A codon encodes an amino acid. Proteins are chain of amino acids.

A large number of nucleotides are linked by bonds to form a polynucleotide, a chain of DNA or RNA. These polynucleotide chains were transformed to ordered fuzzy sets for the purpose of sequence comparison as suggested by Sadegh Zadeh (2000). Further the author showed that a fuzzy polynucleotide is a unique point in a hypercube. The author equipped a metric with these set of polynucleotides to form a Fuzzy Polynucleotide Space. Using the same concept of Fuzzy Polynucleotide Space and simplifying the sequences to codons, Torres and Nieto (2003) tried to define a 12 dimensional metric space which they have also called as Fuzzy Polynucleotide Space in order to compare fuzzy codons. Unfortunately this 12-dimensional metric space as suggested Torres and Nieto was studied and declared faulty by Zadeh (2007). Considering the topological groups under fuzzy setting, the properties of Fuzzy Topological groups and Semi groups were studied by Foster (1979). Ali and Phukan (2013) studied the algebraic and topological properties of the genetic code.

In this paper we have defined a new notion which we have named as fuzzy codon complements and discussed some algebraic and topological properties of these codon complements in fuzzy setting. In section 2, we have discussed some preliminaries to make our work self contained. In section 3, we have defined fuzzy codon complements and discussed some of their properties.

2. Preliminaries

A fuzzy set A of a universal set X is a function

$$\mu_A : X \rightarrow [0, 1]$$

For each $x \in X$, $\mu_A(x)$ is called the membership grade of x in A. For convenience, the fuzzy set as well as the corresponding membership function is represented by A.

For a non empty set X, $I^X = \{ A: X \rightarrow [0,1]\}$.

The elements of I^X are called fuzzy subsets of X. 0_X and 1_X are functions on X identically equal to 0 and 1 respectively.

Given two fuzzy sets A and B, their standard intersection $A \cap B$, standard union $A \cup B$ and standard complement A^C are defined for all $x \in X$ by the equations

$$\mu_{A \cap B}(x) = min[\, \mu_A(x), \mu_B(x) \,]$$

$$\mu_{A \cup B}(x) = max[\, \mu_A(x), \mu_B(x) \,]$$

$$\mu_{A^c}(x) = 1 \quad \mu_A(x)$$

$$A \subset B \text{ if } \mu_A(x) \leq \mu_B(x) \;\; \forall\, x \in X$$

For infinite collection of fuzzy subsets, *min* and *max* are respectively replaced by *infimum* and *supremum*.

A fuzzy point on X is a fuzzy subset

$$P_x^{\alpha}(y) = \begin{cases} \alpha, if \;\; y = x \\ 0, if \; y \neq x \end{cases}$$

Definition 2.1: [5] Let X be a group and G be a fuzzy set in X. Then G is a fuzzy group in X if the following two conditions are obeyed. For every $x, \; y \in X$,

a) $G(x + y) \geq \min\{G(x), G(y)\}$

b) $G(\;\; x) \geq G(x)$

If G satisfies only the first condition then it is a fuzzy semigroup.

Definition 2.2: [5] Let X be a ring and G be a fuzzy set in X. Then G is a fuzzy ring in X if for every $x, y \in X$

i) $G(x \quad y) \geq \min\{G(x), G(y)\}$

ii) $G(x.y) \geq \min\{G(x), G(y)\}$

These two conditions are equivalent to

i) $G(x + y) \geq \min\{G(x), G(y)\}$

ii) $G(x.y) \geq \min\{G(x), G(y)\}$

iii) $G(\;\; x) \geq \; G(x)$

Definition 2.3: [1] A fuzzy metric $\tilde{d}$ on X is a classical metric on $\tilde{X}$ (the set of all fuzzy point of X), satisfying the additional condition $\tilde{d}(P_x^r, P_y^s) = \tilde{d}(P_x^s, P_y^r)$.

Definition 2.4: [5] Let f be a mapping from a set X to Y. Let B be a fuzzy set in Y. Then the inverse image of B, written by $f^{-1}(B)$, is the fuzzy set in X with membership function defined by

$$f^{-1}(B)(x) = B\{f(x)\} \text{ for all } x \in X.$$

Conversely, let A be a fuzzy set in X. Then the image of A, written as $f(A)$, is the fuzzy set in Y with membership function defined by

$$f[A](y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} A(z), & \text{if } f^{-1}(y) \text{ is nonempty} \\ 0, & \text{otherwise} \end{cases}$$

for all $y \in Y$, where $f^{-1}(y) = \{x \mid f(x) = y\}$.

Definition 2.5: [5] A fuzzy topology on a set X is a family $\mathfrak{I}$ of fuzzy sets in X which satisfies the following conditions:

i) For all $c \in I, k_c \in \mathfrak{I}$

ii) If $A, B \in \mathfrak{I}$ then $A \cap B \in \mathfrak{I}$

iii) If $A_i \in \mathfrak{I}$ for all $i \in I$, then $\cup_{i \in I} A_i \in \mathfrak{I}$

The pair $(X, \mathfrak{I})$ is called a fuzzy topological space, or *fts* for short, and the members of $\mathfrak{I}$ are called $\mathfrak{I}$ open fuzzy sets, or, when there is no risk of confusion, simply open fuzzy sets.

In the definition of a fuzzy topology by Chang, the condition i) is just $k_0, k_1 \in \mathfrak{I}$.

Definition 2.6: [5] Let A be a fuzzy set in X and $\mathfrak{I}$ a fuzzy topology on X. Then the induced fuzzy topology on A is the family of fuzzy subsets of A which are the intersections with A of $\mathfrak{I}$ open fuzzy sets in X. The induced fuzzy topology is denoted by $\mathfrak{I}_A$, and the pair $(A, \mathfrak{I}_A)$ is called a fuzzy subspace of $(X, \mathfrak{I})$.

Definition 2.7: [5] Let $\mathfrak{I}$ be a fuzzy topology on a set X. A subfamily   of $\mathfrak{I}$ is base for $\mathfrak{I}$ if and only if each member of $\mathfrak{I}$ can be expressed as the union of members of .

Definition 2.8: [5] Let $\mathfrak{I}$ be a fuzzy topology on a set X and $\mathfrak{I}_A$, the induced fuzzy topology on a fuzzy subset A of X. A subfamily $'$ of $\mathfrak{I}_A$ is base for $\mathfrak{I}_A$ if and only if each member of $\mathfrak{I}_A$ can be expressed as a union of members of $'$.

- *Note:* If   is a base for a fuzzy topology $\mathfrak{I}$ on a set X, then $_A = \{U \cap A | U \in \ \}$ is a base for the induced fuzzy topology $\mathfrak{I}_A$ on the fuzzy subset A.

Next we propose the following definition.

Definition 2.9: Let X be a set. A non empty collection of fuzzy subsets of X (called basis elements) is a basis for a fuzzy topology on X if the following two conditions holds:

i) For each $x \in X$, there is at least one basis element B such that $B(x) > 0$.

ii) We have x such that $B_1(x) > 0$ and $B_2(x) > 0$ where B_1 and B_2 are basis elements. Then there exists a basis element B_3 such that $B_3(x) > 0$ and $B_3 \subset B_1 \cap B_2$.

The fuzzy topology $\mathfrak{I}$ generated by is such that a fuzzy subset U of X is an element of $\mathfrak{I}$ if for each x, for which $U(x) > 0$, there exists $B \in$ such that $B(x) > 0$ and $B \subset U$.

- *Note:* Each basis element is itself an element of $\mathfrak{I}$.

Definition 2.10: [5] Let f be a mapping from a *fts* $(X, \mathfrak{I})$ to a *fts* $(Y, \mathcal{U})$. Let be a base for $\mathcal{U}$. Then f is fuzzy continuous if and only if for each B in , the inverse image $f^{-1}(B)$ is in $\mathfrak{I}$.

Let $(A, \mathfrak{I}_A)$ and $(B, \mathcal{U}_B)$ be the fuzzy subspaces of fuzzy topological spaces $(X, \mathfrak{I})$ and $(Y, \mathcal{U})$ respectively. Let f be a mapping of $(X, \mathfrak{I})$ in to $(Y, \mathcal{U})$, then f is a mapping of $(A, \mathfrak{I}_A)$ into $(B, \mathcal{U}_B)$ if $f(A) \subset B$.

Definition 2.11: [5] Let $(A, \mathfrak{I}_A)$ and $(B, \mathcal{U}_B)$ be the fuzzy subspaces of fuzzy topological spaces $(X, \mathfrak{I})$ and $(Y, \mathcal{U})$ respectively. Then a mapping $f: (A, \mathfrak{F}_A)$ $(B, \mathcal{U}_B)$ is relatively fuzzy continuous if and only if for each open fuzzy set V in $\mathcal{U}_B$, the intersection $f^{-1}(V) \cap A$ is in $\mathfrak{I}_A$.

Proposition 2.12: [5] Let $(A, \mathfrak{I}_A)$ and $(B, \mathcal{U}_B)$ be fuzzy subspaces of fuzzy topological spaces $(X, \mathfrak{I})$ and $(Y, \mathcal{U})$ respectively, and let f be a fuzzy continuous mapping of $(X, \mathfrak{I})$ into a $(Y, \mathcal{U})$ such that $f(A) \subset B$. Then f is relatively fuzzy continuous mapping of $(A, \mathfrak{I}_A)$ into $(B, \mathcal{U}_B)$.

Definition 2.13: [5] Let X be a group and $\mathfrak{I}$ a fuzzy topology on X. Let G be a fuzzy group in X and G be endowed with the induced fuzzy topology $\mathfrak{I}_G$. Then G is a fuzzy topological group in X if and only if it satisfies the following two conditions:

i) The mapping $\alpha: (x, y) \to x + y$ of $(G, \mathfrak{I}_G) \times (G, \mathfrak{I}_G)$ into $(G, \mathfrak{I}_G)$ is relatively fuzzy continuous.

ii) The mapping $\beta: x \to \ x$ of $(G, \mathfrak{I}_G)$ into $(G, \mathfrak{I}_G)$ is relatively fuzzy continuous.

- *Note*: Suppose G is a fuzzy group in a group X. Let $\alpha: (x, y) \to x + y$ be a mapping from $X \times X$ into X and $\beta: x \to \quad x$ be a mapping of X into itself. Then $\alpha[G \times G] \subset G$ and $\beta[G] \subset G$ [5].

If $\Im$ is a fuzzy topology defined on X, then G acquires an induced fuzzy topology $\Im_G$. Also $(G, \Im_G)$ is a fuzzy subspace of the *fts* $(X, \Im)$, and $(G, \Im_G) \times (G, \Im_G)$ a fuzzy subspace of $(X, \Im) \times (X, \Im)$.

Definition 2.14: [4] Let X be a ring and $\Im$ be a fuzzy topology X such that

i) The mapping $\alpha: (x, y) \to x + y$ of $(X, \Im) \times (X, \Im)$ into $(X, \Im)$ is fuzzy continuous.

ii) The mapping $\gamma: (x, y) \to x.y$ of $(X, \Im) \times (X, \Im)$ into $(X, \Im)$ is fuzzy continuous.

iii) The mapping $\beta: x \to \quad x$ of $(X, \Im)$ into $(X, \Im)$ is fuzzy continuous.

The pair $(X, \Im)$ is called fuzzy topological ring.

We now propose the following definition.

Definition 2.15: Let X be a ring and $\Im$ a fuzzy topology on X. Let G be a fuzzy ring in X and G be endowed with the induced fuzzy topology $\Im_G$. Then G is a fuzzy topological ring in X if and only if it satisfies the following three conditions:

i) The mapping $\alpha : (x, y) \to x + y$ of $(G, \Im_G) \times (G, \Im_G)$ into $(G, \Im_G)$ is relatively fuzzy continuous.

ii) The mapping $\gamma : (x, y) \to x.y$ of $(G, \Im_G) \times (G, \Im_G)$ into $(G, \Im_G)$ is relatively fuzzy continuous.

iii) The mapping $\beta : x \to \quad x$ of $(G, \Im_G)$ into $(G, \Im_G)$ is relatively fuzzy continuous.

Definition 2.16: [5] Given a family $\{(X_j, \Im_j)\}$, $j \in J$ be a family of fuzzy topological spaces, and we define their product $\prod_{j \in J}(X_j, \Im_j)$ to be the *fts* $(X, \Im)$, where $X = \prod_{j \in J} X_j$ is the usual set product and $\Im$ is the coarsest topology on X for which the projections p_j of X on to X_j are fuzzy continuous for each $j \in J$. The fuzzy topology $\Im$ is product fuzzy topology on X, and $(X, \Im)$ is a product *fts*.

- *Note*: If $\{X_j\}$, $j = 1,2,\dots,n$, be a finite family of sets and if A_j be a fuzzy set in X_j. The product $A = \prod_{j=1}^{n} A_j$, $j = 1,2,\dots,n$ is the fuzzy set on $X = \prod_{j=1}^{n} X_j$ that has membership function given by

$$\mu_A(x_1, x_2, \dots, x_n) = min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \dots, \mu_{A_n}(x_n)\} \text{ for all } (x_1, x_2, \dots, x_n) \in X.$$

It also follows that if $\{X_j\}$, $j = 1,2,\dots,n$, be a finite family of sets and if X_j has fuzzy topology $\mathfrak{J}_j$, $j = 1,2,\dots,n$, the product fuzzy topology $\mathfrak{J}$ on X has a base, the set of product fuzzy sets of the form $\prod_{j=1}^{n} U_j$ where $U_j \in \mathfrak{J}_j$, $j = 1, 2, \dots, n$.

3. Fuzzy Codon Complement

Let $X = \{A, T, G, C\}$ be the set of nucleotide bases. The crisp codon with different nucleotide bases can be thought of as a set of three nucleotide bases. Then the complement of this crisp codon is a set with only one element or one base. When this complement is viewed as a characteristic function, the membership of the base contained in it has value 1.

Now, if we consider this membership to be in the interval $(0,1]$ by restricting the membership of other members in the base set X to be zero, we get a fuzzy point. Let us name it as fuzzy codon complement.

For example, $Q = ATG\ or\ \{A, T, G\}$ being a codon is a subset of $X = \{A, T, G, C\}$.

Complement of Q, $\bar{Q} = \{C\}$. For this complement we can define a characteristic function $f_{\bar{Q}}: X \to \{0,1\}$ with membership values as follows:

$$f_{\bar{Q}}(A) = 0;$$

$$f_{\bar{Q}}(G) = 0;$$

$$f_{\bar{Q}}(C) = 1;$$

$$f_{\bar{Q}}(T) = 0.$$

Then $\bar{Q}$ is a crisp point.

Now, if we consider the membership of C to be in the interval $(0,1]$, by restricting the membership of other remaining bases to be zero, then we will get a fuzzy point corresponding to codons $ATG, TAG, TGA, GAT, AGT, GTA$. For example, we can define $\bar{Q}: X \to [0,1]$ where

$$\bar{Q}(A) = 0;$$
$$\bar{Q}(G) = 0;$$
$$\bar{Q}(C) = 0.1;$$
$$\bar{Q}(T) = 0.$$

which is a fuzzy codon complement $P_C^{0.1}$.

This is a case, for example, when a base in the sequence is defective and may fit more than one fuzzy class or when a base is not identifiable with certainty or has not yet been identified [6].

The fuzzy codon complements are the only possible fuzzy points on the set of nucleotides X and so the set of all fuzzy codon complements will be denoted by $\tilde{X}$.

4. Properties of Fuzzy Codon Complements

4.1 Fuzzy codon complement as fuzzy group: Based on physio–chemical properties of DNA bases, there are two cyclic groups i) the primal and ii) the dual group, corresponding to the ordered sets $\{A, C, G, T\}$ and $\{T, G, C, A\}$ respectively [2].

Table 1 (A: Primal Algebra **B**: Dual Algebra)

A	+	A	C	G	T	**B**	+	T	G	C	A
	A	A	C	G	T		T	T	G	C	A
	C	C	G	T	A		G	G	C	A	T
	G	G	T	A	C		C	C	A	T	G
	T	T	A	C	G		A	A	T	G	C

Because we have considered the order arrangement of the primal group for defining fuzzy codon complements, we will restrict ourselves to primal algebra. Now for the case of primal group, if we consider a fuzzy set μ then $\mu(A + C) = \mu(C)$.

If $\mu(A) \geq \mu(C)$ then $min\{\mu(A), \mu(C)\} = \mu(C) = \mu(A + C)$.

If $\mu(A) \leq \mu(C)$ then $min\{\mu(A), \mu(C)\} = \mu(A) \leq \mu(C)$.

Thus, in any case, $\mu(A + C) \geq min\{\mu(A), \mu(C)\}$.

Similarly, $\mu(A + G) \geq min\{\mu(A), \mu(G)\}$ and $\mu(A + T) \geq min\{\mu(A), \mu(T)\}$.

Let the following cases be true

$$\mu(C + C) = \mu(G) \geq \mu(C) = min\{\mu(C), \mu(C)\} \tag{4.1}$$

$$\mu(G + G) = \mu(A) \geq \mu(G) = min\{\mu(G), \mu(G)\} \tag{4.2}$$

$$\mu(T + T) = \mu(G) \geq \mu(T) = min\{\mu(T), \mu(T)\} \tag{4.3}$$

$$\mu(C + G) = \mu(T) \geq \mu(C) = min\{\mu(C), \mu(G)\} \tag{4.4}$$

$$\mu(C + T) = \mu(A) \geq \mu(C) = min\{\mu(C), \mu(T)\} \tag{4.5}$$

$$\mu(G + T) = \mu(C) \geq \mu(T) = min\{\mu(G), \mu(T)\} \tag{4.6}$$

Then, from (4.1) to (4.6), we get

$$\mu(A) \geq \mu(T) \text{ and } \mu(T) = \mu(G) = \mu(C) \tag{4.7}$$

We observe from (4.7) that μ can be a fuzzy codon complement P_A^r, $r \in (0,1]$. Thus, this codon complement will form a fuzzy semigroup with respect to primal group. In order to verify for other codon complements we consider the table 2 on the next page.

From Table 2, it is observed that the codon complement P_A^r is the only fuzzy point in X which forms a fuzzy semi group in X. For this fuzzy codon complement

$$P_A^r(\ A) = P_A^r(A),$$

$$P_A^r(\ G) = P_A^r(G),$$

$$P_A^r(\ C) = P_A^r(T) = P_A^r(C),$$

$$P_A^r(\ T) = P_A^r(C) = P_A^r(T)\ .$$

Hence, the fuzzy codon complement P_A^r forms a fuzzy group.

Table 2

s	x	y	$x+y$	$P_s^r(x)$	$P_s^r(y)$	min $\{P_s^r(x), P_s^r(y)\}$	$P_s^r(x+y)$	Whether $min\{P_s^r(x), P_s^r(y)\} \leq P_s^r(x+y)$ (Y/N)
A	A	A	A	r	r	r	r	Y
	A	C	C	r	0	0	0	Y
	A	G	G	r	0	0	0	Y
	A	T	T	r	0	0	0	Y
	C	C	G	0	0	0	0	Y
	C	G	T	0	0	0	0	Y
	C	T	A	0	0	0	r	Y
	G	G	A	0	0	0	r	Y
	G	T	C	0	0	0	0	Y
	T	T	G	0	0	0	0	Y
C	A	A	A	0	0	0	0	Y
	A	C	C	0	r	0	r	Y
	A	G	G	0	0	0	0	Y
	A	T	T	0	0	0	0	Y
	C	C	G	r	r	r	0	N
	C	G	T	r	0	0	0	Y
	C	T	A	r	0	0	0	Y
	G	G	A	0	0	0	0	Y
	G	T	C	0	0	0	r	Y
	T	T	G	0	0	0	0	Y

s	x	y	$x+y$	$P_s^r(x)$	$P_s^r(y)$	$min\{P_s^r(x), P_s^r(y)\}$	$P_s^r(x+y)$	Whether $min\{P_s^r(x), P_s^r(y)\} \leq P_s^r(x+y)$ (Y/N)
G	A	A	A	0	0	0	0	Y
	A	C	C	0	0	0	0	Y
	A	G	G	0	r	0	r	Y
	A	T	T	0	0	0	0	Y
	C	C	G	0	0	0	r	Y
	C	G	T	0	r	0	0	Y
	C	T	A	0	0	0	0	Y
	G	G	A	r	r	r	0	N
	G	T	C	0	0	0	0	Y
	T	T	G	0	0	0	r	Y
T	A	A	A	0	0	0	0	Y
	A	C	C	0	0	0	0	Y
	A	G	G	0	0	0	0	Y
	A	T	T	0	r	0	r	Y
	C	C	G	0	0	0	0	Y
	C	G	T	0	0	0	r	Y
	C	T	A	0	r	0	0	Y
	G	G	A	0	0	0	0	Y
	G	T	C	0	r	0	0	Y
	T	T	G	r	r	r	0	N

* Y stands for yes and N stands for no.

4.2 Fuzzy codon complement as fuzzy ring: In order to prove that P_A^r is a fuzzy ring, it is sufficient to show that

$$min\{P_A^r(x), P_A^r(y)\} \leq P_A^r(x.y).$$

The ring structure on X is given by the following composition table [2]:

Table 3

.	A	C	G	T
A	A	A	A	A
C	A	C	G	T
G	A	G	A	G
T	A	T	G	C

Again we consider the following table:

Table 4

x	y	$x.y$	$P_A^r(x)$	$P_A^r(y)$	min $\{P_A^r(x), P_A^r(y)\}$	$P_A^r(x.y)$	Whether $min\{P_A^r(x), P_A^r(y)\} \leq P_A^r(x.y)$
A	A	A	r	r	r	r	Y
A	C	A	r	0	0	r	Y
A	G	A	r	0	0	r	Y
A	T	A	r	0	0	r	Y
C	C	C	0	0	0	0	Y
C	G	G	0	0	0	0	Y
C	T	T	0	0	0	0	Y
G	G	A	0	0	0	r	Y
G	T	G	0	0	0	0	Y
T	T	C	0	0	0	0	Y

We see from the above table that the fuzzy codon complement P_A^r also forms a fuzzy ring.

4.3 The set of all fuzzy codon complements forms a fuzzy metric space: The function $d: \tilde{X} \times \tilde{X} \to [0, \infty)$ defined as $d\big(P_x^{r_1}, P_y^{r_2}\big) = |r_1 \quad r_2|$ where $P_x^{r_1}$, $P_y^{r_2}$ are fuzzy codon complements is a fuzzy metric as

i) $d\big(P_x^{r_1}, P_y^{r_2}\big) = |r_1 \quad r_2| \geq 0$

ii) $d\big(P_x^{r_1}, P_y^{r_2}\big) + d\big(P_y^{r_2}, P_z^{r_3}\big) \geq d\big(P_x^{r_1}, P_z^{r_3}\big)$

Since $|r_1 \quad r_3| = |r_1 \quad r_2 + r_2 \quad r_3|$
$\leq |r_1 \quad r_2| + |r_2 \quad r_3|$.

iii) $d\big(P_x^{r_1}, P_y^{r_2}\big) = d\big(P_x^{r_2}, P_y^{r_1}\big)$ as $|r_1 \quad r_2| = |r_2 \quad r_1|$.

iv) $d\big(P_x^{r_1}, P_y^{r_2}\big) = d\big(P_y^{r_2}, P_x^{r_1}\big)$.

Hence, d is a fuzzy metric and so $(\tilde{X}, d)$ is a fuzzy metric space.

4.4 A fuzzy topology on the set of bases *X*: Let us consider the following sets of collection of fuzzy points on X:

$$P^0 = \{P_x^r : x \in X \text{ and fixed } r \in (0,1)\},$$

$$P' = P^0 \cup \{P_x^1 : x \in X\},$$

$$P \ = P' \cup \{P_x^0 : x \in X\}.$$

Thus, $P \ = \big\{P_x^t : x \in X \ and \ t \in \{0, r, 1\}\big\}$ where r is an arbitrarily fixed value such that $r \in (0,1)$. We see that P , which is the collection of fuzzy codon complements together with the set $\{P_x^0 : x \in X\}$ is a basis for a fuzzy topology on X because

i) For each $x \in X$, there exists atleast a fuzzy codon complement P_x^t in P such that $P_x^t(x) > 0$.

ii) Let x be such that $P_x^{t_1}(x) > 0$ and $P_x^{t_2}(x) > 0$. In any case, $P_x^{t_1} \cap P_x^{t_2} = P_x^{r_1}, (r_1 = r \ or \ 1)$ which is a member of P and $P_x^{r_1}(x) > 0$.

By considering the arbitrary union of elements of P , we find that P generates a fuzzy topology . Thus $(X,\ )$ is a fuzzy topological space.

4.5 Fuzzy codon complement P_A^r forms a fuzzy topological group: We know that X forms a group as given by composition table 1. Also $(X,\ )$ is a fuzzy topological space and the fuzzy codon complement P_A^r forms a fuzzy group. The basis for is given as $\ = P\ = \{P_x^t : x \in X \ and \ t \in \{0, r, 1\}\}$. Let $M = P_A^m$ for some fixed m, $m \in (0, r]$. Then M is a fuzzy subspace with induced fuzzy topology $_M$. Now,

$$P_A^m \cap P_x^0 = 0_X = P_A^0$$

$$P_A^m \cap P_A^r = P_A^m$$

$$P_A^m \cap P_x^r = 0_X, x \neq A$$

Also, $P_A^m \cap P_A^1 = P_A^m$ and $P_A^m \cap P_x^1 = P_A^0, x \quad A.$

Thus, the basis for this induced fuzzy topology $_M$ is

$$_M = \{P_A^0, P_A^m \text{ for fixed } m \in]0, r]\}.$$

Now, let $\alpha: (x, y) \to x + y$ be a mapping from $X \times X$ into X and $\beta : x \to \quad x$ be a mapping of X into itself. Then α and β are well defined and $\alpha[M \times M] \subset M$ and $\beta[M] \subset M$.

(a) We will first show that the mapping $\alpha : (x, y) \to x + y$ of $(M,\ _M) \times (M,\ _M)$ into $(M,\ _M)$ is relatively fuzzy continuous. We choose $P_A^{s_1}(s_1 = 0, m)$ from $_M$. We will show that $\{\alpha^{-1}(P_A^{s_1}) \cap (M \times M)\}$ is an open set in the product space $M \times M$.

We have,

$$\begin{aligned} L.H.S. &= \{\alpha^{-1}(P_A^{s_1}) \cap (M \times M)\}(x, y) \\ &= min[(P_A^{s_1})\alpha(x, y), min\{M(x), M(y)\}] \\ &= min\{P_A^{s_1}(x + y), M(x), M(y)\} \\ &= min\{P_A^{s_1}(u), M(x), M(y)\} \\ &= \begin{cases} m & \text{if } u = x = y = A, s_1 \neq 0 \\ 0 & ot \ \ erwise \end{cases} \end{aligned}$$

Again,

$$R.H.S. = \left(P_A^{s_1} \times P_A^{m}\right)(x, y)$$
$$= min\{P_A^{s_1}(x), P_A^{m}(y)\}$$
$$= \begin{cases} m & \text{if } x = y = A,\ s_1 \neq 0 \\ 0 & ot\ \ erwise \end{cases}$$

We have the following cases:

Table 5

$Cases$	x	y	$u = x + y$	s_1	$L.H.S$	$R.H.S$
I	A	A	A	m	m	m
II	A	A	A	0	0	0
III	A	C	C	m or 0	0	0
IV	A	G	G	m or 0	0	0
V	A	T	T	m or 0	0	0
VI	C	C	G	m or 0	0	0
VII	C	G	T	m or 0	0	0
$VIII$	C	T	A	m or 0	0	0
IX	G	G	A	m or 0	0	0
X	G	T	C	m or 0	0	0
XI	T	T	G	m or 0	0	0

We see that in all the cases $L.H.S. = R.H.S.$

Hence, we can take $\alpha^{-1}\left(P_A^{s_1}\right) \cap (M \times M) = P_A^{s_1} \times P_A^{m}$.

Since $P_A^{s_1}$, P_A^{m} are the open sets in M, $P_A^{s_1} \times P_A^{m}$ is a base for the product topology on $M \times M$ and so $P_A^{s_1} \times P_A^{m}$ is an open set of $M \times M$. Thus, the mapping $\alpha: (x, y) \to x + y$ of $(M, \ _M) \times (M, \ _M)$ into $(M, \ _M)$ is relatively fuzzy continuous.

(b) We will now show that the mapping $\beta: x \to \ \ x$ of $(M, \ _M)$ into $(M, \ _M)$ is relatively fuzzy continuous. Since $\beta[M] \subset M$, it is sufficient to show that the mapping $\beta: x \to \quad x$ of $(X, \quad)$ into $(X, \quad)$ is fuzzy continuous. We see that the basis for is given as

$$= \{P_x^t : x \in X \; and \; t \in \{0, r, 1\}\}.$$

Let us consider a basis element P_x^t in . Then, we have,

$$\{\beta^{-1}(P_x^t)\}(y) = P_x^t\{\beta(y)\} = P_x^t(\quad y).$$

We have the following table:

Table 6

x	y	y	$P_x^t(\quad y)$	x	y	y	$P_x^t(\quad y)$
A	A	A	t	C	A	A	0
	C	T	0		C	T	0
	T	C	0		T	C	t
	G	G	0		G	G	0
T	A	A	0	G	A	A	0
	C	T	t		C	T	0
	T	C	0		T	C	0
	G	G	0		G	G	t

We can see that $\beta^{-1}(P_x^t) \in$. Hence $\beta^{-1}(P_x^t)$ is an element in . Thus $\beta: x \to \quad x$ of $(M, \quad_M)$ into $(M, \quad_M)$ is relatively fuzzy continuous. Thus, the fuzzy codon complement M is a fuzzy topological group.

4.6 Fuzzy codon complement P_A^m forms a fuzzy topological ring: We know that X forms a ring as given by composition table 2. Also, $(X, \quad)$ is a fuzzy topological space and the fuzzy codon complement P_A^m forms a fuzzy ring. We know that $M = P_A^m$ is a fuzzy subspace with induced fuzzy topology $\quad_M$ and the basis for this induced fuzzy topology $\quad_M$ is $\quad_M = \{P_A^0, P_A^m$ for fixed $m \in (0, r]\}$.

Let $\gamma: (x, y) \to x.y$ be a mapping from $X \times X$ into X. Then γ is well defined on $X \times X$. And we have,

$$\gamma[M \times M](x) = \sup_{(z_1,z_2)\in\gamma^{-1}(x)}[M \times M](z_1, z_2)$$

$$= \sup_{(z_1,z_2)\in\gamma^{-1}(x)} min\{M(z_1), M(z_2)\}$$

$$\leq \sup_{(z_1,z_2)\in\gamma^{-1}(x)} M(z_1.z_2)$$

Thus, $\gamma[M \times M](x) \leq M(x)$ for every $x \in X$.

Hence, γ is well defined on $X \times X$ and $\gamma[M \times M] \subset M$. It is sufficient to show that the mapping $\gamma: (x, y) \rightarrow x.y$ of $(M, \ _M) \times (M, \ _M)$ into $(M, \ _M)$ is relatively fuzzy continuous. We choose $P_A^{s_1}(s_1 = 0, m)$ from $\ _M$. We will show that $\{\gamma^{-1}(P_A^{s_1}) \cap (M \times M)\}$ is an open set in the product space $M \times M$.

We have, $L.H.S. = \{\gamma^{-1}(P_A^{s_1}) \cap (M \times M)\}(x, y)$

$$= min\{\gamma^{-1}(P_A^{s_1})(x, y), (M \times M)(x, y)\}$$

$$= min[(P_A^{s_1})\gamma(x, y), min\{M(x), M(y)\}]$$

$$= min\{P_A^{s_1}(x.y), M(x), M(y)\}$$

$$= min\{P_A^{s_1}(v), M(x), M(y)\}$$

$$= \begin{cases} m & \text{if } v = x = y = A, s_1 \neq 0 \\ 0 & ot\ erwise \end{cases}$$

Again, $R.H.S. = (P_A^{s_1} \times P_A^m)(x, y) = min\{P_A^{s_1}(x), P_A^m(y)\}$

$$= \begin{cases} m & \text{if } x = y = A, s_1 \neq 0 \\ 0 & ot\ erwise \end{cases}$$

We have the following cases:

Table 7

$Cases$	x	y	$v = x.y$	s_1	$L.H.S$	$R.H.S$
I	A	A	A	m	m	m
II	A	A	A	0	0	0
III	A	C	A	$m\ or\ 0$	0	0
IV	A	G	A	$m\ or\ 0$	0	0
V	A	T	A	$m\ or\ 0$	0	0
VI	C	C	C	$m\ or\ 0$	0	0
VII	C	G	G	$m\ or\ 0$	0	0
$VIII$	C	T	T	$m\ or\ 0$	0	0
IX	G	G	A	$m\ or\ 0$	0	0
X	G	T	G	$m\ or\ 0$	0	0
XI	T	T	C	$m\ or\ 0$	0	0

We see that in all the cases, $L.H.S. = R.H.S.$

Thus, we can take

$$\gamma^{-1}\left(P_A^{s_1}\right) \cap (M \times M) = P_A^{s_1} \times P_A^{m}.$$

Hence, the mapping $\gamma: (x, y) \to x.y$ of $(M,\ {}_M) \times (M,\ {}_M)$ into $(M,\ {}_M)$ is relatively fuzzy continuous. Thus, the fuzzy codon complement M also forms a fuzzy topological ring.

5. Observation

i) We see that the only fuzzy codon complement which forms a fuzzy group is the fuzzy point P_A^r in X with support base A. In fact it is also a fuzzy ring, too. The fuzzy codon complement P_A^r, $r < 1$ even forms a fuzzy topological group as well as fuzzy topological ring.

ii) The crisp codons with different bases whose complement is the base A are the six codons as follows: $GUC,\ GCU,\ CUG,\ UCG,\ CGU,\ UGC$. We observe that

$$P = \{GUC, GCU, CUG, UCG, CGU, UGC\}$$

forms a symmetric group $(S_3, \circ)$ and each of the codons codes amino acids such that no two of the amino acids are same. In fact, the same is true if we consider the complement to be G or U.

iii) The crisp codons with different bases whose complement is the base C is an exception because the two of them are stop codons although it do form a symmetric group. This group of crisp codon $\{AGU, GUA, UAG, AUG, UGA, GAU\}$ proves to be quite interesting as it involves start codons as well as stop codons.

iv) On the other hand, the group of crisp codons $\{GUC, GCU, CUG, UCG, CGU, UGC\}$ happens to be interesting again as this is the only group where among the six different amino acids coded by the codons, three are hydrophobic and the other three are hydrophilic.

6. Conclusion

In this paper, we have defined fuzzy codon complements for the codons with different nucleotide bases and studied some of the properties they exihibit. We have defined the fuzzy codon complement as fuzzy points on the set of nucleotide bases. This set of fuzzy points even forms a fuzzy metric space when we define a suitable metric on it. By using these fuzzy points we have developed a fuzzy topology on the set of nucleotide bases. Because the fuzzy codon complement with support A forms a fuzzy group as well as fuzzy ring, it forms a fuzzy topological group as well as fuzzy topological ring. We have developed some mathematical structures on the set of nucleotide bases which may or may not have direct connection to genetics but it suggests a new way of visualizing the tiny, but vital components of molecular biology.

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SOME FIXED POINT RESULTS FOR GENERALIZED $\beta-$GERAGHTY CONTRACTION TYPE MAPS IN *S*-METRIC SPACE

Abstract: In this paper, we introduce the notion of generalized $\beta-$Geraghty contraction type maps and $\beta-$Geraghty contraction type maps in the context of S-metric spaces and establish some fixed point theorems for such maps. Our results (with some modifications) extend the fixed point results of Cho *et al*. [8] in complete *S*-metric spaces. An example is also given to illustrate our result.

Keywords: Metric Space, *S*-Metric Space, Fixed Point, Generalized $\alpha-$Geraghty Contraction Type Map, Generalized $\beta-$Geraghty Contraction Type Map.

Mathematical Subject Classification (2010) No.: 47H10, 54H25.

1. Introduction

The Banach contraction principle is one of the most important and fundamental results in fixed point theory. The study of fixed point problems is indeed a powerful tool in nonlinear analysis and the fixed point theory techniques have very useful applications in many disciplines such as Chemistry, Physics, Biology, Computer Science, Economics, Game Theory and many branches of Mathematics. Due to this, several authors have improved, generalized and extended this basic result of Banach by defining new contractive conditions and replacing the metric space by more general abstract spaces. Among such results was an interesting result by Geraghty [10] which generalized the Banach contraction principle in the setting of a complete metric space by considering an auxiliary function.Then Amini-Harandi and

Emami [2] characterized the result of Geraghty in the context of a partially ordered complete metric space. Caballero *et al*. [6] discussed the existence of a best proximity point of Geraghty contraction. Gordji et al [11] defined the notion of ψ-Geraghty type contraction and obtained results extending the results of Amini-Harandi and Emami [2]. Recently Samet *et al*. [25] defined the notion of α-ψ-contractive mappings and obtained remarkable fixed point results. Inspired by this notion of α-ψ-contractive mappings, Karapinar and Samet [16] introduced the concept of generalized α-ψ-contractive mappings and obtained fixed point results for such mappings. Very recently, Cho *et al*. [8] defined the concept of α-Geraghty contraction type maps in the setting of a metric space starting from the definition of generalized α-Geraghty contraction type maps and proved the existence and uniqueness of fixed point of such maps in the context of a complete metric space.

In this paper, motivated by the results of Cho *et al*. [8], we define generalized β-Geraghty contraction type maps and β-Geraghty contraction type maps in the setting of *S*-metric spaces and obtain the existence and uniqueness of a fixed point of such maps. Our results (with some modifications) extend the fixed point results of Cho *et al*. [8] in complete *S*-metric spaces. We also give an example to illustrate our result.

2. Preliminaries

In this section, we recall some basic definitions and related results on the topic in the literature.

Let be the family of all functions $\beta:[0,\infty)\to[0,1)$ which satisfies the condition

$$\lim_{n\to\infty}\beta(t_n)=1 \ \text{ implies } \ \lim_{n\to\infty}t_n=0.$$

By using such a map, Geraghty proved the following interesting result.

Theorem 2.1: [10] *Let (X, d) be a complete metric space and let* $T:X\to X$ *be a map. Suppose there exists* $\beta\in$ *such that for all x, y* $\in X$*,*

$$d(Tx,Ty)\le\beta(d(x,y))d(x,y).$$

Then T has a unique fixed point $x_*\in X$ *and* $\{T^n x\}$ *converges to* x_* *for each* $x\in X$.

Definition 2.2: [25] Let $T:X\to X$ be a map and $\alpha:X\times X\to\mathbb{R}$ be a function. Then *T* is said to be *α*-admissible if $\alpha(x,y)\geq 1$ implies $\alpha(Tx,Ty)\geq 1$.

Definition 2.3: [14] A map $T:X\to X$ is said to be triangular *α*-admissible if

(T1) *T* is *α*-admissible,

(T2) $\alpha(x,z)\geq 1$ and $\alpha(z,y)\geq 1$ imply $\alpha(x,y)\geq 1$.

Lemma 2.4: [14] *Let* $T:X\to X$ *be a triangular α-admissible map. Assume that there exists* $x_1\in X$ *such that* $\alpha(x_1,Tx_1)\geq 1$. *Define a sequence* $\{x_n\}$ *by* $x_{n+1}=Tx_n$. *Then we have* $\alpha(x_n,x_m)\geq 1$ *for all* $m,n\in$ *with n < m.*

Cho *et al*. [8] defined the concept of generalized *α*-Geraghty contraction type map in the setting of a metric space and proved the following Theorems 2.6., 2.7. and 2.8.

Definition 2.5: [8] Let (*X, d*) be a metric space and $\alpha:X\times X\to\mathbb{R}$ be a function. A map $T:X\to X$ is called a generalized *α*-Geraghty contraction type map if there exists $\beta\in$ such that for all $x,y\in X$,

$$\alpha(x,y)\,d(Tx,Ty)\leq\beta(M(x,y))\,M(x,y),$$

where

$$M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty)\}.$$

Theorem 2.6: [8] *Let* (X,d) *be a complete metric space,* $\alpha:X\times X\to\mathbb{R}$ *be a function and let* $T:X\to X$ *be a map. Suppose that the following conditions are satisfied:*

(1) T *is a generalised* α *-Geraghty contraction type map;*

(2) T *is triangular* α *-admissible;*

(3) *there exists* $x_1\in X$ *such that* $\alpha(x_1,Tx_1)\geq 1$;

(4) T *is continuous.*

Then T *has a fixed point* $x_*\in X$, *and* T *is a Picard operator, that is,* $\{T^n x_1\}$ *converges to* x_*.

Theorem 2.7: [8] *Let* (X,d) *be a complete metric space,* $\alpha : X \times X \to \mathbb{R}$ *be a function and let* $T : X \to X$ *be a map. Suppose that the following conditions are satisfied:*

(1) T *is a generalised* α *- Geraghty contraction type map;*

(2) T *is triangular* α *- admissible;*

(3) *there exists* $x_1 \in X$ *such that* $\alpha(x_1, Tx_1) \geq 1$;

(4) *if* $\{x_n\}$ *is a sequence in X such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x \in X$ *as* $n \to \infty$, *then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, x) \geq 1$ *for all k.*

Then T *has a fixed point* $x_* \in X$, *and* T *is a Picard operator, that is,* $\{T^n x_1\}$ *converges to* x_* .

For the uniqueness of a fixed point of a generalized α -Geraghty contraction type map , Cho et al. [8] considered the following hypothesis.

(H) For all $x, y \in Fix(T)$, there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$. Here $Fix(T)$ denotes the set of fixed points of *T.*

Theorem 2.8: [8] *Adding condition(H) to the hypotheses of Theorem* 2.6.(*resp. Theorem* 2.7.), *we obtain that* x_* *is the unique fixed point of T.*

We also recall *S*-metric space and some of its properties.

Definition 2.9: [24] Let *X* be a nonempty set. An *S*-metric on *X* is a function $S : X \times X \times X \to [0,\infty)$ that satisfies the following conditions, for each *x, y, z, a* ∈ *X*,

(S1) $S(x,y,z) \geq 0$,

(S2) $S(x,y,z) = 0$ if and only if $x = y = z$,

(S3) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair (*X, S*) is called an *S*-metric space.

Example 2.10: [24] Let X be a nonempty set and d be an ordinary metric on X. Then $S(x, y, z) = d(x, z) + d(y, z)$ is an S-metric on X.

Lemma 2.11: [24] *In an S-metric space* (X, S), *we have* $S(x, x, y) = S(y, y, x)$ *for all* $x, y \in X$.

Definition 2.12: [24] Let (X, S) be an S-metric space.

(1) A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$, that is for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \varepsilon$ and we denote this by $\lim_{n\to\infty} x_n = x$.

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if $S(x_n, x_n, x_m) \to 0$ as $n,m \to \infty$, that is for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $S(x_n, x_n, x_m) < \varepsilon$.

(3) An S-metric space (X, S) is said to be complete if every Cauchy sequence is convergent.

Lemma 2.13: [24] *Let* (X, S) *be an S-metric space. If the sequence* $\{x_n\}$ *in* X *converges to* x, *then* x *is unique.*

Lemma 2.14: [24] *Let* (X, S) *be an S-metric space. If the sequence* $\{x_n\}$ *in* X *converges to* x, *then* $\{x_n\}$ *is a Cauchy sequence.*

Lemma 2.15: [24] *Let* (X, S) *be an S-metric space. If there exist sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* X *such that*

$$\lim_{n\to\infty} x_n = x \quad \textit{and} \quad \lim_{n\to\infty} y_n = y,$$

then

$$\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y).$$

Lemma 2.16: [23] *Let* $T : X \to Y$ *be a map from an S-metric space X to an S-metric space Y. The T is continuous at* $x \in X$ *if and only if* $Tx_n \to Tx$ *whenever* $x_n \to x$.

3. Main Results

We now state and prove our main results.

We first introduce some definitions and lemmas.

Let Ω be the family of all functions $\theta:[0,\infty)\to[0,1]$ which satisfies the following conditions

(1) $\theta(t)<1$ for $t>0$, and

(2) $\lim_{n\to\infty}\theta(t_n)=1$ implies $\lim_{n\to\infty}t_n=0$.

Remark 3.1: Here instead of the family we are introducing a more refined family Ω.

Definition 3.2: Let $A:X\to X$ be a map and $\beta:X\times X\times X\to\mathbb{R}$ be a function. Then A is said to be β-admissible if $\beta(x,x,y)\geq 1$ implies $\beta(Ax,Ax,Ay)\geq 1$.

Definition 3.3: A map $A:X\to X$ is said to be triangular β-admissible if

(A1) *A* is β-admissible,

(A2) $\beta(x,x,z)\geq 1$ and $\beta(z,z,y)\geq 1$ imply $\beta(x,x,y)\geq 1$.

Lemma 3.4: *Let $A:X\to X$ be a triangular β-admissible map. Assume that there exists $x_1\in X$ such that $\beta(x_1,x_1,Ax_1)\geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1}=Ax_n$. Then we have $\beta(x_n,x_n,x_m)\geq 1$ for all $m,n\in$ with n < m.*

Proof: Since there exists $x_1\in X$ such that $\beta(x_1,x_1,Ax_1)\geq 1$, we have from (A1) $\beta(x_2,x_2,x_3)=\beta(Ax_1,Ax_1,A^2x_1)\geq 1$. By continuing this process, we get $\beta(x_n,x_n,x_{n+1})\geq 1$ for all $n\in$.

Let us suppose that $m, n \in$ with $n < m$. Since $\beta(x_n, x_n, x_{n+1}) \geq 1$ and $\beta(x_{n+1}, x_{n+1}, x_{n+2}) \geq 1$, from (A2), we get $\beta(x_n, x_n, x_{n+2}) \geq 1$. Again, since $\beta(x_n, x_n, x_{n+2}) \geq 1$ and $\beta(x_{n+2}, x_{n+2}, x_{n+3}) \geq 1$, we deduce that $\beta(x_n, x_n, x_{n+3}) \geq 1$. Continuing the process in this way, we get $\beta(x_n, x_n, x_m) \geq 1$.

Definition 3.5: Let (X, S) be an *S*-metric space and let $\beta : X \times X \times X \to \mathbb{R}$ be a function. Then a map $A : X \to X$ is called a generalized β-Geraghty contraction type map if there exists $\theta \in \Omega$ such that for all $x, y \in X$,

$$\beta(x, x, y) S(Ax, Ax, Ay) \leq \theta(N(x, y)) N(x, y)$$

where $$N(x, y) = \max\{S(x, x, y), S(x, x, Ax), S(y, y, Ay)\}.$$

Theorem 3.6: *Let (X, S) be a complete S-metric space, $\beta : X \times X \times X \to \mathbb{R}$ be a function and let $A : X \to X$ be a map. Suppose that the following conditions hold:*

(i) *A is a generalized β-Geraghty contraction type map;*

(ii) *A is triangular β-admissible;*

(iii) *there exists $x_1 \in X$ such that $\beta(x_1, x_1, Ax_1) \geq 1$;*

(iv) *A is continuous.*

Then A has a fixed point $x^* \in X$ and $\{A^n x_1\}$ converges to x^*.

Proof: Let $x_1 \in X$ be such that $\beta(x_1, x_1, Ax_1) \geq 1$. We construct a sequence of points $\{x_n\}$ in *X* such that $x_{n+1} = Ax_n$ for $n \in$. If $x_n = x_{n+1}$ for some $n \in$, then x_n is a fixed point of *A*. Therefore, we assume that $x_n \neq x_{n+1}$ for all $n \in$.

By hypothesis, $\beta(x_1, x_1, x_2) \geq 1$ and the mapping *A* is triangular β-admissible. Therefore by Lemma 3.4., we have

$$\beta(x_n, x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in$$

Then, we have

$$S\left(x_{n+1}, x_{n+1}, x_{n+2}\right) = S\left(Ax_n, Ax_n, Ax_{n+1}\right) \leq \beta\left(x_n, x_n, x_{n+1}\right) S\left(Ax_n, Ax_n, Ax_{n+1}\right)$$
$$\leq \theta\left(N\left(x_n, x_{n+1}\right)\right) N\left(x_n, x_{n+1}\right) \text{ for all } n \in \quad . \tag{1}$$

Here, we have

$$N\left(x_n, x_{n+1}\right) = \max\left\{S\left(x_n, x_n, x_{n+1}\right), S\left(x_n, x_n, Ax_n\right), S\left(x_{n+1}, x_{n+1}, Ax_{n+1}\right)\right\}$$
$$= \max\left\{S\left(x_n, x_n, x_{n+1}\right), S\left(x_{n+1}, x_{n+1}, x_{n+2}\right)\right\}.$$

If $N\left(x_n, x_{n+1}\right) = S\left(x_{n+1}, x_{n+1}, x_{n+2}\right)$, then from (1) and the definition of θ, we have

$$S\left(x_{n+1}, x_{n+1}, x_{n+2}\right) < S\left(x_{n+1}, x_{n+1}, x_{n+2}\right),$$

which is a contradiction.

Thus, we have

$$S\left(x_{n+1}, x_{n+1}, x_{n+2}\right) \leq \theta\left(N\left(x_n, x_{n+1}\right)\right) N\left(x_n, x_{n+1}\right)$$
$$\leq \theta\left(S\left(x_n, x_n, x_{n+1}\right)\right) S\left(x_n, x_n, x_{n+1}\right)$$
$$< S\left(x_n, x_n, x_{n+1}\right)$$

so that

$$S\left(x_{n+1}, x_{n+1}, x_{n+2}\right) < S\left(x_n, x_n, x_{n+1}\right) \text{ for all } n \in \quad .$$

Thus, the sequence $\left\{S\left(x_n, x_n, x_{n+1}\right)\right\}$ is nonnegative and nonincreasing.

Now, we prove that $S\left(x_n, x_n, x_{n+1}\right) \to 0$ as $n \to \infty$.

It is clear that $\left\{S\left(x_n, x_n, x_{n+1}\right)\right\}$ is a decreasing sequence which is bounded from below. Therefore there exists $r \geq 0$ such that $\lim_{n\to\infty} S\left(x_n, x_n, x_{n+1}\right) = r$. Let us assume that $r > 0$.

We have $\dfrac{S(x_{n+1},x_{n+1},x_{n+2})}{S(x_n,x_n,x_{n+1})} \le \theta(S(x_n,x_n,x_{n+1})) < 1.$

Now by taking limit $n \to \infty$, we have

$$\lim_{n\to\infty} \theta(S(x_n,x_n,x_{n+1})) = 1.$$

By the property of θ, we have

$$\lim_{n\to\infty} S(x_n,x_n,x_{n+1}) = 0 = r \text{, which is a contradiction.}$$

Therefore we have $\lim_{n\to\infty} S(x_n,x_n,x_{n+1}) = r = 0$. (2)

Now we show that the sequence $\{x_n\}$ is a Cauchy sequence. Let us suppose on the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ such that, for all positive integers k, there exist $m_k > n_k > k$ with

$$S(x_{m_k},x_{m_k},x_{n_k}) \ge \varepsilon. \tag{3}$$

Let m_k be the smallest number satisfying the conditions above. Then we have

$$S(x_{m_k-1},x_{m_k-1},x_{n_k}) < \varepsilon. \tag{4}$$

By (3), (4) and (S3), we have

$$\begin{aligned}\varepsilon &\le S(x_{m_k},x_{m_k},x_{n_k})\\ &\le 2S(x_{m_k},x_{m_k},x_{m_k-1}) + S(x_{m_k-1},x_{m_k-1},x_{n_k})\\ &< 2S(x_{m_k-1},x_{m_k-1},x_{m_k}) + \varepsilon\end{aligned}$$

that is, $\varepsilon \le S(x_{m_k},x_{m_k},x_{n_k}) < \varepsilon + 2S(x_{m_k-1},x_{m_k-1},x_{m_k})$ for all $k \in$. (5)

Then in view of (2) and (5), we have

$$\lim_{k\to\infty} S\left(x_{m_k}, x_{m_k}, x_{n_k}\right) = \varepsilon . \tag{6}$$

Again using (S3), we have

$$S\left(x_{m_k}, x_{m_k}, x_{n_k}\right) \le 2S\left(x_{m_k}, x_{m_k}, x_{m_k-1}\right) + S\left(x_{n_k}, x_{n_k}, x_{m_k-1}\right)$$

$$\le 2S\left(x_{m_k}, x_{m_k}, x_{m_k-1}\right) + 2S\left(x_{n_k}, x_{n_k}, x_{n_k-1}\right) + S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}\right)$$

and

$$S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}\right) \le 2S\left(x_{m_k-1}, x_{m_k-1}, x_{m_k}\right) + 2S\left(x_{n_k-1}, x_{n_k-1}, x_{n_k}\right) + S\left(x_{m_k}, x_{m_k}, x_{n_k}\right).$$

Taking limit as $k \to \infty$ and using (2) and (6), we obtain

$$\lim_{k\to\infty} S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}\right) = \varepsilon.$$

By Lemma 3.4., we get $\beta\left(x_{n_k-1}, x_{n_k-1}, x_{m_k-1}\right) \ge 1$. Therefore, we have

$$\begin{aligned} S\left(x_{m_k}, x_{m_k}, x_{n_k}\right) &= S\left(Ax_{m_k-1}, Ax_{m_k-1}, Ax_{n_k-1}\right) \\ &\le \beta\left(x_{n_k-1}, x_{n_k-1}, x_{m_k-1}\right) S\left(Ax_{n_k-1}, Ax_{n_k-1}, Ax_{m_k-1}\right) \\ &\le \theta\left(N\left(x_{n_k-1}, x_{m_k-1}\right)\right) N\left(x_{n_k-1}, x_{m_k-1}\right). \end{aligned}$$

Here we have

$$\begin{aligned} N\left(x_{n_k-1}, x_{m_k-1}\right) &= \max\left\{S\left(x_{n_k-1}, x_{n_k-1}, x_{m_k-1}\right), S\left(x_{n_k-1}, x_{n_k-1}, Ax_{n_k-1}\right), S\left(x_{m_k-1}, x_{m_k-1}, Ax_{m_k-1}\right)\right\} \\ &= \max\left\{S\left(x_{n_k-1}, x_{n_k-1}, x_{m_k-1}\right), S\left(x_{n_k-1}, x_{n_k-1}, x_{n_k}\right), S\left(x_{m_k-1}, x_{m_k-1}, x_{m_k}\right)\right\} \end{aligned}$$

and we see that

$$\lim_{k\to\infty} N\left(x_{n_k-1}, x_{m_k-1}\right) = \varepsilon. \tag{7}$$

Now we have

$$\frac{S\left(x_{n_k}, x_{n_k}, x_{m_k}\right)}{N\left(x_{n_k-1}, x_{m_k-1}\right)} \le \theta\left(N\left(x_{n_k-1}, x_{m_k-1}\right)\right) < 1.$$

Then taking limit as $k \to \infty$ in the above inequality and using (6) and (7), we obtain

$$\lim_{k\to\infty} \theta\left(N\left(x_{n_k-1}, x_{m_k-1}\right)\right) = 1.$$

So, $\lim_{k\to\infty} N\left(x_{n_k-1}, x_{m_k-1}\right) = 0 = \varepsilon$, which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$. As A is continuous, we have $Ax_n \to Ax^*$ i.e. $\lim_{n\to\infty} x_{n+1} = Ax^*$ and so $x^* = Ax^*$. Hence, x^* is a fixed point of A.

In the following Theorem, we replace the continuity of A by a suitable condition.

Theorem 3.7: *Let* (X, S) *be a complete S-metric space,* $\beta : X \times X \times X \to \mathbb{R}$ *be a function and let* $A : X \to X$ *be a map. Suppose that the following conditions hold*:

(i) A *is a generalised* β*-Geraghty contraction type map;*

(ii) A *is triangular* β*-admissible;*

(iii) *there exists* $x_1 \in X$ *such that* $\beta(x_1, x_1, Ax_1) \geq 1$;

(iv) *if* $\{x_n\}$ *is a sequence in* X *such that* $\beta(x_n, x_n, x_{n+1}) \geq 1$ *for all* $n \in$ *and* $x_n \to x \in X$ *as* $n \to \infty$, *then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\beta(x_{n_k}, x_{n_k}, x) \geq 1$ *for all k.*

Then A has a fixed point $x^* \in X$ *and* $\{A^n x_1\}$ *converges to* x^*.

Proof: The proof goes along similar lines of the proof of Theorem 3.6. We conclude that the sequence $\{x_n\}$ defined by $x_{n+1} = Ax_n$ for all $n \in$, converges to a point say $x^* \in X$. By the hypothesis (iv) there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\beta(x_{n_k}, x_{n_k}, x^*) \geq 1$ for all k. Now for all k, we have

$$S\left(x_{n_k+1}, x_{n_k+1}, Ax^*\right) = S\left(Ax_{n_k}, Ax_{n_k}, Ax^*\right)$$
$$\leq \beta\left(x_{n_k}, x_{n_k}, x^*\right) S\left(Ax_{n_k}, Ax_{n_k}, Ax^*\right)$$
$$\leq \theta\left(N\left(x_{n_k}, x^*\right)\right) N\left(x_{n_k}, x^*\right)$$

so that

$$S\left(x_{n_k+1}, x_{n_k+1}, Ax^*\right) \leq \theta\left(N\left(x_{n_k}, x^*\right)\right) N\left(x_{n_k}, x^*\right)$$

On the other hand, we have

$$N\left(x_{n_k}, x^*\right) = \max\left\{S\left(x_{n_k}, x_{n_k}, x^*\right), S\left(x_{n_k}, x_{n_k}, Ax_{n_k}\right), S\left(x^*, x^*, Ax^*\right)\right\}$$
$$= \max\left\{S\left(x_{n_k}, x_{n_k}, x^*\right), S\left(x_{n_k}, x_{n_k}, x_{n_k+1}\right), S\left(x^*, x^*, Ax^*\right)\right\}$$

We suppose that $x^* \neq Ax^*$ so that $S(x^*, x^*, Ax^*) > 0$.Taking limit $k \to \infty$ in the above equality, we get

$$\lim_{k\to\infty} N\left(x_{n_k}, x^*\right) = S\left(x^*, x^*, Ax^*\right).$$

Now we have

$$\frac{S\left(x_{n_k+1}, x_{n_k+1}, Ax^*\right)}{N\left(x_{n_k}, x^*\right)} \leq \theta\left(N\left(x_{n_k}, x^*\right)\right) < 1.$$

And taking limit $k \to \infty$, we get

$\lim_{k\to\infty} \theta\left(N\left(x_{n_k}, x^*\right)\right) = 1$ which implies that $\lim_{k\to\infty} N\left(x_{n_k}, x^*\right) = 0$ i.e.

$S\left(x^*, x^*, Ax^*\right) = 0.$

This is a contradiction. Therefore we must have $x^* = Ax^*$.

For the uniqueness of a fixed point of a generalized β-Geraghty contraction type map, we consider the following hypothesis: (G) For any two fixed points x and y of A, there exists $z \in X$ such that $\beta(x, x, z) \geq 1$, $\beta(y, y, z) \geq 1$ and $\beta(z, z, Az) \geq 1$.

Here we are using a condition stronger than a condition analogous to the condition (H) of Cho *et al*. [8] because we observe that such a condition is not enough.

Theorem 3.8: *Adding condition (G) to the hypotheses of Theorem* 3.6. (*or Theorem* 3.7.), *we obtain that* x^* *is the unique fixed point of A.*

Proof: Due to Theorem 3.6. (or Theorem 3.7.), we obtain that $x^* \in X$ is a fixed point of A. Let $y^* \in X$ be another fixed point of A. Then by hypothesis (G), there exists $z \in X$ such that $\beta(x^*, x^*, z) \geq 1$, $\beta(y^*, y^*, z) \geq 1$ and $\beta(z, z, Az) \geq 1$.

Since A is β - admissible we get

$$\beta(x^*, x^*, A^n z) \geq 1 \text{ and } \beta(y^*, y^*, A^n z) \geq 1 \text{ for all } n \in \quad .$$

Then we have

$$\begin{aligned} S(x^*, x^*, A^{n+1}z) &\leq \beta(x^*, x^*, A^n z) S(Ax^*, Ax^*, AA^n z) \\ &\leq \theta(N(x^*, A^n z)) N(x^*, A^n z), \ \forall n \in N. \end{aligned}$$

And we have

$$\begin{aligned} N(x^*, A^n z) &= \max\{S(x^*, x^*, A^n z), S(x^*, x^*, Ax^*), S(A^n z, A^n z, AA^n z)\} \\ &= \max\{S(x^*, x^*, A^n z), S(x^*, x^*, x^*), S(A^n z, A^n z, A^{n+1} z)\} \\ &= \max\{S(x^*, x^*, A^n z), S(A^n z, A^n z, A^{n+1} z)\}. \end{aligned}$$

By Theorem 3.6. (or Theorem 3.7.) we deduce that the sequence $\{A^n z\}$ converges to a fixed point $z^* \in X$. Then taking limit $n \to \infty$ in the above equality,we get $\lim_{n\to\infty} N(x^*, A^n z) = S(x^*, x^*, z^*)$. Let us suppose that $z^* \neq x^*$. Then we have

$$\frac{S(x^*, x^*, A^{n+1}z)}{N(x^*, A^n z)} \leq \theta(N(x^*, A^n z)) < 1.$$

And taking limit $n \to \infty$, we get $\lim_{n\to\infty} \theta\left(N\left(x^*, A^n z\right)\right) = 1$. This implies $\lim_{n\to\infty} N\left(x^*, A^n z\right) = 0$ i.e. $S\left(x^*, x^*, z^*\right) = 0$, which is a contradiction. Therefore we must have $z^* = x^*$. Similarly, we get $z^* = y^*$. Thus, we have $y^* = x^*$. Hence x^* is the unique fixed point of *A*.

4. Consequences

We start this section with the following definition.

Definition 4.1: Let (X, S) be an *S*-metric space and let $\beta : X \times X \times X \to \mathbb{R}$ be a function. Then a map $A : X \to X$ is called a β-Geraghty contraction type map if there exists $\theta \in \Omega$ such that for all $x, y \in X$,

$$\beta(x, x, y) S(Ax, Ax, Ay) \leq \theta\left(S(x, x, y)\right) S(x, x, y).$$

Theorem 4.2. *Let* (X, S) *be a complete S-metric space,* $\beta : X \times X \times X \to \mathbb{R}$ *be a function and let* $A : X \to X$ *be a map. Suppose that the following conditions hold*:

(i) *A is a* β*-Geraghty contraction type map*

(ii) *A is triangular* β*-admissible*

(iii) *there exists* $x_1 \in X$ *such that* $\beta(x_1, x_1, Ax_1) \geq 1$

(iv) *A is continuous.*

Then A has a fixed point $x^* \in X$ *and* $\{A^n x_1\}$ *converges to* x^*.

Theorem 4.3: *Let* (X, S) *be a complete S-metric space,* $\beta : X \times X \times X \to \mathbb{R}$ *be a function and let* $A : X \to X$ *be a map. Suppose that the following conditions hold*:

(i) *A is a* β*-Geraghty contraction type map*

(ii) *A is triangular* β*-admissible*

(iii) *there exists* $x_1 \in X$ *such that* $\beta(x_1, x_1, Ax_1) \geq 1$

(iv) *if* $\{x_n\}$ *is a sequence in X such that* $\beta(x_n, x_n, x_{n+1}) \geq 1$ *for all* $n \in N$ *and* $x_n \to x \in X$ *as* $n \to \infty$, *then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\beta(x_{n_k}, x_{n_k}, x) \geq 1$ *for all* k.

Then A has a fixed point $x^* \in X$ *and* $\{A^n x_1\}$ *converges to* x^*.

For the uniqueness of a fixed point of a β-Geraghty contraction type map, we consider the following hypothesis (G1) which is weaker than hypothesis (G) : (G1) For any two fixed points *x* and *y* of *A*, there exists $z \in X$ such that $\beta(x, x, z) \geq 1$ and $\beta(y, y, z) \geq 1$.

Theorem 4.4: *If we add condition (G1) to the hypotheses of Theorem* 4.2. (*or Theorem* 4.3.), *we obtain that* x^* *is the unique fixed point of A.*

Proof: Due to Theorem 4.2. (or Theorem 4.3.), we obtain that $x^* \in X$ is a fixed point of *A*. Let $y^* \in X$ be another fixed point of *A*. Then by hypothesis (G1), there exists $z \in X$ such that $\beta(x^*, x^*, z) \geq 1$ and $\beta(y^*, y^*, z) \geq 1$.

Since *A* is β- admissible we get

$$\beta(x^*, x^*, A^n z) \geq 1 \text{ and } \beta(y^*, y^*, A^n z) \geq 1 \text{ for all } n \in \; .$$

Then we have

$$\begin{aligned} S(x^*, x^*, A^{n+1} z) &\leq \beta(x^*, x^*, A^n z) S(Ax^*, Ax^*, AA^n z) \\ &\leq \theta\big(S(x^*, x^*, A^n z)\big) S(x^*, x^*, A^n z) \\ &< S(x^*, x^*, A^n z) \text{ for all } n \in \; . \end{aligned}$$

Thus, the sequence $\{S(x^*, x^*, A^n z)\}$ is nonnegative and non-increasing. Therefore there exists $r \geq 0$ such that $\lim_{n\to\infty} S(x^*, x^*, A^n z) = r$. We show that $r = 0$. We suppose on the contrary that $r > 0$.

We have $\dfrac{S\left(x^*, x^*, A^{n+1}z\right)}{S\left(x^*, x^*, A^n z\right)} \le \theta\left(S\left(x^*, x^*, A^n z\right)\right) < 1.$

Now by taking limit $n \to \infty$, we have

$$\lim_{n\to\infty} \theta\left(S\left(x^*, x^*, A^n z\right)\right) = 1.$$

By the property of θ, we have

$$\lim_{n\to\infty} S\left(x^*, x^*, A^n z\right) = 0 \text{ , which is a contradiction.}$$

Therefore we have $r = 0$.

And this implies that $\lim_{n\to\infty} A^n z = x^*$.

Similarly, we get $\lim_{n\to\infty} A^n z = y^*$. Hence we have $x^* = y^*$.

Here we give an example to illustrate Theorem 4.3.

Example 4.5: Let $X = [0, \infty)$ and let $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in X$. Then (*X*, *S*) is a complete *S*-metric space. And let $\theta(t) = \dfrac{1}{1+t}$ for all $t \ge 0$. Then $\theta \in \Omega$. Let a mapping $A : X \to X$ be defined by

$$Ax = \begin{cases} \dfrac{x}{5} & (0 \le x \le 1), \\ 5x & (x > 1). \end{cases}$$

And let a function $\beta : X \times X \times X \to \mathbb{R}$ be defined by

$$\beta(x, y, z) = \begin{cases} 1 & (0 \le x, y, z \le 1) \\ 0 & \textit{otherwise.} \end{cases}$$

Condition (iii) of Theorem 4.3. is satisfied with $x_1 = 1$. And if $\{x_n\}$ be a sequence in X such that $\beta(x_n, x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then we must have $x \in [0,1]$. Therefore, by definition of the function β we must have $\beta(x_n, x_n, x) \geq 1$. Hence, the condition (iv) of Theorem 4.3. is satisfied.

Let $x, y \in X$ such that $\beta(x, x, y) \geq 1$.

Then we have $x, y \in [0,1]$ and so $Ax \in [0,1]$, $Ay \in [0,1]$ and therefore

$$\beta(Ax, Ax, Ay) = 1.$$

Further if $\beta(x, x, z) \geq 1$ and $\beta(z, z, y) \geq 1$, then $x, y, z \in [0,1]$. Therefore

$$\beta(x, x, y) \geq 1.$$

Hence A is triangular β-admissible and so condition (ii) of Theorem 4.3. is satisfied.

We finally show that condition (i) of Theorem 4.3. is satisfied.

If $0 \leq x, y \leq 1$, then $\beta(x, x, y) = 1$ and we have

$$\begin{aligned} \theta(S(x,x,y))S(x,x,y) - \beta(x,x,y)S(Ax,Ax,Ay) &= \frac{S(x,x,y)}{1+S(x,x,y)} - S\left(\frac{x}{5},\frac{x}{5},\frac{y}{5}\right) \\ &= \frac{2|x-y|}{1+2|x-y|} - 2\left|\frac{x}{5}-\frac{y}{5}\right| \\ &= \frac{|x-y|(8-4|x-y|)}{5(1+2|x-y|)} \geq 0. \end{aligned}$$

Therefore we have

$$\beta(x,x,y)S(Ax,Ax,Ay) \leq \theta(S(x,x,y))S(x,x,y) \text{ for } 0 \leq x, y \leq 1.$$

And if $0 \le x \le 1,\ y > 1$ or $0 \le y \le 1,\ x > 1$ or $x > 1,\ y > 1$, then $\beta(x,x,y) = 0$ and we have $\beta(x,x,y) S(Ax,Ax,Ay) \le \theta(S(x,x,y)) S(x,x,y)$.

Thus, all the conditions of Theorem 4.3. are satisfied and A has a unique fixed point $x^* = 0$.

We also note that if $X = (0,1]$, then X is not complete and A does not have a fixed point in X.

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SOLUTION OF LINEAR AND NON-LINEAR ORDINARY DIFFERENTIAL EQUATIONS USING NATURAL DECOMPOSITION METHOD

Abstract: In the present paper, we present a new method called Natural Decomposition Method. By using this method we solve linear and non-linear differential equations and we also study the properties of Natural Transform. This method gives exact solutions in the form of a rapid convergence series. The Natural Decomposition Method (NDM) is an excellent method to solve non-linear differential equations especially initial and boundary value problems.

Keywords: Natural Transform, Sumudu Transform, Laplace Transform, Non-linear Differential Equations.

Mathematical Subject Classification No.: 35Q61, 44A10, 44A15, 44A20, 44A30, 44A35, 81V10.

1. Introduction

Linear and Non-linear differential equations played important role in pure and applied mathematics. There have been many integral transform methods [5], [6], [7], [8], [9] and [10] exists in the literature for solving PDEs, ODEs and integral equations. The most used one is the Laplace transform [13]. The most recent methods used to solve ODEs and PDEs are the Sumudu transform [1] and Elzaki transform [5], [6], [7], [8], [9] and [10]. Fethi Belgacem and R. Silambarasan [2] used the N-Transform to solve the Maxwell's equations. Zafar H. Khan and Waqar A. Khan [11] used the N-Transform to solve linear differential equations and they presented a table with some properties of the N-Transform of different functions.

In this paper, we use Natural Decomposition Method and apply it to find exact solution of linear and nonlinear ordinary differential equations. We solve some linear and nonlinear ordinary differential equations using Natural Decomposition Method.

2. Definition of N-Transform and its Properties

2.1 N-Transform: The natural transform of the function $f(t)$ for $t \in (\ \ \infty, \infty)$ is defined by

$$N[f(t)] = R(p,v)\int_{-\infty}^{\infty} e^{-pt} f(vt)\,dt\,;p,v(\ \ \infty,\infty),$$

Here $N[f(t)]$ is the natural transformation of the time function $f(t)$ and the variables $p\ and\ v$ are the N-Transform variables. Note that the above equation can be written in the form:

$$N[f(t)] = \int_{-\infty}^{\infty} e^{-pt} f(vt)\,dt\,;p,v \in (\ \ \infty,\infty),$$

$$= \left[\int_{-\infty}^{0} e^{-pt} f(vt)\,dt\,;p,v \in (\ \ \infty,0)\right] + \left[\int_{0}^{\infty} e^{-pt} f(vt)\,dt\,;p,v \in (0,\infty)\right]$$

$$= N^{-}[f(t)] + N^{+}[f(t)]$$

$$= N[f(t)H(\ \ t)] + N[f(t)H(t)]$$

$$= R^{-}(p,v) + R^{+}(p,v)$$

Here $H(.)$ is the Heaviside function.

If the function $f(t)H(t)$ is defined on the positive real axis, with $t \in R$, then we define the Natural transform (N-Transform) on the set

$$A = \left\{\begin{matrix} f(t)\colon \exists, T_1, T_2 > 0\,, suc\ \ t\ \ at\ |f(t)| < Me^{\frac{|t|}{T_j}}, \\ if\ t \in (\ \ 1)^j \times [0,\infty), j \in Z^+ \end{matrix}\right\}$$

$$N[f(t)H(t)] = N^{+}[f(t)] = R^{+}(p,v) = \int_{0}^{\infty} e^{-pt} f(vt)\,dt\,;p,v \in (0,\infty)$$

Here $H(.)$ is the Heaviside function.

2.2 Properties of N-Transform (2.2.1) ***Linearity property*****:** If m and n are any two constants and $f(t)$ and $g(t)$ are functions, then

$$N\{mf(t) + ng(t)\} = mN\{f(t)\} + nN\{g(t)\}$$

2.2.2) ***Change of scale property*****:**

If $N\{f(t)\} = E(p, v)$ then $N\{f(at)\} = \frac{1}{a}E(p, v)$

2.2.3) ***N-Transform of derivatives*****:**

If $N\{f(t)\} = E(p, v)$ then $N\{f'(t)\} = \frac{p}{v}E(p, v) \quad \frac{f(0)}{v}$

2.2.4) ***N-Transform of integrals*****:**

If $N\{f(t)\} = R(p, v)$ then $N\left\{\int_0^t f(x)dx\right\} = \frac{v}{p}R(p, v)$

3. Table of conversation of N-Transform to Laplace and Sumudu Transforms

S. No.	$f(t)$	$N[f(t)]$	$S[f(t)]$	$L[f(t)]$
1	1	$\frac{1}{p}$	1	$\frac{1}{p}$
2	t	$\frac{v}{p^2}$	v	$\frac{1}{p^2}$
3	e^{at}	$\frac{1}{p-av}$	$\frac{1}{1-av}$	$\frac{1}{p-a}$
4	$\frac{1}{\omega}\sin(\omega t)$	$\frac{u}{p^2+\omega^2v^2}$	$\frac{u}{1+\omega^2v^2}$	$\frac{1}{p^2+\omega^2}$
5	$\cos \omega t$	$\frac{p}{p^2+\omega^2v^2}$	$\frac{1}{1+\omega^2v^2}$	$\frac{p}{p^2+\omega^2}$
6	$\cosh t$	$\frac{p}{p^2-v^2}$	$\frac{1}{1-v^2}$	$\frac{p}{p \quad -1}$
7	$\frac{t^{n-1}}{(n-1)!}, n = 1, 2$	$v^{(n-1)}p^{(-n)}$	$v^{(n-1)}$	$p^{(-n)}$
8	$\frac{t^{n-1}}{\Gamma(n)}, n > 0$	$u^{(n-1)}s^{(-n)}$	$u^{(n-1)}$	$s^{(-n)}$
9	$\cos t$	$\frac{p}{p^2+v^2}$	$\frac{1}{1+v^2}$	$\frac{p}{p^2+1}$
10	$\sin t$	$\frac{v}{p^2+v^2}$	$\frac{v}{1+v^2}$	$\frac{1}{p^2+1}$

4. Natural Decomposition Method

Taking the general nonlinear ordinary differential equation of the form:

$$Jv + R(v) + F(v) = g(t), \tag{1}$$

Subject to the initial condition

$$m(0) = \quad (t\,), \tag{2}$$

Here J an operator of the highest derivative is, R is the remainder of the differential operator, $g(t)$ is the homogeneous term and $f(v)$ is the nonlinear term.

Suppose J is a differential operator of the first order, then by taking the N-Transform of the equation (1), we obtain:

$$\frac{sV(p,v)}{v} \quad \frac{V(0)}{v} + N^{+}[R(m)] + N^{+}[F(m)] = N^{+}[g(t)]. \tag{3}$$

By substituting equation (2) into equation (3), then we get:

$$V(p,v) = \frac{h(t)}{p} + \frac{v}{p}N^{+}[g(t)] \quad \frac{v}{p}N^{+}[R(v) + F(v)]. \tag{4}$$

Taking the inverse of the N-Transform of equation (4), we have:

$$m(t) = G(t) \quad N^{-}\left[\frac{v}{p}N^{+}[R(m) + F(m)]\right], \tag{5}$$

Here G (t) is the inventor term.

Now we consider that an infinite series solution of the unknown function $m(t)$ of the form:

$$m(t) = \sum_{n=0}^{\infty} m_n(t). \tag{6}$$

Putting equation (6) in equation (5) then we obtain:

$$\sum_{n=0}^{\infty} m_n(t) = G(t) \quad N^{-1}\left[\frac{v}{p}N^{+}\left[R\sum_{n=0}^{\infty} m_n(t) + \sum_{n=0}^{\infty} A_n(t)\right]\right], \tag{7}$$

Here $A_n(t)$ is an Adomain polynomial which represent the nonlinear term.

Comparing both sides of equation (7), we can easily build the recursive relation as follows:

$$v_0(t) = G(t),$$

$$m_1(t) = \quad N^{-1}\left[\frac{v}{p}N^{+}[Rm_0(t) + A_0(t)]\right],$$

$$m_2(t) = \quad N^{-1}\left[\frac{v}{p}N^{+}[Rm_1(t) + A_1(t)]\right],$$

$$m_3(t) = \quad N^{-1}\left[\frac{v}{p}N^{+}[Rm_2(t) + A_2(t)]\right],$$

Finally, we have the general recursive relation as follows:

$$m_{n+1}(t) = \quad N^{-1}\left[\frac{v}{p}N^{+}[Rm_n(t) + A_n(t)]\right], n \geq 0 \tag{8}$$

Hence, the exact or approximate solution is given by:

$$m(t) = \sum_{n=0}^{\infty} m_n(t). \tag{9}$$

5. Solutions of Some Ordinary Differential Equations

Example 5.1: Solve the first order nonlinear differential equation of the form;

$$\frac{d^2v}{dt^2} + \left(\frac{dv}{dx}\right)^2 + v^2(t) = 1 \quad \cos(t), \tag{2.1}$$

Subject to the initial conditions $v(0) = 1,\ v'(0) = 0.$ (2.2)

Solution: Now we precede the N-Transform to both sides of equation (2.1), we acquire:

$$\frac{p^2V(p,u)}{u^2} \quad \frac{pV(0)}{u^2} \quad \frac{v'(0)}{u} + N^{+}\left[\left(\frac{dv}{dt}\right)^2\right] + N^{+}[v^2(t)] = \frac{1}{p} \quad \frac{p}{p^2 + u^2} \tag{2.3}$$

By exchange equation (2.2) into equation (2.3) we acquire:

$$V(p,u) = \frac{u^2}{p^2} \quad \frac{s}{p^2+u^2} \quad \frac{u^2}{p^2} N^+ \left[\left(\frac{dv}{dt} \right)^2 + v^2(t) \right] \tag{2.4}$$

Now by taking the inverse N-Transform of the equation (2.4), we acquire

$$v(t) = \frac{t^2}{2!} + \cos(t) \quad N^- \left[\frac{u^2}{p^2} N^+ \left[\left(\frac{dv}{dt} \right)^2 + v^2(t) \right] \right] \tag{2.5}$$

We now consider an infinite series solution of the unknown function $v(t)$ of the unknown function $v(t)$ of the form:

$$v(t) = \sum_{n=0}^{\infty} v_n(t) \tag{2.6}$$

By using equation (2.6), we can re-write equation (2.5) as acquire

$$\sum_{n=0}^{\infty} v_n(t) = \frac{t^2}{2!} + \sin(t) \quad N^{-1} \left[\frac{u^2}{p^2} N^+ \left[\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n \right] \right], \tag{2.7}$$

Where A_n and $B_n$are the Adomain polynomials of this nonlinear terms $\left(\frac{dv}{dt}\right)^2$ and $v^2(t)$ accordingly.

Comparing both sides of equation (2.1) we can drive the general reclusive relation as follows:

$$v_0(t) = \frac{t^2}{2!} + \cosh t$$

$$v_1(t) = \quad N^{-1} \left[\frac{u^2}{p^2} N^+ [A_0 + B_0] \right],$$

$$v_2(t) = \quad N^{-1} \left[\frac{u^2}{p^2} N^+ [A_1 + B_1] \right],$$

$$v_3(t) = \quad N^{-1} \left[\frac{u^2}{p^2} N^+ [A_2 + B_2] \right],$$

Therefore, the general recursive relation is given by:

$$v_{n+1}(t) = \quad N^{-1}\left[\frac{u^2}{p^2}N^+[A_n + B_n]\right], \tag{2.8}$$

Then by using the recursive relation derived in equation (2.8), we can compute the remaining components of the unknown function $v(t)$ as follows:

$$v_1(t) = \quad N^{-1}\left[\frac{u^2}{p^2}N^+[A_0 + B_0]\right],$$

$$= \quad N^{-1}\left[\frac{u^2}{p^2}N^+[(v'_0)^2 + (v_0)^2]\right],$$

$$= \quad N^{-1}\left[\frac{u^2}{p^2}N^+[1]\right] +$$

$$= \quad N^{-1}\left[\frac{u^2}{p^2}\right] +$$

$$= \frac{t^2}{2!} +$$

Hence, by the cancellation of the extra terms that appears between $v_0(t)$ and$v_1(t)$, we can see that the non-cancelled term of $v_0(t)$ still satisfies the given differential equation which obtains to an exact solution of the form: $v(t) = \cos(t)$.

Example 5.2: Solve the first order nonlinear differential equation of the form;

$$\frac{d^2v}{dt^2} + \left(\frac{dv}{dx}\right)^2 + v^2(t) = t + \cosh t, \tag{2.9}$$

Subject to the initial conditions

$$v(0) = 1, \qquad v^{'}(0) = 0. \tag{2.10}$$

Solution: Proceeding as in the above manner (Example 5.1), the equation leads to an exact solution of the form: $v(t) = \cosh t$.

7. Conclusion

In this research paper, the Natural decomposition Method was used to solve non-linear ordinary differential equations. This method is an excellent method to obtain the exact solutions of nonlinear differential equations. Using this method, we can find the exact solutions of initial and boundary value problems in the field of science and engineering.

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